

Charge orbits of extremal black holes in five-dimensional supergravityBianca L. Cerchiai,^{1,*} Sergio Ferrara,^{2,3,4,†} Alessio Marrani,^{5,‡} and Bruno Zumino^{6,7,§}¹*Università degli Studi di Milano, Dipartimento di Matematica “Federigo Enriques” via Cesare Saldini 50, I-20133 Milano, Italy*²*Theory Division, CERN, Geneva, Switzerland CH 1211*³*INFN-LNF, via Enrico Fermi 40, I-00044 Frascati, Italy*⁴*Department of Physics and Astronomy, University of California, Los Angeles, California 90095-1547, USA*⁵*Stanford Institute for Theoretical Physics, Stanford University, Stanford, California 94305-4060, USA*⁶*Lawrence Berkeley National Laboratory, Theory Group, Building 50A5104, 1 Cyclotron Road, Berkeley, California 94720-8162, USA*⁷*Department of Physics, University of California, Berkeley, California 94720-8162, USA*

(Received 17 July 2010; published 8 October 2010)

We derive the U -duality charge orbits, as well as the related moduli spaces, of “large” and “small” extremal black holes in nonmaximal *ungauged* Maxwell-Einstein supergravities with symmetric scalar manifolds in $d = 5$ space-time dimensions. The stabilizer groups of the various classes of orbits are obtained by determining and solving suitable U -invariant sets of constraints, both in “bare” and “dressed” charge bases, with various methods. After a general treatment of attractors in real special geometry (also considering nonsymmetric cases), the $\mathcal{N} = 2$ “magic” theories, as well as the $\mathcal{N} = 2$ Jordan symmetric sequence, are analyzed in detail. Finally, the half-maximal ($\mathcal{N} = 4$) matter-coupled supergravity is also studied in this context.

DOI: 10.1103/PhysRevD.82.085010

PACS numbers: 04.65.+e

I. INTRODUCTION

Five-dimensional supergravity theories with nonmaximal supersymmetry ($2 \leq \mathcal{N} < 8$), emerging from Calabi-Yau compactifications of M theory, admit extremal black p -brane solutions in their spectrum [1]. In particular, ungauged theories admit extremal black holes ($p = 0$) and black strings ($p = 1$) which are asymptotically flat and reciprocally related through U duality.¹ These objects have been intensely studied throughout the years, due to the wide range of classical and quantum aspects they exhibit.

For asymptotically flat, spherically symmetric, and stationary solutions, the attractor mechanism [3–6] proved to be a crucial phenomenon, determining, in a universal fashion, the stabilization of scalar fields in the near-horizon geometry in terms of the fluxes of the 2-form field strengths of the Abelian vector fields coupled to the system. Moreover, the attractor mechanism turned out to be important also to unravel dynamical properties such as split attractor flows [7] and wall crossing [8], and to gain insights into the microstate counting analysis (see e.g. [9] and references therein), also in relation to string topologi-

cal partition functions [10] (see also [11] for a recent account and a list of references). In $d = 5$ space-time dimensions, progress has also been achieved with the discovery of new attractor solutions (see e.g. [12]), as well as with the formulation of a first-order formalism governing the evolution dynamics of nonsupersymmetric scalar flows [13].

For supergravity theories with scalar manifolds which are symmetric cosets, the extremal solutions of the ungauged theory can be classified through the orbits of the relevant representation space of the U -duality group, in which the corresponding supporting charges sit. The relation between U -invariant Bogomol’ny-Prasad-Sommerfeld (BPS) conditions and charge orbits in $d = 5$ supergravities has been the subject of various studies throughout the years [14–20].

The present paper extends to $d = 5$ space-time dimensions the four-dimensional investigation of [21].

We derive the U -duality charge orbits, as well as the related moduli spaces, of “large” and “small” extremal black holes and black strings in ungauged Maxwell-Einstein supergravities with symmetric scalar manifolds. The stabilizer groups of the various classes of orbits are obtained by determining and solving suitable U -invariant sets of constraints, both in “bare” and “dressed” charge bases, as well by exploiting İnönü-Wigner (IW) contractions and $SO(1, 1)$ gradings.

It is worth pointing out here that in this paper we will not deal with maximal $\mathcal{N} = 8$, $d = 5$ supergravity, because a complete analysis of extremal black hole attractors and

*Bianca.Cerchiai@unimi.it

†sergio.ferrara@cern.ch

‡marrani@lnf.infn.it

§zumino@thsrv.lbl.gov

¹Here U duality is referred to as the “continuous” version, valid for large values of the charges, of the U -duality groups introduced by Hull and Townsend [2].

their large and small charge orbits is already present in the literature; see e.g. [14–18,20,22–26]. We will just mention such a theory briefly, below Eq. (4.3).

The plan of the paper is as follows.

We start and give a résumé of real special geometry (RSG) in Sec. II, setting up notation and presenting all formulas needed for the subsequent treatment of charge orbits and attractors.

In Sec. III extremal black hole (black string) attractors are studied in full generality within real special geometry. Starting from the treatment of [19], various refinements and generalizations are performed, in particular, addressing the issue of generic, nonsymmetric vector multiplets’ scalar manifolds. In Sec. III A we analyze the various classes of critical points of the effective potential V , also within the so-called “new attractor” approach (see Sec. III A 4). Then, in Sec. III B we compute the higher order covariant derivatives of the previously introduced rank-3 invariant tensor T_{xyz} , which will play a key role in the subsequent developments and results, exposed in Secs. III C and III D, respectively, dealing with generic and homogeneous symmetric real special manifolds. A general analysis of the Hessian matrix of V , crucial in order to establish the stability of considered attractor points, is then performed in Sec. III E.

In Sec. IV all small charge orbits of symmetric, “magic,” real special geometries are explicitly determined and classified, by exploiting the properties of the functional \hat{J}_3 introduced in Sec. III C 3. Note that small charge orbits support nonattractor solutions, which have vanishing Bekenstein-Hawking [27] entropy in the Einsteinian approximation. Nevertheless, they can be treated by exploiting their symmetry properties under U duality.

Section V analyzes the “duality” relating the $\mathcal{N} = 2$ magic theory coupled to 14 Abelian vector multiplets and the $\mathcal{N} = 6$ “pure” supergravity, both based on the rank-3 Euclidean Jordan algebra J_3^{H} and thus sharing the very same bosonic sector.

Then, Sec. VI is devoted to the analysis of the large (Sec. VI A) and small (Sec. VI B) charge orbits of $\mathcal{N} = 2$ Jordan symmetric sequence. Similarly, Sec. VII provides a detailed treatment of the large (Sec. VII A) and small (Sec. VII B) charge orbits of the half-maximal ($\mathcal{N} = 4$) matter-coupled supergravity. The analysis of both Secs. VI and VII is made in the bare charge basis, and various subtleties, related to the reducible nature of the $d = 5$ U -duality group and disconnectedness of orbits in these two theories, are elucidated.

Two appendixes conclude the paper, containing various details concerning the determination of the small orbits in *symmetric* magic real special geometries.

The resolution of U -invariant defining (differential) constraints, both in bare and dressed charge bases, is performed in Appendix A.

Then, in Appendix B we give an equivalent derivation of all small charge orbits of symmetric magic real special geometries, relying on group theoretical procedures, namely, İnönü-Wigner contractions (Appendix B 1) and $SO(1, 1)$ three-grading (Appendix B 2).

Finally, we point out that all results on charge orbits can actually be obtained in various other ways, including the analysis of cubic norm forms of the relevant Jordan systems in $d = 5$; this will be investigated elsewhere.

II. RÉSUMÉ OF REAL SPECIAL GEOMETRY

RSG ([28–33] and references therein) is the geometry underlying the scalar manifold M_5 (with Euclidean signature) of Abelian vector multiplets coupled to the minimal supergravity in $d = 5$ space-time dimensions, namely, to $\mathcal{N} = 2$, $d = 5$ theory.

In the present section, we recall some basic facts about RSG, setting up notation and presenting all formulas needed for the subsequent treatment of charge orbits and attractors. Apart from slight changes in notation, we will adopt the conventions of [19], which are slightly different from the ones used in [34] (see the observations in [34]).

We start by specifying the kind and range of indices being used. The index in the “ambient space” is $i = 0, 1, \dots, n_V$ [in which M_5 is defined through a cubic constraint; see Eq. (2.5) below]. The “0” is the index pertaining to the (bare) $d = 5$ graviphoton, and n_V stands for the number of Abelian vector multiplets coupled to the supergravity multiplet. On the other hand, $x = 1, \dots, n_V$ and $a = 1, \dots, n_V$, respectively, denote “curved” and (local) “flat” coordinates in M_5 .

The metric a_{ij} in the ambient space (named g_{ij} in [34]) can be defined as follows:

$$a_{ij} = -\frac{1}{3} \frac{\partial^2 \log \mathcal{V}(\lambda)}{\partial \lambda^i \partial \lambda^j}, \quad (2.1)$$

where

$$\mathcal{V}(\lambda) \equiv d_{ijk} \lambda^i \lambda^j \lambda^k > 0 \quad (2.2)$$

is the volume of M_5 itself, and $d_{ijk} = d_{(ijk)}$ is a rank-3, completely symmetric, invariant tensor (see further below). In turn, the λ^i ’s are some real functions (with suitable features of smoothness and regularity) of the set of scalars ϕ^x of the theory, coordinatizing M_5 :

$$\lambda^i = \lambda^i(\phi^x). \quad (2.3)$$

They do satisfy the inequality (2.2). As elucidated e.g. in [34], the λ^i ’s are nothing but the (opposite of the) imaginary (“dilaton”) part of the complex scalar fields of the special Kähler geometry (SKG) based on a cubic holomorphic prepotential (usually named d -SKG; see e.g. [32,35]), characterizing the Abelian vector multiplets’ scalar manifold of $\mathcal{N} = 2$ Maxwell-Einstein supergravity in four space-time dimensions. In this respect, the ambient space

in five dimensions is nothing but the ‘‘dilaton sector’’ of the d -SKG in four dimensions.

