On Multimonopole Solutions

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Abstract

We present a class of real solutions that do not fit into the recent analysis of the general n-monopole solution. We also present some perturbative results on a one-parameter family of separated monopoles lying on a plane.

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I. Introduction

In the last few months there has been rapid progress in determining exact multimonopole solutions of the Yang-Mills-Higgs theory in the limit of vanishing Higgs potential. Ward's two-monopole solution [1] triggered the search for more explicit general analytic results. This search culminated in the construction of superimposed axisymmetric multimonopoles of arbitrary charge [2] (later reproduced by different and independent methods by various authors) and in the proof that separated multimonopole solutions can be actually constructed (at least in principle) and possess the expected number of degrees of freedom [3].

In this paper we want to discuss a number of aspects of this problem that have emerged by careful consideration of the previous results, with the aim of achieving a better understanding of the methods involved and their generality. We have specifically addressed two different (and somewhat orthogonal) questions:

a) Is the space of real static solutions completely described by by the formalism used in ref. [3]? (A brief introduction to this formalism is contained in Appendix A which also establishes our notational conventions.)

b) What do the known separated solutions look like, at least for small values of the parameters and in a region where a Taylor series expansion of the solutions is reliable?

Our answer to the first question is that it is possible to identify at least one new class of real static solutions, possessing axial symmetry, for which no representation seems to exist that falls into the classification given in ref. [3]. However, when the request of nonsingularity is enforced, the regular solutions belonging to this class turn out to coincide with the previously found "conventional" solutions. This phenomenon in turn accounts for the possibility of unconventional representations of the known solutions, whose equivalence to the standard representation is by no means mathematically obvious.

Moreover, we have also constructed, at least for the single monopole case, a definitely non-equivalent representation leading nevertheless to gauge-equivalent gauge fields. It is unclear at this stage which could be the "physical" criterion of equivalence leading to the identification of <u>all</u> real static solutions in a <u>unique</u> way - and accounting naturally for all other representations.

For what concerns the explicit analysis of the known solutions, we have first identified the simplest sufficiently general class of separated multimonopoles. This one-parameter class naturally generalizes Ward's separated two-monopole solution [4] and appears to be possibly the only class for which all the constraint conditions may be explicitly solved and the solutions can be written down in closed form.

We have performed a perturbative analysis with the help of the symbol manipulating computer program MACSYMA and managed to identify the zeroes of the Higgs field (i.e. the locations of the monopoles) when up to four monopoles are present. A clear pattern emerges: the monopoles are coplanar but not necessarily collinear and they are "completely" separated, i.e. no residual degeneracy in the locations

survives, even in this simple one-parameter class. It is intriguing to observe that while the main separation axis is orthogonal to the symmetry axis of the limiting (superimposed) axisymmetric configurations, the separation plane however is <u>not</u>: indeed it contains the symmetry axis.

As a final comment on the separated solutions, we want to point at the interesting open problem of analyzing the regime when the separation parameters become very large. While this task is relevant to our complete understanding of the space of parameters for the general solutions, it has so far eluded our analysis for lack of sensible expansion techniques in the domain of interest.

II. Real axi-symmetric solutions

In this section we want to analyze the reality condition and show that a class of real solutions may be constructed, such that it cannot be described in the framework presented by Corrigan and Goddard [3].

The request of axial symmetry is easily implemented by the condition that

$$\Delta_\ell(x) = e^{ix_4} e^{-i\ell\theta} \eta_\ell(s,x_3). \tag{2.1}$$

This condition in turn implies that

$$\Omega(\omega_1,\omega_2,\zeta) = \sum_{\ell=-\infty}^{\infty} \Delta_\ell \zeta^{-\ell} = e^{\omega_1+\omega_2} F(\omega) \tag{2.2}$$

up to equivalence transformations. It is easy to identify the most general generating function Ω that is consistent with Eq. (2.2) and with reality within Corrigan-Goddard's scheme:

$$\Omega^{(n)} = e^{\omega_1+\omega_2} \frac{e^{\omega} + (-1)^n e^{-\omega}}{2 P_n(\omega)} \;;\; P_n(\omega) = \prod_{k=1}^{n} (\omega - q_k) \tag{2.3}$$

where q_k are parameters, independent of ζ. Eq. (2.3) in turn is exactly the generating function for general static and axisymmetric self-dual gauge fields that was derived in ref. [2].

We want to analyze the properties of this class of solutions. Let us first of all notice that

$$\frac{1}{P_n(\omega)} = \sum_{k=1}^{n} \beta_k \frac{1}{\omega - q_k} \quad \text{where } \beta_{k'} = \frac{1}{\prod_{k' \neq k} (q_{k'} - q_k)} \tag{2.4}$$

such that

$$\Delta_\ell = \Sigma_k \beta_k \Delta_\ell^{(k)}, \tag{2.5}$$

where

$$\Delta_\ell^{(k)} = \frac{1}{2\pi i} \oint \frac{e^{\omega} + (-1)^n e^{-\omega}}{2(\omega - q_k)} \zeta^\ell \frac{d\zeta}{\zeta}, \tag{2.6}$$

and the contour integral may be explicitly evaluated to give:

