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Inclusion of the QCD next-to-leading order corrections in the quark-gluon Monte Carlo shower

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A methodology of including QCD next-to-leading-order corrections in the quark-gluon Monte Carlo shower is outlined. The work concentrates on two issues: (i) constructing the leading-order parton-shower Monte Carlo from scratch, such that it rigorously extends collinear factorization into the exclusive (fully unintegrated) one which we call the Monte Carlo factorization scheme; (ii) introducing next-to-leading-order (NLO) corrections to the hard process in this new environment. The presented solution is designed to be extended to the full NLO-level Monte Carlo, including NLO corrections not only in the hard process but in the whole shower. The issue of the difference between the factorization scheme implemented in the Monte Carlo (MC) solution and the standard $\overline{MS}$ scheme is addressed. The principal MC implementation is designed for the electroweak boson production process at the LHC, but in order to discuss universality (process independence), the deep inelastic lepton-hadron scaterring is also brought into the MC framework.

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I. INTRODUCTION

The excellent performance and fast experimental data accumulation of the Large Hadron Collider (LHC) at CERN makes the precise evaluation of the strong interaction effects within perturbative quantum chromodynamics (QCD) [1–3] a more and more important task. The principal role of QCD in hadron collider data analyses (LHC and Tevatron) is to provide precise predictions for the distributions and luminosities of quarks and gluons accompanying the production of heavy particles.

Among the most important theoretical tools of perturbative QCD (pQCD) are factorization theorems [4–6], which reformulate any scattering process in QCD in terms of the on-shell hard-process part convoluted in the light-cone variable with the *ladder parts*, provided a single large-scale Q^2 is involved (short-distance interaction). The hard process is usually treated at a fixed perturbative order, and the ladder parts are resummed to infinite order, for each colored energetic ingoing/outgoing parton. The initial state ladders give rise to the *inclusive* parton distribution functions (PDFs).

The initial state ladder, instead of being the source of the inclusive PDF, can also be modeled using Monte Carlo simulation (including hadronization), as initiated in Refs. [7,8]. Such an implementation of the QCD ladder is referred to as the parton-shower Monte Carlo (PSMC) program. Programs of this kind play enormous practical role in all collider experiments. In today's PSMC applications, the initial ladders are restricted to the leading order (LO). With growing requirements on the quality and precision of pQCD predictions for the LHC experiments, it has become urgent to upgrade PSMC to the same next-to-leading-order (NLO) level which was reached for the inclusive PDFs two decades ago. This is not easy, mainly because factorization theorems of QCD [4–6] were never meant for the Monte Carlo (MC) implementation. They are well suited for the simpler case of the hard process upgraded with finite-order calculations and convoluted with collinear-inclusive PDFs.

There has been, however, significant progress in implementing pQCD in the framework of PSMC, which started with the work of the MC@NLO team [9,10], followed by the development of the POWHEG method [11,12]. In these works, the hard process in PSMC is upgraded to NLO, while the ladder part stays at the LO level; essentially, older solutions and software for the LO PSMC are not modified. This, of course, saves a lot of work, but because of that, the methodology of combining the initial ladder parts and the NLO-corrected matrix element (ME) for the hard process is quite complicated. The solution for this problem is to redesign the basic LO PSMC. This would be too big an investment if simplification of the NLO corrections to the hard-process ME were the only aim. However, this effort is mandatory if we are also aiming to upgrade the ladder parts of PSMC to the complete NLO level.

In this paper, we outline a redesigned LO parton-shower MC and simultaneously present a methodology of including QCD NLO corrections to the hard process which takes advantage of it. Hence, we shall concentrate on two issues: (i) constructing once again the LO parton-shower Monte Carlo from scratch, such that it is based firmly on the rigorous extension of the collinear factorization theorems, which, contrary to the original collinear factorization, is fully exclusive (unintegrated); and (ii) introducing the NLO corrections to the hard process in this new environment. It is natural to expect that the issues of the difference between the factorization scheme implemented in the MC solution and the standard $\overline{MS}$ scheme will have to be addressed. The important point of universality (process independence) will be also discussed extensively. Although the principal aim will be a new MC implementation with the LO ladder (upgradable to NLO) and the NLO hard process for the production of the electroweak (EW) boson in quark-antiquark annihilation, in order to address the issue of universality, and for other practical reasons, the deep inelastic lepton-hadron process will be also brought into consideration. The next step in the project, the upgrade of the ladder part to the NLO level, will be treated in a separate publication [13], although the general method was already outlined in Refs. [14–16]. Many technical details needed for the NLO ladder are provided in Ref. [17], and auxiliary discussions on the soft limit and the choice of the factorization scale in the MC can be found in Refs. [18–20]. The first numerical tests of the discussed method of including the NLO hard-process corrections are presented in Ref. [21].

Let us note that the ongoing effort undertaken in Refs. [22,23] is in some aspects similar to the present work; in particular, the parton shower is also redesigned at the NLO level, and a departure from the standard $\overline{MS}$ factorization scheme is also advocated. These works are extending/exploiting techniques of Refs. [24,25].

The remaining part of the paper is organized as follows: In Sec. II, we discuss collinear factorization in a form suitable for the MC implementation. Section III covers the construction of the LO MC for EW boson production and the new method of introducing the NLO corrections to the hard part of this process. In Sec. IV, we present a similar MC solution for the deep inelastic electron-proton scattering (DIS) process, with the new LO MC modeling of the initial- and final-state ladders and the NLO corrections to the hard process. The issue of universality as well as factorization-scheme dependence are addressed in various steps of this presentation, with the final discussion in Sec. V, where we also give a summary and an outlook on further work.

II. GENERALITIES: COLLINEAR FACTORIZATION

The precise definition of the LO approximation within the factorization of collinear singularities and the resulting distributions implemented in Monte Carlo is a necessary prelude to defining complete NLO distributions, both in the ladder and in the hard process. This is why in the following section we shall define a new LO MC, "anchoring" it in the collinear factorization theorems [4–6] as firmly as we can.

The production process of electroweak bosons W, Z, γ, or of other color-neutral vector particles, in hadron-hadron scattering will be the object of interest in the following discussion. We shall refer to it as the Drell-Yan (DY) process for short. We are going to describe a Monte Carlo algorithm in which two initial-state parton ladders will be modeled up to the LO, and the hard process up to the NLO, with both the hard and ladder parts modeled in a completely exclusive way. In the construction of the MC, we will keep track of the precise relation to the QCD factorization theorems, keeping in mind that the ladder part (parton shower) will be upgraded to the NLO level in the next step. Diagrammatically, we shall temporarily limit ourselves to the C_F^k part in the ladder; that is, to gluon bremsstrahlung. Because of copious soft gluon production, this is the most difficult part in the MC construction of the LO (and later on, of the NLO) MC implementation for the ladder. The addition of more diagrams (quark-gluon transitions, singlet diagrams) will be discussed briefly, but will be treated in a separate publication.

Let us start from the "raw collinear factorization" formula of Ref. [4] (in the axial gauge) illustrated in Fig. 1 in a standard way (cut diagrams). Following closely the notation of Ref. [4], the standard Feynman amplitude for the heavy boson production process is $A^{r_F r_B}_{j_F j_B}(p_F, p_B, q_1, q_2, \Gamma)$, where two incoming partons of types j_F and j_B have spin indices r_F and r_B, the heavy-boson decay lepton momenta

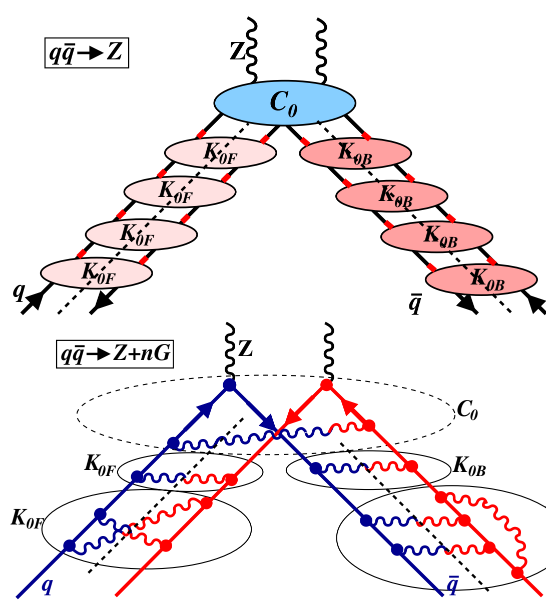


FIG. 1 (color online). The EGMPR [4] factorization for EW boson production. Example 2PI kernels for a quark in the forward hemisphere K_{0F}, an antiquark in the backward hemisphere K_{0B}, and the hard-process part C_0 are delimited by ellipses. The lower figure highlights the use of cut diagram notation by indicating the amplitude in blue (left parts of the ladders) and the complex conjugated amplitude in red (right parts of the ladders).

are q_i, and any number of the emitted on-shell gluons and quarks are collectively denoted as Γ. Also denoting (p_F, p_B, q_1, q_2) collectively as (p, q), the partly integrated cut diagram is defined as

$$\mathcal{M}^{r_F s_F r_B s_B}_{j_F j_B}(p, q) \equiv \sum_{\Gamma} \int d\Gamma\, \delta^{(4)}(p_F + p_B - q_1 - q_2 - p_\Gamma) \times A^{r_F r_B}_{j_F j_B}(p, q, \Gamma) A^{s_F s_B}_{j_F j_B}(p, q, \Gamma)^*. \tag{1}$$

It is related directly to the differential cross section $d\sigma_{j_F j_B}(p, q) = \frac{1}{\text{flux}} \sum_{r_F r_B} \mathcal{M}^{r_F r_B r_F r_B}_{j_F j_B}(p, q) Z_F Z_B$, where $Z_{F,B}$ are wave renormalization factors. The essence of the "raw" factorization theorem in Ref. [4] is that all collinear singularities are located in the ladders with multiple two-particle irreducible (2PI) kernels $K^{r's'rs}_{0j'j}(k, p)$. Suppressing for the moment ladder B and neglecting subscript F, the above statement reads

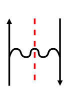
$$\mathcal{M}^{rs}_j(p, q) = C^{rs}_{0j}(p, q) + \sum_{n=1}^{\infty} \prod_{m=1}^{n} \int d^4 p_m \sum_{r_n, s_n, j_n} C_0{}^{r_n s_n}_{j_n}(p_n, q) K_0{}^{r_n s_n\, r_{n-1} s_{n-1}}_{j_n\, j_{n-1}}(p_n, p_{n-1}) \times \sum_{r_{n-1}, s_{n-1}, j_{n-1}} K_0{}^{r_{n-1} s_{n-1}\, r_{n-2} s_{n-2}}_{j_{n-1}\, j_{n-2}}(p_{n-1}, p_{n-2}) \ldots \sum_{r_1, s_1, j_1} K_0{}^{r_1 s_1\, rs}_{j_1\, j}(p_1, p).$$

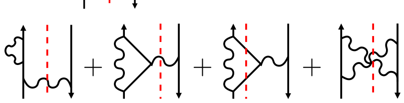
Using the compact matrix notation of Refs. [4,26], the above expression in the case of two ladders reads

$$\sigma = C_0 \frac{1}{1 - \cdot K_{0F}} \frac{1}{1 - \cdot K_{0B}} = \sum_{n_1, n_2 = 0}^{\infty} C_0 (\cdot K_{0F})^{n_1} (\cdot K_{0B})^{n_2}; \tag{2}$$

see also the upper part of Fig. 1 for an equivalent graphical representation. In the lower part of Fig. 1, the above formula is illustrated diagrammatically using the lowest-order bremsstrahlung matrix element where we explicitly indicate the 2PI kernels, with the red part of the diagram representing the conjugate part A^* of Eq. (1). Note that in the above expressions, the phase space of the emitted on-shell partons (cut lines) is integrated over and treated inclusively. In the following discussion, it will be explicit and implemented in the Monte Carlo parton shower.

According to Ref. [4], all collinear singularities in σ of Eq. (2) are coming from (dressed) propagators between kernels K_{0F} (K_{0B}) along the ladders. The 2PI kernels for the initial quark ladder, K_{0F}, and the antiquark ladder, K_{0B}, are expanded to infinite order; see Ref. [4]. In the following practical example, we shall truncate them to the (lowest) first order (LO) or to the second order (NLO), $K^{(2)}_{0F} = K^{[1]}_{0F} + K^{[2]}_{0F}$, taking into account the following $\sim C_F^2$ diagrams:

$K^{(1)}_{0F} = K^{[1]}_{0F} =$  ,

$K^{[2]}_{0F} =$  .

A similar expansion up to the first order (NLO) is done for the hard-process part: $C^{(1)}_0 = C^{[0]}_0 + C^{[1]}_0$. For simplicity, we are omitting the initial quark and antiquark distributions in the beam hadrons, and the flux factor is included in C_0. The dot in the product $A \cdot B$ means full phase-space integration, $\int d^4 q$, over the lines joining two subgraphs in one ladder.

The next step in the classic works of Refs. [4,26] is the introduction of the projection operator $\mathbb{P}$. Its role is to decouple kinematically not only C_0 and the ladder parts, but also the consecutive kernels K_0 along the ladders, such that the integration over light-cone variables and collinear logs becomes manifest and ready for analytical calculations. Formally, Eq. (2) gets transformed (at infinite order) into

$$\sigma = C \otimes \Gamma_F \otimes \Gamma_B = C \frac{1}{1 - \otimes K_F} \frac{1}{1 - \otimes K_B} = \sum_{n_1, n_2 = 0}^{\infty} C (\otimes K_F)^{n_1} (\otimes K_B)^{n_2},$$
$$C = C_0 \frac{1}{1 - \mathbb{R} K_{0F}} \frac{1}{1 - \mathbb{R} K_{0B}}, \qquad K_F = \frac{1}{1 - \otimes \mathbb{P} K_{0F} \frac{1}{1 - \mathbb{R} K_{0F}}},$$
$$K_B = \frac{1}{1 - \otimes \mathbb{P} K_{0B} \frac{1}{1 - \mathbb{R} K_{0B}}}, \qquad \mathbb{R} = (\cdot 1 - \otimes \mathbb{P}), \tag{3}$$

where $A \otimes B$ means convolution in the light-cone variable $\int dz_1 dz_2 \delta(x - z_1 z_2) A(z_1) B(z_2)$, while integration over transverse momenta is traded into $\frac{1}{\varepsilon^k}$ poles of dimensional regularization, extracted (upon integration) by $\mathbb{P}$; see Ref. [26] for details. The projection operator $\mathbb{P}$ used in Ref. [4] is slightly different. However, both approaches are incompatible with any MC implementation, as can be seen from the explicit expansion up to NLO[1]:

$$C = C^{(1)}_0 \Big(1 - \otimes \mathbb{P} K^{[1]}_{0F} - \otimes \mathbb{P} K^{[1]}_{0B}\Big),$$
$$\Gamma_F = \mathbb{1} + \mathbb{P} K^{[1]}_{0F} + \mathbb{P} K^{[2]}_{0F} + \mathbb{P}\Big(K^{[1]}_{0F} \cdot K^{[1]}_{0F}\Big) - \mathbb{P}\Big(K^{[1]}_{0F} \otimes \mathbb{P} K^{[1]}_{0F}\Big) + \Big(\mathbb{P} K^{[1]}_{0F}\Big) \otimes \Big(\mathbb{P} K^{[1]}_{0F}\Big). \tag{4}$$

Why? As we can see, the above is a mixture of the original phase-space integrals like $(K_{0F} \cdot K_{0F})$ and partly integrated integrals like $(\mathbb{P} K_{0F}) \otimes (\mathbb{P} K_{0F})$. Even if we managed

[1]Omitting for simplicity quark wave function renormalization.

somehow to undo the transverse momentum integrations implicit in the $\mathbb{P}$ operator, we would still face huge (double logarithmic) oversubtraction in $\mathbb{P}K_{0F}^{[1]}((\cdot\mathbb{1} - \otimes\mathbb{P})K_{0F}^{[1]})$ compensated by $(\mathbb{P}K_{0F}^{[1]}) \otimes (\mathbb{P}K_{0F}^{[1]})$, which would be deadly for any MC implementation. For an explicit demonstration of this problem, see also the toy model considerations in Ref. [27],[2] or the LO analysis to infinite order in Ref. [16].

The solution of the above oversubtraction problem is well known and already employed in the existing LO parton-shower MC [7,8]—in short, one has to introduce the time-ordered (T.O.) exponent; see also Ref. [28]. Beyond the LO, a collinear factorization formula with a T.O. exponent was outlined in Ref. [16], and we shall adopt it here. We are going to use it for the LO MC ladders combined with the NLO hard-process ME.[3] According to Ref. [16], Eq. (3) is replaced by

$$\sigma_{\rm LO} = \sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty} C_0^{(0)}\{(\mathbb{P}'K_{0F}^{(1)})^{n_1}\}_{\rm T.O.}\{(\mathbb{P}'K_{0B}^{(1)})^{n_2}\}_{\rm T.O.}, \qquad (5)$$

where $K_{0F}^{(1)}$ is the lowest-order 2PI kernel, the same as in Eq. (3); but at the NLO and beyond, it is different; see the definition in Ref. [16]. The above LO process is depicted in Fig. 2. As shown in Ref. [16], the complete and rigorous definition of the new projection operator $\mathbb{P}'$ is not simple. For the present purpose of the two LO ladders and the NLO hard process in EW boson production, we shall define it step by step, starting from the simple cases of zero-, one-, and two-gluon ME, $n_1 + n_2 = 0, 1, 2$, rather than defining it immediately in the full form.

Let us denote the Born-level differential cross section for the EW vector boson production and decay process, $q\bar{q} \to l\bar{l}$, as follows:

$$\frac{d\sigma_B}{d\Omega}(s, \theta).$$

It may also include nonphotonic EW radiative corrections. The above differential cross section is so well known that we may avoid defining its details explicitly. From (two) Feynman diagrams, we obtain the following (exact) LO single-gluon emission differential distribution:

$$d\sigma_1 = \frac{C_F\alpha_s}{\pi}\frac{d\alpha d\beta}{\alpha\beta}\frac{d\varphi}{2\pi}\left[\frac{d\sigma_B(\hat{s},\theta_F)}{d\Omega}\frac{(1-\beta)^2}{2} + \frac{d\sigma_B(\hat{s},\theta_B)}{d\Omega}\frac{(1-\alpha)^2}{2}\right]d\Omega, \qquad (6)$$

[2]In this work, another example of the operator $\mathbb{P}$ is presented, and also the order in factorizing collinear singularities in Eq. (3) is reversed: it starts from the hard process. Nevertheless, it features the same oversubtraction problems that inhibit MC implementation.

[3]Considerations concerning the NLO MC ladder can be found in Refs. [15–17].

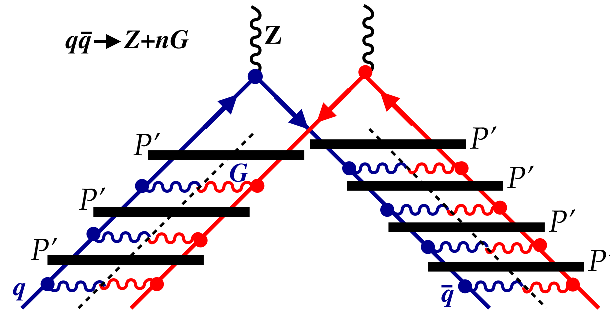


FIG. 2 (color online). The EGMPR factorization for EW boson production; the LO ladder with the projection operators $\mathbb{P}'$ inserted between the LO kernels.

where the Sudakov variables α and β are defined in Appendix A. This elegant formula is valid for any on/off-shell vector particle production, $B = \gamma$, W, Z. The polar angles $\theta_{F,B}$ are defined [29] with respect to $-\vec{p}_{0B}$ and $\vec{p}_{0F}$, respectively, in the rest frame of the B boson (the rest frame of $p_{0F} + p_{0B} - k$).

Two collinear limits—$\beta \to 0$, $1 - \alpha = z =$ const; or $\alpha \to 0$, $1 - \beta = z =$ const in Eq. (6)—are manifest. For instance, in the first case we have

$$d\sigma_1 = \frac{C_F\alpha_s}{\pi}\frac{dz d\beta}{(1-z)\beta}\frac{d\varphi}{2\pi}\frac{1+z^2}{2}\frac{d\sigma_B(zs,\theta_F)}{d\Omega}d\Omega; \qquad (7)$$

that is, the LO kernel $P_{qq}(z) = \frac{2C_F\alpha_s}{\pi}\frac{1+z^2}{2(1-z)}$ shows up as expected. Introducing the $\mathbb{P}'$ projector in this context may look like an overkill, but it will be instructive to explain how it works in this simple case before going to the not-so-obvious case of multiple uses of $\mathbb{P}'$ in the following.