It is now convenient to introduce rescaled variables as follows:

$$\hat{\lambda}^i \equiv \lambda^i \mathcal{V}^{-1/3} \Leftrightarrow d_{ijk} \hat{\lambda}^i \hat{\lambda}^j \hat{\lambda}^k = \mathcal{V}(\hat{\lambda}) = 1. \quad (2.4)$$

Thus, the metric of M_5 is the pullback of a_{ij} on the hypersurface

$$\mathcal{V}(\lambda) \equiv 1 \quad (2.5)$$

in the ambient space, namely,

$$\begin{aligned} g_{xy} &\equiv \hat{\lambda}^i_x \hat{\lambda}^j_y a_{ij} |_{\mathcal{V}(\lambda)=1} \\ &= -\frac{1}{3} \hat{\lambda}^i_x \hat{\lambda}^j_y \frac{\partial^2 \log \mathcal{V}(\lambda)}{\partial \lambda^i \partial \lambda^j} \Big|_{\mathcal{V}(\lambda)=1} = g_{xy}(\hat{\lambda}(\phi)) \\ &= \overset{(\sim)}{g}_{xy}(\phi), \end{aligned} \quad (2.6)$$

where (the semicolon denotes Riemann-covariant differentiation throughout)

$$\hat{\lambda}^i_{;x} \equiv -\sqrt{\frac{3}{2}} \frac{\partial \hat{\lambda}^i}{\partial \phi^x} \equiv -\sqrt{\frac{3}{2}} \hat{\lambda}^i_{,x} = -\sqrt{\frac{3}{2}} \hat{\lambda}^i_{;x}. \quad (2.7)$$

Notice that the constraint (2.4) implies

$$\frac{\partial \mathcal{V}(\hat{\lambda})}{\partial \phi^x} = 3d_{ijk} \hat{\lambda}^i_{,x} \hat{\lambda}^j \hat{\lambda}^k - \sqrt{6} d_{ijk} \hat{\lambda}^i_x \hat{\lambda}^j \hat{\lambda}^k = 0. \quad (2.8)$$

Let us now introduce T_{xyz} , a rank-3, completely symmetric, invariant tensor, related to d_{ijk} through the definition

$$T_{xyz} \equiv \hat{\lambda}^i_x \hat{\lambda}^j_y \hat{\lambda}^k_z d_{ijk} = -\left(\frac{3}{2}\right)^{3/2} \hat{\lambda}^i_{,x} \hat{\lambda}^j_{,y} \hat{\lambda}^k_{,z} d_{ijk} = T_{(xyz)}, \quad (2.9)$$

whose inversion reads

$$d_{ijk} = \frac{5}{2} \hat{\lambda}^i_x \hat{\lambda}^j_y \hat{\lambda}^k_z - \frac{3}{2} \overset{\circ}{a}_{(ij} \hat{\lambda}_{k)} + T_{xyz} \hat{\lambda}^i_x \hat{\lambda}^j_y \hat{\lambda}^k_z, \quad (2.10)$$

where

$$\overset{\circ}{a}_{ij}(\hat{\lambda}) \equiv a_{ij} |_{\mathcal{V}(\lambda)=1}. \quad (2.11)$$

In other words, T_{xyz} is the ϕ -dependent ‘‘dressing’’ [through $\hat{\lambda}^i_{,x}(\phi)$ ’s] of the constant (ϕ -independent) tensor d_{ijk} . It is worth anticipating here that Eqs. (2.9) and (2.10) play the key role to relate the formalism based on the bare charges with the formalism based on the dressed charges (see further below).

T_{xyz} enters the so-called ‘‘RSG constraints,’’ relating in M_5 the Riemann tensor R_{xyzu} to the metric tensor g_{xy} , as follows:

$$\begin{aligned} R_{xyzu} &= \frac{4}{3} (g_{x[u} g_{z]y} + T_{x[u}{}^w T_{z]yw}) \\ &= \frac{4}{3} (g_{x[u} g_{z]y} + T_{xw'[u} T_{z]yw} g^{ww'}). \end{aligned} \quad (2.12)$$

It is worth noticing a direct consequence of such RSG

constraints: the sectional curvature (see e.g. [36,37]) of matter charges in RSG globally vanishes:

$$\mathcal{R}(Z) \equiv R_{xyzw} g^{xx'} g^{yy'} g^{zz'} g^{ww'} Z_{x'} Z_{y'} Z_{z'} Z_{w'} = 0. \quad (2.13)$$

This is trivially due to the symmetry properties of the Riemann tensor R_{xyzw} (which are the ones for a generic Riemann geometry), and it is a feature discriminating RSG from SKG [in which $\mathcal{R}(Z)$ generally does not vanish; see e.g. [38,39]].

As a consequence of the constraints (2.12) (within the metric postulate), the definition of M_5 as a homogeneous symmetric manifold

$$R_{xyz;u} = 0 \quad (2.14)$$

yields

$$\begin{aligned} (T_{xw'[u;T_{z]yw} + T_{xw'[u} T_{z]yw;t}) g^{ww'} \\ = 0 \Leftrightarrow T_{xw'[u} T_{z]yw;t} g^{ww'} = T_{xw'[u} T_{z]y;t}{}^w = 0, \end{aligned} \quad (2.15)$$

which can be solved by

$$T_{xyz;u} = 0. \quad (2.16)$$

Through Eqs. (2.9) and (2.10), and exploiting the constraints imposed by local $\mathcal{N} = 2$ supersymmetry, it can be shown that Eq. (2.16) implies the following relation between the tensors d_{ijk} :

$$\begin{aligned} d^{ijk} d_{j(mn} d_{pq)k} &= \delta^i_{(m} d_{npq)} \Leftrightarrow d_{j(mn} d_{pq)k} d_{rst} \overset{\circ}{a}{}^{sj} \overset{\circ}{a}{}^{tk} \overset{\circ}{a}{}^{ri} \\ &= \delta^i_{(m} d_{npq)}, \end{aligned} \quad (2.17)$$

where the index-raising through the contravariant metric $\overset{\circ}{a}{}^{ij}$ has been made explicit.

III. ATTRACTORS IN RSG

The present section is dedicated to the study of attractors in RSG. This was first treated in [19] (and then reconsidered in [20], in connection to $d = 6$).

Starting from the treatment of [19], we will generalize and elaborate further various results obtained therein.

It is worth recalling that no asymptotically flat dyonic solutions of Einstein equations exist in $d = 5$. Thus, the $d = 5$ asymptotically flat black holes (BHs) can only carry electric charges q_i . Their magnetic duals are the $d = 5$ asymptotically flat black strings, which can only carry magnetic charges p^i .

We will perform all our treatment within the electric charge configuration. Because of the mentioned BH/black string duality, this does not imply any loss of generality. Furthermore, we will study attractors within the *Ansätze* of asymptotical (Minkowski) flatness, staticity, spherical symmetry, and extremality of the BH space-time metric (if no scalars are coupled, this is nothing but the so-called Tangherlini extremal $d = 5$ BH). The near-horizon geome-

try of extremal electric BHs and extremal magnetic black strings, respectively, is $\text{AdS}_2 \times S^3$ and $\text{AdS}_3 \times S^2$.

A. Classes of critical points of V

From the general theory of the attractor mechanism [3–6], the stabilization of scalar fields in proximity to the (unique) event horizon of a static, spherically symmetric, and asymptotically flat extremal BH in $\mathcal{N} = 2$, $d = 5$ Maxwell-Einstein supergravity is described by the critical points of the positive-definite effective potential function

$$\begin{aligned} V &\equiv \hat{a}^{ij} q_i q_j = (\hat{\lambda}^i q_i)^2 + \frac{2}{3} g^{xy} \hat{\lambda}_{,x}^i q_i \hat{\lambda}_{,y}^j q_j \\ &= Z^2 + \frac{2}{3} g^{xy} Z_x Z_y, \end{aligned} \quad (3.1)$$

where the $\mathcal{N} = 2$, $d = 5$ central charge function Z and its Riemann-covariant derivatives (“matter charges”) have been defined as follows:

$$Z \equiv \hat{\lambda}^i q_i, \quad (3.2)$$

$$Z_x \equiv \hat{\lambda}_{,x}^i q_i = Z_{,x} = Z_{,x}. \quad (3.3)$$

The definitions (3.2) and (3.3) can be inverted, obtaining the fundamental identities of RSG (in the electric formulation) [19]:

$$q_i = \hat{\lambda}_i Z - \frac{2}{3} g^{xy} \hat{\lambda}_{i,x} Z_y. \quad (3.4)$$

The identities (3.4) relate the basis of bare (ϕ -independent) electric charges q_i to the basis of dressed (central and matter) charges $\{Z, Z_x\}$, which do depend on the scalars ϕ^x , as yielded by definitions (3.2) and (3.3).