$$\begin{aligned} \Delta_\ell^{(k)} = e^{ix_4} e^{-i\ell\theta} & \left[\frac{e^{r_k}}{2r_k} \left(\frac{x_3 - q_k - r_k}{s}\right)^\ell - \frac{e^{-r_k}}{2r_k} \left(\frac{x_3 - q_k + r_k}{s}\right)^\ell \right] \left(\frac{e^{q_k} - (-1)^n e^{-q_k}}{2}\right) \\ \mp e^{ix_4} e^{-i\ell\theta} & \left[\frac{e^{r_k}}{2r_k} \left(\frac{x_3 - q_k - r_k}{s}\right)^\ell - \frac{e^{-r_k}}{2r_k} \left(\frac{x_3 - q_k + r_k}{s}\right)^\ell \right] \left(\frac{e^{q_k} + (-1)^n e^{-q_k}}{2}\right), \end{aligned} \tag{2.7}$$

where the $\pm$ signs correspond to the regions where $\left| \frac{x_3 - q_k + r_k}{s} \right| \lessgtr 1$, respectively. It is apparent that $\Delta_\ell^{(k)}$ is in general discontinuous and the only way to make it continuous is the request that $F(\omega)$ be an entire function, such that

$$\frac{e^{q_k} + (-1)^n e^{-q_k}}{2} = 0 \;, \qquad \left| \frac{e^{q_k} - (-1)^n e^{-q_k}}{2} \right| = 1 \tag{2.8}$$

In general, however, $\Delta_\ell^{(k)}$ is continuous on the x_3 axis.

$$\Omega(s=0) = e^{ix_4} \frac{e^{x_3} + (-1)^n e^{-x_3}}{2\Pi(x_3 - q_k)} \tag{2.9}$$

$$\Delta_0^{(k)} = e^{ix_4} \frac{e^{x_3} + (-1)^n e^{-x_3}}{2(x_3 - q_k)} \qquad \Delta_\ell^{(k)} = 0. \tag{2.10}$$

The norm of the Higgs field on the x_3 axis is easily evaluated to be

$$h = \left| (\tanh x_3)^{(-1)^n} - \sum \frac{1}{x_3 - q_k} \right| , \tag{2.11}$$

and by dropping exponential terms one also obtains

$$h = 1 - \sum \frac{1}{r_k} + O(e^{-r_j - r_k}). \tag{2.12}$$

These results lead to an identification between these solutions and those found by Lee [5] and Forgács, Horváth, Palla [6] in their approach to the multimonopole problem.

Let's now approach the problem from a different point of view. A fairly general class of complex but entire generating functions is described by:

$$\bar{\Omega} = e^{\omega_1+\omega_2} \frac{1}{2}\int_{-1}^{1} dt\, e^{-t\omega} p(t), \tag{2.13}$$

where p(t) is an arbitrary weight function. Solutions in this class admit the representation

$$\bar{\Delta}_{\pm\ell} = (-1)^\ell e^{ix_4} \frac{e^{\mp i\ell\theta}}{s^\ell} \left(1 \mp \frac{\partial}{\partial x_3}\right)^\ell \bar{\Lambda}_\ell \qquad \ell \ge 0 \tag{2.14}$$

where

$$\bar{\Lambda}_\ell = \frac{1}{2}\int_{-1}^{1} dt\, e^{-tx_3} \left(\frac{s}{2\sqrt{1-t^2}}\right)^\ell I_\ell\left(s\sqrt{1-t^2}\right) p(t) \tag{2.15}$$

It is easy to check that $\bar{\Lambda}_\ell$ satisfy the recursion equations implied by the representation Eq. (2.14) and the recursive definition of the Δ_ℓ:

$$\bar{\Lambda}_\ell = p^{-1}\, \partial_p\, \bar{\Lambda}_{\ell+1} \tag{2.16a}$$

$$[2p^\ell \partial_p (p^{-\ell} \partial_p) + \partial_3 \partial_3]\, \bar{\Lambda}_\ell = \bar{\Lambda}_\ell . \tag{2.16b}$$

Now, let us focus our attention on a special class of weight functions, which we define by analogy with the previously discussed real solutions:

$$p(t) = \sum_{k=1}^{n} \bar{\beta}_k\, e^{q_k t}, \qquad \bar{\beta}_k = (i)^{n+1}(-1)^k \beta_k . \tag{2.17}$$

Eq. (2.17) implies that

$$\bar{\Omega} = e^{\omega_1+\omega_2} \sum_{k=1}^{n} \bar{\beta}_k \frac{\mathrm{sh}(\omega-q_k)}{\omega-q_k} . \tag{2.18}$$

We can compare $\bar{\Omega}$ with Ω and immediately observe that $\bar{\Omega}$ does not define a class of real solutions. However, as we have shown in ref. [2]

$$\bar{\Lambda}_\ell = \Lambda'_\ell - \sum_{k=0}^{\ell-1} \frac{\left(\frac{s^2}{2}\right)^k}{k!} \Lambda'_{\ell-k}(x_3, s=0), \tag{2.19}$$

where

$$\Lambda'_\ell = \sqrt{\frac{\pi}{2}} \sum \bar{\beta}_k \frac{I_{1/2-\ell}(r_k)}{(r_k)^{1/2-\ell}}, \tag{2.20}$$

and it is amazing to observe that Λ'_ℓ satisfies exactly the same Eqs. (2.16) as $\bar{\Lambda}_\ell$ does. Therefore, if we define

$$\Delta'_{\pm\ell} = (-1)^\ell e^{ix_4} \frac{e^{\mp i\ell\theta}}{s^\ell} \left(1 \mp \frac{\partial}{\partial x_3}\right)^\ell \Lambda'_\ell, \tag{2.21}$$

we obtain a new class of solutions that coincide neither with the solutions described by $\bar{\Delta}_\ell$, nor with our original real Δ_ℓ solutions. Actually one may easily evaluate Δ'_ℓ and find that

$$\Delta'^{(k)}_\ell = e^{ix_4} e^{-i\ell\theta} \left[\frac{e^{r_k}}{2r_k}\left(\frac{x_3-q_k-r_k}{s}\right)^\ell - \frac{e^{-r_k}}{2r_k}\left(\frac{x_3-q_k+r_k}{s}\right)^\ell\right]. \tag{2.22}$$

