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ABSTRACT

The vertex function third-order in radiation field is found for an electron in an external constant crossed field of arbitrary strength. It is shown that the radiative interaction "smears" the Airy function which describes the intensity of electron-photon interaction in the external field as a function of the nonconserved component of electron momentum. The qualitative relation $V^{(3)} \sim \alpha \chi^{2/3} V^{(1)}$ between the third-and the first-order vertex functions is found for large values of the dynamic parameter χ. It is shown also that the radiative interaction does not change the order of magnitude of the squared mass transmitted in the vertex. The vertex function satisfies the Ward identity modified by the external field.

VERTEX FUNCTION OF ELECTRON IN A CONSTANT ELECTROMAGNETIC FIELD

D.A.Morozov, N.B.Narozhny, V.I.Ritus

I. Introduction.

Much attention is attracted at present to investigation of radiative corrections to electromagnetic processes in intense external fields, i.e. in fields whose strength is close to the characteristic quantum electrodynamic value

$$F_0 = m^2c^2/e\hbar = 4.4 \times 10^{13}\ \mathrm{Oe}.$$

So, for a constant external field there were calculated mass and polarization operators in the second-order of perturbation theory [I-10], fourth-order corrections to them [II-13] and radiative corrections to some processes [II-13]. The asymptotic behaviours of mass and polarization operators in higher orders of perturbation theory have been also investigated [14-16].

However in all these papers only the diagrammes containing no vertex function were considered since the vertex function has not been calculated yet in electrodynamics with an external field.

In the present paper we are going to bridge the gap and calculate the vertex function in the third order of perturbation theory with respect to the radiation field for an electron in a constant crossed field ($\vec{E} \perp \vec{H}$, $E = H$) of arbitrary strength. Such a field is described by 4-potential[1)]

$$A_\mu = a_\mu (\kappa\cdot x) , \quad \kappa^2 = \kappa a = 0 .$$

It is a low-frequency approximation of a plane-wave field and a good approximation of an arbitrary constant field for ultrarelativistic particles.

2. Momentum representation for a vertex function.

The vertex function of an electron in a constant crossed field can be written in the form

$$V_\mu^{(3)}(x'', y, x') = e^2 \gamma_\nu S^c(x'', y)\gamma_\mu S^c(y, x')\gamma_\nu D^c(x''-x'), \quad (I)$$

where $S^c(x,y)$ is the electron propagation function in a crossed field which was found by Schwinger [17]

$$S^c(x,y) = e^{i\eta} S(x-y) , \quad \eta = \frac{e}{2}(a, x-y)(\kappa, x+y), \quad (2)$$

$$S(z) = \frac{1}{i(4\pi)^2}\int_0^\infty \frac{ds}{s^2}\left[m - i\gamma\pi z\right]\exp\left\{i\frac{z^2}{4s} - is\left(m^2 + \frac{(eFz)^2}{12}\right) + i\frac{e\sigma F}{2}s\right\}, \quad (3)$$

$$\pi_{\alpha\beta}(s) = \frac{1}{2s}\delta_{\alpha\beta} + \frac{e}{2}F_{\alpha\beta} + \frac{s}{6}e^2(FF)_{\alpha\beta} \quad (4)$$

and $D^c(z)$ is a photon propagation function for which we use the proper-time representation

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1) The system of units is used, where $\hbar = c = 1$ and the designations $p_\alpha = (\vec{p}, i p_0)$, $qp = \vec{q}\vec{p} - q_0 p_0$, $p_- = p_0 - p_3$, $p_+ = \frac{1}{2}(p_0 + p_3)$, $\alpha = \frac{e^2}{4\pi\hbar c} = (137{,}03\ldots)^{-1}$.

$$D^c(z) = \frac{1}{i(4\pi)^2}\int_0^\infty \frac{dt}{t^2}\exp\left\{i\frac{z^2}{4t} - i\mu^2 t\right\} . \qquad (5)$$

Here μ is a small photon-mass introduced for elimination of the infrared divergence of the vertex function.

We shall compute the vertex function in the E_p -representation which was introduced by one of the authors [I], and is an analog of the usual Fourier transformation in intense field quantum electrodynamics.

Remember that the basic functions of the E_p -representation have the form

$$E_p(x) = \exp\left\{ipx + i\frac{epa}{2\kappa p}(\kappa x)^2 - i\frac{e^2a^2}{6\kappa p}(\kappa x)^3 + i\frac{e\sigma F}{4\kappa p}(\kappa x)\right\} . \qquad (6)$$

The matrices (6) are the eigenfunctions of the operators [2]

$$-i\partial_1 , -i\partial_2 , i(\partial_0 + \partial_3) , (\gamma\Pi)^2$$

with the eigenvalues

$$p_1 , p_2 , p_- , p^2 .$$

They satisfy the equation

$$\gamma\Pi E_p(x) = E_p(x)\gamma p , \qquad (7)$$

where $\Pi_\mu = -i\partial_\mu - eA_\mu$ is the kinetic momentum operator and γp is a γ -matrix eigenvalue of the operator $\gamma\Pi$.