By recalling definitions (3.2) and (3.3), one obtains that

$$\begin{aligned} Z_{xy} &\equiv Z_{x;y} = Z_{,x;y} = Z_{,x;y} = \hat{\lambda}_{,x;y}^i q_i \\ &= \frac{2}{3} g_{xy} Z - \sqrt{\frac{2}{3}} T_{xyz} g^{zs} Z_s. \end{aligned} \quad (3.5)$$

Therefore, by using Eq. (3.5) the criticality conditions (alias attractor equations) for the effective potential V can be easily computed to be [19]

$$V_x \equiv V_{,x} = V_{;x} = 2 \left(2ZZ_x - \sqrt{\frac{2}{3}} T_{xyz} g^{zs} Z_s \right) = 0. \quad (3.6)$$

A priori, there are three classes of critical points of V which are nondegenerate (i.e. with $V|_{V=0} \neq 0$).

1. $(\frac{1}{2})$ -BPS

This class is defined by the sufficient (but not necessary) criticality constraint

$$Z_x = 0, \quad \forall x, \quad (3.7)$$

implying

$$V = Z^2. \quad (3.8)$$

2. Non-BPS

This class is defined by the constraints

$$Z \neq 0; \quad Z_x \neq 0 \quad \text{for at least some } x\text{'s}, \quad (3.9)$$

which are critical provided the following algebraic constraint among Z and Z_x 's holds:

$$Z_x = \frac{1}{2Z} \sqrt{\frac{3}{2}} T_{xyz} g^{ys} g^{zt} Z_s Z_t. \quad (3.10)$$

At least in symmetric RSG, this implies [19]

$$V = 9Z^2. \quad (3.11)$$

3. Remark

It is worth recalling here the Bekenstein-Hawking entropy-area formula [27], implemented for critical points of V :

$$\frac{S_{\text{BH},d=5}}{\pi} = \frac{A_H}{4\pi} \equiv R_H^2 = (V|_{\partial V=0})^{3/4}. \quad (3.12)$$

The attractor mechanism [3–6] is known to hold only for the so-called “large” BHs, which, through Eq. (3.12), have a nonvanishing Bekenstein-Hawking entropy.

Therefore, attractors *in a strict sense* are given by non-degenerate critical points of V . On the other hand, degenerate critical points of V , namely, critical points such that $V|_{\partial V=0} = 0$, are trivial. Indeed, by virtue of the positive definiteness of V (inherited from the strictly positive definiteness of \hat{a}^{ij} throughout all its domain of definition), it holds that

$$V = 0 \Leftrightarrow q_i = 0 \quad \forall i, \quad (3.13)$$

which is the trivial limit of the theory with *all* (electric) charges switched off.

The same reasoning can be repeated in the magnetic case.

Thus, only large BHs exhibit a (classical) attractor mechanism, implemented through nontrivial (alias nondegenerate) critical points of the effective potential itself [6].

4. “New attractor” approach

Through the so-called new attractor approach [40], an equivalent form of the n_V real criticality conditions (i.e. of the so-called attractor equations) for the various classes of critical points of V can be obtained by plugging the criticality conditions themselves into the $n_V + 1$ real RSG identities² (3.4). By doing so, one, respectively, obtains

²The extra real degree of freedom is only apparent, and it is removed by the homogeneity of degree one of the RSG identities (3.4) under a *real* overall shift of charges

$$q_i \rightarrow \eta q_i, \quad \eta \in \mathbb{R}.$$

(i) BPS attractor equations:

$$q_i = \hat{\lambda}_i Z. \quad (3.14)$$

While Eqs. (3.7) are n_V real differential ones, the $n_V + 1$ real equations (3.14) are purely algebraic.

(ii) Non-BPS attractor equations:

$$q_i = \hat{\lambda}_i Z - \frac{1}{2} \left(\frac{3}{2} \right)^{3/2} \frac{1}{Z} T^{xyz} Z_y Z_z \hat{\lambda}_{i,x}. \quad (3.15)$$

B. Derivatives of T_{xyz}

Now, in order to proceed further, it is convenient to compute the Riemann-covariant derivative of the invariant tensor T_{xyz} , namely, $T_{xyz;w}$, a quantity which will be relevant in the subsequent treatment. By using the definition (2.9), one obtains

$$\begin{aligned} T_{xyz;w} &= T_{(xyz);w} = -\sqrt{6} \left[-\frac{1}{2} g_{(yz} g_{xw)} + T_{r(yz} T_{xw)s} g^{rs} \right] \\ &= T_{(xyz);w}. \end{aligned} \quad (3.16)$$

Consequently, the condition (2.16) for the real special manifold M_5 to be a symmetric coset can be equivalently recast as follows [see e.g. p. 14 of [19] and Eq. (3.2.1.9) of [20]]:

$$T_{r(yz} T_{xw)s} g^{rs} = \frac{1}{2} g_{(yz} g_{xw)}. \quad (3.17)$$

One can then proceed further, and compute $T_{xyz;w;q}$. Starting from Eq. (3.16) one obtains (within the metric postulate)

$$\begin{aligned} T_{xyz;w;q} &= T_{(xyz);w;q} = -2\sqrt{6} T_{r(yz);q} T_{[xw)s} g^{rs} \\ &= -2\sqrt{6} T_{r(yz;q} T_{xw)s} g^{rs} = T_{(xyz);w;q}. \end{aligned} \quad (3.18)$$

Through Eq. (3.16), this result can be further elaborated to give

$$T_{xyz;w;q} = 12 \left[-\frac{1}{2} g_{(yz} T_{xwq)} + T_{(q|vr} T_{p|yz} T_{xw)s} g^{pv} g^{rs} \right]. \quad (3.19)$$

One can now introduce the following rank-5, completely symmetric tensor \tilde{E}_{xyzwq} , which is the ‘‘RSG analogue’’ of the so-called E tensor³ of SKG:

$$\tilde{E}_{xyzwq} \equiv \frac{1}{12} T_{xyz;w;q} = \frac{1}{12} T_{(xyz);w;q} = \tilde{E}_{(xyzwq)}, \quad (3.20)$$

satisfying, by definition, the relation

$$T_{(q|vr} T_{p|yz} T_{xw)s} g^{pv} g^{rs} = \frac{1}{2} g_{(yz} T_{xwq)} + \tilde{E}_{xyzwq}, \quad (3.21)$$

which holds globally in RSG.

³The E tensor of SKG was first introduced in [32], and it has been recently considered in the theory of extremal $d = 4$ BH attractors in [21,38,39,41,42].

By recalling the symmetricity condition (2.16), Eqs. (3.18), (3.19), (3.20), and (3.21) yield

$$\begin{aligned} T_{xyz;w} = 0 &\Rightarrow T_{xyz;w;q} = 0 \Leftrightarrow \tilde{E}_{xyzwq} = 0 \\ &\Leftrightarrow T_{(q|vr} T_{p|yz} T_{xw)s} g^{pv} g^{rs} = \frac{1}{2} g_{(yz} T_{xwq)}. \end{aligned} \quad (3.22)$$

C. Generic RSG

Let us now consider the value of the effective potential V in the various classes of its critical points. By recalling its very definition, (3.1), (3.7), and (3.10) yield the following results.

1. BPS

Recall Eq. (3.8):

$$V = Z^2. \quad (3.23)$$

Through Eq. (3.12), this yields

$$\frac{S_{\text{BH},d=5}}{\pi} = \frac{A_H}{4\pi} \equiv R_H^2 = V^{3/4} = |Z|^{3/2}. \quad (3.24)$$

2. Non-BPS and the dressed charges’ sum rule

$$\begin{aligned} V &= Z^2 + \frac{3}{2} g^{xy} Z_x Z_y \\ &= Z^2 + \frac{3}{8} \frac{1}{Z^2} g^{xy} T_{xzt} T_{wsy} Z^z Z^t Z^w Z^s. \end{aligned} \quad (3.25)$$

By recalling Eq. (3.16), the second term in the right-hand side of Eq. (7.17) can be further elaborated as follows:

$$Z_x Z^x = -\frac{1}{8} \sqrt{\frac{3}{2}} \frac{1}{Z^2} T_{(z|tw;s)} Z^z Z^t Z^w Z^s + \frac{3}{16} \frac{1}{Z^2} (Z_x Z^x)^2, \quad (3.26)$$

yielding ($Z_x Z^x \neq 0$)

$$\frac{3}{2} Z_x Z^x = 8Z^2 + \sqrt{\frac{3}{2}} \frac{T_{(xyz;w)} Z^x Z^y Z^z Z^w}{Z_u Z^u}. \quad (3.27)$$

Consequently, at non-BPS $Z \neq 0$ critical points of V , it generally holds that

$$V = 9Z^2 + \tilde{\Delta}, \quad (3.28)$$

where the real quantity

$$\tilde{\Delta} \equiv \sqrt{\frac{3}{2}} \frac{T_{(xyz;w)} Z^x Z^y Z^z Z^w}{Z_u Z^u} \quad (3.29)$$

has been introduced. This latter is the RSG analogue of the complex quantity Δ introduced in SKG [41] (see also [21,38,39,42]). As Δ enters the dressed charges’ sum rule at non-BPS ($Z \neq 0$) critical points of V_{BH} in SKG [see e.g. Eqs. (282)–(284) of [41]], $\tilde{\Delta}$ enters the dressed charges’ sum rule (3.28) at non-BPS critical points of V in RSG,

which further simplifies to (3.11), *at least* in symmetric RSG (having $\tilde{\Delta} = 0$ globally). Notice that, through Eq. (3.27) and definition (3.29), the (assumed) strictly positive definiteness of g_{xy} (throughout all M_5 and, in particular, in the considered class of critical points of V itself) yields

$$Z^2 + \frac{\tilde{\Delta}}{8} > 0. \quad (3.30)$$

Through Eq. (3.12), Eq. (3.28) yields

$$\frac{S_{\text{BH},d=5}}{\pi} = \frac{A_H}{4\pi} \equiv R_H^2 = V^{3/4} = (9Z^2 + \tilde{\Delta})^{3/4}. \quad (3.31)$$