Eqs. (2.22) are a version of Eqs. (2.7) where the discontinuity has been removed without resorting to a special choice of parameters. While once more the asymptotic behavior is described by Eq. (2.12), it is immediate to compare the solutions on the x_3 axis, where

$$\Delta^{(k)}_0(s=0) = e^{ix_4} \frac{e^{x_3-q_k} - e^{-(x_3-q_k)}}{2(x_3-q_k)} \qquad \Delta^{(k)}_\ell(s=0) \neq 0 \text{ and singular}, \tag{2.23}$$

and see that they describe different field configurations. However, as was first realized by Narain [7], Eq. (2.20) again defines real solutions, whose reality is proven by explicitly finding the gauge transformation that removes all the imaginary components.

We have proven the following fundamental property (see Appendix B).

$$H_{k-\ell}^{n+m\times n+m} = \left[\frac{-2^{2n-2}\, e^{2ix_4}}{(2p\bar{p})^n}\right]^m H_{k-\ell}^{n-m\times n-m} \qquad (2.24)$$

where $H_{k-\ell}^{0\times 0} = 1$ by convention. The special case for $m = 1$ of the identity Eq. (2.24) is the sufficient condition for reality of these solutions, as shown in ref. [8].

By performing a formal summation of the series

$$\sum_{\ell=-\infty}^{\infty} \Delta_\ell\, \zeta^{-\ell} \qquad (2.25)$$

we have managed to obtain a representation for the generating function of these solutions:

$$\Omega'^{(n)} = e^{\omega_1+\omega_2} \sum_{k=1}^{n} \bar{\beta}_k\, \delta(\omega - q_k) \sim e^{\omega_1+\omega_2}\, \delta(P_n(\omega)). \qquad (2.26)$$

However Eq. (2.26), while being a compact and especially useful way of representing these solutions doesn't easily fit into the general transition-matrix description of self-dual gauge fields where Ω is assumed to be an analytic function in a proper sense (and moreover for regularity we expect it to be an entire function) and not a distribution. In turn we could not find any equivalence transformation consistent with Corrigan-Goddard's general scheme and turning $\Omega'^{(n)}$ into a more conventional object.

Needless to say, the special choice of parameters making the solutions regular:

$$q_k = i\pi\left(\frac{n+1}{2} - k\right) \qquad k = 1, \ldots, n \qquad (2.27)$$

also implies $\Delta'_\ell = \Delta_\ell \quad |\ell| < n$ and the equivalence (in a deeper sense than we now understand) of the corresponding $\Omega'^{(n)}$ and $\Omega^{(n)}$.

III. A non-equivalent representation for the single monopole solution

Until now we have assumed that some kind of equivalence may be defined between different representations of a same solution in terms of generating functions. Actually, both the kind of equivalence assumed in Corrigan and Goddard's paper and the extension we have considered in the previous section are characterized by the fact that

$$\Delta'_\ell = \Delta_\ell , \qquad -n < \ell < n \tag{3.1}$$

for the n-th ansatz, which in turn implies that, once a gauge-fixing procedure is defined, such as the one employed by Corrigan et al. in their construction, the gauge fields are uniquely determined.

However, this is not the end of the story. At least for the single monopole solution, starting from the known instanton representation for the solution itself [9], we managed to build up a non-explicitly time-independent generating function such that

$$\Delta'_\ell \neq \Delta_\ell , \tag{3.2}$$

but the resulting gauge fields are gauge-equivalent to the static solution.

Let's recall that a single instanton located at the origin may be described within the $\mathcal{A}_1$ ansatz by choosing

$$\hat{\Omega}^{(1)} = \frac{1}{4\omega_1\omega_2} . \tag{3.3}$$

The generating function for a string of instantons located along the "time" axis at the points $x_4 = 2\pi n$ is then

$$\hat{\Omega} = +\sum_{n=-\infty}^{\infty} \frac{1}{2\omega_1 - 2i\pi n}\,\frac{1}{2\omega_2 - 2i\pi n} = \frac{-1}{2(\omega_1-\omega_2)}\left(\frac{e^{2\omega_1}}{1-e^{2\omega_1}} - \frac{e^{2\omega_2}}{1-e^{2\omega_2}}\right) , \tag{3.4}$$

which is also equal to

$$\hat{\Omega} = +\frac{sh\omega}{2\omega}\,\frac{1}{ch\omega - ch(\omega_1+\omega_2)} . \tag{3.5}$$