The necessary ingredient is a spin projection operator $P_{\rm spin}$, which we define a little bit more rigorously as compared to the Curci-Furmanski-Petronzio work (CFP) [26]. Our $P_{\rm spin}$ acts definitely before the phase-space integration[4]:

$$HP_{\rm spin}K = H\hat{\not q}_i]\!\!\left[\frac{\not n}{4n\cdot q_i}K, \qquad (8)$$

where $\hat{q}_i$ is the on-shell momentum entering the H part, such that it conserves the longitudinal (light cone) component $n\cdot\hat{q}_i = n\cdot q_i$, and for the axial gauge vector n we may take $n = p_{0B}$ or any other lightlike vector, the same for all rungs in a given ladder. The same n is defining transverse polarizations of gluons in the axial gauge and enters a definition of light-cone variables, $x_i = (nq_i)/(np_{0F})$.

In the present case of a single-gluon emission, the action of the $\mathbb{P}'$ projector, inserted in the squared, spin-summed

[4]In CFP, spin projection is part of $H\mathbb{P}K = H\not q_i]P_\varepsilon[\frac{\not n}{4n\cdot q_i}K$, where P_ε extracts the pole part and simultaneously sets $q^2 \to 0$ in H. We have to be more specific about the choice of $\hat{q}$ in the substitution $q \to \hat{q}$, $\hat{q}^2 = 0$.

Feynman diagram[5] between the gluon emission vertex and the $q\bar{q}B$ vertex (where B is the EW vector boson) can be summarized as follows:

(a) Apply $P_{\rm spin}$ to decouple a spinor γ trace into two parts, the hard ME part and the ladder part.

(b) Apply the explicit upper limit of the phase space for an emitted gluon in the transverse momentum; for example, $a < M$, with $a^2 \equiv s\beta/\alpha$.

(c) Take the expression for the hard ME part[6] in the collinear limit $a \to 0$ $(\beta \to 0)$, keeping $\alpha = 1 - z = $ const (or $\alpha + \beta = 1 - z =$ const), and extrapolate it all over the phase space.

(d) Keep unchanged the phase-space integration element and its limits.

Point (c) of the above recipe is the most important and requires more discussion. Finding out the limiting collinear expression is trivial; see Eq. (7). What is nontrivial is the *off-collinear extrapolation* (OCEX) of this formula, out of the $\beta = 0$ point to all noncollinear phase space. The simplest recipe: go back along the $z = 1 - \alpha$ line and use Eq. (7). The formula

$$d\sigma_{\rm LO} = \frac{C_F\alpha_s}{\pi}\frac{d\alpha d\beta}{\alpha\beta}\frac{d\varphi}{2\pi}\frac{1+(1-\alpha)^2}{2} \times \frac{d\sigma_B((1-\alpha)s, \theta_F)}{d\Omega}d\Omega \tag{9}$$

would be acceptable, provided the Born cross section is flat. In the presence of a narrow resonance in $d\sigma_B(\hat{s})$ in the $\hat{s} = s(1-\alpha-\beta) = sz$ variable, this would lead to a disastrous NLO correction $d\sigma_1 - d\sigma_{\rm LO}$, wildly varying over the phase space. This kind of OCEX follows a vertical dashed line in Fig. 3(a).

However, there is a freedom in the off-collinear extrapolation away from the $\beta = 0$ point—we may do it also along the line $x = 1 - \alpha - \beta = $ const:

$$d\sigma_{\rm LO} = \frac{C_F\alpha_s}{\pi}\frac{d\alpha d\beta}{\alpha\beta}\frac{d\varphi}{2\pi}\frac{1+(1-\alpha-\beta)^2}{2} \times \frac{d\sigma_B((1-\alpha-\beta)s, \hat{\theta})}{d\Omega}d\Omega. \tag{10}$$

In Fig. 3(a), this kind of OCEX goes along the curved dashed line. The angle $\hat{\theta}$ also has to be defined within OCEX in some reasonable way which coincides with the correct value θ_F at the $\beta = 0$ point.[7] In a sense, the above is fully compatible with the methodology of calculating the NLO corrections to the EW production process in

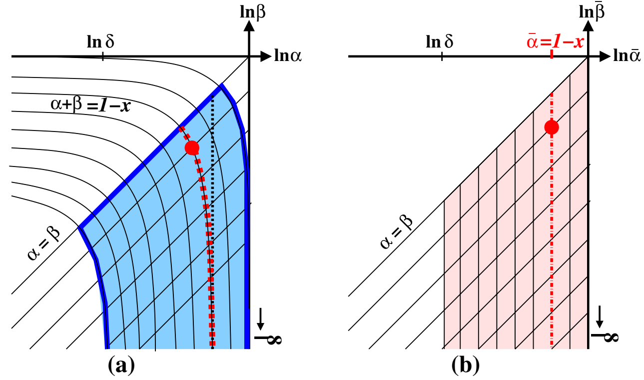


FIG. 3 (color online). The Sudakov plane of single-gluon emission in DY for angular ordering.

Ref. [30], where a light-cone variable of the collinear factorization is also mapped into $x = 1 - \alpha - \beta$. The essential difference is that, with the help of OCEX, we are replacing the traditional collinear PDF of Ref. [30] with the exclusive distribution of Eq. (10). A number of consequences of this replacement will unfold gradually in the following discussion.

For easier generalization to two or more gluons, let us slightly formalize the above as

$$\mathbb{P}'_M = \overleftrightarrow{\rm P\!P}_a \theta_{M>a} P_{\rm spin},$$

where $\overleftrightarrow{\rm P\!P}_a$ takes a collinear limit and implements off-collinear extrapolation on both sides of its location in the Feynman diagrams, without affecting the phase-space integration element:

$$\begin{aligned} C_0^{(0)}\mathbb{P}'_M K_{0F}^{(1)} &= \int \frac{d^3k}{2k^0}K_{0F}^{(1)}(k)\mathbb{P}'_M \\ &\quad \times \int d\tau_2(P-k; q_1, q_2) C_0^{(0)}(q_1, q_2, k) \\ &= \int \frac{2C_F\alpha_s}{\pi}\theta_{a<M}\frac{da}{a}\frac{d\varphi}{2\pi}\frac{d\bar{\alpha}}{\bar{\alpha}}\frac{1+(1-\bar{\alpha})^2}{2}d\Omega_q \\ &\quad \times \frac{d\sigma_B(s(1-\bar{\alpha}), \hat{\theta})}{d\Omega_q}\theta_{s(1-\bar{\alpha})>0}. \end{aligned} \tag{11}$$

Note that $z = 1 - \bar{\alpha} = \hat{s}/s$ is within the proper limits, $0 < z < 1$. The above can be formalized by means of introducing the rescaled four-momentum $\bar{k}^\mu = \lambda^{-1}k^\mu$,

$$\lambda = \frac{\alpha}{\alpha+\beta} = \frac{\bar{\alpha}}{\bar{\alpha}+\bar{\beta}} \le 1,$$

where $\bar{\alpha} = \alpha/\lambda$ and $\bar{\beta} = \beta/\lambda$. The dilatation transformation preserves angles; hence, in the MC, one may generate a and $\bar{\alpha} = 1 - z$ according to

[5] Just one Feynman diagram for gluon emission from the initial quark.

[6] The one-gluon ladder can remain unchanged, but in the case of two gluons in K_0, taking a limiting expression is also done in the ladder.

[7] For instance, as an angle between $\vec{q}_2 - \vec{q}_1$ and $\vec{p}_{0F} - \vec{p}_{0B}$ in the rest frame of the EW boson.

$$C_0^{(0)}\mathbb{P}'_M K_{0F}^{(1)} = \int \frac{d^3\bar{k}}{2\bar{k}^0}\lambda^2 K_{0F}^{(1)}(\bar{k})\mathbb{P}'_M \times \int d\tau_2(P-k;q_1,q_2)C_0^{(0)}(q_1,q_2,k) = \int \frac{2C_F\alpha_s}{\pi}\theta_{a<M}\frac{da}{a}\frac{d\varphi}{2\pi}\frac{dz}{1-z}\frac{1+z^2}{2}d\Omega_q \times \frac{d\sigma_B(sz,\hat{\theta})}{d\Omega_q}\theta_{sz>0}, \quad (12)$$

then construct $\bar{k}^\mu$, and finally rescale it $k=\lambda\bar{k}$. The "barred space" of $\bar{k}^\mu$ depicted in Fig. 3(b) is merely a reparametrization of the true phase space in Fig. 3(a). For a single gluon, the above parametrization of the phase space may look trivial, but for many gluons, it will be useful.

As already said, the main role of the $\mathbb{P}$ operator in Refs. [4,26] is to decouple kinematically the hard process and the ladder. The above $\mathbb{P}'$ does it also, but more gently, protecting the four-momentum conservation. The kinematic decoupling is seen from the phase-space integration

$$C_0^{(0)}\mathbb{P}'_M K_{0F}^{(1)} = \ln\frac{M}{q_0}\int_0^1 dz \frac{2C_F\alpha_s}{\pi}\frac{1+z^2}{2(1-z)}\sigma_B(sx), \quad (13)$$

which provides exactly the same result as the collinear factorization in these classic works where the four-momentum conservation is broken. An additional cutoff, $a>q_0$ (the phase-space boundary in the LO MC), was used in the above.

One may also easily define, within the above scheme, a prototype of a universal inclusive collinear PDF:

$$D(M,x) = \mathbb{P}'_M K_{0F}^{(1)}|_x = \int \frac{d^3\bar{k}}{2\bar{k}^0}K_{0F}^{(1)}(\bar{k})\mathbb{P}'_M\delta_{x=1-\bar{\alpha}} = \ln\frac{M}{q_0}\frac{2C_F\alpha_s}{\pi}\frac{1+x^2}{2(1-x)}, \quad (14)$$

thanks to $\mathbb{P}'_M$ closing the phase space from the above by means of the factorization scale M. As we shall see later on, in the analogous construction for deep inelastic electron-proton scattering, the role of $\mathbb{P}'_M$ is to map the phase space of the ladder into an idealized phase space of $\bar{k}^\mu$, decoupled kinematically from the hard process, thus removing process dependence[8] and gaining universality of the ladder part.

A. NLO correction to the hard process: One real gluon

Having defined the single-gluon ladder parts $C_0^{(0)}\mathbb{P}'K_{0F}^{(1)}$ and $C_0^{(0)}\mathbb{P}'K_{0B}^{(1)}$ in the *exclusive* way, within the same exact phase space where the complete exact single-gluon distribution[9] of Eq. (6),

$$D^{[1]} = C_0^{[1]} + C_0^{[0]}\cdot K_{0F}^{(1)} + C_0^{[0]}\cdot K_{0B}^{(1)}, \quad (15)$$

is defined, it is straightforward to define the LO + NLO factorized hard-process part in the exclusive (unintegrated) manner:

$$\begin{aligned}(C_0^{[0]}+C_0^{[1]})&[1+(1-\mathbb{P}')K_{0F}^{(1)}+(1-\mathbb{P}')K_{0B}^{(1)}]\\ &\simeq C_0^{[0]}+C_0^{[1]}+C_0^{[0]}(1-\mathbb{P}')K_{0F}^{(1)}+C_0^{[0]}(1-\mathbb{P}')K_{0B}^{(1)}\\ &= C_0^{[0]}+C_0^{[1]}+C_0^{[0]}K_{0F}^{(1)}+C_0^{[0]}K_{0B}^{(1)}\\ &\quad - C_0^{[0]}\mathbb{P}'K_{0F}^{(1)} - C_0^{[0]}\mathbb{P}'K_{0B}^{(1)}\\ &= C_0^{[0]}+D^{[1]}-C_0^{[0]}\mathbb{P}'K_{0F}^{(1)}-C_0^{[0]}\mathbb{P}'K_{0B}^{(1)}. \end{aligned} \quad (16)$$

The above difference of the exact and approximate MEs at the level of the integrand reads

$$\begin{aligned}C_0^{[1]}+C_0^{[0]}(1-\mathbb{P}')K_{0F}^{(1)}+C_0^{[0]}(1-\mathbb{P}')K_{0B}^{(1)} &= D^{[1]}-C_0^{[0]}\mathbb{P}'K_{0F}^{(1)}-C_0^{[0]}\mathbb{P}'K_{0B}^{(1)}\\ &= \int\frac{d\alpha d\beta}{\alpha\beta}\frac{d\varphi}{2\pi}d\Omega_q\frac{2C_F\alpha_s}{\pi}\Big\{\Big[\frac{(1-\beta)^2}{2}\frac{d\sigma_B}{d\Omega_q}(\hat{s},\theta_F)+\frac{(1-\alpha)^2}{2}\frac{d\sigma_B}{d\Omega_q}(\hat{s},\theta_B)\Big]\\ &\quad -\theta_{\alpha>\beta}\frac{1+(1-\alpha-\beta)^2}{2}\frac{d\sigma_B}{d\Omega_q}(\hat{s},\hat{\theta})-\theta_{\alpha<\beta}\frac{1+(1-\alpha-\beta)^2}{2}\frac{d\sigma_B}{d\Omega_q}(\hat{s},\hat{\theta})\Big\}\\ &= \int dx C_{2r}(x)\sigma_B(sx), \end{aligned} \quad (17)$$

where the $C_{2r}(x)$ function is calculated in Appendix B.

Note that in the above integral, the phase space is covered by the LO distribution completely, without any gap or overlap, provided the factorization scale variable M in both hemispheres is adjusted conveniently. Similarly, the entire integrand of the NLO correction is also defined all over the single-gluon phase space.

[8] A phase-space shape (upper limits) usually depends on the process type.

[9] Here $C_0^{[1]}$ is the interference term, while the other two terms are amplitude squares.

B. LO two-gluon $C_0^{(0)}\mathbb{P}'K_{0F}^{(0)}\mathbb{P}'K_{0F}^{(0)}$: A prelude for the LO ladder MC

Let us take the exact integrated distribution for the ladder diagram (no projections) with two-gluon emission,

$$\begin{aligned}\sigma_2 &= C_0^{(0)}\cdot K_{0F}^{(1)}\cdot K_{0F}^{(1)}\\ &= \int dx_1 \frac{d^3k_1}{2k_1^0}\frac{d^3k_2}{2k_2^0} d\tau_2(P-k_1-k_2;q_1,q_2)\\ &\quad\times \rho_B(p_{0F},p_{0B},k_1,k_2,q_1,q_2)s\delta_{sx_1=(P-k_1)^2},\end{aligned} \tag{18}$$

in which we have introduced explicitly a variable for the effective mass squared

$$\hat{s}_1 = sx_1 = (q_1+q_2+k_2)^2 = (P-k_1)^2$$

of the final state system after emitting the gluon k_1.

Let us start with the same operation of parametrization of the phase space in terms of $k_1 = \lambda_1\bar{k}_1$ as in the previous case of the single gluon, for a gluon at the end of the ladder:

$$\begin{aligned}C_0^{(0)}K_{0F}^{(1)}K_{0F}^{(1)} &= \int dx_1 \frac{d^3\bar{k}_1}{2\bar{k}_1^0}\frac{d^3k_2}{2k_2^0} d\tau_2(P-k_1-k_2;q_1,q_2)\\ &\quad\times \lambda_1^2\rho_B(p_{0F},p_{0B},\lambda_1\bar{k}_1,k_2,q_1,q_2)\delta_{x_1=1-\bar{\alpha}_1},\\ \lambda_1 &= \frac{s(1-x_1)}{2P\bar{k}_1} = \frac{\bar{\alpha}_1}{\bar{\alpha}_1+\bar{\beta}_1}.\end{aligned} \tag{19}$$

The factor λ_1^2 from the phase space is compensated for by a similar factor in the single-gluon distribution, as it is for the single-gluon case. No approximations or projections are present yet.

Insertion of the first $\mathbb{P}'$ requires examination of the collinear limit $a_1 \to 0$ ($\bar{\beta}_1 \to 0$) while keeping $\bar{\alpha}_1 =$ const; in this limit we also have $\beta_1 \to 0$, $\bar{\alpha}_1 \to \alpha_1$, and $\lambda_1 \to 1$. The spin projection operator of Eq. (8) is also used. The collinear limit is well known:

$$\begin{aligned}\lim_{a_1\to 0} a_1^2\rho_B(\lambda\bar{k}_1,k_2,\ldots) &= \frac{1}{\bar{\alpha}_1^2}\frac{2C_F\alpha_s}{\pi^2}\frac{1+(1-\bar{\alpha}_1)^2}{2}\\ &\quad\times\rho_1(x_1p_{0F},p_{0B};k_2,q_1,q_2),\end{aligned} \tag{20}$$

where ρ_1 is the already discussed distribution of the single-gluon emission ρ of Eq. (6), in the reduced center-of-mass system of $sx_1 = (x_1p_{0F}+p_{0B})^2$, provided we rename $k \to k_2$.

The formula obtained above, valid originally at the collinear point $a_1 = 0$, is now extrapolated to the off-collinear phase space using $\bar{k}_1$:

$$\begin{aligned}\sigma_{2F} &= C_0^{(0)}K_{0F}^{(1)}\mathbb{P}'K_{0F}^{(1)} = \int dx_1\frac{d^3\bar{k}_1}{2\bar{k}_1^0}\lambda_1^2\frac{d^3k_2}{2k_2^0}d\tau_2(P-\lambda_1\bar{k}_1-k_2;q_1,q_2)\frac{2C_F\alpha_s}{\pi^2}\theta_{a_2>a_1}\\ &\quad\times\frac{1+(1-\bar{\alpha}_1)^2}{2a_1^2\bar{\alpha}_1^2}\rho_1(x_1p_{0F},p_{0B};k_2,q_1,q_2)\delta_{x_1=1-\bar{\alpha}_1}\\ &= \int dx_1\delta_{x_1=z_1}\frac{C_F\alpha_s}{\pi}\frac{\bar{P}(z_1)}{1-z_1}\frac{da_1}{a_1}\frac{d\varphi_1}{2\pi}\theta_{a_2>a_1}d\tau_3(P-\lambda_1\bar{k}_1;q_1,q_2,k_2)\rho_1(z_1p_{0F},p_{0B};k_2,q_1,q_2),\end{aligned} \tag{21}$$

where the LO splitting kernel for the first emission of k_1,

$$P_{qq}^{(0)}(z) = \frac{\bar{P}(z)}{1-z} = \frac{1+z^2}{2(1-z)}$$

is factorized off explicitly, and the factorization scale for $\mathbb{P}'_{M_1}$ is just $M_1 = a_2$ of the gluon in the next $K_{0F}(k_2)$.

This fact—that the factorization scale for the first emission is defined to be a_2 of the second emission—is the essential difference from the standard EGMPR/CFP scheme [4,26], where $a_i < \mu$ for both emissions and therefore $a_2 \to 0$ is not blocked by a_1 as it is here. The EGMPR arrangement has the advantage of being similar to the system of UV subtractions [26], but it causes oversubtractions, unfriendly for the MC implementation. We assume implicitly a cutoff regularizing the $a_1 \to 0$ limit, for instance $a_1 > a_0$. Note that although the distribution of a_1 seems to be simple, we cannot perform $\int\frac{da_1}{a_1}$ to get a pure log, because the upper limit $a_2 > a_1$ is still nontrivial, and we have to wait until the next simplifications due to insertion of the second $\mathbb{P}'$ before getting the pure log from the integration.