The matrices $E_p(x)$ satisfy the orthogonality and

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[2] Noninvariant notation is always made in the coordinate system with axis 3 along the vector $\vec{K}$, where $K_+ = K_0 = K_3$, $K_1 = K_2 = K_- = 0$.

completeness relations

$$\begin{aligned} &\int d^4x\, \bar{E}_q(x)\, E_p(x) = (2\pi)^4\, \delta(q-p)\,, \\ &\int \frac{d^4p}{(2\pi)^4}\, E_p(x)\, \bar{E}_p(y) = \delta(x-y)\,, \\ &\qquad \bar{E}_p(x) = \gamma_4\, E_p^{+}(x)\, \gamma_4\,. \end{aligned} \tag{8}$$

E_p -transform of the vertex function (I) will be written in the form

$$V_\mu^{(3)}(q,p,l) = \int d^4x'\, d^4x''\, d^4y\, \bar{E}_q(x'')\, V_\mu^{(3)}(x'', y, x')\, E_p(x')\, e^{ily}. \tag{9}$$

We may rewrite the formula (9) in the form

$$V_\mu^{(3)}(q,p,l) = \int d^4y\, \bar{E}_q(y)\, \Lambda_\mu^{(3)}(\tilde{q}(y), \tilde{p}(y))\, E_p(y)\, e^{ily}, \tag{10}$$

if introduce the integration variables

$$z'' = x'' - y\,, \quad z' = y - x'\,, \quad y\,, \tag{11}$$

and use the relation

$$E_p(y+z) = E_p(y)\, E_{\tilde{p}(y)}(z)\, e^{ie(az)(\kappa y)}, \tag{12}$$

which follows directly from the explicit expression (6) for the Ep-function. Here,

$$\tilde{p}_\mu(y) = p_\mu - e a_\mu (\kappa y) + \frac{e(pa)}{\kappa p}(\kappa y)\kappa_\mu - \frac{e^2 a^2}{2\kappa p}(\kappa y)^2 \kappa_\mu \tag{13}$$

is a classical kinetic momentum of a charged particle at the point y_- with the initial value p_μ at the point $y_- = 0$,

$$\Lambda^{(3)}_\mu(\tilde{q},\tilde{p}) = e^2 \int d^4z' d^4z'' \bar{E}_{\tilde{q}}(z'')\gamma_\nu S(z'')\gamma_\mu S(z')\gamma_\nu E_{\tilde{p}}(-z')\cdot \tag{14}$$
$$\cdot D^c(z''+z')\exp\left\{ \frac{ie}{2}\left[(az'')(\kappa z'') - (az')(\kappa z')\right]\right\} ,$$

and $S(z)$ is a diagonal part of the electron propagator (3).

The representation (13), (14) obtained for $V^{(3)}_\mu(q,p,l)$ is a Fourier integral with respect to the argument l . By shifting the integration variable y_- and using the relation (II) one can obtain for $V^{(3)}_\mu(q,p,l)$ the representation (13) with another function $\Lambda^{(3)}_\mu$ which differs from (14) and depends explicitly on l . This representation will not be already a Fourier-integral with respect to l , but it will allow to simplify the dependence of the function $\Lambda^{(3)}_\mu$ on $\tilde{q}$, $\tilde{p}$, i.e. on the integration variable in (13).

Formula (13) differs from the E_p -representation of the point vertex by the substitution $\gamma_\mu \to \Lambda_\mu$. Therefore it is the function $\Lambda_\mu(\tilde{q},\tilde{p})$ that determines the correction to the vertex function in the E_p -representation:

$$\Gamma_\mu(\tilde{q},\tilde{p}) = \gamma_\mu + \Lambda_\mu(\tilde{q},\tilde{p}) . \tag{15}$$

As is seen from (14) and (15), the vertex function in the E_p -representation depends on the y -coordinate of photon absorption through the classical kinetic electron momenta $\tilde{p}_\mu(y)$ and $\tilde{q}_\mu(y)$ before and after absorption, more precisely through

$\gamma\tilde p$, $\gamma\tilde q$. These latter γ -matrix invariants may in some sense be regarded as eigenvalues of the operator $\gamma\Pi$. This statement follows directly from the equation (7) which can be presented in the form, see ref. [18], page 9,

$$\gamma\Pi E_p(x) = \gamma\tilde p(x) E_p(x) , \qquad (16)$$

so that

$$\gamma\tilde p(x) E_p(x) = E_p(x)\gamma p . \qquad (17)$$

The equality (17) can also be written in the form

$$\bar E_p(x)\gamma\tilde p(x) E_p(x) \equiv e^{-i\frac{e\sigma F}{4\kappa p}(\kappa x)}\gamma\tilde p(x) e^{i\frac{e\sigma F}{4\kappa p}(\kappa x)} = \gamma p , \qquad (18)$$

and, therefore, a transition from the kinetic momentum $\gamma\tilde p$ to the quantum numbers γp or vice versa is none other than Lorentz transformation.

The function $\Lambda_\mu(\tilde q,\tilde p)$ considered as a function of the quantum numbers $\tilde q$ and $\tilde p$ plays an independent role, the coordinate dependence of $\tilde q$ and $\tilde p$ being disregarded. We shall show this on an example of the fourth-order vertex correction to the mass operator, which in the E_p -representation has the form

$$M_\nu^{(4)}(q,p) = \int d^4x\, d^4x'\, \bar E_q(x) M_\nu^{(4)}(x,x') E_p(x') , \qquad (19)$$

where

$$M_\nu^{(4)}(x,x') = ie^2\int d^4y\, d^4y'\, \gamma_\mu S^c(x,y)\gamma_\nu S^c(y,y')\gamma_\mu S^c(y',x')\gamma_\nu D^c(x-y') D^c(y-x') \qquad (20)$$

Using the evident equality $E_p(x)\bar{E}_p(x) = 1$, we shall rewright (19) in the form

$$M_v^{(4)}(q,p) = \int d^4x\, \bar{E}_q(x) E_p(x) \int d^4x' \bar{E}_p(x)\, M_v^{(4)}(x,x') E_p(x'), \tag{21}$$

and show that the integral

$$M_v^{(4)}(p) = \int d^4x' \bar{E}_p(x)\, M_v^{(4)}(x,x') E_p(x') \tag{22}$$

does not depend on x .