3. The functional \hat{I}_3

Within a generic RSG, let us now consider the function

$$\hat{I}_3 \equiv \frac{1}{6}Z^3 - \frac{3}{8}ZZ_xZ^x - \frac{1}{4}\sqrt{\frac{3}{2}}T_{xyz}Z^xZ^yZ^z. \quad (3.32)$$

In general, \hat{I}_3 is a diffeomorphism- and symplectic-invariant function of the scalars ϕ^x in M_5 , or equivalently a functional of the dressed charges $\{Z, Z_x\}$ in M_5 . Its derivative reads [recalling Eq. (3.16)]

$$\begin{aligned} \hat{I}_{3,w} &= \hat{I}_{3;w} = -\sqrt{\frac{3}{2}}T_{xyz;w}Z^xZ^yZ^z \\ &= -\frac{1}{2}Z_xZ^xZ_w + \frac{1}{3}g^{rs}(T_{rzy}T_{xws} + T_{rzx}T_{yws} \\ &\quad + T_{rzw}T_{xys})Z^xZ^yZ^z. \end{aligned} \quad (3.33)$$

From the definition (3.29), it thus follows that

$$\tilde{\Delta} = -\frac{\hat{I}_{3,x}Z^x}{Z_yZ^y}. \quad (3.34)$$

The computation of \hat{I}_3 and $\hat{I}_{3,x}$ [respectively given by Eqs. (3.32) and (3.33)] at the various classes of critical points of V [specified by Eqs. (3.7), (3.8), (3.9), and (3.10)] respectively yield the following results.

BPS:

$$\hat{I}_3 = \frac{1}{6}Z^3, \quad (3.35)$$

$$\hat{I}_{3,x} = 0. \quad (3.36)$$

Thus, by recalling Eqs. (3.23) and (3.24), it follows that

$$\frac{S_{\text{BH},d=5}}{\pi} = \frac{A_H}{4\pi} \equiv R_H^2 = |Z|^{3/2} = V^{3/4} = \sqrt{6}|\hat{I}_3|^{1/2}. \quad (3.37)$$

Non-BPS:

Eq. (3.27) and definition (3.29) yield

$$Z_xZ^x = \frac{16}{3}Z^2 + \frac{2}{3}\tilde{\Delta}. \quad (3.38)$$

On the other hand, by recalling Eqs. (3.10) and (3.16), the term $T_{xyz}Z^xZ^yZ^z$ can be further elaborated at non-

BPS $Z \neq 0$ critical points of V as follows:

$$T_{xyz}Z^xZ^yZ^z = -\frac{1}{2\sqrt{6}}\frac{(Z_xZ^x)}{Z}\left(\tilde{\Delta} - \frac{3}{2}Z_yZ^y\right). \quad (3.39)$$

Thus, definition (3.32) yields the following expression of \hat{I}_3 at non-BPS $Z \neq 0$ critical points of V :

$$\hat{I}_3 = -\frac{9}{2}Z^3\left(1 + \frac{7}{6}\frac{\tilde{\Delta}}{3^2Z^2}\right) \Leftrightarrow \frac{\tilde{\Delta}}{3^2Z^2} = -\frac{6}{7}\left(\frac{2}{9}\frac{\hat{I}_3}{Z^3} + 1\right). \quad (3.40)$$

Thus, by recalling Eqs. (3.28) and (3.31), it follows that

$$\begin{aligned} \frac{S_{\text{BH},d=5}}{\pi} &= \frac{A_H}{4\pi} \equiv R_H^2 = (9Z^2 + \tilde{\Delta})^{3/4} = V^{3/4} \\ &= \frac{3^{3/2}}{7^{3/4}}|Z|^{3/2}\left(1 - \frac{4}{3}\frac{\hat{I}_3}{Z^3}\right)^{3/4}, \end{aligned} \quad (3.41)$$

thus necessarily yielding

$$\frac{3}{4} > \frac{\hat{I}_3}{Z^3}. \quad (3.42)$$

D. Symmetric RSG and large charge orbits

Let us now consider the case in which ⁴

$$M_5 = \frac{G_5}{H_5} = \frac{G_5}{\text{MCS}(H_5)} \quad (3.43)$$

is a symmetric coset.

At least in this case, d_{ijk} is the unique G_5 -invariant, rank-3, completely symmetric tensor, whereas T_{xyz} is the unique H_5 -invariant, rank-3, completely symmetric tensor.

Magic symmetric M_5 's are reported in Table I (see e.g. [32] and references therein; see also [44] for a brief review and a list of references).

Besides these four isolated cases, there are two infinite sequences of other symmetric real special manifolds, namely, the so-called Jordan symmetric sequence

$$M_{J,5,n} \equiv SO(1,1) \times \frac{SO(1,n)}{SO(n)}, \quad n = n_V - 1 \in \mathbb{N}, \quad (3.44)$$

and the non-Jordan symmetric sequence [45]

$$M_{nJ,5,n} \equiv \frac{SO(1,n)}{SO(n)}, \quad n = n_V \in \mathbb{N}, \quad (3.45)$$

n_V being the number of Abelian vector supermultiplets coupled to the $\mathcal{N} = 2$, $d = 5$ supergravity one.

The sequence (3.45) is the only (sequence of) symmetric RSG which is not related to Jordan algebras of degree

⁴“MCS” is an acronym for maximal compact subgroup (with symmetric embedding). Unless otherwise noted, all considered embeddings are symmetric. Moreover, the subscript “max” denotes the maximality of the embedding throughout.

TABLE I. Homogeneous symmetric real special vector multiplets' scalar manifolds M_5 of $\mathcal{N} = 2$, $d = 5$ magic supergravity. M_5 's also are (1) the non-BPS $Z \neq 0$ moduli spaces of $\mathcal{N} = 2$, $d = 4$ special Kähler symmetric vector multiplets' scalar manifolds [23], and (2) the large $\frac{1}{2}$ -BPS charge orbits $\mathcal{O}_{\text{BPS,large}}$ of $\mathcal{N} = 2$, $d = 5$ Maxwell-Einstein supergravity itself [19]. The large non-BPS $Z \neq 0$ charge orbits $\mathcal{O}_{n\text{BPS,large}} = M_5^*$ (see e.g. Table 5 of [43] and references therein) and the related non-BPS $Z \neq 0$ moduli spaces $\mathcal{M}_{n\text{BPS,large}}$ are reported, as well. The rank r of the orbit is defined as the minimal number of charges defining a representative solution. As observed in [23], for magic supergravities $n_V = \dim_{\mathbb{R}} M_5 = 3q + 2$, whereas $\dim_{\mathbb{R}} \mathcal{M}_{n\text{BPS,large}} = 2q$, and $\text{Spin}(1+q) \subset \tilde{h}_5$. See text for more details.

$J_3^{\mathbb{A}}$	$M_5 = \frac{G_5}{H_5} = \frac{\text{Str}_0(J_3)}{\text{Aut}(J_3)}$ $= \mathcal{O}_{\text{BPS,large}}$ $r = 3$ $H_5 \equiv mcs(G_5)$	$M_5^* = \frac{G_5}{\tilde{H}_5}$ $= \mathcal{O}_{n\text{BPS,large}}$ $r = 3$	$\mathcal{M}_{n\text{BPS,large}} = \frac{\tilde{H}_5}{\tilde{h}_5}$ $\tilde{h}_5 \equiv mcs(\tilde{H}_5)$
$J_3^{\text{O}}, q = 8$	$\frac{E_{6(-26)}}{F_{4(-52)}}$	$\frac{E_{6(-26)}}{F_{4(-20)}}$	$\frac{F_{4(-20)}}{SO(9)}$
$J_3^{\text{H}}, q = 4$	$\frac{SU^*(6)}{USp(6)}$	$\frac{SU^*(6)}{USp(4,2)}$	$\frac{USp(4,2)}{USp(4) \times USp(2)}$
$J_3^{\text{C}}, q = 2$	$\frac{SL(3, \mathbb{C})}{SU(3)}$	$\frac{SL(3, \mathbb{C})}{SU(2,1)}$	$\frac{SU(2,1)}{SU(2) \times U(1)}$
$J_3^{\text{R}}, q = 1$	$\frac{SL(3, \mathbb{R})}{SO(3)}$	$\frac{SL(3, \mathbb{R})}{SO(2,1)}$	$\frac{SO(2,1)}{SO(2)}$

three. It is usually denoted by $L(-1, n-1)$ in the classification of homogeneous Riemannian d spaces (see e.g. [32] and references therein). It will not be further considered here, because it does not correspond to symmetric spaces in four dimensions.