Let us now insert the second $\mathbb{P}'$ into $C_0^{(0)}\mathbb{P}'K_{0F}^{(1)}\mathbb{P}'K_{0F}^{(1)}$. Again, we examine the limit $a_2 \to 0$, keeping $a_2/a_1 =$ const. While taking this limit, we keep $\bar{\alpha}_2 = x_1 - x_2 =$ const, such that $\hat{s} = (P-k_1-k_2)^2 =$ const, in addition to the previous $\hat{s}_1 =$ const. More precisely, we start by introducing

$$\begin{aligned}\hat{s} &= \hat{s}_2 = sx = sx_2 = (P-k_1-k_2)^2\\ &= \hat{s}_1 - 2(P-k_1)\cdot k_2\end{aligned}$$

as an integration variable:

$$\begin{aligned}\sigma_{2F} &= \int_{a_2>a_1} dx_1dx_2\frac{d^3\bar{k}_1}{2\bar{k}_1^0}\frac{d^3k_2}{2k_2^0}d\tau_2(P-\lambda_1\bar{k}_1-k_2;q_1,q_2)\\ &\quad\times\frac{2C_F\alpha_s}{\pi^2}\frac{\bar{P}(1-\bar{\alpha}_1)}{a_1^2\bar{\alpha}_1^2}\rho_1(x_1p_{0F},p_{0B};k_2,q_1,q_2)\\ &\quad\times\delta_{x_1=1-\bar{\alpha}_1}s\delta(sx_2-sx_1+2(P-k_1)\cdot k_2).\end{aligned} \tag{22}$$

Next, we perform the same transformations on δ functions accompanied by the rescaling $k_2 = \lambda_2 \bar{k}_2$:

$$\begin{aligned} s\delta(sx_2 - sx_1 + 2(P-k_1)\cdot k_2) &= \int dY s\delta(sx_2 - sx_1 + 2(P-k_1)\cdot k_2) Y^{-2} 2p_{0B}\cdot k_2 \delta(sx_2 - sx_1 + 2p_{0B}\cdot k_2 Y^{-1}) \\ &= \int dY s\delta(sx_2 - sx_1 + 2(P-k_1)\cdot Y\bar{k}_2) Y^{-1} 2p_{0B}\cdot \bar{k}_2 \delta(sx_2 - sx_1 + 2p_{0B}\cdot \bar{k}_2) \\ &= \frac{\lambda_2^{-1} 2p_{0B}\cdot \bar{k}_2}{2(P-k_1)\cdot \bar{k}_2} \delta\Big(x_2 - x_1 + \frac{2p_{0B}\cdot \bar{k}_2}{s}\Big) = \delta\Big(x_2 - x_1 + \frac{2p_{0B}\cdot \bar{k}_2}{s}\Big) = \delta_{x_1 - x_2 = \bar{\alpha}_2}, \end{aligned} \quad (23)$$

where

$$\lambda_2(\bar{k}_1, \bar{k}_2) = \frac{s(x_1 - x_2)}{2(P-k_1)\cdot \bar{k}_2} = \frac{s(x_1 - x_2)}{2(P - \lambda_1(\bar{k}_1)\bar{k}_1)\cdot \bar{k}_2}.$$

Note that the scaling factor $\lambda_2 \to 1$ in the collinear limit $\bar{\beta}_2 \to 0$.

Let us stress that the integral under consideration is now transformed into a new equivalent form, but the limit $a_2 \to 0$ is yet to be taken. In the transformed variables, the integral reads

$$\begin{aligned} \sigma_{2F} = \int_{a_2 > a_1 > a_0} & dx_1 dx_2 \frac{d^3\bar{k}_1}{2\bar{k}_1^0} \frac{d^3\bar{k}_2}{2\bar{k}_2^0} \\ & \times d\tau_2(P - \lambda_1\bar{k}_1 - \lambda_2\bar{k}_2; q_1, q_2) \frac{2C_F\alpha_s}{\pi^2} \frac{\bar{P}(1-\bar{\alpha}_1)}{a_1^2\bar{\alpha}_1^2} \\ & \times \lambda_2^2 \rho_1(x_1 p_{0F}, p_{0B}; \lambda_2\bar{k}_2, q_1, q_2) \delta_{x_1 = 1-\bar{\alpha}_1} \delta_{x_1 - x_2 = \bar{\alpha}_2}. \end{aligned} \quad (24)$$

Now we are ready to take the limit $a_2 \to 0$, keeping $a_1/a_2 = \mathrm{const}$ (also $a_0 \to 0$) and $\bar{\alpha}_i = \mathrm{const}$:

$$\begin{aligned} C_0^{(0)} \mathbb{P}' K_{0F}^{(1)} \mathbb{P}' K_{0F}^{(1)} = \int_{M > a_2 > a_1} & dx_1 dx_2 \frac{d^3\bar{k}_1}{2\bar{k}_1^0} \frac{d^3\bar{k}_2}{2\bar{k}_2^0} \\ & \times d\tau_2(P - \lambda_1\bar{k}_1 - \lambda_2\bar{k}_2; q_1, q_2) \\ & \times \frac{2C_F\alpha_s}{\pi^2} \frac{\bar{P}(x_1)}{\bar{\alpha}_1^2 a_1^2} \frac{2C_F\alpha_s}{\pi^2} \frac{\bar{P}(x_2/x_1)}{\bar{\alpha}_2^2 a_2^2} \\ & \times \frac{d\sigma_B}{d\Omega}(sx_2, \hat{\theta}) \delta_{x_1 = 1-\bar{\alpha}_1} \delta_{x_2 = 1-\bar{\alpha}_1-\bar{\alpha}_2}. \end{aligned} \quad (25)$$

Note that the λ_2^2 factor from the phase space and the matrix element cancels out as before. The above formula is the principal result of this subsection. It defines the double use of $\mathbb{P}'$, the transformation $k_i(\bar{k}_j)$, and its inverse $\bar{k}_j(k_i)$, $i, j = 1, 2$. Note that in the above formulas we could use the variables $z_1 = 1 - \bar{\alpha}_1 = x_1$ and $z_2 = (1 - \bar{\alpha}_1 - \bar{\alpha}_1)/(1 - \bar{\alpha}_1) = x_2/x_1$ instead of $\bar{\alpha}_i$, $i = 1, 2$. In the following discussion, we may find it useful to switch to the z_i, $i = 1, 2$ variables.

With the global factorization scale M inserted at the end of the LO ladder, the transverse-plane integration $\int_{M > a_2 > a_1 > a_0} \frac{da_2}{a_2} \frac{da_1}{a_1} = \frac{1}{2!} \ln^2 \frac{M}{a_0}$ now decouples and provides a pure double log[10]:

$$C_0^{(0)} \mathbb{P}' K_{0F}^{(1)} \mathbb{P}' K_{0F}^{(1)} = \frac{1}{2!} \ln^2 \frac{M}{a_0} \Big(\frac{2C_F\alpha_S}{\pi}\Big)^2 \times \int_0^1 dx [P_{qq} \otimes P_{qq}]_{2R}(x) \sigma_B(sx), \quad (26)$$

where

$$4[P_{qq}^{(0)} \otimes P_{qq}^{(0)}]_{2R}(z) = \frac{1+z^2}{1-z}\Big[4\ln\frac{1}{\delta} + 4\ln(1-z)\Big] + (1+z)\ln z - 2(1-z)$$

is just a double convolution of the LO kernel with the IR regularization $\alpha_i > \delta$.

The distribution of Eq. (25) is easy to generate in the Monte Carlo simulation. First, one generates a_i and $\bar{\alpha}_i$, paying attention to the constraint $x = x_2 = 1 - \bar{\alpha}_1 - \bar{\alpha}_2$, and $\bar{k}_i$ are constructed in the laboratory frame. Then λ_i are calculated, and the rescaling $\bar{k}_i^\mu \to k_i^\mu$ is done (in two steps). Finally, in the frame $P - k_1 - k_2$, one generates q_i according to the Born differential distribution. The phase-space boundary $\hat{s} \ge 0$ is obeyed automatically. In the MC simulation, the soft IR regulator $z_i = x_i/x_{i-1} < 1 - \delta$ will be introduced and the overall virtual Sudakov form factor $\exp(-\frac{2C_F\alpha_s}{\pi} \ln\frac{M}{q_0} \ln\frac{1}{\delta})$ will also be supplemented.

The above example demonstrates the most important features of the $\mathbb{P}'$ projector (see Ref. [16] for more details). In particular, the following lessons are to be learnt:

(a) The phase space parametrization $k_i \to \bar{k}_i$ plays an important role in $\mathbb{P}'$, as it is instrumental in implementing off-collinear extrapolation, and also helps in taking the collinear limit in the first place.

(b) The rescaled four-momenta $\bar{k}_i$ violate four-momentum conservation, similarly to the four-momenta after the action of the kinematical projector of Refs. [4,27]. In our case, however, the off-collinear extrapolation is *effectively undoing* this kinematical projection[11] and allows us to operate in

[10] With our phase-space parametrization in terms of $\bar{k}_i$, the above mechanism of producing pure logs is a general phenomenon, because M is always at the end of the ladder, and the ladder has a built-in time-ordered exponential.

[11] The undoing is "effective" because the kinematical projection of Refs. [4,27] in our methodology is never done. A clever parametrization of the phase space is the only thing really done.

the original phase space, with four-momentum conservation untouched.

(c) A nice accident of the Jacobian $|\partial(\bar{k}_1, \bar{k}_2)/\partial(k_1, k_2)|$ being compensated by the matrix element is generally not guaranteed. However, if it were not true, we would have to impose this by hand, such that a pure logarithm resulted from the phase-space integration, like in Eq. (26), similarly to the definition of the collinear counterterm in Refs. [9,31], for example.

(d) The role of the parametrization $k_i \to \bar{k}_i$ in assuring universality (process independence) of the ladder parts will be clarified once the MC for the DY and DIS processes with NLO corrections to the hard part are defined; see below.

(e) The phase-space parametrization in terms of $\bar{k}_i$ will also be used inside K_0 to parametrize the two-gluon phase space; for instance, for the two-gluon crossed diagram in the NLO ladder.

(f) The soft eikonal limit is protected by $\mathbb{P}'$, because rescaling $\bar{k}_i \to k_i$ preserves it.

For a better (complete) understanding of the construction of $\mathbb{P}'$, one needs to examine in fine detail the case of two gluons in the middle of the ladder; for instance,[12] in $C_0\mathbb{P}'K_{0F}(1-\mathbb{P}')K_{0F}$, which provides the NLO correction to the evolution kernel.

III. MONTE CARLO FOR EW BOSON PRODUCTION

The insertion of the $\mathbb{P}'$ operator into the LO gluonstrahlung ladder with any number of gluons can be done in similar fashion to Sec. II B to obtain a distribution ready for the LO MC modeling of the production process of the EW boson with multiple gluons emitted from the incoming quarks.

A. Simplified single-ladder case

We start with the gluonstrahlung ladder in just one hemisphere in order to avoid algebraic complications of the two-ladder case (to be dealt with in the next subsection):

$$C_0^{(0)} \cdot \Gamma_F = \sum_{n=1}^{\infty} \{C_0^{(0)}(\mathbb{P}'K_{0F}^{(1)})^n\}_{\text{T.O.}}$$
$$= e^{-S_F} \sum_{n=0}^{\infty} \int dx \bigg(\prod_{i=1}^{n} d^3\mathcal{E}(\bar{k}_i)\theta_{\eta_i<\eta_{i-1}} \frac{2C_F\alpha_s}{\pi^2}\bar{P}(z_i)\bigg)$$
$$\times d\tau_2\bigg(P - \sum_{j=1}^{n} k_j; q_1, q_2\bigg)\theta_{\Xi<\eta_n}\delta_{x=\prod_{j=1}^{n} z_j}$$
$$\times \frac{d\sigma_B}{d\Omega}(sx, \hat{\theta}), \tag{27}$$

where

$$k_i = \lambda_i \bar{k}_i, \qquad \lambda_i = \frac{s(x_{i-1} - x_i)}{2(P - \sum_{j=1}^{i-1} k_j)\cdot \bar{k}_i},$$
$$x_i = 1 - \sum_{j=1}^{i} \bar{\alpha}_i = \prod_{j=1}^{i} z_j, \qquad z_i = \frac{x_i}{x_{i-1}},$$

and $\bar{P}(z) = \frac{1}{2}(1+z^2)$. The a ordering, $M > a_i > a_{i-1}$ $i = 1, \ldots, n$, is rephrased into an equivalent rapidity ordering, $\Xi < \eta_i < \eta_{i-1} < \eta_0$, where $M = \sqrt{s}e^{-\Xi}$; see the definitions of the phase-space integration element $d^3\mathcal{E}(k)$ and other kinematic notations in Appendix A. The T.O. subscript stands for the time-ordering exponential structure in the factorization scale; see Ref. [16] for a general definition. The S_F function is the usual MC Sudakov form factor depending on the shape of the IR boundary, $\bar{\alpha}_i \to 0$. The factorization scale is now defined as the minimum rapidity Ξ for gluons in the F hemisphere (maximum rapidity for gluons in the B hemisphere). For the moment, Ξ is a free parameter, to be defined more precisely later on.

Note that the definition of λ_i is *recursive*; that is, to define λ_i, one must know λ_{i-1}. In a typical MC event, the first λ's, corresponding to very collinear gluons, will be very close to 1 ($\lambda_i \simeq 1$); only the last λ's, corresponding to *noncollinear*, *nonsoft* gluons—i.e., close to the hard process—will be rescaled by a significant $\lambda_i \neq 1$ factor.

Similarly to the case of two gluons in Eq. (26), the transverse integration decouples and is feasible analytically:

$$C_0^{(0)} \cdot \Gamma_F = \int_0^1 dx G_F(M, x)\sigma_B(sx),$$
$$G_F(M, x) = e^{-\frac{2C_F\alpha_S}{\pi}\ln\frac{1}{\Delta}\ln\frac{M}{q_0}}\bigg\{\delta_{x=1} + \sum_{n=1}^{\infty}\frac{1}{n!}\bigg(\frac{2C_F\alpha_S}{\pi}\bigg)^n$$
$$\times \ln^n \frac{M}{q_0}[P_{qq}^{(0)}]^{\otimes n}(x)\bigg\}, \tag{28}$$

where the IR regularization $(1-z_i) < \Delta$ is used in the n-times convolution of the LO kernel $[P_{qq}^{(0)}]^{\otimes n}$. It should be stressed that the above LO formula represents the LO MC without any approximation.

The inclusive (bare) PDF $G_F(M, x)$ of the MC obeys by construction the LO evolution equation:

$$\frac{\partial}{\partial \ln M} G_F(M, x) = [\mathcal{P}_{qq}^{(0)} \otimes G_F(M)](x), \tag{29}$$

where $\mathcal{P}_{qq}^{(0)}(z) = \frac{2C_F\alpha_S}{\pi}[\frac{1+z^2}{2(1-z)}]_+$. This is essentially due to the use of the T.O. exponent in Eq. (27).

B. Two-ladder LO case

Let us now consider the case of two ladders. In the backward (B) hemisphere, x_i are related to $\bar{\beta}_i$ (instead of $\bar{\alpha}_i$), and the evolution runs towards larger rapidity; otherwise, all algebraic structure is the same. Again, we first express the LO MC master formula in terms of rescaled

[12] This task is pursued in separate works [13,17].

four-momenta $\bar{k}_i = \lambda_i k_i$, but we postpone the definition of λ_i, as it will be a little bit special. We propose the following multigluon distribution to be implemented in the LO approximation[13]:

$$
\begin{aligned}
C_0^{(0)} \cdot \Gamma_F^{(1)} \cdot \Gamma_B^{(1)} &= \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} [C_0^{(0)} (\cdot \mathbb{P}' K_{0F}^{(1)})^{n_1} (\cdot \mathbb{P}' K_{0B}^{(1)})^{n_2}]_{\mathrm{T.O.}} \\
&= \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \int dx_F dx_B e^{-S_F} \int_{\Xi<\eta_{n_1}} \Big(\prod_{i=1}^{n_1} d^3\mathcal{E}(\bar{k}_i) \theta_{\eta_i<\eta_{i-1}} \frac{2C_F\alpha_s}{\pi^2} \bar{P}(z_{Fi}) \Big) \delta_{x_F = \prod_{i=1}^{n_1} z_{Fi}} \\
&\quad \times e^{-S_B} \int_{\Xi>\eta_{n_2}} \Big(\prod_{j=1}^{n_2} d^3\mathcal{E}(\bar{k}_j) \theta_{\eta_j>\eta_{j-1}} \frac{2C_F\alpha_s}{\pi^2} \bar{P}(z_{Bj}) \Big) \delta_{x_B = \prod_{j=1}^{n_2} z_{Bj}} d\tau_2 \Big(P - \sum_{j=1}^{n_1+n_2} k_j; q_1, q_2 \Big) \frac{d\sigma_B}{d\Omega}(s x_F x_B, \hat{\theta}).
\end{aligned}
\tag{30}
$$

In the above, we understand that the F part is the forward part of the phase space $\eta_{0F} > \eta_i > \Xi$ and the B part is the backward part $\Xi > \eta_i > \eta_{0B}$. The rapidity boundary Ξ between the F and B parts is kept as a free parameter as long as possible, to be adjusted later on. In particular, $\Xi = 0$ is perfectly legal and may serve as the first choice before something better is found.[14] The variables z_{Fi} and z_{Bj} could be defined similarly to the single-hemisphere case, provided we perform kinematical mappings asymmetrically, first in one hemisphere and then in the other one. We shall do it below, however, in a more sophisticated, symmetric way. The definition of z_{Fi} and z_{Bj} will result from that.

The angle $\hat{\theta}$ can be defined with respect to any reasonable z axis in the rest frame of $P - \sum k_j$; for instance, along $\vec{p}_{0F} - \vec{p}_{0B}$ in this frame. The boundary between the F and B phase spaces is at the rapidity Ξ. It is also understood that the differential cross section of Eq. (30) is implicitly convoluted with some initial quark distributions at η_{0F} and η_{0B}, and the appropriate boost is done from the reference frame of the quark-antiquark frame $p_{0F} + p_{0B}$ to the laboratory frame.

Again, in Eq. (30), phase space can be integrated analytically over transverse momenta, providing the classical factorization formula

$$
C_0^{(0)} \Gamma_F^{(1)} \Gamma_B^{(1)} = \int_0^1 dx_F dx_B G_F(\Xi, x_F) G_B(\Xi, x_B) \sigma_B(s x_F x_B),
\tag{31}
$$

where $G_B(\Xi, x) = G_F(\Xi, x)$. The remarkable feature is that the above LO formula represents the exact LO MC without any approximations.

We could define the dilatation parameters λ_i (recursively) first for one ladder and then for the other, but this solution would be asymmetric. Instead, we define the dilatation parameters in $k_i = \lambda \bar{k}_i$ in a more sophisticated way, for both hemispheres simultaneously. For that purpose, we introduce a new ordering (indexing) of gluons according to the distance $|\eta_i - \Xi|$ from the rapidity Ξ of the EW boson.[15] Formally, we define a permutation

$$
\pi = \{\pi_1, \pi_2, \ldots, \pi_{n_1+n_2}\}
$$

of all gluons in the $F + B$ phase space, such that

$$
|\eta_{\pi_i} - \Xi| > |\eta_{\pi_{i-1}} - \Xi|, \qquad i = 1, \ldots, n_1 + n_2.
$$

With the help of the above simultaneous "double ordering" in the F and B hemispheres, we define in a *recursive* way

$$
k_{\pi_i} = \lambda_i \bar{k}_{\pi_i}, \qquad \lambda_i = \frac{s(x_{i-1} - x_i)}{2(P - \sum_{j=1}^{i-1} k_{\pi_j}) \cdot \bar{k}_{\pi_i}}, \qquad i = 1, 2, \ldots, n_1 + n_2,
\tag{32}
$$

where x_i now is

$$
x_i = \prod_{j=1}^{i} z_{(F,B)\pi_j},
$$

where (F, B) means that we insert in the above product either z_{Fj} or z_{Bj}, depending whether π_j points to the F or B region. The parameter λ_i is defined in Eq. (32) recursively by means of solving the following equation:

$$
\bar{s}_i = s x_i = \Big(P - \sum_{j=1}^{i} k_{\pi_j} \Big)^2 = \Big(P - \sum_{j=1}^{i} \lambda_j \bar{k}_{\pi_j} \Big)^2.
$$

The LO Monte Carlo algorithm using the above algebraic framework can be described step by step as follows:

(1) The variables x_F and x_B are generated with the help of the FOAM program [32], then parton (gluon)

[13]The reader should keep in mind that the above is for the "primordial" quark and antiquark initial beams, and their distributions in hadrons will be added in the MC program.

[14]In practice, the choice of Ξ may be quite complicated; for instance, it can be correlated with the rapidity position of the EW boson. In the NLO correction calculation, for a single gluon, $\Xi = 0$ should be set.

[15]Which is also the phase-space boundary between the F and B ladders.

multiplicities $n_{1,2}$ and variables z_{Fi} and z_{Bj} are generated using the constrained MC (CMC) algorithm [33].[16] The Ξ variable is set.

(2) The four-momenta $\bar{k}_i^\mu$ are defined separately in the F and B parts of the phase space using the CMC module, with the corresponding constraints: $1-\sum_{j\in F}\bar{\alpha}_j = \prod_{j\in F} z_{Fj} = x_F$ and $1-\sum_{j\in B}\bar{\beta}_j = \prod_{j\in B} z_{Bj} = x_B$.

(3) The permutation π with simultaneous ordering in $F+B$ space is established.