Eq. (3.4) is not equivalent, in the sense of Eq. (3.1), to the standard representation

$$\Omega = \frac{e^{2\omega_1} - e^{2\omega_2}}{2(\omega_1 - \omega_2)} = \frac{sh\omega}{\omega}\, e^{\omega_1+\omega_2} \tag{3.6}$$

We can, however, compare the gauge fields, by noticing that in the $\mathcal{A}_1$ ansatz they may be written in the compact form

$$A^a_\mu = -\bar{\eta}^a_{\mu\nu}\,\partial_\nu \ln \Delta_0 \tag{3.7}$$

and that

$$\hat{\Delta}_0 = \frac{shr}{2r}\,\frac{1}{chr - \cos x_4} ; \qquad \Delta_0 = \frac{shr}{r}\, e^{ix_4} . \tag{3.8}$$

It is easy to recognize that the gauge transformations [9]

$$U^{\pm}(\theta) = \exp \pm i\vec{T}\cdot\hat{r}\,\theta , \qquad \theta = (\tan^{-1} \frac{\sin x_4\, shr}{chr \cos x_4 - 1} - ir) \tag{3.9}$$

turns the two field configurations into each other. We think it's intriguing to recognize that two different generating functions for which no superficial equivalence can be identified may generate the same self-dual field configuration. Apparently, equivalence in the gauge theory sense is a deeper concept than the mathematical equivalence used in classifying the transition matrices.

IV. Multimonopoles close together on a plane

In this section we study multimonopole solutions that are situated close together on a plane. That the monopoles are actually separated is verified by computing the zeros of the Higgs field which can be interpreted physically as the "location" of the monopoles in space. This is to be contrasted with the axially symmetric multimonopoles [1,2] which represent superimposed monopoles with the Higgs field vanishing at only one point, the origin $x_1 = x_2 = x_3 = 0$.

The separated n-monopole solution, to be discussed below, depends continuously on a single real parameter d such that as $d \to 0$ we regain the axially symmetric n-monopole solution [2]. For $d \neq 0$ the regularity of the solution can be insured provided d is sufficiently small, because the $d = 0$ solution is known to be regular and by continuity this will be true for d close to zero.

The $d = 0$ axially symmetric n-monopole solution Higgs field has an n-th order zero at the origin. For sufficiently small $d \neq 0$ we find that the Higgs field acquires n simple zeros at n distinct points in space, close to the origin and on a plane [note that we always use dimensionless space coordinates where length scales are measured in units of (gauge coupling constant x vacuum expectation value of Higgs field)$^{-1}$]. We will present explicit perturbative results for the zeros of the Higgs field to lowest order in d for n = 3, 4. This perturbative calculation, though conceptually trivial, is in practice extremely cumbersome and had to be done with the help of the symbol manipulating computer program MACSYMA.

Motivated by Ward's 2-monopole solution [4] we take the transition matrix to be equivalent to:

$$G(\omega,\zeta) = \begin{bmatrix} \dfrac{e^{f} + (-1)^{n}\, e^{f}}{K} & (-1)^{n}\, \zeta^{n}\, e^{-f} \\ \zeta^{-n}\, e^{-f} & K\, e^{-f} \end{bmatrix} \tag{4.1a}$$

$$f \equiv \omega/\varepsilon, \quad \varepsilon \equiv \frac{1}{2\pi}\int_0^{2\pi} \left[\frac{1+(d/2)(\zeta-\zeta^{-1})}{1+id\,\sin\psi}\right]^{1/2} d\psi \tag{4.1b}$$

$$K = \frac{2}{(n-1)!\,\pi^{n-1}} \prod_{k=1}^{n} (\omega - \varepsilon z_k), \quad z_k = i\pi[(\tfrac{n+1}{2}) - k]. \tag{4.1c}$$

Note that as $d \to 0$, $\varepsilon \to 1$ and we regain the axially symmetric n-monopole solution [2]. With respect to the Corrigan-Goddard constraint equations [3], Eq. (4.1) is the only solution we have been able to construct explicitly; in particular, (4.1) is the most general solution with f depending linearly on ω.

It is easy to see that Eq. (4.1) satisfies Ward's reality requirements (A.8d, e), whereas to check that it also satisfies (A.8a, b, c) we proceed as follows. Considered as a function of the complex parameter ζ, $1/\varepsilon$ is analytic except for branch points at $\zeta = 0, \infty, [-d^{-1} \pm (1+d^{-2})^{1/2}]$. Therefore $1/\varepsilon$ is analytic within an annular region, which contains the circle $|\zeta| = 1$, so that it can be expanded in a Laurent series:

$$1/\varepsilon = 1 + g^{+} - g^{-}, \quad g^{\pm} \equiv \pm \sum_{n=1}^{\infty} a_{\pm n}\, \zeta^{\pm n} \tag{4.2a}$$

$$a_{\pm n} \equiv \left[\frac{1}{2\pi}\int_0^{2\pi} (1+id\,\sin\psi)^{-1/2}\, d\psi\right]^{-1} \frac{1}{2\pi}\int_0^{2\pi} e^{\mp in\chi}(1+id\,\sin\chi)^{-1/2}\, d\chi. \tag{4.2b}$$

