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PSEUDOCCLASSICAL DESCRIPTION OF WEYL PARTICLES

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A B S T R A C T

We present a pseudoclassical description of a massless spinning particle belonging to the representation $(\frac{1}{2}, 0)$ of $SL(2, \mathbb{C})$. We discuss the path integral quantization of the theory and the coupling with Yang-Mills and gravitational fields.

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In this letter we present a "pseudoclassical" ¹⁾ description of a spin $\frac{1}{2}$ massless particle. Whereas a massless Dirac particle has been already described in the literature ²⁾, such a description has not been given for a Weyl particle, that is, for a particle belonging to the $(\frac{1}{2}, 0)$ representation of $SL(2, C)$.

A Weyl particle is described by the following wave equation

$$\hat{p} (1 - \gamma_5) \psi = 0 \quad (1)$$

The wave equation corresponds to a constraint in the pseudoclassical description. A convenient form for such a constraint is

$$p_\mu \xi^\mu + \frac{i}{3} \epsilon^{\mu\nu\rho\sigma} p_\mu \xi_\nu \xi_\rho \xi_\sigma \approx 0 \quad (2)$$

where the ξ_μ are Grassmann variables. Equation (2), plus the mass-shell condition $p^2 = 0$, can be derived as first class constraints from the following Lagrangian

$$L(\xi) = -\frac{i}{2} \xi_\mu \dot{\xi}^\mu + \frac{1}{4\lambda_1} \dot{x}^2 + \frac{\lambda_2}{2\lambda_1} \dot{x}_\mu \left(\xi^\mu + \frac{i}{3} \epsilon^{\mu\nu\rho\sigma} \xi_\nu \xi_\rho \xi_\sigma \right) \quad (3)$$

Here λ_1 and λ_2 are Lagrangian multipliers, with λ_2 being an odd quantity, and the conjugate momentum p^μ is given by

$$p^\mu = -\frac{\partial L}{\partial \dot{x}_\mu} = -\frac{1}{2\lambda_1} \left[\dot{x}^\mu + \lambda_2 \left(\xi^\mu + \frac{i}{3} \epsilon^{\mu\nu\rho\sigma} \xi_\nu \xi_\rho \xi_\sigma \right) \right] \quad (4)$$

Furthermore, we get the following second class constraints

$$\chi^\mu = \pi^\mu - \frac{i}{2} \xi^\mu \approx 0 \quad (5)$$

$$\pi^\mu = \frac{\partial L}{\partial \dot{\xi}^\mu}$$

Defining in the usual way the Dirac brackets with respect to Eq. (5), we obtain

$$\{\xi^\mu, \xi^\nu\}^* = i g^{\mu\nu}$$

$$\{x^\mu, p^\nu\}^* = -g^{\mu\nu} \quad (6)$$

The quantization of the theory gives ¹⁾

$$[\xi^\mu, \xi^\nu]_+ = -g^{\mu\nu} \quad (7)$$

It follows that ξ^μ can be represented as

$$\xi^\mu = \frac{i}{\sqrt{2}} \gamma^\mu \quad (8)$$

By inserting this expression in Eq. (2), we get the condition (1) on the physical states.

The descriptions of a Weyl particle and of a massless Dirac particle are closely related. To be more precise, it is easy to show that the two theories coincide at the classical level. Indeed, performing the following canonical transformation in the Lagrangian (3)

$$\xi^\mu = \theta^\mu - \frac{i}{3} \epsilon^{\mu\nu\rho\sigma} \partial_\nu \theta_\rho \theta_\sigma \quad (9)$$

we get

$$L(\xi) = L(\theta) - \frac{d}{dz} F \quad (10)$$

$$F = \frac{1}{12} \epsilon^{\mu\nu\rho\sigma} \theta_\mu \partial_\nu \theta_\rho \theta_\sigma$$

where

$$L(\theta) = -\frac{i}{2} \theta_\mu \dot{\theta}^\mu + \frac{1}{4\lambda_1} \dot{x}^2 + \frac{\lambda_2}{2\lambda_1} \dot{x}_\mu \theta^\mu \quad (11)$$

is the Lagrangian describing a massless Dirac particle ²⁾. Furthermore, the transformation (9) is an isomorphism of Grassmann algebras :

$$\begin{aligned} \xi^\mu \xi^\nu &= \theta^\mu \theta^\nu \\ \xi^\mu \xi^\nu \xi^\rho &= \theta^\mu \theta^\nu \theta^\rho \quad \text{etc.} \end{aligned} \quad (12)$$

We see that the two theories are completely equivalent at the classical level. However, it is not so in the quantum case. In fact, the transformation (9) does not preserve the spectrum because it gives

$$Q^{\mu} = \frac{i}{\sqrt{2}} \gamma^{\mu} (1 - \gamma_5) \quad (13)$$

from which $Q^2 = 0$. This situation is typical for non-linear transformations because they can give rise to quantum ordering problems. In fact, in the present case, the origin of the phenomenon is that the classical generator F of the transformation satisfies $F^2 = 0$, whereas the quantum generator $\tilde{F} = -(i/2)\gamma_5$ satisfies $\tilde{F}^2 = -\frac{1}{4}$.