G_5 and H_5 can, respectively, be interpreted as the reduced structure group Str_0 and the automorphism group Aut of the corresponding Euclidean Jordan algebra of degree three (see e.g. [46] for a recent review, and references therein):

$$M_5 = \frac{G_5}{H_5} = \frac{\text{Str}_0(J_3)}{\text{Aut}(J_3)}. \quad (3.46)$$

Furthermore (*at least*⁵) in symmetric RSG, due to Eqs. (2.16) and (3.33), it holds that

$$\hat{I}_{3,x} = \hat{I}_{3;x} = 0. \quad (3.47)$$

In other words, \hat{I}_3 is independent of all scalars ϕ^x . Furthermore,

$$\hat{I}_3 = I_3, \quad (3.48)$$

where I_3 is the unique cubic invariant of the relevant electric (irreducible) representation [(ir)repr.] \mathbf{R}_Q of $d = 5$ U -duality G_5 , defined by (7.2). As mentioned above, d_{ijk}

⁵Notice that, from Eq. (3.33), it follows that

$$\hat{I}_{3,w} = 0 \Leftrightarrow T_{xyz;w} Z^x Z^y Z^z = 0,$$

for which (2.16) is a solution.

is G_5 invariant in all RSG, whereas d^{ijk} is G_5 invariant *at least* in symmetric RSG.

In this framework, by virtue of the relations (7.27) and (7.31), the Bekenstein-Hawking entropy-area formula (3.12) can be completed as follows [recall Eq. (3.48)]:

$$\begin{aligned} \frac{S_{\text{BH},d=5}}{\pi} &= \frac{A_H}{4\pi} \equiv R_H^2 = (V|_{\partial V=0})^{3/4} = \sqrt{6} |I_3|^{1/2} \\ &= \sqrt{6} |\hat{I}_3|^{1/2}. \end{aligned} \quad (3.49)$$

Furthermore, in RSG based on symmetric cosets $\frac{G_5}{H_5}$, the representation space of the irrepr. of G_5 in which the (electric *or* magnetic) charges sit admits a stratification in disjoint orbits [15,19]. Such orbits are homogeneous, in some cases symmetric, manifolds.

The charge orbits supporting nondegenerate (in the sense specified above; see the end of Sec. III A) critical points of V are called large orbits, because they correspond to the previously introduced class of large BHs with nonvanishing Bekenstein-Hawking entropy area [see Eq. (3.12)]. On the other hand, charge orbits corresponding to small BHs (having vanishing Bekenstein-Hawking entropy area) are correspondingly dubbed small orbits.

In the treatment of symmetric RSG performed in the present subsection, only large orbits, first found in [19], are considered.

In Sec. IV, through the properties of the function \hat{I}_3 defined by Eq. (3.32), the stabilizers of all small charge orbits of symmetric RSG will be derived, by suitably solving G_5 -invariant (sets of) defining differential constraints, as well as by performing suitable group theoretical procedures.

We can now specialize the results obtained in Sec. III C and in Sec. III C 3 to magic symmetric RSG. The detailed treatment of $\mathcal{N} = 2$ Jordan symmetric sequence (3.44) will be given in Sec. VI. Actually, the large charge orbits of (3.44) have already been considered in [19] (see also [23] for the study of corresponding moduli spaces), but in Sec. VI the treatment is further refined.

1. BPS

Equations (3.35) and (3.48) yield

$$\hat{I}_3 = \frac{1}{6} Z^3 = I_3, \quad (3.50)$$

and thus

$$\begin{aligned} \frac{S_{\text{BH},d=5}}{\pi} &= \frac{A_H}{4\pi} \equiv R_H^2 = (V|_{\partial V=0})^{3/4} = \sqrt{6} |I_3|^{1/2} \\ &= \sqrt{6} |\hat{I}_3|^{1/2} = |Z|^{3/2}. \end{aligned} \quad (3.51)$$

Such a large BH is supported by (electric) charges belonging to the large charge orbit (homogeneous symmetric manifold) [19]

$$\mathcal{O}_{\text{BPS,large}} = \frac{G_5}{H_5} = M_5. \quad (3.52)$$

The compactness of H_5 yields the absence of a moduli space related to $\frac{1}{2}$ -BPS large attractor solutions, a fact that can be seen also from the expression of the Hessian matrix of V evaluated along the BPS criticality constraints (3.7) [see Eq. (3.72) below].

It is worth remarking that M_5 's also are the non-BPS $Z \neq 0$ moduli spaces of $\mathcal{N} = 2$, $d = 4$ special Kähler symmetric vector multiplets' scalar manifolds [23].

Notice that, in general,

$$\dim_{\mathbb{R}} M_5 = n_V. \quad (3.53)$$

As observed in [23], for magic supergravities (based on Euclidean Jordan algebras of degree three $J_3^{\mathbb{A}}$ over the division algebras \mathbb{A}), it holds that

$$\begin{aligned} \dim_{\mathbb{R}} M_5 &= 3q + 2, \\ q &\equiv \dim_{\mathbb{R}}(\mathbb{A} = \mathbb{O}, \mathbb{H}, \mathbb{C}, \mathbb{R}) = (8, 4, 2, 1). \end{aligned} \quad (3.54)$$

2. Non-BPS

Equations (3.40) and (3.48) yield

$$\hat{I}_3 = -\frac{9}{2}Z^3 = I_3. \quad (3.55)$$

Indeed, from its very definition, in this framework it globally holds that

$$\tilde{\Delta} = 0, \quad (3.56)$$

and thus [recall Eq. (3.11)]

$$\frac{3}{2}Z_x Z^x = 8Z^2 \Leftrightarrow V = 9Z^2. \quad (3.57)$$

Through Eq. (3.49), it thus follows that

$$\begin{aligned} \frac{S_{\text{BH},d=5}}{\pi} &= \frac{A_H}{4\pi} \equiv R_H^2 = (V|_{\partial V=0})^{3/4} = \sqrt{6}|I_3|^{1/2} \\ &= \sqrt{6}|\hat{I}_3|^{1/2} = 3^{3/2}|Z|^{3/2}. \end{aligned} \quad (3.58)$$

Such a large BH is supported by (electric) charges belonging to the large charge orbit (homogeneous symmetric manifold)

$$\mathcal{O}_{n\text{BPS,large}} = \frac{G_5}{\tilde{H}_5} = M_5^*, \quad (3.59)$$

where \tilde{H}_5 is the unique noncompact, real form of $H_5 = \text{mcs}(G_5)$ which admits a maximal symmetric embedding into G_5 :

$$G_5 \supseteq_{\max} \tilde{H}_5. \quad (3.60)$$

The homogeneous symmetric pseudo-Riemannian manifold M_5^* is the “* version” of M_5 , obtained through time-like $d = 6 \rightarrow 5$ reduction from the corresponding anomaly-free uplifted $\mathcal{N} = (1, 0)$, $d = 6$ chiral theory

(see e.g. Table 5 of [43], and references therein). Notice that Eq. (3.59) yields

$$\mathcal{O}_{n\text{BPS,large}} = \mathcal{O}_{\text{BPS,large}}^*, \quad (3.61)$$

in the sense we have just specified.

The noncompactness of \tilde{H}_5 implies the existence of a non-BPS moduli space [23]

$$\mathcal{M}_{n\text{BPS,large}} \equiv \frac{\tilde{H}_5}{\text{MCS}(\tilde{H}_5)} \equiv \frac{\tilde{H}_5}{\tilde{h}_5}. \quad (3.62)$$

As observed in [23], for magic supergravities it holds that (see e.g. Table 8 of [44], and references therein)

$$\dim_{\mathbb{R}} \mathcal{M}_{n\text{BPS,large}} = 2q; \quad \text{Spin}(1+q) \subset \tilde{h}_5, \quad (3.63)$$

where $\text{Spin}(1+q)$ is the spin group in $1+q$ dimensions. Notice that $2q$ is the number of $d = 6$ (scalarless) vector multiplets needed for an anomaly-free uplift of the considered $\mathcal{N} = 2$, $d = 5$ magic Maxwell-Einstein supergravity to the corresponding $\mathcal{N} = (1, 0)$ chiral quarter-minimal magic supergravity in $d = 6$ (see e.g. Sec. 5 of [20], and references therein).

Thus, by recalling (3.54), the number $\#$ of “nonflat” scalar degrees of freedom along $\mathcal{O}_{n\text{BPS,large}}$ is

$$n_{\text{BPS, large}} \equiv \dim_{\mathbb{R}} M_5 - \dim_{\mathbb{R}} \mathcal{M}_{n\text{BPS,large}} = q + 2. \quad (3.64)$$

The large non-BPS $Z \neq 0$ charge orbits $\mathcal{O}_{n\text{BPS,large}} = M_5^*$, and the related non-BPS $Z \neq 0$ moduli spaces $\mathcal{M}_{n\text{BPS,large}}$ for magic models are reported in Table I. Furthermore, it should be recalled that the Jordan symmetric sequence (3.44) is related to the reducible rank-3 Euclidean Jordan algebra $\mathbb{R} \oplus \Gamma_{1,n}$, where $\Gamma_{1,n}$ is the rank-2 Jordan algebra with a quadratic form of Lorentzian signature $(1, n)$, i.e. the Clifford algebra of $O(n, 1)$ [49].