We can now define the variables:

$$f_1 \equiv \omega_1 + \omega g^+, \quad f_2 \equiv \omega_2 + \omega g^-, \quad f = f_1 - f_2 = \omega/\varepsilon, \tag{4.3}$$

so that f_1 (f_2) is analytic away from $\zeta = \infty$ ($\zeta = 0$). It is then easy to check that:

$$G(\omega,\zeta) = \begin{bmatrix} e^{-f_2} & 0 \\ 0 & e^{f_2} \end{bmatrix} \begin{bmatrix} \zeta & \tilde{\Omega} \\ 0 & \zeta^{-n} \end{bmatrix} \begin{bmatrix} 0 & -e^{f_1} \\ e^{-f_1} & K\zeta^n e^{-f_1} \end{bmatrix} \tag{4.4a}$$

$$\tilde{\Omega} \equiv K^{-1} \, [e^{2f_1} + (-1)^n \, e^{2f_2}], \tag{4.4b}$$

so that Ward's reality requirements (A.8a, b, c) are also satisfied. From Eq. (4.4b) and Eq. (A.7c) one can compute the $\tilde{\Delta}$'s after which one computes ${}_n\tilde{\phi}$, ${}_n\tilde{\rho}$, ${}_n\tilde{\tilde{\rho}}$ using Eq. (A.7a, b). Finally one can compute the Higgs field, in Yang's R gauge, using Eqs. (A.9).

When $d = 0$ the n-monopole solution becomes axially symmetric and the Higgs field has an n-th order zero at the origin, where it behaves as [10]:

$$d = 0: \; h_1 \sim s^n \cos(n\theta), \; h_2 \sim s^n \sin(n\theta), \; h_3 \sim x_3 \tag{4.5}$$

and $s\,e^{i\theta} \equiv x_1 + ix_2$. Therefore to investigate the zeroes of the Higgs field for sufficiently small $d \neq 0$ we need to compute the Higgs field to order n in the variables x_1, x_2, x_3 and d which are treated as small deviations from the origin and $d = 0$ respectively. For this purpose we directly expand $\tilde{\Omega}$ in Eq. (4.4b) to (n+1)th order in x_1, x_2, x_3 and d. Using Eq. (A.7c) and the formula

$$\frac{1}{2\pi i} \oint \zeta^k \frac{d\zeta}{\zeta} = \delta_{k,0}. \tag{4.6}$$

We can then compute the $\tilde{\Delta}$'s to (n+1)th order. Finally, Eq. (A.9) gives the Higgs field to n-th order. The results of our calculations for n = 3, 4 are as follows:

<u>n = 3:</u>

$$h_1 = \frac{1}{96}\left(1+\frac{3}{\pi^2}\right) d^2 s \cos\theta + \left(\frac{1}{90} - \frac{1}{6\pi^2}\right) s^3 \cos(3\theta)$$

$$h_2 = \frac{1}{96}\left(1+\frac{3}{\pi^2}\right) d^2 s \sin\theta + \left(\frac{1}{90} - \frac{1}{6\pi^2}\right) s^3 \sin(3\theta)$$

$$h_3 = \left(\frac{1}{180} - \frac{1}{2\pi^4}\right)(4x_3^3 - 6s^2 x_3) + \left(\frac{1}{6} - \frac{1}{\pi^2}\right)\left[2x_3\left(-1+\frac{d^2}{8}\right) + x_1 d\right]$$

zeroes: $x_1 = x_2 = x_3 = 0$; $x_2 = 0$, $x_1 = \pm\gamma_0 d$, $x_3 = \pm\gamma_0 d^2/2$.

$$\gamma_0 = \left[\frac{15}{16}\left(\frac{3+\pi^2}{15-\pi^2}\right)\right]^{1/2}$$

<u>n = 4:</u>

$$h_1 = -\frac{3d^4}{2048} + \left(-\frac{1}{128} + \frac{5}{48\pi^2}\right) d^2 s^2 \cos(2\theta) + \left(-\frac{1}{24} + \frac{25}{54\pi^2} - \frac{5}{9\pi^4}\right) s^4 \cos(4\theta)$$

$$h_2 = \left(-\frac{1}{128} + \frac{5}{48\pi^2}\right) d^2 s^2 \sin(2\theta) + \left(-\frac{1}{24} + \frac{25}{54\pi^2} - \frac{5}{9\pi^4}\right) s^4 \sin(4\theta)$$

$$h_3 = \left(\frac{1}{12} - \frac{656}{81\pi^4}\right)(4x_3^3 - 6s^2 x_3 - 12x_3^2 dx_1 + 3s^2 dx_1) + \left(\frac{1}{2} - \frac{40}{9\pi^2}\right)\left[2x_3\left(-1+\frac{d^2}{8}\right) + x_1\left(d - \frac{3d^3}{8}\right)\right]$$

zeroes: $x_2 = 0$, $x_1 = \pm\gamma_+ d$, $x_3 = \pm\gamma_+ d^2/2$; $x_2 = 0$, $x_1 = \pm\gamma_- d$, $x_3 = \pm\gamma_- d^2/2$

$$\gamma_\pm = \frac{3\pi}{4\sqrt{2}}\left\{\frac{40 - 3\pi^2 \pm (1120 + 160\pi^2 - 27\pi^4)^{1/2}}{120 - 100\pi^2 + 9\pi^4}\right\}^{1/2}$$

As asserted, the zeroes of the Higgs field are at n distinct points on the $x_2 = 0$ plane and close to the origin $x_1 = x_2 = x_3 = 0$. It is important to note that to the order we are working h_1, h_2, and h_3 are real - a situation which we know cannot be true to all orders.