The path integral quantization of this model presents some interest due to the trilinear terms in the Lagrangian. The functional integral can be easily performed if we take the arbitrary functions λ_1 and λ_2 to be constant. In fact, in such a case, λ_2 plays the role of a coupling constant and we can evaluate the functional integral by making a perturbative expansion. However, $\lambda_2^2 = 0$ and therefore only the zero and the first order terms contribute to the final result. This allows us to make the calculation exactly.

As usual ³⁾ it is convenient to introduce complex combinations

$$\begin{aligned} \eta_1 &= \frac{1}{\sqrt{2}} (\xi^0 + \xi^3), & \bar{\eta}_1 &= -\frac{1}{\sqrt{2}} (\xi^0 - \xi^3) \\ \eta_2 &= \frac{1}{\sqrt{2}} (\xi^1 + i \xi^2), & \bar{\eta}_2 &= \frac{1}{\sqrt{2}} (\xi^1 - i \xi^2) \end{aligned} \quad (14)$$

Then the amplitude will be given by

$$\begin{aligned} &\langle X_f, \bar{\eta}_f, z_f | X_i, \eta_i, z_i \rangle = \\ &= \int_{X_i, \eta_i}^{X_f, \bar{\eta}_f} \mathcal{D}(x) \mathcal{D}(\eta, \bar{\eta}) \exp \left\{ \frac{1}{2} \sum_{\alpha} \bar{\eta}_{\alpha, f} \gamma_{\alpha} (z_f) + \frac{1}{2} \sum_{\alpha} \bar{\eta}_{\alpha} (z_i) \eta_{\alpha, i} + \right. \\ &+ i \int_{z_i}^{z_f} dz \left[\frac{i}{2} \sum_{\alpha} (\bar{\eta}_{\alpha} \dot{\eta}_{\alpha} - \dot{\bar{\eta}}_{\alpha} \eta_{\alpha}) - \frac{1}{2\lambda_1} \sum_{\alpha} \dot{\bar{X}}_{\alpha} \dot{X}_{\alpha} - \right. \\ &\left. \left. - \frac{\lambda_2}{2\lambda_1} \sum_{\alpha} \left(\dot{X}_{\alpha} \bar{\eta}_{\alpha} e^{-2 \sum_{\beta} \bar{\eta}_{\beta} \gamma_{\beta}} + \dot{\bar{X}}_{\alpha} \eta_{\alpha} e^{2 \sum_{\beta} \bar{\eta}_{\beta} \gamma_{\beta}} \right) \right] \right\} \quad (15) \end{aligned}$$

where x_α is defined in terms of the four-vector x_μ in the way given by Eq. (14).

After performing the integral over x^μ we get

$$\begin{aligned} \langle X_f, \bar{\eta}_f, z_f | X_i, \eta_i, z_i \rangle &= - \frac{i}{16 \pi^2 \lambda_1^2 (\Delta z)^2} e^{\frac{i}{4\lambda_1} \frac{z^2}{\Delta z}} \\ &\cdot \exp \left\{ \frac{\lambda_2}{2\lambda_1 \Delta z} \int_{z_i}^{z_f} dz \left[\sum_{\alpha} (z_\alpha \frac{\delta}{\delta J_\alpha} - \bar{z}_\alpha \frac{\delta}{\delta \bar{J}_\alpha}) - \right. \right. \\ &\left. \left. - 2 \sum_{\alpha, \beta} (\bar{z}_\beta \frac{\delta}{\delta \bar{J}_\beta} + z_\beta \frac{\delta}{\delta J_\beta}) \frac{\delta}{\delta J_\alpha} \frac{\delta}{\delta \bar{J}_\alpha} \right] \right\} K(J, \bar{J}) \Big|_{J=\bar{J}=0} \end{aligned} \quad (16)$$