Making use of the E_p -representation of the electron propagator and the representation (13) for the function $V_\mu^{(3)}(q,p,\ell)$ we obtain

$$M_v^{(4)}(p) = -\frac{ie^2}{(2\pi)^8} \int d^4\ell\, d^4f\, d^4y\, \frac{e^{-i\ell(x-y)}}{\ell^2 - i\varepsilon}\, \bar{E}_p(x) \gamma_\mu E_f(x) \cdot \tag{23}$$

$$\cdot \frac{1}{m + i\gamma f - i\varepsilon}\, \bar{E}_f(y) \Lambda_\mu^{(3)}(\tilde{f}(y), \tilde{p}(y)) E_p(y) .$$

By virtue of relations (II), (I8) we have

$$\bar{E}_p(x)\gamma_\mu E_f(x) \frac{1}{m + i\gamma f} \bar{E}_f(y) = \bar{E}_p(y) \bar{E}_{\tilde{p}(y)}(x-y) \gamma_\mu E_{\tilde{f}(y)}(x-y) \frac{1}{m + i\gamma \tilde{f}(y)} . \tag{24}$$

Therefore the expression (23) can be reduced to the form

$$M_v^{(4)}(p) = -\frac{ie^2}{(2\pi)^8} \int d^4y\, \bar{E}_p(y) \left\{ \int d^4\ell\, d^4\tilde{f}\, \frac{e^{-i\ell z}}{\ell^2 - i\varepsilon}\, \bar{E}_{\tilde{p}(y)}(z) \gamma_\mu E_{\tilde{f}}(z) \cdot \right. \tag{25}$$

$$\left. \cdot \frac{1}{m + i\gamma\tilde{f} - i\varepsilon}\, \Lambda_\mu^{(3)}(\tilde{f}, \tilde{p}(y)) \right\} E_p(y) ,$$

if in curly brackets one passes over from the integration variable f to $\tilde{f} = \tilde{f}(y)$ taking into account that the Jacobian here is equal to I and designating $x - y = z$.

After integrating over the virtual momenta, the expression in curly brackets in (25) is a γ -matrix invariant depending on $\gamma\tilde{p}(y)$, γz , σF only. The Lorentz rotation operation (18) transforms $\gamma\tilde{p}(y)$ into γp , γz into $\gamma\tilde{z}$, where

$$\tilde{z}_\mu = z_\mu - \frac{ky}{kp} e(Fz)_\mu + \frac{(ky)^2}{2(kp)^2} e^2 (FFz)_\mu \tag{26}$$

and does not change the invariant σF . Therefore

$$\bar{E}_p(y)\{\ \}E_p(y) = \int d^4\ell\, d^4 f \frac{e^{-i\ell\tilde{z}}}{\ell^2 - i\varepsilon} \bar{E}_p(\tilde{z})\gamma_\mu E_{\tilde{f}}(\tilde{z}) \frac{1}{m + i\gamma\tilde{f} - i\varepsilon} \Lambda^{(3)}_\mu(\tilde{f}, p). \tag{27}$$

If in the expression (25) one now goes over from the integration variable y to $\tilde{z}$ (the Jacobian is equal to unity) and takes away the unnecessary "tilda" signs in $\tilde{f}$, $\tilde{z}$, one finally obtains

$$M^{(4)}_v(p) = -\frac{ie^2}{(2\pi)^8} \int \frac{d^4 f\, d^4 \ell}{\ell^2 - i\varepsilon} V^{(1)}_\mu(p, f, -\ell) \frac{1}{m + i\gamma f - i\varepsilon} \Lambda^{(3)}_\mu(f, p). \tag{28}$$

So, $M^{(4)}_v(p)$ does not depend on the coordinate x , and by virtue of (8) for the mass operator (21) there appears the following expression

$$M^{(4)}_v(q, p) = (2\pi)^4 \delta(q - p) M^{(4)}_v(p), \tag{29}$$

which should be expected also from the general considerations [I].

An analogous representation can also be obtained for the second-order mass operator

$$M_V^{(4)}(p) = -\frac{ie^2}{(2\pi)^8}\int \frac{d^4f\,d^4\ell}{\ell^2 - i\varepsilon}\, V_\mu^{(4)}(p,f,-\ell)\,\frac{1}{m + i\gamma f - i\varepsilon}\,\gamma_\mu \,. \qquad (30)$$

Comparing formulae (28) and (30) we see that the transition from the second-order mass operator to the vertex correction (28) is realized by a replacement of the matrix γ_μ by $\Lambda_\mu^{(3)}$, the correction to the vertex function $\Lambda_\mu^{(3)}$ being independent on the coordinates and being considered as a function of the quantum numbers f and p .

Note that the representations of the mass operator (28), (30) are remarkable by being close to vacuum representations since there is only one vertex that is "dressed" in E_p-functions. This closeness reached by the E_p - representation technique facilitates essentially the interpretation and the calculations.

3. Calculation of the vertex function.

Before passing over to a direct calculation of the vertex function Λ_μ in the third order of the perturbation theory, we would like to recall that the representation (14) is not unique. Choosing integration variables in the formula (9) different from (10), one can, for example, obtain for the vertex function Λ_μ a representation depending explicitly on ℓ . In the absence of an external field this arbitrariness corresponds to the use of the conservation laws. In our case the situation is however less trivial since in a constant field we have only three conserved quantum number and various representations of Λ_μ differ from each other by the γ -matrix structure. Transition from one representation to another requires integration by parts over the variable y_- in (13), and therefore when calculating $\Lambda_\mu^{(3)}$ it is convenient to proceed directly from formula (13). We shall try to represent the vertex function in the form of an integral over electron and photon proper times, where the field-independent part of the phase is expressed in terms of $\tilde{q}^2 = q^2$, $\tilde{p}^2 = p^2$ and ℓ^2, and in the γ -matrix structure the matrices $\gamma\tilde{q}$ stand on the left and the matrices $\gamma\tilde{p}$ on the right.