E. Hessian matrix of V

From its very definition (3.1), the first derivative of V reads [recall Eq. (3.6)]

$$V_x \equiv V_{,x} = V_{;x} = 2 \left(2ZZ_x - \sqrt{\frac{3}{2}} T_{xyz} g^{ys} g^{zt} Z_s Z_t \right). \quad (3.65)$$

By further differentiating, the global expression of the real Hessian $n_V \times n_V$ matrix of V in a generic RSG can be computed as follows:

$$\begin{aligned} V_{x;y} &= V_{;x;y} \\ &= \frac{8}{3} g_{xy} (Z^2 - \frac{3}{8} Z_w Z^w) + 2Z_x Z_y - 8\sqrt{\frac{2}{3}} Z T_{xyz} Z^z \\ &\quad + 2(T_{xys} T_{rzw} + 4T_{xzt} T_{yws}) g^{rs} Z^z Z^w \\ &= V_{(x;y)}, \end{aligned} \quad (3.66)$$

where Eqs. (3.5) and (3.16) have been used.

On the other hand, by recalling definitions (3.20) and (3.33), it can be computed that

$$\begin{aligned} \hat{I}_{3;x;y} = & -3\sqrt{\frac{3}{2}}\left(4\tilde{E}_{xyzwr}Z^r + \frac{2}{3}ZT_{xyz;w}\right. \\ & \left. - \sqrt{\frac{2}{3}}T_{zws;x}T_{ys'r}g^{ss'}Z^r\right)Z^zZ^w. \end{aligned} \quad (3.67)$$

Then, further elaboration of Eq. (3.66) is possible for $Z \neq 0$. Indeed, in such a case Eq. (3.67) implies that [recall Eq. (3.16)]

$$\begin{aligned} T_{xzw;y}Z^zZ^w = & -\frac{1}{\sqrt{6}}\frac{1}{Z}\hat{I}_{3;x;y} - \frac{6}{Z}\tilde{E}_{xyzwr}Z^zZ^wZ^r \\ & + \frac{1}{2Z}(Z_wZ^w)T_{xyz}Z^z + \frac{1}{Z}Z_yT_{xzw}Z^zZ^w \\ & - \frac{1}{Z}(T_{xrp}T_{yr's}T_{tzw} + 2T_{xwp}T_{yr's}T_{tzt}) \\ & \times g^{rr'}g^{tpt}Z^sZ^zZ^w. \end{aligned} \quad (3.68)$$

Notice that the symmetry properties $\hat{I}_{3;x;y} = \hat{I}_{3;(x;y)}$ and $T_{xzw;y}Z^zZ^w = T_{(xzw;y)}Z^zZ^w$ are not manifest, respectively, from Eqs. (3.67) and (3.68), due to the presence of \tilde{E}_{xyzwr} , $T_{xyz;w}$, and $\hat{I}_{3;x;y}$ itself. By plugging Eq. (3.68) back into Eq. (3.66), the following result is achieved:

$$\begin{aligned} V_{x;y} = & V_{;x;y} \\ = & 4Z_xZ_y + \frac{8}{3}Z^2g_{xy} - 8\sqrt{\frac{2}{3}}ZT_{xyz}Z^z + \frac{1}{Z}\hat{I}_{3;x;y} \\ & + \frac{6\sqrt{6}}{Z}\tilde{E}_{xyzwr}Z^zZ^wZ^r - \sqrt{\frac{3}{2}}\frac{1}{Z}(Z_wZ^w)T_{xyz}Z^z \\ & - \frac{\sqrt{6}}{Z}Z_yT_{xzw}Z^zZ^w + \frac{\sqrt{6}}{Z}(T_{xrp}T_{yr's}T_{tzw} \\ & + 2T_{xwp}T_{yr's}T_{tzt})g^{rr'}g^{tpt}Z^sZ^zZ^w \\ & + 4T_{xzw}T_{ysw'}g^{ww'}Z^zZ^s, \end{aligned} \quad (3.69)$$

holding true for $Z \neq 0$. Once again, notice that the symmetry property $V_{x;y} = V_{(x;y)}$ is not manifest from Eq. (3.69), due to the presence of \tilde{E}_{xyzwr} and $\hat{I}_{3;x;y}$.

By inserting the global condition (2.16) into Eq. (3.66), one obtains that

$$\begin{aligned} V_{x;y} = & V_{;x;y} \\ = & 4Z_xZ_y + \frac{8}{3}Z^2g_{xy} - 8\sqrt{\frac{2}{3}}ZT_{xyz}Z^z \\ & + 4T_{xzw}T_{ysw'}g^{ww'}Z^zZ^s \\ \equiv & V_{;x;y,\text{symm}}. \end{aligned} \quad (3.70)$$

This is the global expression of the real Hessian $n_V \times n_V$ matrix of V (at least) in symmetric RSG, and indeed it matches the result given by Eq. (5-1) of [19] (see also [20]). Thus, Eqs. (3.66) and (3.70) yield the following result:

$$\begin{aligned} V_{;x;y} = & V_{;x;y,\text{symm}} - g_{xy}(Z_wZ^w) - 2Z_xZ_y \\ & + 2(2T_{xwz}T_{ysz'} + T_{xyz}T_{swz'})g^{zz'}Z^wZ^s. \end{aligned} \quad (3.71)$$

1. Evaluation at critical points of V

We will now proceed to evaluate the Hessian matrix of V given by Eq. (3.66) in the various classes of critical points of V itself, as given by Eqs. (3.7), (3.8), (3.9), and (3.10).

BPS.—The necessary and sufficient BPS criticality constraints (3.7) plugged into Eq. (3.66) yield

$$V_{;x;y} = \frac{8}{3}g_{xy}Z^2. \quad (3.72)$$

Equation (3.72) holds for a generic RSG, and it matches the result given by Eq. (5-2) of [19]. For a strictly positive-definite g_{xy} (as it is usually assumed), it implies that the Hessian matrix of V at its BPS critical points has *all* strictly positive eigenvalues.

As mentioned above, the lack of Hessian massless modes at $\frac{1}{2}$ -BPS critical points of V determines the absence of a moduli space in BPS attractor solutions, which thus have *all* scalar fields ϕ^x stabilized at the (unique) event horizon of the considered (electric) $d = 5$ extremal BH.

Non-BPS.—It is worth noticing here that Eq. (3.10) yields

$$Z_xZ^x = \sqrt{\frac{3}{2}}\frac{1}{2Z}T_{xyz}Z^xZ^yZ^z. \quad (3.73)$$

By recalling the dressed charges' sum rule given by Eq. (3.27) and definition (3.29), Eq. (3.73) implies

$$\frac{32}{3}Z^2 + \tilde{\Delta} = \sqrt{\frac{3}{2}}\frac{1}{2Z}T_{xyz}Z^xZ^yZ^z. \quad (3.74)$$

On the other hand, by using Eq. (3.16), one can compute also that

$$Z_xZ^x = -\frac{1}{8}\sqrt{\frac{3}{2}}\frac{1}{2Z^2}T_{xyz;w}Z^xZ^yZ^zZ^w + \frac{3}{16}\frac{1}{Z^2}(Z_xZ^x)^2. \quad (3.75)$$

By dividing by $Z_xZ^x \neq 0$, one then obtains the dressed charges' sum rule given by Eq. (3.27). However, one can also interpret Eq. (3.75) as a quadratic equation in the unknown Z_xZ^x , obtaining the result

$$0 < Z_xZ^x = \frac{8}{3}Z^2 \pm \sqrt{\frac{64}{9}Z^4 - \frac{2}{3}\hat{I}_{3;x}Z^x}. \quad (3.76)$$

When $\hat{I}_{3;x} = 0$ (i.e.—at least—for symmetric RSG), Eq. (3.76) consistently yields [19]

$$\frac{3}{2}Z_xZ^x = 8Z^2. \quad (3.77)$$

IV. SMALL CHARGE ORBITS AND MODULI SPACES IN SYMMETRIC MAGIC RSG

In the treatment of symmetric RSG performed in Sec. III D, only large charge orbits, supporting solutions to the corresponding attractor equations (and first found in [19]; see also [20]), have been considered.

In the present section, by exploiting the properties of the functional \hat{I}_3 introduced in Sec. III C 3, all small charge orbits of magic symmetric RSG will be explicitly determined through the resolution of G_5 -invariant defining (differential) constraints both in bare and dressed charge bases, as well as through group theoretical techniques.

By definition, $\hat{I}_3 [= I_3$ in symmetric RSG, as discussed in Sec. III D; see Eq. (3.48)] vanishes for all small charge orbits. Consequently, such orbits do not support solutions to the attractor equations [alias criticality conditions of the effective potential V ; see Eqs. (3.7), (3.8), (3.9), and (3.10), or Eqs. (3.14) and (3.15) in the so-called new attractor approach]. In other words, the (classical) attractor mechanism does not hold for small charge orbits, which indeed do support BH states which are intrinsically quantum, in the sense that the effective description through Einstein supergravity fails for them.

Besides the condition of vanishing \hat{I}_3 , further conditions, formulated in terms of derivatives of \hat{I}_3 in some charge basis, may be needed to fully characterize the class of small orbits under consideration. It is worth pointing out here that the (sets of) G_5 -invariant constraints which define small charge orbits in homogeneous symmetric real special manifolds $\frac{G_5}{H_5}$ are characterizing equations for charges (in both bare and dressed bases), but they actually are identities in *all* scalar fields ϕ^x , and thus they hold globally in $\frac{G_5}{H_5}$. This is to be contrasted with large charge orbits, which are defined through the attractor equations themselves, which are, at the same time, characterizing equations for charges (in both bare and dressed bases) and stabilization equations for the scalars ϕ^x at the event horizon of the extremal BH.