(4) Using P and $\bar{k}_{\pi_1}$, the rescaling parameter λ_1 is calculated, and then $k_{\pi_1} = \lambda_1 \bar{k}_{\pi_1}$ is set. At this stage $(P-k_{\pi_1})^2 = sx_1$, where $x_1 = z_{\pi_1} = 1-\bar{\alpha}_{\pi_1}$ or $x_1 = z_{\pi_1} = 1-\bar{\beta}_{\pi_1}$, depending on whether k_{π_1} was in the F or B part of the phase space.

(5) Using $P-k_{\pi_1}$ and $\bar{k}_{\pi_2}$, the parameter λ_2 is found and $k_{\pi_2} = \lambda_2 \bar{k}_{\pi_2}$ is set, enforcing $(P-k_{\pi_1}-k_{\pi_2})^2 = sx_2 = sz_{\pi_1}z_{\pi_2}$. The recursive procedure continues until the last gluon.

(6) In the rest frame of $\hat{P} = P - \sum_j k_{\pi_j}$, the lepton four-momenta q_1^μ and q_2^μ are generated according to the Born angular distribution.

The definition of z_{Fj} and z_{Bj} in terms of $\bar{\alpha}_j$, $j\in F$ and $\bar{\beta}_j$, $j\in B$ follows from the above algorithm and is more complicated than in the case of one hemisphere. The main advantage of the above scenario is that this way the kinematics of the two hemispheres are interrelated very gently, starting from very collinear gluons (for which $\lambda_i \simeq 1$) and finishing with the least collinear ones, next to the hard-process EW boson. (The angular ordering is the same for k_i and $\bar{k}_i$.)

As compared to Ref. [34], where a similar algorithm based on CMC [33] and rescaling of the four-momenta was proposed, the present algorithm does the "rescaling"[17] in a more sophisticated way. The rescaling affects mainly the hard, noncollinear gluons, not the soft and collinear ones, while the rescaling in Ref. [34] was global, similarly to the global manipulations on the four-momenta (boosts and rescaling) used in other parton-shower MCs [7,8] in order to impose four-momentum conservation. Moreover, kinematic parametrization of the phase space in the present MC is based on the projector $\mathbb{P}'$ of the collinear factorization (instead of being *ad hoc*), which is essential for completing the NLO.

C. Real NLO correction to the hard process

The NLO correction in the EW boson production hard process (nonsinglet) will be implemented using a single "monolithic" MC weight, which reweighs the LO distributions defined in the previous subsection. The real emission part of the NLO correction in this weight comes from the integrand of Eq. (17), which we rewrite as follows:

$$C^{[1]} = C_0^{[1]} + C_0^{[0]}(1-\mathbb{P}')K_{0F}^{(1)} + C_0^{[0]}(1-\mathbb{P}')K_{0B}^{(1)} = \int d^3\mathcal{E}(k)d\Omega_q \frac{2C_F\alpha_s}{\pi^2}\tilde{\beta}_1(\hat{p}_F,\hat{p}_B;q_1,q_2,k);$$

$$\tilde{\beta}_1(\hat{p}_F,\hat{p}_B;q_1,q_2,k) = \left[\frac{(1-\beta)^2}{2}\frac{d\sigma_B}{d\Omega_q}(\hat{s},\theta_F) + \frac{(1-\alpha)^2}{2}\frac{d\sigma_B}{d\Omega_q}(\hat{s},\theta_B)\right] - \theta_{\alpha>\beta}\frac{1+(1-\alpha-\beta)^2}{2}\frac{d\sigma_B}{d\Omega_q}(\hat{s},\hat{\theta}) - \theta_{\alpha<\beta}\frac{1+(1-\alpha-\beta)^2}{2}\frac{d\sigma_B}{d\Omega_q}(\hat{s},\hat{\theta}). \tag{33}$$

In the following use of the function $\tilde{\beta}_1(\hat{p}_F,\hat{p}_B,\ldots)$, defined in Eq. (33), the vectors $\hat{p}_F$ and $\hat{p}_B$ ($\hat{p}_{F,B}^2 = 0$) result from the last insertion of $\mathbb{P}'$ before the hard process, and they are defined in the rest frame of $\hat{P} = q_1 + q_2$ to determine $\hat{\theta} = \angle(\vec{q}_1, \vec{p}_{0F})$ in the LO part of the Born cross section. On the other hand, the angles θ_F and θ_B are defined with respect to the original $-\vec{p}_{0B}$ and $\vec{p}_{0F}$ in this frame. They will all coincide when all gluons become collinear.

The distribution of the MC with the LO + NLO hard process is now defined as follows:

$$C^{(1)}\Gamma_F^{(1)}\Gamma_B^{(1)} = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\int dx_F dx_B e^{-S_F}\int_{\Xi<\eta_{n_1}}\left(\prod_{i=1}^{n_1} d^3\mathcal{E}(\bar{k}_i)\theta_{\eta_i<\eta_{i-1}}\frac{2C_F\alpha_s}{\pi^2}\bar{P}(z_{Fi})\right)\delta_{x_F=\prod_i z_{Fi}} e^{-S_B} \times \int_{\Xi>\eta_{n_2}}\left(\prod_{j=1}^{n_2} d^3\mathcal{E}(\bar{k}_j)\theta_{\eta_j>\eta_{j-1}}\frac{2C_F\alpha_s}{\pi^2}\bar{P}(z_{Bj})\right)\delta_{x_B=\prod_j z_{Bj}} d\tau_2\left(P-\sum_{j=1}^{n_1+n_2}k_j;q_1,q_2\right)\frac{d\sigma_B(sx_Fx_B,\hat{\theta})}{d\Omega}W_{\rm MC}^{\rm NLO}, \tag{34}$$

[16] In the CMC algorithm, parton multiplicity is generated as the first variable, contrary to the Markovian and backward evolution algorithms, where parton multiplicity is generated at the end of the MC algorithm.

[17] The "rescaling" in our case is merely a synonym for parametrization of the exact phase space in terms of the rescaled four-momenta.

where the MC weight is

$$W_{\rm MC}^{\rm NLO} = 1 + \Delta_{S+V} + \sum_{j\in F} \frac{\tilde{\beta}_1(\hat{s}, \hat{p}_F, \hat{p}_B; a_j, z_{Fj})}{\bar{P}(z_{Fj}) d\sigma_B(\hat{s}, \hat{\theta})/d\Omega} + \sum_{j\in B} \frac{\tilde{\beta}_1(\hat{s}, \hat{p}_F, \hat{p}_B; a_j, z_{Bj})}{\bar{P}(z_{Bj}) d\sigma_B(\hat{s}, \hat{\theta})/d\Omega}, \tag{35}$$

with the NLO soft + virtual correction Δ_{V+S} to be defined separately in the following subsection.

Our construction of the above MC weight of Eq. (35) is quite similar to that proposed in Ref. [35] for the NLO corrections in the middle of the ladder.[18] The difference is that the proposal of Ref. [35] was based entirely on the Bose-Einstein symmetrization of the multigluon emission in the LO MC, and the resulting weight was more complicated, while the MC weight of Eq. (35) is significantly simpler and algebraically similar to that of Ref. [36] (albeit for the collinear rather than soft gluon resummation).

The meaning of the arguments in $\tilde{\beta}_1$ is such that in Eq. (33) $\hat{s} = s x_F x_B$, and the three angles $\hat{\theta}$, θ_F, and θ_B have already been explained above. What remains to be specified is the definition of α_j and β_j in terms of the variables a_j, z_j in the $j \in F, B$ parts of the phase space, as follows:

$$\alpha_j = 1 - z_{Fj}, \qquad \beta_j = \alpha_j a_j^2/a_\Xi^2 \quad \text{for } j \in F,$$
$$\beta_j = 1 - z_{Bj}, \qquad \alpha_j = \beta_j a_\Xi^2/a_j^2 \quad \text{for } j \in B,$$

where the rapidity η_j is translated properly into a_j, and the rapidity Ξ corresponds to a_Ξ.

As compared to earlier attempts to consistently implement the NLO corrections to the hard process in the parton-shower MC, the proposals of Refs. [34,37] were going in a similar direction. However, the present work differs from these works in three important points: (i) the virtual corrections Δ_{S+V} are added here; (ii) the method of combining the NLO correction with the LO MC proposed here is systematic and NLO complete; (iii) the treatment of the kinematics is compatible with the principles of the collinear factorization.

Comparison of our methodology with the well-established methods of MC@NLO [9] and POWHEG [11] is done in Sec. III F.

D. NLO analytical factorization formula

For the LO MC defined in Eq. (30) we have seen that, without compromising the exact phase space for multiple gluons (keeping the four-momentum conservation), we could get the contributions of ladder parts to factorize off, exactly as in the traditional collinear factorization (in which the four-momentum is not conserved). The above seems to be almost miraculous in view of the complicated nature of the exact complete phase space and the fact that no approximation was done.

What is even more amazing is that the same nice, exact factorization is also true for the MC with the NLO-corrected hard process included, according to Eq. (34). The key point is that analytical integration of the phase space, again without any approximation, in Eq. (34) is feasible and leads to a simple, familiar result:

$$C^{(1)}\Gamma_F^{(1)}\Gamma_B^{(1)} = \int_0^1 dx_F dx_B dz G_F(\Xi, x_F) G_B(\Xi, x_B) \times \sigma_B(s z x_F x_B)\{\delta_{z=1}(1 + \Delta_{S+V}) + C_{2r}(z)\}, \tag{36}$$

where $C_{2r}(z) = \frac{2C_F\alpha_s}{\pi}[-\frac{1}{2}(1-z)]$; see Appendix B. Two LO PDFs, $G_F(\Xi, x_F)$ and $G_B(\Xi, x_B)$, are those of Eq. (31). The algebraic proof of the above formula can be found in Appendix C.[19]

Note that as the LO PDFs $G_{F,B}(\Xi, x)$ are, by construction in the collinear factorization scheme, specific for the MC, it will be the same in the MC for the DIS process—the scheme dependence of physical observables will cancel as usual—for instance, while transferring experimental knowledge on the parton distributions from the DIS process to the DY process (or vice versa).

Let us comment on the z-dependent $C_{2r}(z)$ term in Eq. (36), since it is different from what we see in Ref. [30],[20] where it is simply absent. It is not present there because it is compensated by the twin terms $\sim -\frac{1}{2}\varepsilon(1-z)\frac{1}{\varepsilon}\delta(1-y^2)$ originating from the γ traces and located at $y = \pm 1$; that is, exactly at the collinear poles (for gluons strictly parallel with the quark). This cancellation is not disturbed by the $\overline{MS}$ collinear counterterm subtraction. In our MC factorization scheme, this term is included in the counterterm; hence, the net contribution from one of these terms, contrary to $\overline{MS}$, stays uncanceled in Eq. (36).

E. NLO soft + virtual corrections

Let us consider the unsubtracted results of the Altarelli-Ellis-Martinelli (AEM) work [30],[21] in which real and virtual single-gluon emission diagrams are combined and integrated over the phase space, keeping the variable $z = x = \hat{s}/s = 1 - \alpha - \beta$ under control:

[18] The MC weight of Ref. [35], when applied to the hard process, would also render complete NLO in the hard process and provide the same inclusive LO + NLO cross section.

[19] A quite similar formula for introducing the NLO correction to the LO kernel in the middle of the ladder was already tested numerically in the MC exercise with four-digit precision; see Ref. [16].

[20] It is also absent in QED, in the well-known formula of Bonneau-Martin [38].

[21] See Eq. (89) there.

$$\sigma_{\rm DY}^{\rm AEM} = \int_0^1 dz f_{\rm DY}^{\rm AEM}(z)\sigma_B(zs),$$
$$f_{\rm DY}^{\rm AEM}(z) = \delta(1-z) + \delta(1-z)\frac{C_F\alpha_s}{\pi}\left(\frac{2}{3}\pi^2 - \frac{7}{4}\right) + 2\frac{C_F\alpha_s}{\pi}\left(\frac{\hat{s}}{\mu^2}\right)^{\varepsilon}\left(\frac{\bar{P}(z)}{1-z}\right)_+\left(\frac{1}{\varepsilon}+\omega_2\right) + \frac{C_F\alpha_s}{\pi}\left\{4\bar{P}(z)\frac{\ln(1-z)}{1-z} - 2\bar{P}(z)\frac{\ln z}{1-z}\right\}_+, \quad (37)$$

where $\bar{P}(z) = \frac{1}{2}(1+z^2)$, $\omega_2 = \gamma_E - \ln 4\pi$, and the incoming quark and antiquark are massless and on shell. The NLO real single-gluon emission distribution entering in the above AEM result is identical to $D^{[1]}$ of Eq. (15), and $\sigma_{\rm DY}^{\rm AEM}$ includes all (gluonstrahlung) virtual corrections, including the quark wave-function renormalization constant Z_F. Since $\sigma_{\rm DY}^{\rm AEM}$ is gauge invariant, the calculation of Eq. (37) is done in Ref. [30] in the convenient Landau gauge.[22]

In the formal, standard MS methodology, one subtracts from $f_{\rm DY}^{\rm AEM}(z)$ two LO collinear counterterms

$$K_F^{MS}(z) + K_B^{MS}(z) = 2K_F^{MS}(z) = 2\frac{C_F\alpha_s}{\pi}\left(\frac{\bar{P}(z)}{1-z}\right)_+\frac{1}{\varepsilon}, \quad (38)$$

in order to avoid double counting with the ladder and/or experimental PDF, thus obtaining $C_0(1 - \mathbb{P}K_{0F} - \mathbb{P}K_{0B})$. This gives rise to the standard subtracted DY analog of the DIS coefficient function in the MS scheme:

$$f_{\rm DY}^{\rm MS}(z) = f_{\rm DY}^{\rm AEM}(z) - K_F^{MS}(z) - K_B^{MS}(z) = \delta(1-z) + \delta(1-z)\frac{C_F\alpha_s}{\pi}\left(\frac{2}{3}\pi^2 - \frac{7}{4}\right) + 2\frac{C_F\alpha_s}{\pi}\left(\frac{\bar{P}(z)}{1-z}\right)_+\left(\ln\frac{\hat{s}}{\mu^2}+\omega_2\right) + \frac{C_F\alpha_s}{\pi}\left\{4\bar{P}(z)\frac{\ln(1-z)}{1-z} - 2\bar{P}(z)\frac{\ln z}{1-z}\right\}_+. \quad (39)$$

Note that the $\ln\frac{\hat{s}}{\mu^2}+\omega_2$ term in the above equation will be absent if the relation $\hat{s} = \mu^2 e^{-\omega_2}$ is adopted,[23] as in Refs. [39,40]. It is well known [30] that the numerically dominant term $\frac{\ln(1-z)}{(1-z)_+}$ in the above function is correcting for the misrepresentation of the soft gluon behavior and incorrect phase-space limits of the $\overline{MS}$ dimensional regularization (subtraction) scheme. In our MC scheme, this term will not be present (it gets transferred to the ladder).

The plus regularization $(\ldots)_+$ of the IR singularity in Eq. (38), in the diagrammatic approach of the CFP [26], comes from Z_F in the axial gauge. It is also shown by CFP that there is a diagram-per-diagram correspondence between Feynman diagrams in the axial gauge and the diagrams of the operator product expansion (OPE) [41,42].[24]

In the context of the subtraction/factorization scheme aligned with the MC, we should apply Eq. (17); that is, in order to avoid double counting with the ladder (PDF), we should subtract from Eq. (6) the LO contribution of the MC:

$$\rho_{1Fc}(\alpha,\beta) = \frac{1}{\beta}\frac{C_F\alpha_s}{\pi}\frac{\bar{P}(1-\alpha-\beta)}{\alpha}\theta_{\beta<\alpha}.$$

However, in order to combine it properly with the virtual corrections, the above distribution should be extrapolated to $n = 4+2\varepsilon$ and integrated over the phase space. This is done for the F hemisphere in the following (keeping in mind that $1-\alpha-\beta = 1-\bar{\alpha}$):

$$K_F^{\rm MC}(z,\varepsilon) = \frac{C_F\alpha_s}{\pi}\int\frac{d\bar{\alpha}d\bar{\beta}}{\bar{\alpha}\bar{\beta}}\int d\Omega_{2+2\varepsilon}\left(\frac{\hat{s}\,\bar{\alpha}\,\bar{\beta}}{z\mu^2}\right)^{\varepsilon}\bar{P}(1-\bar{\alpha},\varepsilon) \times\theta_{\bar{\beta}<\bar{\alpha}}\delta_{\bar{\alpha}=1-z}\theta_{\bar{\alpha}>\delta} - \delta(1-z)S_{\rm MC}(s,\varepsilon)$$
$$= \frac{C_F\alpha_s}{\pi}\left(\frac{\hat{s}}{z\mu^2}\right)^{\varepsilon}\frac{\Omega_{2+2\varepsilon}}{\varepsilon}\frac{\bar{P}(z,\varepsilon)}{(1-z)^{1-2\varepsilon}}\theta_{1-z>\delta} - \delta(1-z)S_{\rm MC}(s,\varepsilon)$$
$$= \frac{C_F\alpha_s}{\pi}\left(\frac{\bar{P}'(z,\varepsilon)}{1-z}\left[\frac{1}{\varepsilon}+\omega_2+\ln\frac{\hat{s}}{z\mu^2}\right]\right)_+, \quad (40)$$

where the $\mathcal{O}(\varepsilon)$ contribution from the γ trace is added to the LO kernel,

$$\bar{P}'(z,\varepsilon) = \bar{P}(z)(1+2\varepsilon\ln(1-z)) + \frac{1}{2}\varepsilon(1-z)^2,$$
$$\bar{P}(z,\varepsilon) = \bar{P}(z) + \frac{1}{2}\varepsilon(1-z)^2.$$

The overall plus prescription is coming from the first-order expansion of the Sudakov form factor in the MC in $n = 4+2\varepsilon$ dimensions and from the usual sum rule $\int dz K_F^{\rm MC}(z,\varepsilon) = 0$ (treating $\hat{s}$ as z independent):

$$S_{\rm MC}(\hat{s},\varepsilon) = \frac{C_F\alpha_s}{\pi}\frac{\Omega_{2+2\varepsilon}}{\varepsilon}\int_0^{1-\delta}dz\frac{\bar{P}(z,\varepsilon)}{(1-z)^{1-2\varepsilon}}\left(\frac{\hat{s}}{z\mu^2}\right)^{\varepsilon}, \quad (41)$$

with both $\mathcal{O}(\varepsilon)$ terms in $\bar{P}'(z,\varepsilon)$ necessarily participating in the $(\ldots)_+$ prescription.[25]

[22] In the Landau gauge, $Z_F = 1$ up to the NLO level.
[23] This corresponds to the use of $\frac{1}{\varepsilon}+\omega_2$ instead of the pure pole in the counterterm and setting $\hat{s} = \mu^2$.

[24] In this sense, the axial gauge is implicitly present in Eq. (38), while the unsubtracted Eq. (37) is gauge invariant.
[25] In the MC practice, regularization of $1/(1-z)$ is done with some cutoff $1-z>\delta$ rather than with ε of the dimensional regularization.

The expression of Eq. (37) is the result of the following phase-space integration:

$$f_{\rm DY}^{\rm AEM}(z,\varepsilon)=\frac{C_F\alpha_s}{\pi}\int\frac{d\alpha d\beta}{\alpha\beta}\int d\Omega_{2+2\varepsilon}\left(\frac{s\alpha\beta}{\mu^2}\right)^{\varepsilon}\times\rho_1(\alpha,\beta)\delta_{1-z=\alpha+\beta}\theta_{\alpha+\beta>\delta}-\delta(1-z)U_{S+V}, \quad (42)$$

where $\rho_1(\alpha,\beta)$ is the one-real-gluon distribution, and U_{S+V} sums up the real soft gluon $\alpha+\beta>\delta$ and vertex virtual contribution. The complete single-gluon contribution (including the expanded Sudakov form factor) of the LO MC reads

$$\begin{aligned}K_F^{\rm MC}(z,\varepsilon)+K_B^{\rm MC}(z,\varepsilon)&=\frac{C_F\alpha_s}{\pi}\int\frac{d\bar\alpha d\bar\beta}{\bar\alpha\,\bar\beta}d\Omega_{2+2\varepsilon}\left(\frac{\hat s\,\bar\alpha\,\bar\beta}{z\mu^2}\right)^{\varepsilon}[\bar P(1-\bar\alpha,\varepsilon)\\&\quad\times\theta_{\bar\beta<\bar\alpha}\delta_{1-z=\bar\alpha}+\bar P(1-\bar\beta,\varepsilon)\theta_{\bar\beta>\bar\alpha}\delta_{1-z=\bar\beta}]\\&\quad\times\theta_{1-z>\delta}-\delta(1-z)2S_{\rm MC}(s,\varepsilon).\end{aligned} \quad (43)$$

The difference between the complete NLO of Eq. (37) and the above LO MC contribution, after partial phase-space integration, reads

$$\begin{aligned}f_{\rm DY}^{\rm MC}(z)&=f_{\rm DY}^{\rm AEM}(z,\varepsilon)-K_F^{\rm MC}(z,\varepsilon)-K_B^{\rm MC}(z,\varepsilon)\\&=-\frac{C_F\alpha_s}{\pi}(1-z)+\delta(1-z)\Delta_{V+S},\\\Delta_{V+S}&=\frac{C_F\alpha_s}{\pi}\left(\frac{2}{3}\pi^2-\frac{7}{4}\right)+\frac{C_F\alpha_s}{\pi}\frac{1}{2}\\&=\frac{C_F\alpha_s}{\pi}\left(\frac{2}{3}\pi^2-\frac{5}{4}\right).\end{aligned} \quad (44)$$

From the above, we are able to determine the z-independent soft + virtual correction Δ_{V+S} in the NLO MC weight.[26] The above does not include any singular terms like $\ln(1-z)/(1-z)_+$, as advertised earlier.