When choosing the coordinate system and the gauge so that the vector $\vec{a}$ be directed along the axis 1 and the vector $\vec{K}$ along the axis 3, we represent (13) after integration over the coordinates y_1, y_2, $y_+ = \frac{1}{2}(y_0 + y_3)$ in the form

$$V_\mu^{(3)}(q,p,\ell) = (2\pi)^3 \delta(q_1 - p_1 - \ell_1)\delta(q_2 - p_2 - \ell_2)\delta(q_- - p_- - \ell_-) \cdot \tag{31}$$

$$\cdot \int_{-\infty}^{\infty} dy_- e^{if(\varphi)} e^{-i\frac{e\sigma F}{4\kappa q}\varphi} \Lambda_\mu^{(3)}(\tilde{q}(y), \tilde{p}(y)) e^{i\frac{e\sigma F}{4\kappa p}\varphi},$$

where

$$f(\varphi) = -\tau\varphi + \frac{1}{2}\alpha\varphi^2 - \frac{4\beta}{3}\varphi^3 , \quad \varphi = \kappa y = -\kappa_+ y_- ,$$

$$\tau = \frac{q_+ - p_+ - \ell_+}{\kappa_+} , \quad \alpha = \frac{eqFp}{(\kappa q)(\kappa p)} , \quad 8\beta = \frac{(eF\ell)^2}{(\kappa q)(\kappa p)(\kappa \ell)} . \tag{32}$$

Let us pay attention to the fact that as soon as the external field is switched off, the integral over y_- immediately gives $2\pi\,\delta(q_+ - p_+ - l_+)\,\Lambda_\mu^{vac}$ and leads to the conservation law $p_\alpha + l_\alpha = q_\alpha$ for all the four components of the momenta. If in formula (31) the radiation interaction is switched off, i.e. if one changes $\Lambda_\mu^{(3)}$ by γ_μ, then the $V_\mu^{(3)}$ will be transformed to vertex function $V_\mu^{(4)}$ determined by formulae (22), (23) of ref. [19]. In the presence of the field 1,2 and "-" components of the momenta are conserved, while "+" component of the momentam conjugate to the "-" coordinate is not converved - the kinetic momentum of an electron depend on its "-" coordinate and interaction depends on the coordinate y_- - of photon absorption. Therefore, after integration over y_- the final momentum is not completely determined by the initial momenta p , l , the conservation law is valid only for three components of the four, see formula (31). At the same time if we regard Λ_μ as a slowly changing function of y_- and abstract ourselves from spin effects, then due to oscillation of the function $e^{if(\varphi)}$ the largest contribution to the integral (31) is made by the vicinity of the points $\varphi = \varphi_c$, where $f'(\varphi_c) = 0$. Since

$$-f'(\varphi) = \tau - \alpha\varphi + 4\beta\varphi^2 = \frac{1}{\kappa_+}\left(\tilde{q}_+(y) - \tilde{p}_+(y) - l_+\right), \qquad (33)$$

this is the vicinity of the points for which the conservation law is valid for the remaining "+" - components of the momenta.

For the first-order vertex function $V_\mu^{(1)}$ for which $\Lambda_\mu^{(1)} = \gamma_\mu$, the integral over y_- in (31) is reduced essentially to the Airy function $\Phi(z)$ with the argument

$$z = (4\beta)^{2/3}\left[(\tau/4\beta) - (\alpha/8\beta)^2\right], \qquad (34)$$

which replaces $2\pi\,\delta(q_+ - p_+ - l_+)$ -function. Here we abstract ourselves from the general factor and the spin structure, leading to the items with the I-st and the 2-nd derivatives of the Airy functions.

Note that the argument z is gauge invariant and its magnitude determines the "detuning" $q_+ - p_+ - l_+$ as a function of the field intensity $F_{\mu\nu}$ and the values of conserved components of the momenta q, p, l. The quantity $(q-p-l)_+$ can be also considered as the squared mass transmitted in the charged or the neutral channels since with the help of the conservation laws it can be represented in the form

$$(q-p-l)_+ = \frac{(p+l)^2 - q^2}{2q_-} = \frac{l^2 - (q-p)^2}{2l_-} .$$

Thus, the electron-photon interaction is intensive only if is not very large in magnitude, i.e. if the conservation law for "+" components of the momenta is violated not very much. Radiative interaction makes Λ_μ a complicated function of y_- and modifies essentially the Airy function, see formula (54).

Making use of the representations (2), (5) for Green's functions and the explicit form (6) of the E_p-functions, we shall integrate over z', z'' in (14) with the help of the formula

$$J = \int \prod_{i=1}^{n} d^4x^{(i)} \exp\left\{-iq^{(i)}_\alpha x^{(i)}_\alpha + \frac{i}{4} x^{(i)}_\alpha W^{ik}_{\alpha\beta} x^{(k)}_\beta\right\} g(kx^{(i)}) =$$

$$= \frac{i^n (4\pi)^{2n}}{\sqrt{\det W}} \exp\left\{-iq^{(i)}_\alpha (W^{-1})^{ik}_{\alpha\beta} q^{(k)}_\beta\right\} g\left[2k_\alpha (W^{-1})^{im}_{\alpha\beta} q^{(m)}_\beta\right] , \qquad (35)$$

where W - 4 x 4 - matrix for the Lorentz indices α, β and $n \times n$ -matrix for the indices i, k which numerates

4-coordinates of integration and 4-momenta, W^{-1} is the inverse matrix for W and g is an arbitrary function of the variables $x_-^{(k)}$, k = I,2, ... , n.

In our case

$$W^{ik}_{\alpha\beta} = W_0^{ik}\,\delta_{\alpha\beta} + W_1^{ik}\,\frac{e}{kp}\,\tilde{p}_\lambda\left(F_{\lambda\alpha}k_\beta + F_{\lambda\beta}k_\alpha\right) + \tag{36}$$

$$+ W_2^{ik}\,\frac{e}{kq}\,\tilde{q}_\lambda\left(F_{\lambda\alpha}k_\beta + F_{\lambda\beta}k_\alpha\right) + W_3^{ik}\,e^2(FF)_{\alpha\beta}\,,$$

where two-rowed matrices W_n have the form

$$W_0 = \begin{pmatrix}\omega_1^{-1} & t^{-1}\\ t^{-1} & \omega_2^{-1}\end{pmatrix},\; W_1 = \begin{pmatrix}-1 & 0\\ 0 & 0\end{pmatrix},\; W_2 = \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix},\; W_3 = \frac{1}{3}\begin{pmatrix}s_1 & 0\\ 0 & s_2\end{pmatrix}, \tag{37}$$

s_1 , s_2 and t are proper times of electrons with momenta p , q and a photon, respectively, $\omega_i^{-1} = s_i^{-1} + t^{-1}$.