As it is well known [23], at non-BPS $Z \neq 0$ critical points of V , some scalars are actually unstabilized at the (unique) event horizon of the corresponding large extremal BH solutions. Such unstabilized ϕ^x 's span the moduli space $\mathcal{M}_{n\text{BPS,large}}$ [given by Eq. (3.62)], associated with a hidden compact symmetry of the non-BPS $Z \neq 0$ attractor equations themselves, which can be traced back to the noncompactness of the stabilizer of the non-BPS $Z \neq 0$ large charge orbit $\mathcal{O}_{n\text{BPS,large}}$ [see Eq. (3.59), to be contrasted with Eq. (3.52)].

The small charge orbits are homogeneous manifolds of the form

$$\mathcal{O}_{\text{small}} = \frac{G_5}{\mathcal{S}_{\text{max}} \rtimes \mathcal{T}}, \quad (4.1)$$

where \rtimes denotes the semidirect group product throughout,

and \mathcal{T} is the non-semisimple part of the stabilizer of $\mathcal{O}_{\text{small}}$, which in all symmetric RSG (with some extra features characterizing the symmetric Jordan sequence; see Sec. VI) can be identified with an Abelian translational subgroup of G_5 itself.

One can also associate a moduli space with small charge orbits, by observing that the noncompactness of $\mathcal{S}_{\text{max}} \rtimes \mathcal{T}$ yields the existence of a corresponding moduli space defined as⁶

$$\mathcal{M}_{\text{small}} \equiv \frac{\mathcal{S}_{\text{max}}}{\text{MCS}(\mathcal{S}_{\text{max}})} \rtimes \mathcal{T}. \quad (4.2)$$

Note that, differently from large orbits, for small orbits there also exists a moduli space $\mathcal{M}_{\text{small}} = \mathcal{T}$ when \mathcal{S}_{max} is compact. As found in [47,48] for large charge orbits of the $\mathcal{N} = 2, d = 4$ *stu* model, and recently proved in a model-independent way in [49], the moduli spaces of charge orbits are defined all along the scalar flows, and thus they can be interpreted as moduli spaces of unstabilized scalars at the event horizon (*if any*) of the extremal BH, as well as moduli spaces of the Arnowitt-Deser-Misner (ADM) mass of the extremal BH at spatial infinity. In the small case, the interpretation at the event horizon breaks down, simply because such a horizon does not exist at all (*at least* in the Einsteinian supergravity approximation).

In general, the number $\#$ of nonflat scalar degrees of freedom supported by a (large or small) charge orbit \mathcal{O} with associated moduli space \mathcal{M} is defined as follows:

$$\# \equiv \dim_{\mathbb{R}} \mathcal{M}_{d=5} - \dim_{\mathbb{R}} \mathcal{M}. \quad (4.3)$$

As an example, let us briefly consider the maximal $\mathcal{N} = 8, d = 5$ supergravity, whose large and small charge orbits have been classified in [15]. The scalar manifold of the theory is

$$\mathcal{M}_{\mathcal{N}=8,d=5} = \frac{E_{6(6)}}{USp(8)}, \quad \dim_{\mathbb{R}} = 42. \quad (4.4)$$

- (1) The unique large charge orbit is $\frac{1}{8}$ -BPS:

$$\mathcal{O}_{(1/8)\text{-BPS}} = \frac{E_{6(6)}}{F_{4(4)}}, \quad \dim_{\mathbb{R}} = 26, \quad (4.5)$$

with corresponding moduli space [23]

$$\mathcal{M}_{(1/8)\text{-BPS}} = \frac{F_{4(4)}}{USp(6) \times USp(2)}, \quad (4.6)$$

$$\dim_{\mathbb{R}} = 28.$$

Thus, the number of nonflat directions along $\mathcal{O}_{(1/8)\text{-BPS}}$ reads

⁶We thank M. Trigiante for a discussion on the ‘‘flat’’ directions of small charge orbits.

$$\begin{aligned} \#_{(1/8)\text{-BPS}} &\equiv \dim_{\mathbb{R}} M_{\mathcal{N}=8,d=5} - \dim_{\mathbb{R}} \mathcal{M}_{(1/8)\text{-BPS}} \\ &= 14. \end{aligned} \quad (4.7)$$

Since the charge orbit is large, $\#_{(1/8)\text{-BPS}}$ also expresses the actual number of scalar degrees of freedom which are stabilized in terms of the electric (magnetic) charges in the near-horizon geometry of the extremal black hole (black string) under consideration.

(2) The small $\frac{1}{4}$ -BPS orbit is

$$\mathcal{O}_{(1/4)\text{-BPS}} = \frac{E_{6(6)}}{SO(5,4) \rtimes \mathbb{R}^{16}}, \quad \dim_{\mathbb{R}} = 26, \quad (4.8)$$

with corresponding moduli space

$$\begin{aligned} \mathcal{M}_{(1/4)\text{-BPS}} &= \frac{SO(5,4)}{SO(5) \times SO(4)} \rtimes \mathbb{R}^{16}, \\ \dim_{\mathbb{R}} &= 36. \end{aligned} \quad (4.9)$$

Thus, the number of nonflat directions along $\mathcal{O}_{(1/4)\text{-BPS}}$ reads

$$\#_{(1/4)\text{-BPS}} \equiv \dim_{\mathbb{R}} M_{\mathcal{N}=8,d=5} - \dim_{\mathbb{R}} \mathcal{M}_{(1/4)\text{-BPS}} = 6. \quad (4.10)$$

(3) The small $\frac{1}{2}$ -BPS orbit is

$$\mathcal{O}_{(1/2)\text{-BPS}} = \frac{E_{6(6)}}{SO(5,5) \rtimes \mathbb{R}^{16}}, \quad \dim_{\mathbb{R}} = 17, \quad (4.11)$$

with corresponding moduli space

$$\begin{aligned} \mathcal{M}_{(1/2)\text{-BPS}} &= \frac{SO(5,5)}{SO(5) \times SO(5)} \rtimes \mathbb{R}^{16} \\ &= M_{(2,2),d=6} \rtimes \mathbb{R}^{16}, \\ \dim_{\mathbb{R}} &= 41, \end{aligned} \quad (4.12)$$

where $M_{(2,2),d=6}$ is the scalar manifold of maximal (nonchiral) supergravity in $d = 6$. Thus, the number of nonflat directions along $\mathcal{O}_{(1/2)\text{-BPS}}$ reads

$$\#_{(1/2)\text{-BPS}} \equiv \dim_{\mathbb{R}} M_{\mathcal{N}=8,d=5} - \dim_{\mathbb{R}} \mathcal{M}_{(1/2)\text{-BPS}} = 1. \quad (4.13)$$

As we will point out more than once in the treatment below, result (4.13) expresses the pretty general fact that the unique nonflat direction along maximally supersymmetric (namely, $\frac{1}{2}$ -BPS) charge orbits is the Kaluza-Klein radius in the dimensional reduction $d = 6 \rightarrow d = 5$.

In the treatment of Sec. IV A, the G_5 -invariant constraints defining all classes of small charge orbits in *all* symmetric RSG will be derived. Then they will be solved both in bare and dressed charge bases in Appendix A.

Furthermore, in Appendix B the origin of small charge orbits (and, in particular, of \mathcal{T}) will be elucidated through group theoretical procedures [namely, İnönü-Wigner contractions [50,51] and $SO(1,1)$ three-grading].

While the treatment of Sec. IV A holds for *all* symmetric RSG, the treatments given in Appendixes A and B strictly fit only the isolated cases of symmetric RSG provided by the so-called magic symmetric RSG's [28–31]. The main results of Appendixes A and B are reported in Tables III and IV [the symmetric Jordan sequence (3.44) is considered in Sec. VI]. In the magic octonionic case $J_3^{\mathbb{O}}$ ($q = 8$), the results of [15] are matched.

Below we summarize the main results of Appendixes A and B.

(i) The small lightlike BPS charge orbit ($\dim_{\mathbb{R}} = 3q + 2$)

$$\mathcal{O}_{\text{lightlike,BPS}} = \frac{G_5}{(SO(q+1) \times \mathcal{A}_q) \rtimes \mathbb{R}^{\text{spin}(q+1), \text{spin}(Q_q)}}, \quad (4.14)$$

with

$$\mathcal{S}_{\text{max,lightlike,BPS}} = SO(q+1) \times \mathcal{A}_q, \quad (4.15)$$

$$\mathcal{T}_{\text{lightlike,BPS}} = \mathbb{R}^{\text{spin}(q+1), \text{spin}(Q_q)}. \quad (4.16)$$

Q_q and \mathcal{A}_q , a further factor group in \mathcal{S}_{max} , are given by Table II. Furthermore, we define

$$\text{spin}(q+1) \equiv \dim_{\mathbb{R}}(\mathbf{Spin}(q+1)), \quad (4.17)$$

$$\text{spin}(Q_q) \equiv \dim_{\mathbb{R}}(\mathbf{Spin}(Q_q)), \quad (4.18)$$

with $\mathbf{Spin}(q+1)$ and $\mathbf{Spin}(Q_q)$, respectively, denoting the spinor irreps. in $q+1$ and Q_q dimensions. It is worth remarking that \mathcal{A}_q is independent of the space-time dimension ($d = 3, 4, 5, 6$) in which the quarter-minimal symmetric magic (Maxwell-Einstein) supergravity (classified by $q = 8, 4, 2, 1$) is considered. It also holds that

$$d = 5, 6: \hat{G}_{\text{cent}} = SO(1,1) \times SO(q-1) \times \mathcal{A}_q, \quad (4.19)$$

$$d = 3, 4: \hat{G}_{\text{cent}} = G_{\text{paint}} = SO(q) \times \mathcal{A}_q, \quad (4.20)$$

TABLE II. Q_q and \mathcal{A}_q for the various $\mathcal{N} = 2$, $d = 5$ magic supergravities (based on $J_3^{\mathbb{A}}$, $\mathbb{A} = \mathbb{O}, \mathbb{H}, \mathbb{C}, \mathbb{R}$), classified by $q \equiv \dim_{\mathbb{R}} \mathbb{A} = 8, 4, 2, 1$.

q	Q_q	\mathcal{A}_q
8
4	2	$SO(3)$
2	2	$SO(2)$
1

TABLE III. Small lightlike charge orbits $\mathcal{O}_{\text{lightlike,BPS}}$ and $\mathcal{O}_{\text{lightlike},n\text{BPS}}$ (with associated moduli spaces) in symmetric magic RSG.