Appendix A. Multimonopole Formalism

Let us define in four dimensional Euclidean space (x_1,x_2,x_3,x_4) the SU(2) gauge potentials A_μ^a when a = 1,2,3 and μ = 1,2,3,4. The gauge field strength is defined by:

$$F_{\mu\nu}^a \equiv \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + \epsilon^{abc} A_\mu^b A_\nu^c \,. \tag{A.1}$$

Multimonopole solutions with magnetic charge n = 1,2,3,... may be found within the framework described in ref. [8]. This means that we want to solve the self-duality equations:

$$F_{\mu\nu}^a = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma} F_{\rho\sigma}^a \tag{A.2}$$

(our convention is $\epsilon_{1234} \equiv +1$) with the requirement that A_μ^a be static (independent of x_4), real and regular. We also require that $A_4^a A_4^a \to 1 - \frac{2n}{r} + O(r^{-2})$ as $r \equiv (x_1^2 + x_2^2 + x_3^2)^{1/2} \to \infty$ (note that A_4^a is just the Higgs field). Provided these conditions are met the solutions will correspond to finite energy $E = 1/4 \int F_{\mu\nu}^a F_{\mu\nu}^a d^3x = 4\pi n$ magnetic monopole solutions with magnetic charge n.

Yang [11] has shown that by introducing complex coordinates

$$\sqrt{2}p = x_1 + ix_2 \quad \sqrt{2}\bar{p} = x_1 - ix_2 \quad \sqrt{2}q = x_3 - ix_4 \quad \sqrt{2}\bar{q} = x_3 + ix_4, \tag{A.3}$$

and by choosing a certain gauge, the R gauge, any solution of Eq. (A.2) can be brought to the following form: $(A_\mu \equiv \frac{\sigma^a}{2i} A_\mu^a)$:

$$A_u = \begin{pmatrix} \frac{-\phi_u}{2\phi} & 0 \\ \frac{\rho_u}{\phi} & \frac{\phi_u}{2\phi} \end{pmatrix} \qquad A_{\bar{u}} = \begin{pmatrix} \frac{\phi_{\bar{u}}}{2\phi} & \frac{-\bar{\rho}_{\bar{u}}}{\phi} \\ 0 & \frac{-\phi_{\bar{u}}}{2\phi} \end{pmatrix} \tag{A.4}$$

where u = p,q and $\phi,\rho,\bar{\rho}$ satisfy the following coupled equations:

$$(\partial_p \partial_{\bar{p}} + \partial_q \partial_{\bar{q}})\, \ln\phi + \phi^{-2}(\rho_p \bar{\rho}_{\bar{p}} + \rho_q \bar{\rho}_{\bar{q}}) = 0 \tag{A.5a}$$

$$(\phi^{-2}\rho_p)_{\bar{p}} + (\phi^{-2}\rho_q)_{\bar{q}} = 0 \qquad (\phi^{-2}\bar{\rho}_{\bar{p}})_p + (\phi^{-2}\bar{\rho}_{\bar{q}})_q = 0 \tag{A.5b}$$

On the other hand, Ward [12], using techniques of algebraic geometry and twistor theory, showed that all information of self-dual gauge fields can be "coded" into the structure of complex analytic vector bundles that are specified by a transition matrix G. In general there is no known procedure for explicitly extracting A_μ from G. However, Atiyah and Ward [13] argued that if the transition matrix G is of the following form:

$$\tilde{G}^{(n)}(\omega_1,\omega_2,\zeta) = \begin{pmatrix} \zeta^n & \tilde{\Omega}^{(n)}(\omega_1,\omega_2,\zeta) \\ 0 & \zeta^{-n} \end{pmatrix}, \tag{A.6}$$

where $\sqrt{2}\omega_1 = (\bar{q} - p\zeta)$, $\sqrt{2}\omega_2 = -(q + \bar{p}\zeta^{-1})$ and ζ is a complex parameter, then one can systematically find A_μ. Corrigan et al. [14], working in Yang's R gauge, started from Eq. (A.6) and found the following solutions of Eq. (A.5) for any n = 1,2,3,...

$$_n\tilde{\phi} = \frac{\tilde{H}^{n\times n}_{k-\ell}}{\tilde{H}^{n-1\times n-1}_{k-\ell}} \qquad _n\tilde{\rho} = (-1)^n \frac{\tilde{H}^{n\times n}_{k-\ell-1}}{\tilde{H}^{n-1\times n-1}_{k-\ell}} \qquad _n\tilde{\bar{\rho}} = (-1)^{n+1} \frac{\tilde{H}^{n\times n}_{k-\ell+1}}{\tilde{H}^{n-1\times n-1}_{k-\ell}} \tag{A.7a}$$

where

$$\tilde{H}^{j\times j}_{k-\ell+m} = \begin{vmatrix} \tilde{\Delta}_m & \cdots & \tilde{\Delta}_{m-j+1} \\ \vdots & \ddots & \\ \tilde{\Delta}_{m+j-1} & \cdot & \tilde{\Delta}_m \end{vmatrix} \tag{A.7b}$$

$$\tilde{\Delta}_\ell = \frac{1}{2\pi i} \oint \tilde{\Omega}^{(n)}(\omega_1,\omega_2,\zeta)\, \zeta^\ell \frac{d\zeta}{\zeta}, \tag{A.7c}$$

so that

$$\partial_p\tilde{\Delta}_\ell = -\partial_{\bar{q}}\tilde{\Delta}_{\ell+1}, \qquad \partial_q\tilde{\Delta}_\ell = \partial_{\bar{p}}\tilde{\Delta}_{\ell+1}. \tag{A.7d}$$

It remains to determine which of the above solutions indeed give static, real and regular gauge fields.