The inverse matrix W^{-1} also has the structure (36) with matrix coefficients u_n instead of W_n :

$$u_0 = W_0^{-1} = \frac{1}{\det W_0}\begin{pmatrix}\omega_2^{-1} & -t^{-1}\\ -t^{-1} & \omega_1^{-1}\end{pmatrix}, \tag{38}$$

$$u_{1,2} = -W_0^{-1}W_{1,2}W_0^{-1}\,,\quad u_3 = -W_0^{-1}W_3W_0^{-1} - W_0^{-1}\sigma_3W_0^{-1}\sigma_3W_0^{-1},$$

where σ_3 is a Pauli matrix, and

$$\det W_0 = \frac{1}{s_1 s_2} + \frac{1}{t}\left(\frac{1}{s_1} + \frac{1}{s_2}\right). \tag{39}$$

After integration, (14) can be written in the form

$$\Lambda_\mu^{(3)}(\tilde{q}(y), \tilde{p}(y)) = \frac{\alpha}{4\pi i} \iiint_0^\infty \frac{ds_1 ds_2 dt}{s_1^2 s_2^2 t^2} e^{-i(m^2 s_1 + m^2 s_2 + \mu^2 t)} Q_\mu \cdot \tag{40}$$

$$\cdot \exp\left\{-iqW^{-1}q + i\frac{4e^2a^2}{3\kappa q}\left[\kappa(W^{-1}q)^{(2)}\right]^3 + i\frac{4e^2a^2}{3\kappa p}\left[\kappa(W^{-1}q)^{(1)}\right]^3\right\},$$

where

$$Q_\mu = e^{-i\frac{e\sigma F}{2\kappa q}\kappa(W^{-1}q)^{(2)}} \gamma_\nu \left[m + i\gamma\pi(s_2)\frac{\overleftrightarrow{\partial}}{\partial q^{(2)}}\right] e^{i\frac{e\sigma F}{2}s_2} \gamma_\mu \cdot \tag{41}$$

$$\left[m + i\gamma\pi(s_1)\frac{\overleftrightarrow{\partial}}{\partial q^{(1)}}\right] e^{i\frac{e\sigma F}{2}s_1} \gamma_\nu \, e^{-i\frac{e\sigma F}{2\kappa p}\kappa(W^{-1}q)^{(1)}},$$

the two-way arrow over the differentiation operators indicates that they act on the functions both on the right and on the left of them.

Differentiation being carried out, one should put

$$q^{(1)} = \tilde{p}(y), \quad q^{(2)} = \tilde{q}(y).$$

Expression (40) possesses a rather complicated dependence on $\tilde{p}$, $\tilde{q}$ and therefore on y_-. This dependence can be essentially simplified in the following ways. After substituting expression (40) into formula (31) and making the following replacement

$$z_- = y_- + \frac{2 s_1 s_2}{s_1 + s_2 + t}\, l_- \,, \tag{42}$$

$$s_1 = \frac{1}{2}\,\omega(1+v)(1+\eta)\,, \quad s_2 = \frac{1}{2}\,\omega(1+v)(1-\eta)\,, \quad t = \omega\,\frac{1+v}{v}\,, \tag{43}$$

one can rewright formula (31) in the form

$$V_\mu^{(3)}(q,p,l) = \int d^4 z\, \bar{E}_q(z)\, \Lambda_\mu^{(3)}\big(\tilde{q}(z), \tilde{p}(z), l\big) E_p(z)\, e^{ilz}\,, \tag{44}$$

where the transformed vertex function $\Lambda_\mu^{(3)}(\tilde{q}, \tilde{p}, l)$ depends now on l but for that it has a very simple dependence on z_- :

$$\Lambda_\mu^{(3)}(\tilde{q},\tilde{p},l) = \frac{\alpha}{4\pi i}\int_0^\infty d\omega \int_0^\infty \frac{dv}{1+v}\int_{-1}^{1} d\eta\, e^{-i\omega S - i\frac{v\omega^3}{3}R - \frac{1}{2}\omega^2 v(1-\eta^2)e\tilde{q}F\tilde{p}}\, Q_\mu \,, \tag{45}$$

$$S = m^2(1+v_\lambda) + \frac{1}{2}\,q^2(1-\eta) + \frac{1}{2}\,p^2(1+\eta) + \frac{1}{4}\,l^2 v(1-\eta^2)\,, \tag{46}$$

$$\begin{aligned} R = {} & \frac{1}{2}(eFq)^2(1-\eta)\left[1+\frac{1}{2}\eta(1+\eta)(1-v)\right] + \frac{1}{2}(eFp)^2(1+\eta)\,\cdot \\ & \cdot\left[1-\frac{1}{2}\eta(1-\eta)(1-v)\right] + \frac{1}{4}(eFl)^2(1-\eta^2)\left[v-2+\frac{1}{4}(1-\eta^2)(1+v)^2\right]. \end{aligned} \tag{47}$$

The variable $v_\lambda = v + \lambda \frac{1+v}{v}$, $\lambda = (\mu/m)^2$ includes the photon mass. The phase of the integrand depends on z_- through $e\tilde{q}F\tilde{p} = eqFp + e^2 lFFz$ only linearly.