where

$$\begin{aligned} z &= X_f - X_i \\ \Delta z &= z_f - z_i \end{aligned}$$

$$\begin{aligned} K(J, \bar{J}) &= \int_{\eta_i}^{\bar{\eta}_f} \mathcal{D}(\eta, \bar{\eta}) \exp \left\{ \frac{1}{2} \sum_{\alpha} \bar{\eta}_{\alpha, f} \eta_{\alpha} (z_f) + \right. \\ &+ \frac{1}{2} \sum_{\alpha} \bar{\eta}_{\alpha} (z_i) \eta_{\alpha, i} + i \int_{z_i}^{z_f} dz \left[\frac{i}{2} \sum_{\alpha} (\bar{\eta}_{\alpha} \dot{\eta}_{\alpha} - \dot{\bar{\eta}}_{\alpha} \eta_{\alpha}) + \right. \\ &\left. \left. + \sum_{\alpha} (\bar{\eta}_{\alpha} J_{\alpha} + \bar{J}_{\alpha} \eta_{\alpha}) \right] \right\} \end{aligned} \quad (17)$$

Evaluating the generating functional and doing the functional derivatives, we get

$$\begin{aligned} \langle X_f, \bar{\eta}_f, z_f | X_i, \eta_i, z_i \rangle &= - \frac{i}{16 \pi^2 \lambda_1^2 (\Delta z)^2} \\ &\cdot \exp \left\{ \frac{i}{4\lambda_1} \frac{z^2}{\Delta z} - \frac{i \lambda_2}{2\lambda_1} e^{\sum_{\alpha} \bar{\eta}_{\alpha, f} \eta_{\alpha, i}} \right. \\ &\cdot \left[\sum_{\alpha} \bar{\eta}_{\alpha, f} z_{\alpha} \left(e^{-2 \sum_{\beta} \bar{\eta}_{\beta, f} \eta_{\beta, i}} + 1 \right) + \right. \\ &\left. \left. + \sum_{\alpha} \eta_{\alpha, i} \bar{z}_{\alpha} \left(e^{2 \sum_{\beta} \bar{\eta}_{\beta, f} \eta_{\beta, i}} - 1 \right) \right] \right\} \end{aligned} \quad (18)$$

It is interesting to notice that the functional integral is "almost" saturated by the classical solution. In fact, the argument of the exponential in (18) is the classical action [plus the surface terms in (15)] apart from the linear term $\sum_{\alpha} (\bar{\eta}_{\alpha, f} z_{\alpha} - \eta_{\alpha, i} \bar{z}_{\alpha})$. The origin of this term can be traced back to the fact that the contraction $\bar{\eta}(\tau)\eta(\tau)$ is not zero for the particular boundary conditions defining the integral (15).

Therefore, the only effect of the interaction is to perform the shift $\bar{\eta}_{\alpha, f} \eta_{\alpha, i} \rightarrow \bar{\eta}_{\alpha, f} \eta_{\alpha, i}^{-\frac{1}{2}}$ in the trilinear terms contained in the classical action.

In order to get the physical kernel we must integrate Eq. (18) over $\lambda_1 \Delta\tau$ and $\lambda_2 \Delta\tau$ ³⁾; furthermore, going to the momentum space and to a spinor basis ³⁾, we get

$$K_{phys.}(p', p) = -\frac{i}{\sqrt{2}} \frac{\hat{p}(1-\gamma_5)}{p^2 + i\epsilon} \delta^4(p' - p) \quad (19)$$

where we have used the representation (8) for the ξ^{μ} operators.

After the discussion of the quantum mechanical aspects of the model, let us go back to some pseudoclassical considerations. The presence of two first class constraints implies that the theory must be invariant under two local transformations. Indeed, we have invariance under reparametrization in the τ variable and under "local" supersymmetry. In particular, by using the results of Ref. 2) for the Lagrangian (11) we find that the Lagrangian (3) is invariant under

$$\begin{aligned} \delta \xi^{\mu}(\tau) = \epsilon(\tau) \left[\frac{1}{2\lambda_1} (\dot{X}^{\mu} + \lambda_2 \xi^{\mu}) - \frac{i\lambda_2}{3\lambda_1} \epsilon^{\mu\nu\sigma} \xi_{\nu} \xi_{\sigma} - \right. \\ \left. - \frac{i}{2\lambda_1} \epsilon^{\mu\nu\sigma} \dot{X}_{\nu} \xi_{\sigma} \right] \end{aligned} \quad (20)$$

$$\delta X^{\mu}(\tau) = i \epsilon(\tau) \left[\xi^{\mu} + \frac{i}{3} \epsilon^{\mu\nu\sigma} \xi_{\nu} \xi_{\sigma} \right]$$

$$\delta \lambda_1 = -\epsilon(\tau) \lambda_2$$

$$\delta \lambda_2 = -i \dot{\epsilon}(\tau)$$

where $\epsilon(\tau)$ is an odd parameter.