Ward [1] has shown that in order for the gauge fields resulting from Eq. (A.7) to be static and in some gauge real, it is sufficient to assume that there exist two 2×2 matrices $\tilde{Q}_L$ and $\tilde{Q}_R$ such that $(\omega \equiv \omega_1 - \omega_2)$:

$$\tilde{Q}_L\,\tilde{G}(\omega_1,\omega_2,\zeta)\,\tilde{Q}_R = G(\omega,\zeta) \tag{A.8a}$$

$\tilde{Q}_L$ is analytic away from $\zeta = 0$ (A.8b)

$\tilde{Q}_R$ is analytic away from $\zeta = \infty$ (A.8c)

$$\det G(\omega,\zeta) = +1 \tag{A.8d}$$

$$[G(\omega,\zeta)]^+ = G(\omega^*,-\zeta^{*-1}) \tag{A.8e}$$

There is no known simple criterion that insures the regularity of the gauge fields but a necessary condition [2] is that $G(\omega,\zeta)$ must be an entire function of the space coordinates x_1, x_2, x_3, x_4.

Finally we note that in Yang's R gauge (A.4) the Higgs field A_4^a has the following components:

$$h_1 \equiv A_4^1 = \frac{1}{2_n\tilde{\phi}}\{({}_n\tilde{\tilde{\rho}} - {}_n\tilde{\rho}) + \partial_3({}_n\tilde{\rho} + {}_n\tilde{\tilde{\rho}})\} \tag{A.9a}$$

$$h_2 \equiv A_4^2 = \frac{1}{i2_n\tilde{\phi}}\{-({}_n\tilde{\rho} + {}_n\tilde{\tilde{\rho}}) + \partial_3({}_n\tilde{\rho} - {}_n\tilde{\tilde{\rho}})\} \tag{A.9b}$$

$$h_3 \equiv A_4^3 = -\partial_3 \,\ell n\, {}_n\tilde{\phi}. \tag{A.9c}$$

Appendix B. Identities between determinants

We want to show that, when

$$\Delta_\ell = \sum_{k=1}^{n} \bar{\beta}_k\, \Delta_\ell^{(k)} e^{ix_4}, \quad \Delta_\ell^{(k)} = \Delta_{\ell,k+} + \Delta_{\ell,k-} \tag{B.1}$$

where

$$\Delta_{\ell,k,\varepsilon_k} = \left[\frac{e^{\varepsilon_k r_k}}{2\varepsilon_k r_k}\left(\frac{x_3 - q_k - \varepsilon_k r_k}{\sqrt{2}\,p}\right)^{\ell}\right] \quad \varepsilon_k = \pm 1, \tag{B.2}$$

and

$$\bar{\beta}_k = \frac{(i)^{n+1}\,(-1)^k}{\prod_{k'\neq k}(q_{k'} - q_k)}, \tag{B.3}$$

then

$$H_{k-\ell}^{(n+m)} = \left(-\frac{2^{2n-2}\,e^{2ix_4}}{(2p\bar{p})^n}\right)^m H_{k-\ell}^{(n-m)}. \tag{B.4}$$

In order to prove Eq. (B.4) let's first observe that, by exploiting standard properties of determinants, we can write the expansion

$$H_{k-\ell}^{(n+m)} = e^{i(n+m)x_4} \sum_{(i_1\varepsilon_1\ldots i_{n+m}\varepsilon_{n+m})} D^{(n+m)}_{\{i_1\varepsilon_1\ldots i_{n+m}\varepsilon_{n+m}\}}, \tag{B.5}$$

where $\{i_1\varepsilon_1\ldots i_{n+m}\varepsilon_{n+m}\}$ means permutation over all indices for the determinants

$$D^{(n+m)} = \begin{vmatrix} \Delta_0 & i_1\varepsilon_1 & \cdots & \Delta_{-n-m+1} & i_1\varepsilon_1 \\ \Delta_1 & i_2\varepsilon_2 & \cdots & \Delta_{-n-m+2} & i_2\varepsilon_2 \\ \vdots & & \ddots & \vdots & \\ \Delta_{n+m-1} & i_{n+m}\varepsilon_{n+m} & & \Delta_0 & i_{n+m}\varepsilon_{n+m} \end{vmatrix} \tag{B.6}$$

Let's observe that whenever the index $i_k = i_{k'}$ and $\varepsilon_k = \varepsilon_{k'}$ the corresponding determinant vanishes, such that we can write the further decomposition:

$$H_{k-\ell}^{(n+m)} = e^{i(n+m)x_4} \sum_{\ell=m}^{\leq \frac{n+m}{2}} \sum_{\substack{(i_1\varepsilon_1 \ldots i_{n+m-2\ell}\varepsilon_{n+m-2\ell}) \\ i_k \neq i_{k'}}} D_{\ell(i_1\varepsilon_1 \ldots i_{n+m-2\ell}\varepsilon_{n+m-2\ell})}^{(n+m)}, \qquad (B.7)$$

where

$$D_{\ell(i_1\varepsilon_1 \ldots i_{n+m-2\ell}\varepsilon_{n+m-2\ell})}^{(n+m)} = \sum_{\substack{(j_1 \ldots j_\ell) \\ j_k \neq i_{k'}}} D_{\{i_1\varepsilon_1 \ldots i_{n+m-2\ell}\varepsilon_{n+m-2\ell}\ j_1, 1\ j_1, -1 \ldots j_\ell, 1\ j_\ell, -1\}}^{(n+m)}. \qquad (B.8)$$