$$Q_\mu(\tilde{q},\tilde{p},l) = -\left(m^2 - i\frac{v}{\omega(1+v)^2}\right)\tilde{\gamma}_\mu - \frac{2+2v+v^2}{4(1+v)^2}\left[e\sigma F, \tilde{\gamma}_\mu\right]_+ - \tag{48}$$
$$- i\frac{\omega v}{8(1+v)^2}\left(2+v+v\frac{1+v}{3}\right) e^2 \sigma F \gamma_\mu \sigma F +$$
$$+ \left[\tilde{q}A\gamma - lB(\eta)\gamma\right]\tilde{\gamma}_\mu\left[\gamma A\tilde{p} + \gamma B(-\eta)l\right] -$$
$$- im\left[\tilde{q}A\gamma - lB(\eta)\gamma\right]\tilde{\gamma}'_\mu - im\left[\tilde{q}A'\gamma + lB'(-\eta)\gamma\right]\tilde{\gamma}_\mu -$$
$$- im\tilde{\gamma}'_\mu\left[\gamma Ap + \gamma B(-\eta)l\right] - im\tilde{\gamma}_\mu\left[\gamma A'p - \gamma B'(\eta)l\right],$$

$$\tilde{\gamma}_\mu = \exp\left\{-i\frac{e\sigma F}{4}\omega\left[2+v(1+\eta)\right]\right\}\gamma_\mu \exp\left\{-i\frac{e\sigma F}{4}\omega\left[2+v(1-\eta)\right]\right\}, \tag{49}$$

$$\tilde{\gamma}'_\mu = \exp\left\{i\frac{e\sigma F}{4}\omega v(1-\eta)\right\}\gamma_\mu \exp\left\{i\frac{e\sigma F}{4}\omega v(1+\eta)\right\},$$

and the matrices A, A', $B(\eta)$, $B'(\eta)$ are determined by the equalities

$$A = \frac{1}{1+v}\left[1 + \omega(2+v)eF + \frac{\omega^2}{3}(6+4v+v^2)e^2FF\right], \tag{50}$$

$$A' = \frac{1}{1+v}\left[1 - \omega v eF + \frac{\omega^2}{3}v(v-2)e^2FF\right], \tag{51}$$

$$B(\eta) = \frac{2+v(1-\eta)}{2(1+v)} + \frac{\omega}{4(1+v)}\left[v^2(1-\eta^2) + v(1-\eta)^2 + 4(2+v)\right]eF +$$
$$+ \frac{\omega^2}{4(1+v)}\left[8 + 4v\eta + 3v(1-\eta)^2 + \frac{2}{3}v(1+v)(1+\eta) + v(1-v+v^2)\frac{\eta(1-\eta^2)}{3}\right]e^2FF, \tag{52}$$

$$B'(\eta) = \frac{v(1-\eta)}{2(1+v)}\left\{1 + \frac{\omega}{2}\left[2+(1-v)(1+\eta)\right]eF +\right.$$
$$\left. + \frac{\omega^2}{2}\left[(2-v)(1+\eta) - \frac{2v}{3} - v(1-\eta^2) + \frac{1}{3}(1-\eta)^2 + v(v-4)\frac{\eta(1+\eta)}{3}\right]e^2FF\right\}. \tag{53}$$

When transforming Q_μ we have performed the above-mentioned integration by parts over the variable y_- in (31) with the help of the equality (33).

Note that the matrices A , A' have a simple physical meaning, namely

$$A_{\alpha\beta}\, q_\beta = \pi_{\alpha\beta}\, z_\beta^{eff} \quad , \quad A'_{\alpha\beta}\, q_\beta = \pi^*_{\alpha\beta}\, z_\beta^{eff} \, ,$$

where the matrix $\pi_{\alpha\beta}$ is determined in (4), $\pi^*_{\alpha\beta}$ differs from it in the sign of the charge or the field strength and z_{eff} is an effective value of the relative coordinate of the classical electron with the momentum q [5].

The field being switched off, the vertex function (45) is transformed into the vacuum one obtained first by Karplus and Kroll [20] , see also ref. [21].

As has already been mentioned, for the first-order vertex function $V_\mu^{(1)}$ the integral over y_- in the representation (31) is reduced, in effect, to the Airy function $\Phi(z)$ with the argument (34). From formula (45) it follows that radiative interaction "smears" the $\Phi(z)$ replacing it by the function

$$\frac{dm^2}{4\pi i}\int_0^\infty d\omega \int_0^\infty \frac{dv}{1+v}\int_{-1}^{1} d\eta \, e^{-i\omega S - \frac{i}{3}\omega^3 v R}\,\Phi(z-\zeta), \qquad (54)$$

where z is the same argument (34) and ζ is a correction depending on ω, v, η :

$$\zeta = \omega^2 v(1-\eta^2)(4\beta)^{2/3}(\kappa q)(\kappa p). \qquad (55)$$

<u>4. Properties of the vertex function in intense fields.</u>

Let us consider two properties of the calculated vertex function which differ essentially from the properties of the corresponding vacuum function.

The interaction of particles with an external field is characterised by Lorentz- and gauge-invariant parameters

$$\chi' = \frac{\sqrt{(eFq)^2}}{m^3}, \quad \chi = \frac{\sqrt{(eFp)^2}}{m^3}, \quad \varkappa = \frac{\sqrt{(eFl)^2}}{m^3}, \qquad (56)$$

which obey a conservation law depending on a channel

$$\chi + \varkappa = \chi', \quad \chi + \chi' = \varkappa . \qquad (57)$$

It follows already from the first-order vertex function $\Gamma_\mu^{(1)} \sim \Phi(z)$ and the structure of the argument z , that a squared mass of order

$$q^2-(p+l)^2 \sim m^2\chi'^{2/3}\left(\frac{\varkappa}{\chi}\right)^{1/3}, \quad (q+p)^2-l^2 \sim m^2\varkappa^{2/3}\left(\frac{\varkappa}{\chi\chi'}\right)^{1/3} \tag{58}$$

can be transmitted at the vertex. The transmitted squared mass doesn't depend on m and vanishes the field being switched off.