Let us discuss now the possible couplings with Yang-Mills and gravitational fields. The internal degrees of freedom are described in terms of Grassmann variables θ_{α} [see Ref. 4]. The coupling with Yang-Mills fields

is then obtained by the minimal substitution. However, in order to preserve the first class character of the constraints, we must also insert a non-minimal coupling⁴⁾. The correct Lagrangian satisfying this requirement turns out to be

$$L = L_{\text{free}} + \frac{i}{2} \sum_{\alpha=1}^N (\partial_{\alpha}^* \dot{\theta}_{\alpha} - \dot{\theta}_{\alpha}^* \partial_{\alpha}) - g A_{\mu}^a I_a \dot{x}^{\mu} + g \lambda_1 F_{\mu\nu}^a I_a S^{\mu\nu} \quad (21)$$

where L_{free} is the Lagrangian (3), the second term is the kinetic term for the internal degrees of freedom, and

$$I_a = \theta^* \tau_a \theta$$

$$[\tau_a, \tau_b]_- = i f_{ab}^c \tau_c \quad (22)$$

$$S^{\mu\nu} = -\frac{i}{2} [\xi^{\mu}, \xi^{\nu}]_-$$

The τ_a 's are the generators of the gauge group in the representation defined by the θ_{α} 's.

In the case of a gravitational field, it is convenient to introduce vierbein fields G_{μ}^A and their inverses H_A^{μ} ⁵⁾. Then, the interacting Lagrangian is given by

$$L = -\frac{i}{2} \eta_{AB} \xi^A \left[\dot{\xi}^B + (A_{\mu})^B{}_C \dot{x}^{\mu} \xi^C \right] + \frac{1}{4\lambda_1} \eta_{AB} \left[G_{\mu}^A \dot{x}^{\mu} + \lambda_2 \left(\xi^A + \frac{i}{3} \epsilon^{ACDE} \xi_C \xi_D \xi_E \right) \right] \cdot \left[G_{\nu}^B \dot{x}^{\nu} + \lambda_2 \left(\xi^B + \frac{i}{3} \epsilon^{BFGH} \xi_F \xi_G \xi_H \right) \right] \quad (23)$$

where $(A_{\mu})^A{}_B$ is the spin connection which, in the absence of torsion, can be expressed in terms of the Christoffel symbols

$$(A_{\mu})^A{}_B = H_B^{\nu} (G_{\nu}^A \Gamma_{\nu\mu}^{\sigma} - G_{\nu,\mu}^A) \quad (24)$$

From (23) we get the equations of motion :

$$\frac{D}{D\tau} G_{\mu} = \frac{i}{2} \dot{x}^{\nu} (R_{\mu\nu})^A{}_{.B} \theta^B \theta_A \quad (25)$$

$$\frac{D}{D\tau} \theta^A = -i \lambda_2 G_{\mu} \eta^{AB} H_B^{\mu} \quad (26)$$

where

$$G_{\mu} = -\frac{1}{2\lambda_1} g^{\mu\nu} (\dot{x}^{\nu} + \lambda_2 H_A^{\nu} \theta^A) \quad (27)$$

The relation between θ^A and ξ^A is given in Eq. (9), $(R_{\mu\nu})^A{}_{.B}$ is the Riemann tensor and $D/D\tau$ means the covariant derivative. Equation (25) is completely independent of the choice of the arbitrary functions λ_1 and λ_2 ; therefore, the four-momentum does not undergo a parallel transport. On the contrary, the equations of motion for x^{μ} are gauge dependent

$$\begin{aligned} \ddot{x}^{\mu} + \Gamma_{\nu\sigma}^{\mu} \dot{x}^{\nu} \dot{x}^{\sigma} = & -i \lambda_1 g^{\mu\nu} \dot{x}^{\sigma} (R_{\nu\sigma})^A{}_{.B} \theta^B \theta_A - \\ & - 2 \lambda_1 g^{\mu\nu} G_{\nu} - \lambda_2 H_A^{\mu} \theta^A \end{aligned} \quad (29)$$

As shown by Galvao and Teitelboim⁶⁾, due to the particular kinematics of the massless case, it is possible to choose a gauge fixing and a particular co-ordinate system, in such a way that x^{μ} satisfies the geodesic equation

$$\ddot{x}^{\mu} + \Gamma_{\nu\sigma}^{\mu} \dot{x}^{\nu} \dot{x}^{\sigma} = 0 \quad (30)$$

However, due to our previous observation about Eq. (25), it should be clear that in the massless case also there is an intrinsic coupling between spin and curvature of space-time. In fact, although such a coupling can be minimized by a clever choice of the gauge, it will always manifest itself by forbidding the parallel transport for the four-momentum.

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