By trivially generalizing a formula by Narain [7] we can now show that:

$$D_{\{i_1\varepsilon_1 \ldots i_{n+m-2\ell}\varepsilon_{n+m-2\ell}\ j_1 \ldots j_\ell\}}^{(n+m)} = \prod_{t=1}^{\ell} \left[\frac{\bar{\beta}_{j_t}^2}{k} \prod_{k=1}^{n+m-2\ell} (q_{j_t} - q_{i_k})^2 \prod_{r \neq t} (q_{j_t} - q_{j_r})^2 \right] \left(-\frac{2}{p\bar{p}} \right)^{\ell(n+m-\ell)} \times \qquad (B.9)$$

$$D_{\{i_1\varepsilon_1 \ldots i_{n+m-2\ell}\varepsilon_{n+m-2\ell}\}}^{(n+m-2\ell)}.$$

Let's now use the property that defines real solutions:

$$\bar{\beta}_k^2 = \frac{(-1)^{n+1}}{\prod_{k' \neq k} (q_{k'} - q_k)^2} \qquad (B.10)$$

and define the complementary roots $q_{\mu_1} \ldots q_{\mu_{\ell-m}}$ such that $\mu_k \neq i_{k'}$, $\mu_k \neq j_{k'}$. By substituting Eq. (B.10) into Eq. (B.9) it is immediate to obtain

$$H_{k-\ell}^{(n+m)} = e^{i(n+m)x_4} \sum_{\ell=m}^{\leq \frac{n+m}{2}} \sum_{(i_1\varepsilon_1 \ldots i_{n+m-2\ell}\varepsilon_{n+m-2\ell})} \sum_{\substack{(j_1 \ldots j_\ell) \\ (\mu_1 \ldots \mu_{\ell-m})}} \left[\frac{(-1)^{(n+1)\ell}}{\prod_{t=1}^{\ell} \prod_{s=1}^{\ell-m} (q_{j_t} - q_{\mu_s})^2} \right] 2^{-2\ell} \left(-\frac{2}{p\bar{p}} \right)^{\ell(n+m-\ell)} D_{\{i_1\varepsilon_1 \ldots i_{n+m-2\ell}\varepsilon_{n+m-2\ell}\}}^{(n+m-2\ell)}. \qquad (B.11)$$

Let's now perform the change of variables $\ell' = \ell - m$ in Eq. (B.11), thus obtaining

$$H_{k-\ell}^{(n+m)} = e^{i(n+m)x_4} \sum_{\ell'=0}^{\leq \frac{n-m}{2}} \sum_{(i_1\varepsilon_1 \ldots i_{n-m-2\ell'}\varepsilon_{n-m-2\ell'})} \sum_{\substack{(j_1 \ldots j_{\ell'+m}) \\ (\mu_1 \ldots \mu_{\ell'})}} \left[\frac{(-1)^{(n+1)(\ell'+m)}}{\prod_{t=1}^{\ell'+m} \prod_{s=1}^{\ell'} (q_{j_t} - q_{\mu_s})^2} \right] 2^{-2\ell' - 2m} \left(-\frac{2}{p\bar{p}} \right)^{(\ell'+m)(n-\ell')} D_{\{i_1\varepsilon_1 \ldots i_{n+m-2\ell'}\varepsilon_{n+m-2\ell'}\}}^{(n+m-2\ell')}. \qquad (B.12)$$

However by repeating the previous arguments we could have obtained

$$H_{k-\ell}^{(n-m)} = e^{i(n-m)x_4} \sum_{\ell=0}^{\leq \frac{n-m}{2}} \sum_{(i_1\varepsilon_1 \ldots i_{n-m-2\ell}\varepsilon_{n-m-2\ell})} \sum_{\substack{(j_1 \ldots j_\ell) \\ (\mu_1 \ldots \mu_{\ell+m})}} \left[\frac{(-1)^{(n+1)\ell}}{\prod_{t=1}^{\ell} \prod_{s=1}^{\ell+m} (q_{j_t} - q_{\mu_s})^2} \right] 2^{-2\ell} \left(-\frac{2}{p\bar{p}} \right)^{\ell(n-m-\ell)} D_{\{i_1\varepsilon_1 \ldots i_{n-m-2\ell}\varepsilon_{n-m-2\ell}\}}^{(n-m-2\ell)}, \qquad (B.13)$$

and by observing that j_t and μ_s enter in a symmetric way in Eqs. (B.12) and (B.13) and can therefore be interchanged, we recognize by direct comparison of the last two equations that

$$H_{k-\ell}^{(n+m)} = \left[-\frac{2^{2n-2} e^{2ix_4}}{(2p\bar{p})^n} \right]^m H_{k-\ell}^{(n-m)}, \qquad H^{(0)} = 1. \qquad (B.14)$$

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