When the field strength or the momenta of the particles are large, at least two of the parameters χ, χ', $\varkappa$ are large too. Then the transmitted squared mass is much larger than the electron mass and increases with the field strength. If, for example, $\chi' \sim \chi \sim \varkappa \gg 1$, the transmitted squared mass is of order $(eFp)^{2/3}$ for any channel.

Radiative interaction doesn't change the relation (58) since the effective values of the integration variables in (54) are of such order that ζ is always less or of order 1.

Even of greater interest is the asymptotic behaviour of the vertex function $\Gamma^{(3)}_\mu$ in the limit of very strong field. We will consider a simple case when $\chi' \sim \chi \gg 1$, $\varkappa \sim 1$ and all the particles are on the mass shell. Besides we put two parameters

$$\rho = \frac{eqFp}{x m^4 \varkappa}, \quad \tau = \frac{epF^*q}{m^4\varkappa},$$

where $x = ea/m$, to be equal to zero.

The parameter ρ has the meaning of the center of the vertex function formation region and due to the uniformity of the field all matrix elements including the vertex don't depend on it. Therefore the condition $\rho = 0$ is quite natural. In the coordinate system with $\vec{a}$ directed along the axis 1 and $\vec{K}$ along the axis 3 the parameter τ has the meaning $\tau \sim q_2/m$ and is definitely of order 1 in matrix elements. We put it to zero for the sake of simplicity.

In the case under consideration the effective values of the integration variables in formula (45) are

$$\omega_{eff} \sim \chi^{-2/3} \ , \quad v_{eff} \sim 1 \ .$$

Then for the quantities S (16) and R (47) we have

$$S \approx m^2 v_\lambda \ , \quad R \approx m^6 \chi^2$$

The argument of the Airy functions (54) in our approximation is of order

$$z - \zeta \approx (-\zeta) \sim \chi^{-2/3} \ll 1$$

and integrals over ω are reduced, in effect, to the well-known special functions [19]

$$f(y) = i \int_0^\infty dt\, e^{-i\left(yt + \frac{t^3}{3}\right)} \ , \quad f_1(y) = \int_y^\infty dt \left[f(t) - t^{-1} \right]$$

and their derivatives with the argument $y = (v/\chi)^{2/3} \ll 1$. Integrals over η and v are trivial and we can finally write for $V_\mu = V_\mu^{(0)} + V_\mu^{(1)}$ the following:

$$V_\mu \approx (2\pi)^3 \delta(q_1 - p_1 - l_1)\delta(q_2 - p_2 - l_2)\delta(q_- - p_- - l_-) \cdot$$
$$\cdot \frac{2m}{eF}\left(\frac{2\chi^2}{\varkappa}\right)^{1/3} \left\{ v_1 \Phi(0)\gamma_\mu - v_2 i \Phi'(0) \left(\frac{2\chi^2}{\varkappa}\right)^{1/3} \frac{e(\gamma F)_\mu}{m^2 \chi} + \right. \tag{59}$$
$$\left. + v_3 i \Phi'(0) \left(\frac{\varkappa^2}{4\chi}\right)^{1/3} \frac{ie\gamma_5(\gamma F^*)_\mu}{m^2\chi} - v_4 \Phi(0) \frac{e^2(\gamma FF)_\mu}{2m^4\chi^2} + b_\mu \right\},$$

where

$$v_1 \approx 1 - \frac{\alpha}{2\pi} \ln\left(\lambda \chi^{1/3}\right) ,$$

$$v_2 \approx 1 - \frac{\alpha}{2\pi} \left[\frac{7}{3} \ln \chi - 2 \ln \lambda \right] ,$$

$$v_3 \approx 1 - \alpha \left(\frac{2\chi}{\text{æ}} \right)^{2/3} \frac{5(1+i\sqrt{3})}{54\sqrt{3}} \; \frac{\Gamma^2\left(\frac{1}{3}\right)}{\Gamma\left(\frac{2}{3}\right)} ,$$

$$v_4 \approx 1 - \alpha (3\chi)^{2/3} \frac{14(1-i\sqrt{3})}{81\sqrt{3}} \; \Gamma\left(\frac{2}{3}\right) ,$$

and b_μ denotes a vector which doesn't appear in the first-order vertex function and has a rather combersome structure. We will write down here only those terms in b_μ which do not decrease in the limit $\chi \gg 1$.

$$b_\mu \approx \frac{3\alpha}{2\pi m^2} \Phi(0) \ln\left(\frac{\chi^{2/3}}{\lambda}\right) p_\mu \frac{e^2(\gamma F F p)}{m^4 \chi^2} +$$

$$+ \frac{2\alpha}{27\sqrt{3} m^2} \left(\frac{6}{\text{æ}}\right)^{1/3} (1+i\sqrt{3}) \Gamma\left(\frac{1}{3}\right) \Phi'(0) \frac{e(pF)_\mu}{m^2\chi} \Bigg\{ 3m + i\gamma p -$$

$$- 3 \frac{1+i\sqrt{3}}{2} (3\chi)^{1/3} \frac{\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{1}{3}\right)} \left[\frac{e\sigma F}{m\chi} + \frac{11}{6} \frac{ie\gamma_5(\gamma F^* p)}{m^2\chi} \right] \Bigg\} -$$

$$- \frac{i\alpha}{6\pi m^2} \Phi(0) \frac{e^2(pFF)_\mu}{m^4\chi^2} \Bigg\{ 4m + i\gamma p + 9 \left(\frac{6\chi^2}{\text{æ}}\right)^{1/3} \ln\left(\frac{\chi^{2/3}}{\lambda}\right) \frac{\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{1}{3}\right)} \frac{e(\gamma F p)}{m^2\chi}$$

$$- \frac{2\pi}{9} \frac{1+i\sqrt{3}}{\sqrt{3}} (3\chi)^{1/3} \Gamma\left(\frac{1}{3}\right) \left[\frac{e\sigma F}{m\chi} + \frac{5}{3} \frac{ie\gamma_5(\gamma F^* p)}{m^2\chi} \right] \Bigg\} .$$