The difference between the standard MS function of Eq. (39) and that of Eq. (44) is entirely due to the difference between the MS counterterm of Eq. (38) and the MC counterterm of Eq. (40),

$$\begin{aligned}f_{\rm DY}^{MS}(z)-f_{\rm DY}^{\rm MC}(z)&=-2K_F^{MS}(z,\varepsilon)+2K_F^{\rm MC}(z,\varepsilon)\\&=\frac{C_F\alpha_s}{\pi}(1-z)_++2\frac{C_F\alpha_s}{\pi}\left(\frac{\bar P(z)}{1-z}\right)_+\\&\quad\times\left(\ln\frac{\hat s}{\mu^2}+\omega_2\right)+\frac{C_F\alpha_s}{\pi}\bigg\{4\bar P(z)\\&\quad\times\frac{\ln(1-z)}{1-z}-2\bar P(z)\frac{\ln z}{1-z}\bigg\}_+,\end{aligned} \quad (45)$$

and it represents clearly the difference between the MS and MC factorization schemes.

[26]The last term in Δ_{V+S} is due to the plus prescription in the $(1-z)_+$ part of the MC counterterm of Eq. (40).

One may ask how to interpret this change from the MS factorization scheme to the MC factorization scheme—in particular, how unique the modified MC counterterms of Eq. (40) are. One may answer this question in two complementary ways. One way is that the new MC counterterm of Eq. (40) represents just the collinear limit of the *exact* matrix element in $n=4+2\epsilon$ dimensions (keeping higher order terms in ε) in the sense of the $\mathbb{P}'$ projection operator. This definition has to be supplemented with the plus prescription in the soft limit or, alternatively, by saying that Z_F, which in CFP (MS) provides for plus prescription, is replaced by the Sudakov form factor. This approach represents an effort in combining the best from the two, the collinear and soft resummation. Another way of addressing this question is to say that the real backbone in the collinear factorization is OPE, with CFP providing a solid bridge to OPE, and the only thing that has to be explained and kept track of is the difference between CFP and MC (in a similar way to finite UV renormalization). This approach was already advocated in Refs. [4,43] and in other papers [39], where factorization-scheme dependence was discussed. In our approach, we are using both ways of addressing the above question.

The related question is whether the counterterm of Eq. (40) is universal. Basically, the answer is that it is universal thanks to the fact that it is defined in terms of the $\bar k^\mu=\lambda k^\mu$ variables. In other words, the kinematic mapping, inherent in the new $\mathbb{P}'$ operator, should remove the hard-process dependence on the side of the ladder, in the same way as the pole-part operation in CFP [26] or $P_{\rm kin}$ of Ref. [4]. To be completely certain that the above aim of the universality of the new MC factorization scheme is achieved, in the next section we shall define a similar MC scheme for the DIS process; define and use the collinear counterterm of this MC scheme; and in Sec. IV I, we shall check the validity of the factorization-scheme-independent relation (DY − 2 × DIS) of Ref. [30] between the coefficient functions of DY and DIS, both taken in the MC factorization scheme.

F. Differences compared with POWHEG and MC@NLO methods

In this subsection, we outline the main differences of our method compared to the well-established approaches of POWHEG [11] and MC@NLO [9] used today to combine the NLO-corrected hard process with the LO parton shower.

The first and most obvious difference between our method and those of POWHEG and MC@NLO is the use of different factorization schemes. In our approach, we use a factorization scheme [16,17] designed especially for MC simulations, whereas POWHEG and MC@NLO use the standard $\overline{MS}$ scheme. This allows them to use the standard $\overline{MS}$ collinear PDFs directly, while we need additional work here.[27]

[27]One possibility is refitting PDFs, which should not be too complicated, as the difference between the MC and $\overline{MS}$ schemes on the inclusive level is small.

Moreover, we build the LO parton-shower MC from scratch, whereas POWHEG and MC@NLO profit from the well-established (unmodified) LO MC programs.

At first it may seem that these general features result in unnecessary complications in our approach; however, profits are more important, especially if we have in mind the construction of the fully NLO parton-shower MC (with NLO corrections not only in the hard process but also in the ladder parts). Our method features the following:

(i) A simple and positive MC weight implementing the NLO on top of the LO MC; see Ref. [21] (MC@NLO features negative weights).

(ii) No need to correct for the difference in the collinear counterterm between the LO MC and the standard $\overline{MS}$ scheme.

(iii) Virtual + soft corrections Δ_{V+S} that are completely kinematics independent—all annoying $d\Sigma^{c\pm}$ contributions of MC@NLO are gone.

(iv) Built-in resummation of the $\frac{\ln^n(1-x)}{1-x}$ terms.

(v) Direct relation to the collinear factorization procedures.

Note also that in the presented method, there is no need to define the hardest emission, as in POWHEG, as it is automatically included into the sum over spectator gluons in the formula for the MC weight in Eq. (35). In fact, we can explicitly see that the dominant contribution is from the "hardest" (in k_T) gluon[28]; for numerical illustration, see Ref. [21]. This allows us to avoid truncated/vetoed gluons needed in POWHEG methodology in case of angular ordering.

A detailed comparison of the MC@NLO and POWHEG methods themselves can be found in Refs. [44,45].

IV. DEEP INELASTIC ELECTRON-PROTON SCATTERING

As already said, the process of DIS is included in the scope of this article because it is an important source of information on parton distributions in a proton, and by comparing the DIS and DY processes, the question of universality in the collinear factorization implemented in the MC can be fully discussed.

In the following subsections, we shall first introduce kinematics, phase space, and notation for one-real-gluon emission. Next, we shall define the multigluon LO MC distribution with initial-state radiation (ISR) and final-state radiation (FSR) LO ladders and the LO matrix element for the hard process for electron-hadron DIS. Analytical integration of the MC distribution will lead to the familiar formula for the structure function F_2 in the form of the convolution of PDF with the Born cross section. Then we shall give a close simple formula for the MC weight, implementing the NLO correction to the hard process. The analytical integration will again be possible, giving the structure functions F_2 and F_1 in the form of the convolution of PDF with the NLO coefficient function. Of course, the above NLO coefficient function will be in the MC factorization scheme, but we shall see that universality is preserved by means of checking the validity of the factorization-scheme-independent relation DY − 2 × DIS of Ref. [30] between the coefficient functions of the DY and DIS processes.

A. One-real-gluon distribution and kinematics

The Born differential cross section of the electron-quark scattering $e(p_1) + q(q_1) \to e(p_2) + q(q_2)$ in terms of the standard variables[29] $s = 2p_1q_1$, $t = 2p_1p_2$, $u = 2p_1q_2$ reads

$$d\sigma_B = \frac{\alpha^2}{s} d\left(\frac{t}{s}\right) d\varphi Q_q^2 \frac{s^2+u^2}{t^2}, \tag{46}$$

where Q_q is the quark charge. Next, consider the process with the emission of an additional gluon from the quark line:

$$e(p_1) + q(q_1) \to e(p_2) + q(q_2) + g(k).$$

The differential distribution in this case reads

$$d\sigma_1 = \frac{Q_q^2 \alpha_{\rm QED}^2}{s} d\left(\frac{t}{s}\right) d\varphi \frac{s^2+u_1^2+s_1^2+u^2}{2tt_1} \frac{d\psi}{2\pi} \times \frac{C_F\alpha_s}{\pi} \frac{d\alpha d\beta}{\alpha\beta} \frac{t_1}{t}. \tag{47}$$

The additional invariants $s_1 = 2p_2q_2$, $u_1 = 2q_1p_2$, $t_1 = 2q_1q_2$ are introduced in this case. The factor $\frac{t_1}{t}$ is the Jacobian due to parametrization of the phase space in terms of the rescaled Sudakov variables [46,47]:

$$\alpha = \frac{2kq_2}{t_1 + 2kq_1}, \qquad \beta = \frac{2kq_1}{t_1 + 2kq_1}. \tag{48}$$

The angle ψ is the azimuthal angle of $\vec{k}$ around the z axis in the Breit frame of $Q = q_2 + k - q_1$; that is, where $Q^0 = 0$, with an additional requirement that $\vec{q}_1$ be parallel to the z axis. We call this the reference frame B; see Fig. 4.

Yet another Breit frame B_1 is marked in Fig. 4, that of $Q_1 = q_2 - q_1$, with the z axis along $\vec{q}_1$. It will be used in the MC and in the analytical calculations. Note that the integration is over the angle ψ of the k, q_1, q_2 plane as a whole around $\vec{\Pi}$, while another azimuthal angle ϕ_1 of $\vec{k}$ in the B_1 frame is frozen at zero. Note that the standard Sudakov variables are

[28]This is just a relabeling according to k_T; we do not need to change previously generated, angular-ordered gluons. It is only exploited here for the purpose of efficient evaluation of the NLO MC weight.

[29]We omit the minus sign in the variables like t and u with respect to the standard notation.

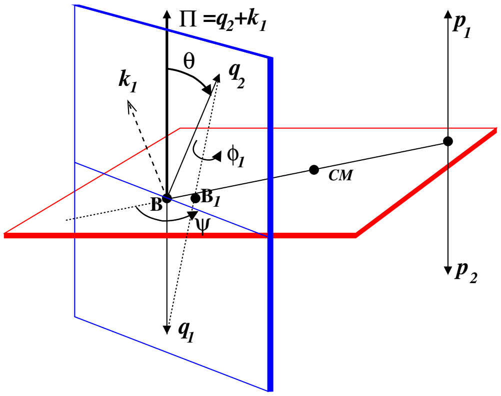


FIG. 4 (color online). Kinematics of one-gluon emission in the Breit frame.

$$\alpha' = \frac{kq_2}{q_1 q_2}, \qquad \beta' = \frac{kq_1}{q_1 q_2}, \qquad t = t_1(1 - \alpha' + \beta'). \tag{49}$$

They are not convenient, because $\beta' \in (0, \infty)$, and the following transformation is mandatory[30]:

$$\alpha = \frac{\alpha'}{1+\beta'}, \qquad \beta = \frac{\beta'}{1+\beta'}, \qquad \alpha' = \frac{\alpha}{1-\beta},$$
$$\beta' = \frac{\beta}{1-\beta}, \qquad 0 < \alpha \le 1 - \frac{t}{s}, \qquad 0 < \beta \le 1,$$
$$t_1 = t\frac{1-\beta}{1-\alpha}. \tag{50}$$

B. Bjorken variables, structure functions, collinear limits

The standard Bjorken variables are

$$x_B \equiv \frac{t}{2q_1 Q} = \frac{|Q^2|}{2q_1 Q}, \qquad 1 \ge x_B > 0,$$
$$y_B \equiv \frac{Qq_1}{p_1 q_1} = \frac{t}{s x_B}. \tag{51}$$

In the case of a single gluon, they are expressed as follows:

$$x_B = \frac{t}{2q_1 Q} = \frac{1 - \alpha' + \beta'}{1+\beta'} = 1 - \alpha,$$
$$y_B = \frac{2q_1 Q}{2q_1 p_1} = \frac{t}{s(1-\alpha)}. \tag{52}$$

The reader should keep in mind that, for simplicity, x_B is the fraction of the parton momentum in the initial quark.[31]

Let us recall the definitions of the standard deep inelastic structure functions in terms of the above Bjorken variables:

$$\frac{d^2\sigma}{dt dx_B} = \frac{2\pi\alpha_{\rm QED}^2 Q_q^2}{t^2} x_B^{-1}\{y_B^2 2x_B F_1(x_B) + 2(1-y_B)F_2(x_B)\}$$
$$= \frac{2\pi\alpha_{\rm QED}^2 Q_q^2}{t^2} x_B^{-1}\{[1+(1-y_B)^2]F_2(x_B) - y_B^2 x_B F_L(x_B)\}$$
$$= \frac{2\pi\alpha_{\rm QED}^2 Q_q^2}{t^2} x_B^{-1}\{[1+(1-y_B)^2]2x_B F_1(x_B) + 2(1-y_B)x_B F_L(x_B)\}, \tag{53}$$

where we have employed the standard definition $2xF_1 \equiv F_2 - xF_L$. In the LO case, the Callan-Gross relation $2xF_1 = F_2$ is fulfilled, and the longitudinal structure function $F_L = 0$ (it will receive a nonzero contribution at NLO). The LO relation to the parton distribution function (luminosity) is $2F_1(x) = F_2/x = \mathrm{PDF}(x)$.

It is instructive to investigate the collinear ISR and FSR limits. The slightly reorganized single-gluon emission distribution reads

$$d\sigma_1 = \alpha_{\rm QED}^2 Q_q^2 \frac{dt}{t^2} d\varphi \frac{d\psi}{2\pi} \frac{C_F\alpha_s}{\pi} \frac{d\alpha d\beta}{\alpha\beta} W,$$
$$W = \frac{s^2 + u_1^2 + s_1^2 + u^2}{2s^2}. \tag{54}$$

The soft limit is already manifest in the eikonal phase-space factor $\frac{d\alpha d\beta}{\alpha\beta}$. The following explicit expressions for the invariants in terms of our Sudakov variables are useful:

$$\frac{t_1}{s} = (1-\beta)y_B,$$
$$\frac{u_1}{s} = 1 - y_B,$$
$$\frac{u}{s} = \frac{s_1}{s} - (1-\alpha-\beta)y_B,$$
$$\frac{s_1}{s} \simeq (1-\alpha)(1-\beta) + \alpha\beta(1-y_B) + 2\cos\psi\sqrt{(1-\alpha)(1-\beta)\alpha\beta(1-y_B)}. \tag{55}$$

In the FSR collinear limit, where $\alpha \simeq 0$, $\beta \simeq 1 - z$, $k \simeq q_2(1-z)/z$, and $y_B \simeq y_0 = t/s$, we have

$$s^2 + u_1^2 \simeq s^2 + (s-t)^2,$$
$$s_1^2 + u^2 \simeq (s^2 + (s-t)^2)(1-\beta)^2,$$
$$W \simeq \frac{1+(1-\beta)^2}{2}\frac{s^2+(s-t)^2}{s^2}$$
$$= \frac{1+z^2}{2}[1+(1-y_0)^2]. \tag{56}$$

In the ISR collinear limit, where $\beta \simeq 0$, $\alpha \simeq 1 - z$ and $k \simeq (1-z)q_1$, we have

[30] In the collinear limit $k \simeq q_2\beta/(1-\beta)$, with $z = 1 - \beta$ being the light-cone variable in the LO splitting kernel.

[31] Returning to the normal definition in the MC (a fraction of the hadron momentum) is quite trivial.

$$
\begin{aligned}
s^2 + u_1^2 &\simeq s^2 + \left(s - \frac{t}{1-\alpha}\right)^2, \\
s_1^2 + u^2 &\simeq [s(1-\alpha)]^2 + [s(1-\alpha) - t]^2, \\
W &\simeq \frac{1+(1-\alpha)^2}{2}[1+(1-y_B)^2] \\
&= \frac{1+z^2}{2}[1+(1-y_B)^2]. \qquad (57)
\end{aligned}
$$

C. Bare structure functions for single-gluon emission

Our immediate aim is now to reproduce the well-known [26,48,49] result for the NLO correction to the $F_2(x)$ structure function by means of integration of the one-gluon phase space [the NLO correction to $F_L(x)$ will also be found]. The aim is to test our Monte Carlo phase-space parametrization, prepare ground for determining the soft + virtual correction in the MC, and put FSR under control.

The unsubtracted (bare) contribution to $F_2(x)/x$, corrected due to the real-gluon emissions plus the vertex correction [Eq. (59) in Ref. [30]], can be rewritten as

$$
C^{\rm AEM}_{2,\rm bare}(z) = \delta(1-z) + \frac{C_F\alpha_s}{\pi}\left\{P_{qq}(z)\left[\frac{1}{\varepsilon} + \omega_2\right] + P_{qq}(z)\ln\frac{t(1-z)}{z\mu^2} - \frac{3}{4}\frac{1}{1-z} + \frac{1}{2}(3+2z)\right\}_+, \qquad (58)
$$

where $P_{qq}(z) = \frac{1+z^2}{2(1-z)}$, and $\omega_2 = \gamma_E - \ln(4\pi)$ comes from the $\Omega_{2+2\varepsilon} = 2\pi(1+\varepsilon\omega_2+\cdots)$ expansion. The baryon number conservation sum rule $\int_0^1 dz C^{\rm AEM}_{2,\rm bare}(z) = 1$ holds explicitly.

The standard NLO $\overline{MS}$ correction C_2^s to the $z^{-1}F_2(z)$ form factor is obtained simply by means of subtracting the $\overline{MS}$ collinear counterterm[32] $\frac{1}{\varepsilon}\{P_{qq}(z)\}_+$ (i.e., the pole part). The formula of Ref. [48] to be reproduced reads

$$
\begin{aligned}
\Delta F_2^{\rm NLO}(x_B) = C_2^s(x_B) &= \frac{C_F\alpha_s}{\pi}\left\{P_{qq}(x_B)\left[\ln\frac{t(1-x_B)}{\mu^2 x_B} + \omega_2\right] - \frac{3}{4}\frac{1}{1-x_B} + \frac{3}{2} + x_B\right\}_+ \\
&= \frac{C_F\alpha_s}{\pi}\left\{P_{qq}(x_B)\left[\ln\frac{t(1-x_B)}{\mu^2 x_B} + \omega_2\right] - \frac{3}{4}\frac{1}{1-x_B} + 1 + \frac{3}{2}x_B\right\}_+ + \frac{C_F\alpha_s}{\pi}\left\{\frac{1-x_B}{2}\right\}_+. \qquad (59)
\end{aligned}
$$

The last term in the nonsingular part, $(1-x_B)/2$, is due to the ε term from the γ trace for the intital-state collinear singularity and ω_2 from the n-dimensional phase space.

We start from the unsubtracted (bare) DIS distribution coming from two-real-gluon emission diagrams from the quark line plus the vertex virtual correction in $n = 4 + 2\varepsilon$ dimensions:

$$
\begin{aligned}
\frac{d^2\sigma}{dt dx_B} &= \frac{2\pi\alpha_{\rm QED}^2 Q_q^2}{t^2}[(1-U)\delta_{x_B=1} + \bar{G}_2(x_B, y_B)], \\
\bar{G}_2(x_B, y_B) &= \int d\alpha d\beta \int \frac{d\Omega^{\psi}_{2+2\varepsilon}}{2\pi}\rho_2(\alpha,\beta)\delta_{x_B=1-\alpha}, \\
\rho_2(\alpha,\beta) &= \frac{C_F\alpha_s}{\pi}\frac{1}{\alpha\beta}W(\alpha,\beta,y_B,\varepsilon)\left(\frac{t\alpha\beta}{\mu^2(1-\alpha)(1-\beta)}\right)^{\varepsilon}\theta_{1>\alpha>\delta}\theta_{1>\beta} \\
W(\alpha,\beta,y_B,\varepsilon) &= \frac{s^2+u_1^2+s_1^2+u^2}{2s^2} + \varepsilon\frac{s^2+u_1^2}{s^2}\frac{(t-t_1)^2}{t^2},
\end{aligned}
\qquad (60)
$$

where we have reinstalled in W the ε term from the γ trace.[33]

The real emission phase space is explicitly integrated for $\alpha > \delta$, where $\delta \ll 1$ is an IR cutoff. The above phase-space division is graphically shown in Fig. 5. Note that the $\alpha > \delta$ part of the phase space, which we are going to integrate over includes not only the hard collinear ISR but also the hard collinear FSR. The constant U must include the vertex correction summed with the soft real emission $\alpha < \delta$. For determining U, it will be enough to know [48,49] that the F_2 part of the distribution in Eq. (60) fulfills exactly the Adler sum rule in the dimensional regularization, and in this way we may omit the details of its calculation. The complicated phase-space factor is simply due to the fact that the transverse momentum of the gluon in the Breit frame is

$$
k_T^2 = |\mathbf{k}|^2 = t_1\alpha'\beta' = \frac{t\alpha\beta}{(1-\alpha)(1-\beta)}.
$$

[32] We subtract a pure pole as in the original CFP work and not $(\frac{1}{\varepsilon}+\omega_2)\{P_{qq}(z)\}_+$, as is a common practice nowadays.

[33] Only for the ISR collinear singularity; the one for FSR falls into the U factor.

Finally, $\int d\Omega^{\psi}_{2+2\varepsilon}$ is the n-dimensional extension of $\int_0^{2\pi} d\psi$.

In the CFP scheme, the ISR collinear singularity upon integration gives rise to the LO pole part

$$C_0^{[0]}\mathbb{P}K_{0I} = C_0^{[0]}\Gamma_I^{[1]}(x_B) = \frac{1}{\varepsilon}[1+(1-y_B)^2]\frac{C_F\alpha_s}{\pi}\left(\frac{\bar{P}(x_B)}{1-x_B}\right)_+,$$

where K_{0I} is the lowest-order 2PI kernel for the ISR ladder, and the plus prescription comes from Z_F, as usual. The subtracted hard-process matrix element in the CFP scheme is $C_0^{[1]} - C_0^{[0]}\mathbb{P}K_{0I}$. We shall calculate it with the help of the usual counterterm technique. The ISR collinear/soft counterterm (SCC) we define as follows:

$$\rho_{2c}(\alpha,\beta) = [1+(1-y_B)^2]\frac{C_F\alpha_s}{\pi}\frac{\bar{P}(1-\alpha)}{\alpha} \times \beta^{\varepsilon-1}B^{-\varepsilon}\theta_{\beta<B(\alpha)}\theta_{1>\alpha>\delta},$$
$$B(\alpha) = \frac{\mu^2(1-\alpha)}{t\alpha}. \quad (61)$$

It is defined such that it integrates to the pure pole part exactly:

$$C_0^{[0]}\Gamma_I^{[1]}(x_B) \equiv \int d\alpha d\beta \rho_{2c}(\alpha,\beta)\delta_{1-x_B=\alpha}.$$

In Fig. 5, we have also marked the integration area for the above counterterm. As we see, in this area the upper phase-space integration limit from energy-momentum conservation is replaced by the limit on the gluon transverse momentum equal (approximately) to μ.

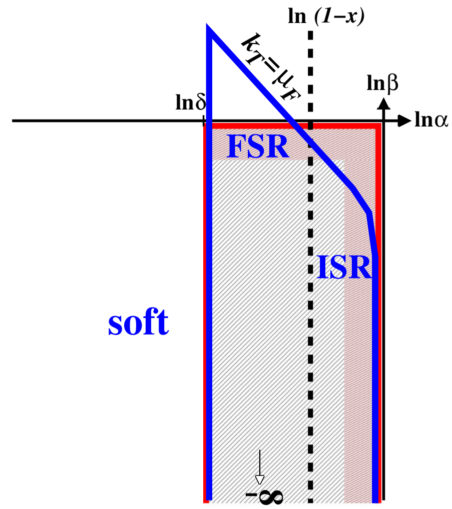


FIG. 5 (color online). The logarithmic Sudakov plane for single-gluon emission in the DIS process.

With the help of the above ISR collinear counterterm, our task is reduced to calculating the subtracted DIS distribution in $n=4$ dimensions:

$$G_2(x_B,y_B) = \int d\alpha d\beta \int_0^{2\pi}\frac{d\psi}{2\pi}[\rho_2(\alpha,\beta) - \rho_{2c}(\alpha,\beta)]\delta_{x_B=1-\alpha}, \quad (62)$$

except for the trivial ε term in W, which contributes $\frac{C_F\alpha_s}{\pi}(\frac{1-x_B}{2})_+$, to be added at the end. The same holds true with the similar $\sim\varepsilon\omega_2$ term from the phase space.

The integration can be summarized as follows:

$$\frac{d^2\sigma^{\rm NLO}_{\rm subt}}{dt dx_B} = \frac{2\pi\alpha^2_{\rm QED}Q_q^2}{t^2}\frac{2C_F\alpha_s}{\pi^2}\int d^3\mathcal{E}(k_1)[W(\alpha,\beta,y_B) - W_0(y_B)\bar{P}(1-\alpha)\theta_{\beta_1<B(\alpha_1)}]\delta_{1-x_B=\alpha_1}$$
$$= \frac{2\pi\alpha^2_{\rm QED}Q_q^2}{t^2}([1+(1-y_B)^2]C_2^s(x_B) - y_B^2C_L(x_B)),$$
$$C_2^s(x_B) = \frac{C_F\alpha_s}{\pi}\left\{P_{qq}(x_B)\left[\ln\frac{t(1-x_B)}{\mu^2x_B}+\omega_2\right] - \frac{3}{4}\frac{1}{1-x_B}+1+\frac{3}{2}x_B\right\}_+ + \frac{C_F\alpha_s}{\pi}\left\{\frac{1-x_B}{2}\right\}_+,$$
$$C_L(x_B) = \frac{C_F\alpha_s}{\pi}x_B, \qquad W_0(y) \equiv 1+(1-y)^2, \quad (63)$$

where the plus prescription is provided by the virtual corrections. We have also included the ε contribution from the γ trace and ω_2 from the phase space. As we see, $C_2^s(z)$ is equal to the finite part of $C^{\rm AEM}_{2,\rm bare}(z)$ of Eq. (58); thus, we have reproduced the classic result [48], as promised.

In the MC scheme, the ISR counterterm $C_0^{(0)}\mathbb{P}'K_{0I}$ is defined as the single-gluon distribution which is extrapolated to $n=4+2\varepsilon$ dimensions and integrated over the phase space:

$$K_I(z,\varepsilon) = \frac{C_F\alpha_s}{\pi}\int\frac{d\alpha d\beta}{\alpha\beta}\int d\Omega_{2+2\varepsilon}\left(\frac{t\alpha\beta}{(1-\alpha)\mu^2}\right)^{\varepsilon}\bar{P}(1-\alpha,\varepsilon)\theta_{\beta<\alpha}\delta_{1-z=\alpha}\theta_{\alpha>\delta} - \delta_{z=1}S_I$$
$$= \frac{C_F\alpha_s}{\pi}\left(\frac{t}{z\mu^2}\right)^{\varepsilon}\frac{\Omega_{2+2\varepsilon}}{\varepsilon}\frac{\bar{P}'(z,\varepsilon)}{(1-z)^{1-2\varepsilon}}\theta_{1-z>\delta} - \delta(1-z)S_I = \frac{C_F\alpha_s}{\pi}\left(\frac{\bar{P}'(z,\varepsilon)}{1-z}\left[\frac{1}{\varepsilon}+\omega_2+\ln\frac{t}{z\mu^2}\right]\right)_+, \quad (64)$$

where $\bar{P}'(z,\varepsilon) = \bar{P}(z)(1 + 2\varepsilon \ln(1-z)) + \frac{1}{2}\varepsilon(1-z)^2$. The source of the plus prescription in this case is the MC Sudakov form factor calculated in $n = 4 + 2\varepsilon$ in such a way that the sum rule $\int dz K_I(z,\epsilon) = 0$ is preserved also in n dimensions:

$$S_I = \frac{C_F\alpha_s}{\pi}\frac{\Omega_{2+2\varepsilon}}{\varepsilon}\int_0^{1-\delta} dz \frac{\bar{P}'(z,\varepsilon)}{(1-z)^{1-2\varepsilon}}\left(\frac{t}{z\mu^2}\right)^{\varepsilon},$$

hence two $\mathcal{O}(\varepsilon)$ terms in $\bar{P}'(z,\varepsilon)$ necessarily participate in the $(\ldots)_+$ prescription.

Subtracting $K_I(z,\varepsilon)$ of Eq. (64) from the complete $\mathcal{O}(\alpha^1)$ result of Eq. (58) gives us the following coefficient function:

$$C_2^{\rm MC}(z) = \frac{C_F\alpha_s}{\pi}\left\{-\frac{1+z^2}{2(1-z)}\ln(1-z) - \frac{3}{4}\frac{1}{1-z} + 1 + \frac{3}{2}z\right\}_+ \tag{65}$$

in the MC factorization scheme, with the angular ordering.

The most important part of the difference between the above MC structure functions and the $\overline{MS}$ variant is coming from the different cutoff in the ISR counterterms:

$$\begin{aligned}\frac{2\pi\alpha_{\rm QED}^2 Q_q^2}{t^2}&\frac{C_F\alpha_s}{\pi}\int d^3\mathcal{E}(k_1)W_0(y_B)[-\bar{P}(1-\alpha)\theta_{\beta_1<B(\alpha_1)}\\&+\bar{P}(1-\alpha)\theta_{\beta_1<\alpha_1}]\delta_{1-x_B=\alpha_1}\\&=\frac{2\pi\alpha_{\rm QED}^2 Q_q^2}{t^2}W_0(y_B)\frac{C_F\alpha_s}{\pi}\Big[2P_{qq}(x_B)\ln(1-x_B)\\&+P_{qq}(x_B)\ln\frac{t}{\mu^2 x_B}\Big].\end{aligned} \tag{66}$$

A few comments are in order:

(i) Why not k^--ordering? In such a DIS-like factorization scheme,[34] the term $(\frac{\ln(1-x)}{1-x})_+$ would have been gone from Eq. (65). FSR could be treated in the DIS MC without the LO resummation, with the unexponentiated FSR NLO corrections. However, if the universality is to be maintained, and the same k^- ordering is applied to the W/Z production process, that would either mean asymmetric treatment of the emission from the quark and antiquark lines or a large double-logarithmic dead zone in the corresponding LO MC, between the ISR and FSR phase spaces. Both options are unacceptable.

(ii) Is there also a kinematic mapping involved in the above $\mathbb{P}'$, like in the previous W/Z production process? Yes, it is implicitly included in the definition of the α and β variables in Eq. (50), where dilatation using the factor $1/(1+\beta')$ is seen.

(iii) From the point of view of the MC, the above considerations are incomplete, as they still keep FSR in the inclusive/integrated form.

[34]In the DIS factorization scheme, $C_2 = 0$ exactly, while in the k^- ordering it would only be less singular.

D. DIS multigluon LO Monte Carlo

Let us start with the raw distribution for n gluons, the $(\alpha_s C_F)^n$ part only, relevant for the LO MC:

$$e(p_1) + q(q_1) \to e(p_2) + q(q_2) + g(k_1) + g(k_2) + \cdots + g(k_n).$$

The corresponding differential distribution reads

$$d\sigma_n = Q_q^2\alpha_{\rm QED}^2 dt d\varphi \frac{W}{tt_1}\frac{d\psi}{2\pi}\left(\frac{C_F\alpha_s}{\pi}\right)^n\left(\prod_{i=1}^n \frac{d\alpha_i d\beta_i}{\alpha_i\beta_i}\frac{d\phi_i}{2\pi}\right)\times\delta_y\left(\sum_j \vec{k}_j\right)\frac{t_1}{t}, \tag{67}$$

where W is a mild function to be defined later on. The invariants $s_1 = 2p_2q_2$, $u_1 = 2q_1p_2$, $t_1 = 2q_1q_2$ are the same as previously. The factor $\frac{t_1}{t}$ is again the Jacobian due to the parametrization of the phase space in terms of the Sudakov variables [50]; see below. The angle ψ is the azimuthal angle of $\vec{k}$ around the z axis in the Breit frame of $Q = q_2 + k - q_1$ with $Q^0 = 0$, with the additional requirement that $\vec{q}_1$ be parallel to the z axis. We call this reference frame B. Another Breit frame, B_1, is used in the MC, that of $Q_1 = q_2 - q_1$ with the z axis also along $\vec{q}_1$ (and $\vec{q}_2$). The illustration of the kinematics in Fig. 4 is still valid, provided we replace k_1 with $\sum_j k_j$.

The integration is done over the angle ψ of the (Π, q_1, q_2) plane as a whole around $\vec{\Pi} = \vec{q}_2 + \sum_j \vec{k}_j$, while there is a single restriction on n azimuthal angles ϕ_i of $\vec{k}_i$ in the B_1 frame—namely, the vector $\sum \vec{k}_j$ must be coplanar with p_1 and p_2.

The standard Sudakov variables are

$$\alpha_i' = \frac{k_i q_2}{q_1 q_2}, \qquad \beta_i' = \frac{k_i q_1}{q_1 q_2},$$
$$t = t_1\left(1 - \sum_j \alpha_j' + \sum_j \beta_j'\right) - K^2, \qquad K = \sum_j k_j. \tag{68}$$

Next, we transform them as follows [50]:

$$\alpha_i = \frac{\alpha_i'}{1+\sum_j \beta_j'}, \qquad \beta_i = \frac{\beta_i'}{1+\sum_j \beta_j'}, \qquad \alpha_i' = \frac{\alpha_i}{1-\sum_j \beta_j},$$
$$\beta_i' = \frac{\beta_i}{1-\sum_j \beta_j}, \qquad 0 < \sum_j \alpha_j \le 1 - \frac{t}{s}, \qquad 0 < \sum_j \beta_j \le 1. \tag{69}$$

The Bjorken variable x_B (of a parton in the initial quark) can be expressed in terms of the Sudakov variables. Using $Q = K + q_2 - q_1$ and $K = \sum_j k_j$, we obtain

$$x_B = \frac{2q_1q_2 + 2q_1K - 2q_2K - K^2}{2q_1q_2 + 2q_1K} = \frac{1 + \sum_j \beta'_j - \sum_j \alpha'_j}{1 + \sum_j \beta'_j} - \tilde{K}^2 = 1 - \sum_j \alpha_j - \tilde{K}^2 = \frac{t}{t_1(1 + \sum_j \beta'_j)}. \tag{70}$$

In the NLO world, the term

$$\tilde{K}^2 = \frac{K^2}{2q_1Q} = \frac{2\sum_{i>j} k_i \cdot k_j}{2q_1Q}$$

can be either omitted or taken care of in the collinear limit. Note that $\tilde{K}^2$ is absent in the case of the single-gluon calculation of the NLO coefficient function.

The fully differential distribution for emitting n gluons in the LO MC for DIS we define as follows:

$$d\sigma_n = Q_q^2 \alpha_{\rm QED}^2 dt d\varphi \frac{1}{t^2} e^{-S} \frac{d\psi}{2\pi} \delta_y\Big(\sum_j \vec{k}_j\Big) \times \Big(\prod_{i=1}^n \frac{C_F \alpha_s}{\pi} \frac{d\alpha_i d\beta_i}{\alpha_i \beta_i} \frac{d\phi_i}{2\pi} \bar{P}(\hat{z}_i) \theta_{a_i > a_{i-1}}\Big) \times \theta_{\sum \alpha_i < 1} \theta_{\sum \beta_i < 1}. \tag{71}$$

The key objects to be defined are the variables $\hat{z}_j$ and the Sudakov form factor S. For this LO modeling of the gluonstrahlung, we use ordering according to the factorization scale (evolution) variable

$$a_i^2 = t\frac{\beta_i}{\alpha_i} \in \Big(t\Delta, \frac{t}{\Delta}\Big), \tag{72}$$

which is the variable of the angular ordering of the MC.

The ISR part of the Sudakov plane (the blue trapezoid in Fig. 6) contains the gluons $\mathcal{I} = (1, 2, 3, \ldots m)$ which have $\beta_i/\alpha_i < e^{\Xi}$, and the FSR part (the red trapezoid in Fig. 6) hosts the gluons $\mathcal{F} = (m+1, m+2, \ldots, n)$ which have $\beta_i/\alpha_i > e^{\Xi}$. We shall indicate that the gluon j belongs to one of these two subsets by $j \in \mathcal{I}$ or $j \in \mathcal{F}$. The variable $\hat{z}_j$ of the ISR or FSR gluon is defined in terms of either α's or β's:

$$\text{for } j \in \mathcal{I}\text{: } \hat{z}_j = z_j^{\mathcal{I}} = \frac{x_j^{\mathcal{I}}}{x_{j-1}^{\mathcal{I}}}, \qquad x_j^{\mathcal{I}} \equiv 1 - \sum_{i=1}^j \alpha_i,$$
$$\text{for } j \in \mathcal{F}\text{: } \hat{z}_j = z_j^{\mathcal{F}} = \frac{x_j^{\mathcal{F}}}{x_{j-1}^{\mathcal{F}}}, \qquad x_j^{\mathcal{F}} \equiv 1 - \sum_{i=m+1}^j \beta_i. \tag{73}$$

The Sudakov form factor S is the integral over the area in the logarithmic Sudakov plane available for the real emission in the step-by-step Markovian process. This area is visualized in Fig. 6 as a shaded polygon. The rapidity Ξ defines the boundary between the ISR and FSR emissions according to the corresponding LO distribution and can be treated as an arbitrary parameter; for example, $\Xi = 0$ in Fig. 6(a) is an acceptable LO choice. In fact, another more clever choice of Ξ, like the one indicated in Fig. 6(b), can be made, for instance, within the Markovian LO MC algorithm. It is also possible to switch from one value of Ξ to another in the final stage of the LO MC by means of reweighting MC events. In view of the above flexibility, we leave the exact definition of Ξ to a later stage of the MC code implementation.

E. Structure function for LO MC

The standard double-differential distribution of the DIS process, as realized in our LO MC, is obtained by inserting the δ function defining x_B:

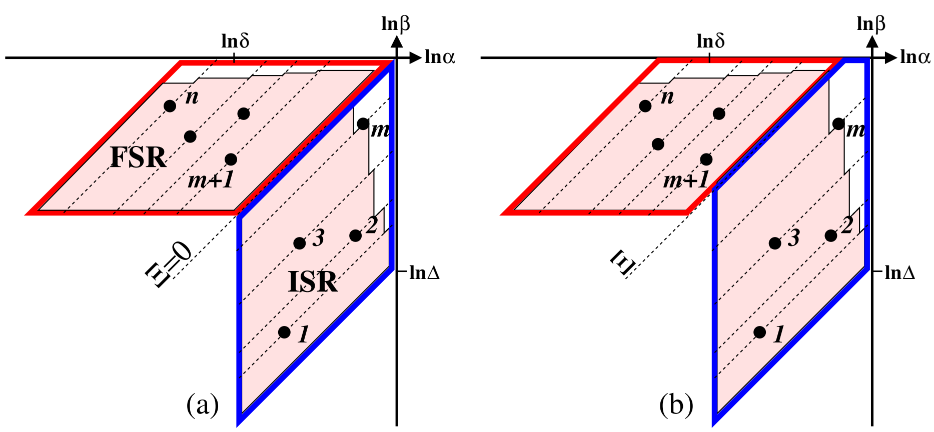


FIG. 6 (color online). The Sudakov plane of the LO MC for DIS. The shaded area denotes the integration domain for the Sudakov form factor S of the LO MC. Rapidity equal to Ξ marks the boundary between ISR and FSR.

$$\frac{d^2\sigma^{\rm LO}}{dtdx_B} = \frac{2\pi Q_q^2\alpha_{\rm QED}^2}{t^2} W_0(y_B) \times \sum_{n=0}^{\infty}\left(\prod_{i=1}^{n}\frac{2C_F\alpha_s}{\pi^2}\int d^3\mathcal{E}(k_i)\bar{P}(z_i^k)\theta_{a_i>a_{i-1}}\right) \times e^{-S}\delta_{\sum_j \vec{k}_j^y=0}\theta_{\sum\alpha_i<1}\theta_{\sum\beta_i<1}\delta_{x_B=1-\sum\alpha_j-\tilde{K}^2}. \tag{74}$$

The above distribution is directly implementable in the MC form; for instance, using the Markovian algorithm, and the above double-differential distribution of x_B and t is coming just from histogramming, using MC events with all four-momenta of all leptons, quarks, and gluons explicitly defined.