One can see from the expressions for the scalar coefficients v_K of the vertex function that some of them are expanded with the parameter $\alpha \ln \chi$, and some with the parameter $\alpha \chi^{2/3}$. Hence; we may say that in the limit of large χ there exists a qualitative relation

$$V^{(3)} \sim \alpha \chi^{2/3} \, V^{(1)} \tag{60}$$

between the third- and the first-order vertex functions. This relation remains to be true when $\chi \sim \chi' \sim \varkappa \gg 1$, but the explicit expression for V_μ in this case is much more complicated. The relation (60) is an indication that $\alpha \chi^{2/3}$ is a universal parameter of the perturbation theory in intense-field quantum electrodynamics in the high energy or high field strength limit. Some considerations for 2/3-power, but not logarithmic, dependence of the parameter on χ were first discussed by one of the authors [1] and then confirmed in [14] .

5. Ward identity

The vertex function must satisfy the generalized Ward identity first found by Fradkin [22] . It is convinient for us to use this identity in the following representation (see Mitter's work [23])

$$l_\mu V_\mu^{(3)}(q,p,l) = i\left[I(q,p,l)M^{(2)}(p) - M^{(2)}(q)I(q,p,l)\right], \tag{61}$$

where $M^{(2)}(q)$ is the second-order mass operator, and the matrix I (q,p,l) has the representation

$$I(q,p,l) = \int d^4x\, \bar{E}_q(x)\, E_p(x)\, e^{ilx}. \qquad (61')$$

It is a generalization of the four-dimensional $(2\pi)^4\delta(q-p-l)$ - function which describes the momentum conservation in the vertex for the case of a nonzero external field. Hence, $I(q,p,l)$ differs from $V_\mu(q,p,l)$ by a replacement of Λ_μ by unit, see (13), (31).

In the limit $l \to 0$ one can obtain from the identity(61) the following differential Ward identity

$$\left[e^{-i\frac{eF}{\kappa p}\frac{\partial}{\partial s}}\right]_{\mu\nu} \Lambda_\nu(p+s\kappa, p, 0)\Big|_{s=0} = -i\frac{\partial M(p)}{\partial p_\mu} - \qquad (62)$$

$$-i\frac{\kappa_\mu\kappa_\nu}{4(\kappa p)^2}\frac{\partial M}{\partial p_\nu}e\sigma F - \frac{(eFp)_\mu\kappa_\nu\kappa_\rho}{2(\kappa p)^2}\frac{\partial^2 M}{\partial p_\nu\partial p_\rho} + i\frac{e^2(FF)_{\mu\nu}\kappa_\rho\kappa_\lambda}{6(\kappa p)^2}\frac{\partial^3 M}{\partial p_\nu\partial p_\rho\partial p_\lambda}.$$

After direct calculations one can make sure that the obtained vertex function (45) and the mass operator of an electron in a crossed field [12] satisfies the relation (62). This is a good verification of the calculations made above. The choice of integration variables (13) in the vertex function is just determined by the requirement that at $l = 0$ they coincide with the integration variables in the expression for the mass operator [12]. It is readily seen that at $l = 0$ the integral over y_- in (45) gives δ-function for "+"-components of the momentum, the phase of the integrad in (45) depends no longer on the variable $\eta = (s_2-s_1)/(s_2+s_1)$, which has the meaning of a relative difference of the electrons proper times, and coincides with the phase of the second-order mass operator [12] with the only natural difference that in the vertex variables

$$\omega^{-1} = (S_1+S_2)^{-1} + t^{-1}, \quad v = (S_1+S_2)/t$$

the summ S_1+S_2 replaces the proper time S of the electron in analogous variables of the mass operator . The variable ω can be referred to as the proper time of the vertex function.

Note that for the representation (25) of the vertex function Λ_μ in addition to (62) there holds the relation

$$K_\mu \frac{\partial^2 \Lambda_\mu (P+SK, P, 0)}{\partial S^2}\Bigg|_{S=0} = -\frac{i}{3} K_\mu \frac{\partial^2}{\partial S^2}\left[\frac{\partial M(P+SK)}{\partial P_\mu}\right]_{S=0}. \tag{63}$$

The obtained relation for the vertex function contains, of course, a logarithmic divergence in the proper time and requires regularization. Since the presence of an external field does not introduce additional, as compared with the vacuum, ultraviolet divergences, then to regularize the vertex function (45) it is sufficient to subtract from it the value it has at $\ell = 0$, $\gamma\tilde{q} = \gamma\tilde{p} = im$ and F=0

$$\Lambda_\mu^{(3)}(\tilde{q}, \tilde{q}, 0)\Big|_{\gamma\tilde{q}=im, F=0} = L^{(3)} \gamma_\mu, \tag{64}$$

$$L^{(3)} = \frac{\alpha}{4\pi i}\int_0^\infty d\omega \int_0^\infty \frac{dv}{1+v}\int_{-1}^{1} d\eta\, e^{-im^2\omega v_\lambda}\left[\frac{iv}{\omega(1+v)^2} + m^2\frac{2+2v-v^2}{(1+v)^2}\right]. \tag{65}$$

In this case for the regularized vertex function we have

$$\Lambda^{(3)}_{R\mu}(\tilde{q},\tilde{p},l) = \Lambda^{(3)}_{\mu}(\tilde{q},\tilde{p},l) - L^{(3)}\gamma_{\mu} . \qquad (66)$$

Introducing for the variable ω the lower integration kimit $\omega_0 \to 0$, and carrying out integration in (65), we obtain for $L^{(3)}$ the value

$$L^{(3)} = \frac{\alpha}{2\pi}\left(\frac{1}{2}\ln\frac{1}{i\gamma m^2\omega_0} + \ln\lambda + 2\right), \quad \gamma = 1.781\ldots \qquad (67)$$

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