On the other hand, we may explicitly show analytically that the above distribution is proportional to PDF convoluted with the coefficient function. This is done by means of inserting into the integrand $1 = \prod_{i=1}^{n}(\theta_{a_i>a_\Xi} + \theta_{a_i<a_\Xi})$ after expanding/reordering the sums of the integrals. The distribution of Eq. (74) almost factorizes into the ISR and FSR parts:

$$\frac{d^2\sigma}{dtdx_B} = \frac{2\pi Q_q^2\alpha_{\rm QED}^2}{t^2} W_0(y_B)\left\{\left[\sum_{n=0}^{\infty}\left(\prod_{i=1}^{n}\frac{2C_F\alpha_s}{\pi^2}\int d^3\mathcal{E}(k_i)\bar{P}(z_i^{\mathcal{I}})\theta_{a_\Xi\geq a_i>a_{i-1}}\right)e^{-S_I}\right] \times\left[\sum_{n'=0}^{\infty}\left(\prod_{i=1}^{n'}\frac{2C_F\alpha_s}{\pi^2}\int d^3\mathcal{E}(k_i)\bar{P}(z_i^{\mathcal{F}})\theta_{a_i>a_{i-1}\geq a_\Xi}\right)e^{-S_F}\theta_{\sum\beta_{i\in\mathcal{F}}<1-\beta_I}\right]\theta_{\sum_{\mathcal{I}+\mathcal{F}}\alpha_i<1}\delta_{x_B=1-\sum_{\mathcal{I}+\mathcal{F}}\alpha_j}\right\}, \tag{75}$$

where we have split the Sudakov form factor into the ISR and FSR parts, $S = S_I + S_F$ and $\beta_I = \sum\beta_{i\in\mathcal{I}}$. In the above equation, we have also neglected $\tilde{K}^2$ in $x_B = 1 - \sum_j\alpha_j - \tilde{K}^2$. This is well justified at LO, but it turns out that it can be done at NLO as well. The alternative solution would be to make special effort in parametrizing the phase space (part of the definition of the $\mathbb{P}'$ operator) to "protect" x_B as it was done for $\hat{x} = \hat{s}/s$ in the W/Z production process. We have decided that this is not worth the effort, as the dependence on x_B of the differential distributions is relatively mild. We may come back to this idea if an additional justification is found.

Altogether, the final LO formula can be written as a convolution of the PDF for ISR and the the resummed "coefficient function" $C_{\mathcal{F}}(z_F)$ for FSR:

$$\frac{d^2\sigma}{dtdx_B} = \frac{2\pi Q_q^2\alpha_{\rm QED}^2}{t^2} W_0(y_B)\int dx_I dz_F\delta_{x_B=x_Iz_F}D_{\mathcal{I}}(\Xi, x_I)C_{\mathcal{F}}(z_F),$$
$$D_{\mathcal{I}}(\Xi, x_I) = e^{-S_I}\sum_{n=0}^{\infty}\left(\prod_{i=1}^{n}\frac{2C_F\alpha_s}{\pi^2}\int d^3\mathcal{E}(k_i)\bar{P}(z_i^{\mathcal{I}})\theta_{a_\Xi\geq a_i>a_{i-1}}\right)\delta_{x_I=1-\sum_{j\in\mathcal{I}}\alpha_j}, \tag{76}$$
$$C_{\mathcal{F}}(z_F) = e^{-S_F}\left\{\delta_{1=z_F} + \sum_{n'=1}^{\infty}\left(\prod_{i=1}^{n'}\frac{2C_F\alpha_s}{\pi^2}\int d^3\mathcal{E}(k_i)\bar{P}(z_i^{\mathcal{F}})\theta_{a_i>a_{i-1}\geq a_\Xi}\right)\delta_{1-z_F=x_I^{-1}\sum_{j\in\mathcal{F}}\alpha_j}\right\}, \qquad \int_0^1 dz_F C_{\mathcal{F}}(z_F) \equiv 1.$$

The interesting pure FSR object $C_{\mathcal{F}}(x)$ is probing the FSR evolution variable, instead of the FSR light-cone variable. In the LO version, it is enough to keep only the trivial $C_{\mathcal{F}}(x) = \delta(1-x)$ term, while for our NLO purpose it is enough to retain only one more easily calculable term, $n' = 1$, $\Xi = 0$:

$$C_{\mathcal{F}}^{(1)}(x) = \delta(1-x) + \frac{C_F\alpha_s}{\pi}\left(\int_0^1\int_0^1\frac{d\alpha_1 d\beta_1}{\alpha_1\beta_1}\bar{P}(1-\beta_1)\theta_{\alpha_1<\beta_1}\delta_{1-x-\alpha_1}\right)_+ = \delta(1-x) + \frac{C_F\alpha_s}{\pi}\left(-\frac{\ln(1-x)}{1-x} - \frac{3}{4}\frac{1}{1-x} + \frac{1}{4}(3+x)\right)_+. \tag{77}$$

Note that the above reproduces the bulk of the coefficient function of Eq. (65); that is, terms like $(\frac{\ln(1-x)}{1-x})_+$ and $(\frac{1}{1-x})_+$.

The MC initial-state PDF obeys the LO DGLAP evolution equation (limited to the nonsinglet gluonstrahlung):

$$2t\frac{\partial}{\partial t}D_I(t,x) = \frac{\partial}{\partial\Xi}D_I(t,x) = \int dzdx\delta_{x_I=xz}\frac{C_F\alpha_s}{\pi}\left(\frac{\bar{P}(z)}{1-z}\right)_+ D_I(t,x), \tag{78}$$

and the same is true for the structure function $2F_1 = C_{\mathcal{F}}\otimes D_I$:

$$2t\frac{\partial}{\partial t}F_1(t,x)=\frac{\partial}{\partial\Xi}F_1(t,x)$$
$$=\int dz dx\delta_{x_I=xz}\frac{C_F\alpha_s}{\pi}\left(\frac{\bar{P}(z)}{1-z}\right)_+F_1(t,x). \tag{79}$$

F. Exclusive ISR and FSR subtractions in DIS

The two soft counterterms, for ISR and FSR, can be identified in the fully differential distribution of the single real gluon in the LO MC:

$$d\sigma_1^{\rm MCLO}=Q_q^2\alpha_{\rm QED}^2dtd\varphi\frac{1}{t^2}\frac{d\psi}{2\pi}\delta_y(\vec{k}_1)\frac{C_F\alpha_s}{\pi}\frac{d\alpha_1 d\beta_1}{\alpha_1\beta_1}\frac{d\phi_1}{2\pi}$$
$$\times W_0(y_B)\{\bar{P}(1-\alpha_1)\theta_{\beta_1<\alpha_1}+\bar{P}(1-\beta_1)\theta_{\beta_1>\alpha_1}\}, \tag{80}$$

where we define the $y_B=\frac{t}{s(1-\alpha_1)}$ variable, and the Born spin factor is $W_0(y)=1+(1-y)^2$. On the other hand, the NLO-complete unsubtracted distribution is

$$d\sigma_1^{\rm NLO}=Q_q^2\alpha_{\rm QED}^2dtd\varphi\frac{1}{t^2}\frac{d\psi}{2\pi}\delta_y(\vec{k}_1)\frac{C_F\alpha_s}{\pi}$$
$$\times\frac{d\alpha_1 d\beta_1}{\alpha_1\beta_1}\frac{d\phi_1}{2\pi}W(\alpha_1,\beta_1,y_B),$$
$$W(\alpha_1,\beta_1,y_B)\equiv\frac{s^2+u_1^2+s_1^2+u^2}{2s^2}. \tag{81}$$

See Eq. (55) for explicit Mandelstam invariants.

For the MC, we shall use the subtracted distribution with both the ISR and FSR counterterms:

$$d\sigma_1^{\Delta\rm NLO}=d\sigma_1^{\rm NLO}-d\sigma_1^{\rm MCLO}$$
$$=Q_q^2\alpha_{\rm QED}^2dtd\varphi\frac{1}{t^2}\frac{d\psi}{2\pi}\delta_y(\vec{k}_1)\frac{C_F\alpha_s}{\pi}$$
$$\times\frac{d\alpha_1 d\beta_1}{\alpha_1\beta_1}\frac{d\phi_1}{2\pi}\tilde{\beta}_1(k),$$
$$\tilde{\beta}_1(k)=\tilde{\beta}_I\theta_{\beta_1<\alpha_1}+\tilde{\beta}_F\theta_{\beta_1>\alpha_1},$$
$$\tilde{\beta}_I(\alpha_1,\beta_1,y_B)=W(\alpha_1,\beta_1,y_B)-W_0(y_B)\bar{P}(1-\alpha_1),$$
$$\tilde{\beta}_F(\alpha_1,\beta_1,y_B)=W(\alpha_1,\beta_1,y_B)-W_0(y_B)\bar{P}(1-\beta_1), \tag{82}$$

which defines (up to NLO) the following expression:

$$\hat{C}_{\Delta\rm NLO}(z,y_B/z)=\frac{C_F\alpha_s}{\pi}\int_0^1\int_0^1\frac{d\alpha_1 d\beta_1}{\alpha_1\beta_1}\frac{d\psi}{2\pi}\{\tilde{\beta}_I\theta_{\beta_1<\alpha_1}$$
$$+\tilde{\beta}_F\theta_{\beta_1>\alpha_1}\}\delta_{1-z=\alpha_1}$$
$$=\frac{C_F\alpha_s}{\pi}W_0(y_B)\left[\frac{1}{2}(1+z)\ln(1-z)\right.$$
$$\left.+\frac{5}{4}z+\frac{1}{4}\right]-y_B^2z, \tag{83}$$

to be used in the numerical tests of the MC implementations.

G. Exclusive NLO correction to the hard process in DIS MC

In the following, we propose a MC weight which upgrades the MC with the LO hard process and LO evolution kernels to the MC with the NLO hard process and LO evolution kernels. The distribution in the LO + NLO MC reads

$$d\sigma_n^{\rm NLO}=Q_q^2\alpha_{\rm QED}^2\frac{dt}{t^2}d\varphi\frac{d\psi}{2\pi}\delta_y\left(\sum_j\vec{k}_j\right)$$
$$\times\left(\prod_{i=1}^n\frac{C_F\alpha_s}{\pi}\frac{d\alpha_i d\beta_i}{\alpha_i\beta_i}\frac{d\phi_i}{2\pi}\bar{P}(z_i^k)\theta_{a_i>a_{i-1}}\right)$$
$$\times\theta_{\sum\alpha_i<1}\theta_{\sum\beta_i<1}e^{-S}W_0(y_B)w_{\rm MC}^{\Delta\rm NLO}, \tag{84}$$

where the key element is the following MC weight:

$$w_{\rm MC}^{\Delta\rm NLO}=[1+\Delta_{S+V}]+\sum_{j\in\mathcal{I}}\frac{\tilde{\beta}_1(\alpha_j',\beta_j',y_B)}{W_0(y_B)\bar{P}(z_j)}$$
$$+\sum_{j\in\mathcal{F}}\frac{\tilde{\beta}_1(\alpha_j'',\beta_j'',y_B)}{W_0(y_B)\bar{P}(z_j)}$$
$$=[1+\Delta_{S+V}]+\sum_{j\in\mathcal{I}}\frac{\tilde{\beta}_I(a_j,z_j^{\mathcal{I}},y_B)}{W_0(y_B)\bar{P}(z_j^{\mathcal{I}})}$$
$$+\sum_{j\in\mathcal{F}}\frac{\tilde{\beta}_F(a_j,z_j^{\mathcal{F}},y_B)}{W_0(y_B)\bar{P}(z_j^{\mathcal{F}})}, \tag{85}$$

which adds the missing NLO correction of the real emission type and also includes Δ_{S+V}, representing the remaining NLO virtual + soft corrections.

The important point is the definition of the variables α_i', β_i' and α_i'', β_i'' in terms of a_i and z_i, in the presence of many "spectator LO gluons." An extrapolation of the one-gluon matrix element over all the multigluon phase space is an inevitable feature of any scheme combining the fixed-order ME with the resummed ME, and there is always certain freedom in doing that. The above extrapolation is done in terms of z_j and a_j. In the ISR part of the sum, we proceed such that first we define $\alpha_j'=1-z_j^{\mathcal{I}}$, and next from the evolution scale $a_j^2/a_\Xi^2=\beta_j'/\alpha_j'$ we calculate β_j' (apparently we proceed as if there were no spectator gluons). In the FSR part, we proceed similarly; i.e., using $z_j^{\mathcal{F}}$, we define $\beta_I''=1-z_j^{\mathcal{F}}$. Next, from the evolution scale $a_j^2/a_\Xi^2=\beta_j''/\alpha_j''$ we calculate α_j''.

H. Analytical integration of DIS MC distributions and determining Δ_{S+V}

A remarkable feature of the complicated multigluon distribution defined within the exact phase space (with full energy-momentum conservation) is that it can be integrated analytically. This integration result will help us to determine the NLO soft + virtual correction Δ_{S+V} and will also be used in the numerical cross-check of the MC code.

The result of the analytical phase space integration for the DIS MC reads

$$\frac{d^2\sigma^{\rm NLO}_{\rm MC}}{dtdx_B} = \frac{2\pi Q_q^2\alpha^2_{\rm QED}}{t^2}\int dx_I dz \delta_{x_B=x_I z} D_I(t,x_I)[W_0(y_B)(1+\Delta_{S+V})\delta_{1=z} + \bar{C}_I(z,y_B) + \bar{C}_F(z,y_B) + W_0(y_B)C^{[1]}_{\mathcal{F}}(z)_+], \tag{86}$$

where

$$\begin{aligned}\bar{C}_I(z,y_B) &= \frac{2C_F\alpha_s}{\pi^2}\int d^3\mathcal{E}(k)\tilde{\beta}_I(\alpha,\beta,y_B)\theta_{\beta<\alpha}\delta_{1-z=\alpha}\\ &= \frac{2C_F\alpha_s}{\pi^2}\int d^3\mathcal{E}(k)[W(\alpha,\beta,y_B) - W_0(y_B)\bar{P}(1-\alpha)]\theta_{\beta<\alpha}\delta_{1-z=\alpha},\\ \bar{C}_F(z,y_B) &= \frac{2C_F\alpha_s}{\pi^2}\int d^3\mathcal{E}(k)\tilde{\beta}_F(\alpha,\beta,y_B)\theta_{\beta>\alpha}\delta_{1-z=\alpha}\\ &= \frac{2C_F\alpha_s}{\pi^2}\int d^3\mathcal{E}(k)[W(\alpha,\beta,y_B) - W_0(y_B)\bar{P}(1-\beta)]\theta_{\beta>\alpha}\delta_{1-z=\alpha},\end{aligned} \tag{87}$$

and

$$\begin{aligned}C^{[1]}_{\mathcal{F}}(z)_+ &= \frac{2C_F\alpha_s}{\pi^2}\int d^3\mathcal{E}(k)\bar{P}(1-\beta)\theta_{\alpha<\beta}\delta_{1-z=\alpha}\theta_{\alpha>\delta} - \delta_{z=1}S_F(\delta), \qquad \int_0^1 dz C^{[1]}_{\mathcal{F}}(z)_+ = 0,\\ S_F(\delta) &= \frac{2C_F\alpha_s}{\pi^2}\int d^3\mathcal{E}(k)\bar{P}(1-\beta)\theta_{\alpha<\beta}\theta_{\alpha>\delta} = \int_0^{1-\delta} dz C^{[1]}_{\mathcal{F}}(z).\end{aligned} \tag{88}$$

The plus prescription for $C^{[1]}_{\mathcal{F}}(z)_+$ is provided by the Sudakov form factor of the MC. $\bar{C}_I(z,y_B)$ and $\bar{C}_F(z,y_B)$ are completely finite/regular, without any $(\ldots)_+$ parts. The IR regulator δ will drop out at the end.

Let us now find out Δ_{S+V} of the MC by means of comparing/matching the first-order Eq. (58) and/or Eq. (65) with Eq. (86), truncated also to the first order. Going back for a moment to $n = 4 + 2\varepsilon$, we find out that the first-order bare PDF of the LO MC is

$$D_I(t,x_I)|_{1\,\rm st\,ord} = \delta(1-x_I) + K_I(x_I,\varepsilon), \qquad \int_0^1 dz K_I(z,\varepsilon) = 0.$$

Also, as anticipated, the contribution $C^{[1]}_{\mathcal{F}}(z)$ cancels the counterterm in $\bar{C}_F(z,y_B)$:

$$\begin{aligned}W_0(y_B)C^{[1]}_{\mathcal{F}}(z) + \bar{C}_F(z,y_B) &= \bar{D}_F(z,y_B,\delta) - W_0(y_B)S_F(\delta)\delta(1-z),\\ \bar{D}_F(z,y_B,\delta) &= \frac{2C_F\alpha_s}{\pi^2}\int d^3\mathcal{E}(k)\delta_{1-z=\alpha}\theta_{1-z>\delta}W(\alpha,\beta,y_B)\theta_{\beta>\alpha}.\end{aligned} \tag{89}$$

Altogether, the first-order truncation of Eq. (86) reads

$$\begin{aligned}\frac{d^2\sigma^{\rm NLO}_{\rm MC(1)}}{dtdx_B} &= \frac{2\pi Q_q^2\alpha^2_{\rm QED}}{t^2}\{\delta_{1=x_B}W_0(y_B)[1+\Delta_{S+V} - S_F(\delta)] + W_0(y_B)K_I(x_B,\varepsilon) + \bar{C}_I(x_B,y_B) + \bar{D}_F(x_B,y_B,\delta)\},\\ \bar{C}_I(z,y_B) + \bar{D}_F(z,y_B,\delta) &= \frac{2C_F\alpha_s}{\pi^2}\int d^3\mathcal{E}(k)\{W(\alpha,\beta,y_B) - W_0(y_B)\bar{P}(1-\alpha)\theta_{\beta_1<\alpha_1}\}\delta_{1-z=\alpha}\theta_{1-z>\delta}\\ &= [W_0(y_B)\theta_{1-z>\delta}\bar{C}^s_2(z) - y_B^2 C_L(z)],\end{aligned} \tag{90}$$

where

$$\bar{C}^s_2(z) = \frac{C_F\alpha_s}{\pi}\left\{\frac{1+z^2}{2(1-z)}\ln\frac{1}{1-z} - \frac{3}{4}\frac{1}{1-z} + 1 + \frac{3}{2}z\right\}. \tag{91}$$

Remembering that [cf. Eq. (65)]

$$C^{\rm MC}_2(z) = (\bar{C}^s_2(z))_+ = \theta_{1-z>\delta}\bar{C}^s_2(z) - \delta_{z=1}T(\delta), \qquad T(\delta) = \int_0^{1-\delta} dx \bar{C}^s_2(x),$$

we finally get

$$\frac{d^2\sigma^{\rm NLO}_{\rm MC(1)}}{dtdx_B} = \frac{2\pi Q_q^2\alpha_{\rm QED}^2}{t^2}\{\delta_{1=x_B}W_0(y_B)[1+\Delta_{S+V}-S_F(\delta) + T(\delta)] + W_0(y_B)K_I(x_B,\varepsilon) + W_0(y_B)C_2^{\rm MC}(x_B) - y_B^2 C_L(x_B)\}. \quad (92)$$

Comparing this with the NLO-complete (real + virtual) calculation [e.g., Eq. (63)], we see that the matching with the above MC implementation dictates the following relation (the Adler sum rule for F_2):

$$\Delta_{S+V} = S_F(\delta) - T(\delta) = \int_0^1 dz[C^{[1]}_{\mathcal{F}}(z) - \bar{C}_2^s(z)]. \quad (93)$$

The above is finite in the $\delta \to 0$ limit. This is not surprising, because $C^{[1]}_{\mathcal{F}}(z)$ integrates the FSR counterterm, while $\bar{C}_2^s(x)$ comes from the ISR-subtracted exact ME—they both coincide in the FSR collinear limit, while the ISR collinear singularity is already removed from $\bar{C}_2^s(x)$.

Summarizing, the complete analytical result for the structure function from the DIS Monte Carlo (angular ordering) defined in Eq. (84) takes the following final form:

$$\frac{d^2\sigma^{\rm NLO}_{\rm DIS}}{dtdx_B} = \frac{2\pi\alpha_{\rm QED}^2 Q_q^2}{t^2}\int dxdz\delta_{x_B=xz}D_I(t,x)[W_0(y_B) \times(1+\Delta_{S+V})\delta_{1=z} + W_0(y_B)C_2^{\rm MC}(z) - y_B^2 C_L(z)],$$
$$C_2^{\rm MC}(z) = \frac{C_F\alpha_s}{\pi}\left\{-\frac{1+z^2}{2(1-z)}\ln(1-z) - \frac{3}{4}\frac{1}{1-z} + 1 + \frac{3}{2}z\right\}_+$$
$$C_L(z) = \frac{C_F\alpha_s}{\pi}z. \quad (94)$$

The above formula is "ready to go" for numerical comparison with the Monte Carlo.

The virtual + soft correction Δ_{S+V} is given by Eq. (93), more precisely

$$\Delta_{S+V} = \frac{C_F\alpha_s}{\pi}\int_0^1 dz\left\{-\frac{\ln(1-z)}{1-z} - \frac{3}{4}\frac{1}{1-z} + \frac{3}{4} + \frac{1}{4}z + \frac{1+z^2}{2(1-z)}\ln(1-z) + \frac{3}{4}\frac{1}{1-z} - 1 - \frac{3}{2}z\right\} = \frac{C_F\alpha_s}{\pi}\int_0^1 dz\left\{-\frac{1+z}{2}\ln(1-z) - \frac{1}{4} - \frac{5}{4}z\right\} = 0. \quad (95)$$

The above is just the result of the rigorous NLO calculation.

Notice also that the MC result features in a natural way the exponentiation of the distributions like

$$f(z) = \delta(1-z) + \frac{C_F\alpha_s}{\pi}\left(\frac{\ln(1-z)}{1-z}\right)_+ \simeq \frac{C_F\alpha_s}{\pi}\frac{\ln(1-z)}{(1-z)}e^{-\frac{C_F\alpha_s}{\pi}\frac{1}{2}\ln^2(1-z)},$$
$$\int_0^1 dz f(z) = 1.$$

Such an exponentiation can be included in the analytical formula.

Last but not least, let us write explicitly the difference between the coefficient functions of the standard $\overline{MS}$ factorization scheme of Eq. (59) and the MC factorization scheme of Eq. (94)[35]:

$$\Delta C_2(z) = C_2^s(z) - C_2^{\rm MC}(z) = \frac{C_F\alpha_s}{\pi}\left\{\frac{1+z^2}{2(1-z)}\ln\frac{(1-z)^2}{z} + \frac{1-z}{2}\right\}_+. \quad (96)$$

The above function should be used to correct the existing $\overline{MS}$ PDFs before using it to fix input in our MC. Alternatively, the coefficient function of Eq. (94) should be used to fit DIS experimental data with the PDF function compatible with the presented MC.[36]

I. Factorization-scheme-independent relation between DY and DIS processes

In spite of the change of factorization scheme in the MC, the factorization-scheme-independent and regularization-independent relation of AEM [Eq. (91) in Ref. [30]] should be reproduced exactly, if we claim to protect the universality. Let us verify it. The original AEM relation reads[37]

$$\Delta_q^{\rm AEM}(z) = f_{q,DY} - 2f_{q,2} = \frac{C_F\alpha_s}{\pi}\left[\delta_{z=1}\left(\frac{2}{3}\pi^2 + \frac{1}{2}\right) + \frac{3}{2}\frac{1}{(1-z)_+} + (-3-2z) + (1+z^2)\left(\frac{\ln(1-z)}{1-z}\right)_+\right] = \frac{C_F\alpha_s}{\pi}\left[\delta_{z=1}\left(\frac{2}{3}\pi^2 - \frac{7}{4}\right) + \frac{3}{2}\frac{1}{(1-z)_+} + (-3-2z)_+ + \left((1+z^2)\frac{\ln(1-z)}{1-z}\right)_+\right]. \quad (97)$$

Using the result of the analytical integration of the DIS MC, Eq. (94),

$$C_2^s(z) = \frac{C_F\alpha_s}{\pi}\left\{-\frac{1+z^2}{2(1-z)}\ln(1-z) - \frac{3}{4}\frac{1}{1-z} + 1 + \frac{3}{2}z\right\}_+,$$

and the analogous analytical result for the DY MC of Eq. (44),

$$C_2(z) = \delta_{z=1}\frac{C_F\alpha_s}{\pi}\left(\frac{2}{3}\pi^2 - \frac{7}{4}\right) + \frac{C_F\alpha_s}{\pi}[-(1-z)_+],$$

[35] Here, the usual $\overline{MS}$ assignment $t = \mu^2 e^{-\omega_2}$ is done; see Refs. [39,40].

[36] Similar corrections also have to be determined for the NLO-inclusive kernels, once the NLO corrections are included in the ladder part of the MC; see the first incomplete results in Ref. [17].

[37] Using again $\int_0^1 dz(1+z^2)(\frac{\ln(1-z)}{1-z})_+ = \frac{7}{4}$.

we obtain from our two MC implementations the same result:

$$C_2(z) - 2C_2^s(z) = \Delta_q^{\mathrm{AEM}}(z). \tag{98}$$

In this way, we have reproduced the AEM [30] result for the MC factorization scheme, confirming its universality. The above agreement with the AEM result is easily traced back to the fact that it holds already for the difference of the unsubtracted coefficient functions[38]

$$f_{\mathrm{DY}}^{\mathrm{AEM}}(z) - 2C_{2,\mathrm{bare}}^{\mathrm{AEM}}(z) = \Delta_q^{\mathrm{AEM}}(z) + 2\frac{C_F\alpha_s}{\pi}\ln\frac{\hat{s}}{t} \tag{99}$$

[cf. Eqs. (37) and (58)], and because in both MC formulas the same ISR counterterm of Eqs. (40) and (64) is subtracted.[39] In particular, terms due to the $\varepsilon\frac{1-z}{2}$ component in the γ trace present in the DY and DIS coefficient functions necessarily cancel out.

It is fair to mention that in Ref. [30], the relation of Eq. (97) is treated as the pQCD result for the coefficient function of the DY process in the DIS factorization scheme. On the other hand, this relation can be turned into an experimentally testable relation between the structure functions of the DY and DIS processes, testing the important principle of universality (process independence) of collinear singularities in pQCD predictions, independently of any particular choice of factorization scheme and the PDFs.

V. SUMMARY AND OUTLOOK

We have presented a complete method for implementing NLO corrections to the hard process in the LO MC for DY and DIS processes. This method was originally developed for introducing NLO corrections in the ladder MC [15,16]; therefore, it is well suited to be extended to include NLO corrections in both hard-process and ladder parts.

The presented method is based on a new factorization scheme [16,17] extending the collinear factorization theorems [4,26] to the fully exclusive (unintegrated) form, which can serve as a base for the MC distributions. All differences between the $\overline{MS}$ and this new MC scheme are kept under strict control, and we elaborate on that in quite some detail. In particular, we make a powerful cross-check of the whole MC factorization scheme by showing (analytically) that the NLO MC results reproduce the factorization-scheme-independent relation of Altarelli-Ellis-Martinelli [30] between the Drell-Yan and DIS processes; see Eq. (98).

The main practical results of this work are the multiparton distributions of Eq. (34) and (84) for the EW boson production in hadron-hadron collision and electron-hadron deep inelastic scattering, respectively, which are ready for Monte Carlo implementation. These distributions feature the NLO corrections in the hard-process part and the LO pQCD evolution in two multiparton ladder parts. The NLO corrections to the hard process are introduced by means of a single MC weight on top of the LO distributions—it is, therefore, critical that the LO MC cover the multiparton phase space without any gaps or overlaps. This is achieved by means of using the angular ordering, which is also essential for good control of the soft gluon behavior beyond LO, already in the LO MC. The correct soft limit also assures good behavior of the MC weight; weights are positive and small (peaked near 1). For the weight distributions and other numerical cross-checks of the presented method, we refer the reader to Ref. [21].

In our opinion, this work solves the main obstacles on the way to the NLO MC, based rigorously on the new MC factorization scheme. There are still many less important problems to be solved on the way to the practical level; i.e., the construction of a MC program applicable in the LHC data analysis. Let us signal some of these problems and their solutions: (i) For simplicity in our formulas, we have omitted the initial PDF of the quark in the hadron at the low factorization scale $Q \sim 1$ GeV. This can be easily included in the MC. (ii) If we are aiming for a fully NLO MC, the ladder parts have to be upgraded to the NLO level, and this work is already well advanced [14–16]. (iii) In the presented MC scheme, the QCD coupling was constant and nonrunning. It is quite trivial to make it running within the LO MC. It will be less trivial, but also profitable, to disentangle the running-coupling effect from the NLO corrections in the MC implementation of the NLO ladder. This problem is under study and will be treated in a separate publication. (iv) All the MC distributions presented in this work are defined for quarks and gluons; hence, in the practical level MC code they will be subject to a hadronization procedure, using one of the existing MC tools, such as HERWIG [8] or PYTHIA [7].

Obviously, the proposed scheme of implementing the NLO corrections to the hard process combined with the MC parton showers (ladders) is different from the existing ones. In Sec. III F, we comment on the differences between our scheme and those of MC@NLO [9] and POWHEG [11]. More systematic comparisons with these and other schemes [22,23] will be done separately, at the time of the numerical MC implementation.

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[38]The last term is, of course, absent for the usual assignment $\hat{s} = t = \mu^2$.

[39]The ISR counterterm is defined in the DIS and DY processes at the exclusive level, involving $\mathbb{P}'$ and kinematic mapping, so the statement that "it is the same" is more nontrivial than in the case of the CFP-inclusive counterterms.

The presented method of implementing the NLO corrections to the hard process does not have any principal limitations—it can be extended to more diagrams and other processes. However, at the practical level, its application requires that the LO parton shower provide for the full coverage (no gaps or dead zones) of the hard-process phase space relevant at the NLO level. This requirement is typically not fulfilled by the classic MC parton showers like HERWIG or PYTHIA. It is not excluded that the modernized version of these MCs may provide for better phase-space coverage, notably by using tools developed for the MC@NLO and POWHEG implementations. Otherwise, the LO parton shower has to be reconstructed, for instance, using the scheme proposed in the present work. (This may turn out to be mandatory for implementing NLO corrections in the ladder parts of the parton-shower MC.)

Summarizing, this work represents an important step into a new area in the pQCD calculations for hadron colliders in the MC form, in which the NLO corrections are implemented in both the hard process and the ladder parts in a completely exclusive (unintegrated) way, in full compatibility with the redefined, fully exclusive pQCD factorization.

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APPENDIX A: KINEMATICS OF THE EW BOSON PRODUCTION PROCESS

Let us consider the case of a single real (not necessarily collinear) gluon emission, relevant to the NLO level description of the hard process,

$$q(p_{0F}) + \bar{q}(p_{0B}) \to l^-(q_1) + l^+(q_2) + g(k), \qquad \text{(A1)}$$

which is the classic EW vector boson production process in the annihilation of the quark-antiquark pair (the Drell-Yan process in case of γ^*) decaying into a lepton pair. Note that in the definition below we omit the distribution of the quark (antiquark) in the proton. This can be always added easily in the MC.

The following kinematical variables are used in this work:

$$\begin{aligned}
&x = \frac{\hat{s}}{s} = \frac{(P-k)^2}{P^2} = 1 - \alpha - \beta, \qquad P = p_{0F} + p_{0B},\\
&Q = \hat{P} = p_{0F} + p_{0B} - k, \qquad s = P^2,\\
&\hat{s} = \hat{P}^2 = Q^2 = (P-k)^2 = P^2 - 2k\cdot P,\\
&\alpha = \frac{2k\cdot p_{0B}}{P^2}, \qquad \beta = \frac{2k\cdot p_{0F}}{P^2}. \qquad \text{(A2)}
\end{aligned}$$

The most important of them is the invariant mass squared $\hat{s}$ of the produced colorless boson.

For the emitted gluons, we are using dimensionless *eikonal phase space* parametrized in terms of various variables:

$$\begin{aligned}
d^3\mathcal{E}(k) &= \frac{d^3k}{2k^0}\frac{1}{\mathbf{k}^2} = \frac{1}{2}\frac{dk^+}{k^+}\frac{d^2\mathbf{k}}{\mathbf{k}^2} = \frac{\pi}{2}\frac{d\phi}{2\pi}\frac{d\alpha}{\alpha}\frac{d\mathbf{a}^2}{\mathbf{a}^2}\\
&= \frac{\pi}{2}\frac{d\phi}{2\pi}\frac{d\alpha}{\alpha}\frac{d\beta}{\beta} = \pi\frac{d\phi}{2\pi}\frac{d\alpha}{\alpha}d\eta = \pi\frac{d\phi}{2\pi}\frac{d\beta}{\beta}d\eta,
\end{aligned} \qquad \text{(A3)}$$

where $\mathbf{k} = (k^1, k^2)$ is a transverse Cartesian two-vector ($k_T^2 = |\mathbf{k}|^2 = s\alpha\beta$), and the Sudakov (light cone) variables are

$$k^\pm = k^0 \pm k^3, \qquad \alpha = \frac{2k^+}{\sqrt{s}}, \qquad \beta = \frac{2k^-}{\sqrt{s}}.$$

Moreover, we introduce the variable $\mathbf{a} \equiv \mathbf{k}/\alpha$, and the conventional rapidity variable η is defined as

$$\eta = \frac{1}{2}\ln\frac{\alpha}{\beta} = \frac{1}{2}\ln\frac{k^+}{k^-} = -\ln\frac{|\mathbf{a}|}{\sqrt{s}}, \qquad a = |\mathbf{a}| = e^{-\eta}\sqrt{s}.$$

Multiparticle phase space is defined as

$$d\tau_n(P; p_1, p_2, \ldots, p_n) = \delta^{(4)}\Big(P - \sum_{i=1}^n p_i\Big)\prod_{i=1}^n \frac{d^3p_i}{2p_i^0}. \qquad \text{(A4)}$$

The two-dimensional phase space for massless particles is then

$$d\tau_2(Q; q_1, q_2) = \frac{1}{2}d\Omega. \qquad \text{(A5)}$$

APPENDIX B: ONE-REAL-GLUON NLO CORRECTION, ANALYTICAL INTEGRATION

We are going to integrate analytically the one-real-gluon NLO correction as defined in Eq. (17). The contribution from the F hemisphere is easily calculable:

$$\begin{aligned}\frac{1}{2}C_{2r}(x) &= \frac{C_F\alpha_s}{\pi}\int_0^\infty d\alpha\int_0^\infty d\beta\Big[\frac{\bar{P}(x)-\alpha\beta}{\alpha\beta}\theta_{\alpha+\beta<1}\theta_{\beta<\alpha}\delta_{x=1-\alpha-\beta} - \frac{\bar{P}(x)}{\alpha\beta}\theta_{\beta<\alpha}\delta_{x=1-\alpha}\Big]\\
&= \frac{C_F\alpha_s}{\pi}\Big[\bar{P}(x)\int_\Delta^{(1-x)/2} d\beta\frac{1}{(1-x-\beta)\beta} - \int_0^{(1-x)/2} d\beta - \bar{P}(x)\int_\Delta^{(1-x)} d\beta\frac{1}{(1-x)\beta}\Big]\\
&= \frac{C_F\alpha_s}{\pi}\frac{\bar{P}(x)}{(1-x)}\Big[\int_0^{(1-x)/2}\frac{d\beta}{1-x-\beta} + \int_\Delta^{(1-x)/2}\frac{d\beta}{\beta} - \int_\Delta^{(1-x)}\frac{d\beta}{\beta}\Big] - \frac{C_F\alpha_s}{\pi}\frac{1-x}{2}\\
&= -\frac{C_F\alpha_s}{\pi}\frac{1-x}{2}. \end{aligned} \tag{B1}$$

APPENDIX C: INCLUSIVE NLO FACTORIZATION FORMULA FOR DY MC

We are going to prove the formula of Eq. (36), representing the MC with two LO ladders and the NLO-corrected hard process, by means of reorganizing the phase-space integration of Eq. (34). Let us consider the part of the total cross section of Eq. (34) proportional to the term $j \in F$ in the MC weight of Eq. (35). The summation and integration over the "spectator" LO gluons in the B part of the phase space can be easily folded into the LO PDF. What remains to be considered is the following sum of integrals:

$$\begin{aligned}\sigma_I^{\rm NLO} &= \int dx_F dx_B \sum_{n_1=1}^\infty e^{-S_F}\int_{\Xi<\eta_{n_1}}\Big(\prod_{i=1}^{n_1} d^3\mathcal{E}(\bar{k}_i)\theta_{\eta_i<\eta_{i-1}}\frac{2C_F\alpha_s}{\pi^2}\bar{P}(z_{Fi})\Big)\sum_{j\in F}\frac{\tilde{\beta}_1(\hat{s},\hat{p}_F,\hat{p}_B;a_j,z_{Fj})}{\bar{P}(z_{Fj})}G_B(\Xi,x_B)\delta_{x_F=\prod_i z_{Fi}}\\
&= \int dx_F dx_B\sum_{n_1=1}^\infty e^{-S_F}\sum_{j=1}^{n_1}\int\Big(\prod_{i=1,i\neq j}^{n_1} d^3\mathcal{E}(\bar{k}_i)\theta_{\eta_i<\eta_{i-1}}\frac{2C_F\alpha_s}{\pi^2}\bar{P}(z_{Fi})\Big)\\
&\quad\times\int d^3\mathcal{E}(\bar{k}_j)\theta_{\eta_{j+1}<\eta_j<\eta_{j-1}}\tilde{\beta}_1(\hat{s},\hat{p}_F,\hat{p}_B;a_j,z_{Fj})G_B(\Xi,x_B)\delta_{x_F=\prod_i z_{Fi}}. \end{aligned} \tag{C1}$$

The essential step in transforming each jth term is relabeling the gluons $i \to i'$ such that $i' = i$ for $i = 1, 2, \ldots, j-1$, and $i' = i-1$ for $i = j+1, \ldots, n_1$; hence $i' = 1, 2, \ldots, n_1 - 1$ without any gap, and finally $i = j$ is relabeled as $j' = 0$. Using the symmetry of the integrand, integrals over $k_{i'}$ can be pulled out, and the sum over adjacent integration ranges of $k_{j'} = k_0$ is factorized off:

$$\begin{aligned}\sigma_I^{\rm NLO} &= \int dx_F dx_B\delta_{x=x_Fx_B}\sum_{n_1=1}^\infty e^{-S_F}\int\Big(\prod_{i'=1}^{n_1-1} d^3\mathcal{E}(\bar{k}_{i'})\theta_{\eta_{i'}<\eta_{i'-1}}\frac{2C_F\alpha_s}{\pi^2}\bar{P}(z_{Fi'})\Big)\\
&\quad\times\sum_{i'=1}^{n_1-1}\int d^3\mathcal{E}(\bar{k}_0)\theta_{\eta_{i'}<\eta_0\le\eta_{i'-1}}\tilde{\beta}_1(\hat{s},\hat{p}_F,\hat{p}_B;a_0,z_{F0})G_B(\Xi,x_B)\delta\Big(x_F - z_{F0}\prod_{i'=1}^{n_1-1} z_{Fi'}\Big). \end{aligned} \tag{C2}$$

The sum over the adjacent integration intervals is combined into a single integral

$$\int_0^{a_1}\tilde{\beta}_1 da_0 + \int_{a_1}^{a_2}\tilde{\beta}_1 da_0 + \int_{a_2}^{a_3}\tilde{\beta}_1 da_0\cdots + \int_{a_{n_1-2}}^{a_{n_1-1}}\tilde{\beta}_1 da_0 = \int_0^{a_{n_1-1}}\tilde{\beta}_1 da_0$$

and factorized off, while the remaining integrals over the spectator gluons $i' = 1, 2, \ldots, n_1 - 1$ give rise to the LO PDF:

$$\begin{aligned}\sigma_I^{\rm NLO} &= \int dx_F dx'_F dx_B\Big\{\sum_{n_1=1}^\infty e^{-S_F}\int\Big(\prod_{i'=1}^{n_1-1} d^3\mathcal{E}(\bar{k}_{i'})\theta_{\eta_{i'}<\eta_{i'-1}}\frac{2C_F\alpha_s}{\pi^2}\bar{P}(z_{Fi'})\Big)\delta_{x'_F=\prod_{i'} z_{Fi'}}\Big\}\\
&\quad\times\int d^3\mathcal{E}(\bar{k}_0)\tilde{\beta}_1(\hat{s},\hat{p}_F,\hat{p}_B;a_0,z_{F0})G_B(\Xi,x_B)\delta_{x_F=z_{F0}x'_F}\delta_{x=x_Fx_B}\\
&= \int dx_B dx'_F dz_{F0} G_F(\Xi,x'_F)G_B(\Xi,x_B)\frac{1}{2}C_{2r}(z_{F0})\sigma_B(sx)\delta_{x=x_Bx'_Fz_{F0}}, \end{aligned} \tag{C3}$$

where we have replaced the integration variable x_F with $z_{F0} = x_F/x'_F$. In the last step, we were able to use the integral defined in Eq. (17) and evaluated in Eq. (B1).

The other part of the total cross section of Eq. (34) proportional to the term $j \in B$ in the MC weight of Eq. (35) gives the same result. For the LO part times $(1 + \delta_{S+V})$, we use Eq. (31).

As already noted, the key part of the above algebra is reminiscent of that in Ref. [36], except that here the resummed singularities are in the angle, while in Ref. [36] they are in the energy variable.

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