$J_3^{\mathbb{A}}$ (+ rel. data)	$\mathcal{O}_{\text{lightlike,BPS}}, r=2$	$\mathcal{M}_{\text{lightlike,BPS}}$	$\mathcal{O}_{\text{lightlike},n\text{BPS}}, r=2$	$\mathcal{M}_{\text{lightlike},n\text{BPS}}$
$\mathbb{A} = \mathbb{O}, q = 8$ Spin(9) = 16 $\#_{\text{light,BPS}} = 10$ $\#_{\text{light},n\text{BPS}} = 2$	$\frac{E_{6(-26)}}{SO(9) \rtimes \mathbb{R}^{16}}$	\mathbb{R}^{16}	$\frac{E_{6(-26)}}{SO(8,1) \rtimes \mathbb{R}^{16}}$	$\frac{SO(8,1)}{SO(8)} \rtimes \mathbb{R}^{16}$
$\mathbb{A} = \mathbb{H}, q = 4$ $\mathcal{A}_4 = SO(3),$ $Q_4 = 2$ Spin(5) = 4 Spin(Q_4) = 2 $\#_{\text{light,BPS}} = 6$ $\#_{\text{light},n\text{BPS}} = 2$	$\frac{SU^*(6)}{(SO(5) \times SO(3)) \rtimes \mathbb{R}^{(4,2)}}$	$\mathbb{R}^{(4,2)}$	$\frac{SU^*(6)}{(SO(4,1) \times SO(3)) \rtimes \mathbb{R}^{(4,2)}}$	$\frac{SO(4,1)}{SO(4)} \rtimes \mathbb{R}^{(4,2)}$
$\mathbb{A} = \mathbb{C}, q = 2$ $\mathcal{A}_2 = SO(2)$ $Q_2 = 2,$ Spin(3) = 2, Spin(Q_2) = 2 $\#_{\text{light,BPS}} = 4$ $\#_{\text{light},n\text{BPS}} = 2$	$\frac{SL(3, \mathbb{C})}{(SO(3) \times SO(2)) \rtimes \mathbb{R}^{(2,2)}}$	$\mathbb{R}^{(2,2)}$	$\frac{SL(3, \mathbb{C})}{(SO(2,1) \times SO(2)) \rtimes \mathbb{R}^{(2,2)}}$	$\frac{SO(2,1)}{SO(2)} \rtimes \mathbb{R}^{(2,2)}$
$\mathbb{A} = \mathbb{R}, q = 1$ Spin(2) = 2 $\#_{\text{light,BPS}} = 3$ $\#_{\text{light},n\text{BPS}} = 2$	$\frac{SL(3, \mathbb{R})}{SO(2) \rtimes \mathbb{R}^2}$	\mathbb{R}^2	$\frac{SL(3, \mathbb{R})}{SO(1,1) \rtimes \mathbb{R}^2}$	$SO(1,1) \rtimes \mathbb{R}^2$

where the groups \hat{G}_{cent} and G_{paint} are usually introduced in the treatment of supergravity billiards and timelike reductions (for a recent treatment and a set of related references, see e.g. [43]; see also Table V therein, for subtleties concerning the case $q = 8$ in

$d = 5, 6$). The moduli space corresponding to (4.14) is purely translational:

$$\mathcal{M}_{\text{lightlike,BPS}} = \mathbb{R}^{(\text{spin}(q+1), \text{spin}(Q_q))}, \quad (4.21)$$

with real dimension

TABLE IV. Small critical charge orbit $\mathcal{O}_{\text{critical,BPS}}$ (with associated moduli space $\mathcal{M}_{\text{critical,BPS}}$) in symmetric magic RSG.

$J_3^{\mathbb{A}}$ (+ rel. data)	$\mathcal{O}_{\text{critical,BPS}}, r=1$	$\mathcal{M}_{\text{critical,BPS}}$
$\mathbb{A} = \mathbb{O}, q = 8$ Spin(9) = 16 $\#_{\text{crit,BPS}} = 1$	$\frac{E_{6(-26)}}{SO(9,1) \rtimes \mathbb{R}^{16}}$	$\frac{SO(9,1)}{SO(9)} \rtimes \mathbb{R}^{16}$
$\mathbb{A} = \mathbb{H}, q = 4$ $\mathcal{A}_4 = SO(3), Q_4 = 2$ Spin(5) = 4, Spin(Q_4) = 2 $\#_{\text{crit,BPS}} = 1$	$\frac{SU^*(6)}{(SO(5,1) \times SO(3)) \rtimes \mathbb{R}^{(4,2)}}$	$\frac{SO(5,1)}{SO(5)} \rtimes \mathbb{R}^{(4,2)}$
$\mathbb{A} = \mathbb{C}, q = 2$ $\mathcal{A}_2 = SO(2),$ $Q_2 = 2,$ Spin(3) = 2, Spin(Q_2) = 2 $\#_{\text{crit,BPS}} = 1$	$\frac{SL(3, \mathbb{C})}{(SO(3,1) \times SO(2)) \rtimes \mathbb{R}^{(2,2)}}$	$\frac{SO(3,1)}{SO(3)} \rtimes \mathbb{R}^{(2,2)}$
$\mathbb{A} = \mathbb{R}, q = 1$ Spin(2) = 2 $\#_{\text{crit,BPS}} = 1$	$\frac{SL(3, \mathbb{R})}{SO(2,1) \rtimes \mathbb{R}^2}$	$\frac{SO(2,1)}{SO(2)} \rtimes \mathbb{R}^2$

TABLE V. \mathcal{N} -dependent supersymmetry-preserving features of large and small charge orbits of the irrepr. **15** of the $d = 5$ U -duality group $SU^*(6)$, related to $J_3^{\mathbb{H}}$. This corresponds to two “twin” theories, sharing the same bosonic sector: an $\mathcal{N} = 2$ Maxwell-Einstein theory and the $\mathcal{N} = 6$ “pure” theory. The subscript “ H ” stands for “(evaluated at the) horizon.”

$J_3^{\mathbb{H}}$	$\mathcal{N} = 2$	$\mathcal{N} = 6$
$\frac{SU^*(6)}{USp(6)},$ large, $I_3 \neq 0$	$\frac{1}{2}$ -BPS	n BPS, $Z_{AB,H} = 0,$ $X_H \neq 0$
$\frac{SU^*(6)}{USp(4,2)},$ large, $I_3 \neq 0$	n BPS, $Z_H \neq 0$	$\frac{1}{6}$ -BPS, $Z_{AB,H} \neq 0,$ $X_H \neq 0$
$\frac{SU^*(6)}{(SO(5) \times SO(3)) \rtimes \mathbb{R}^{(4,2)}},$ small, $I_3 = 0$	$\frac{1}{2}$ -BPS	$\frac{1}{6}$ -BPS
$\frac{SU^*(6)}{(SO(4,1) \times SO(3)) \rtimes \mathbb{R}^{(4,2)}},$ small, $I_3 = 0$	n BPS	$\frac{1}{3}$ -BPS
$\frac{SU^*(6)}{(SO(5,1) \times SO(3)) \rtimes \mathbb{R}^{(4,2)}},$ small, $\partial I_3 = 0$	$\frac{1}{2}$ -BPS	$\frac{1}{2}$ -BPS

$$\text{spin}(q+1) \cdot \text{spin}(Q_q) = 2q. \quad (4.22)$$

Thus, by recalling (3.54), the number $\#$ of scalar degrees of freedom on which the ADM mass depends along $\mathcal{O}_{\text{lightlike,BPS}}$ is [recall Eq. (3.64)]

$$\begin{aligned} \#_{\text{light,BPS}} &\equiv \dim_{\mathbb{R}} M_5 - \dim_{\mathbb{R}} \mathcal{M}_{\text{lightlike,BPS}} \\ &= 3q + 2 - (\text{spin}(q+1) \cdot \text{spin}(Q_q)) \\ &= q + 2. \end{aligned} \quad (4.23)$$

By recalling Eq. (3.63), it is worth noting that $\mathcal{M}_{n\text{BPS,large}}$ and $\mathcal{M}_{\text{lightlike,BPS}}$ have the same real dimension, but they are completely different, as yielded by Eqs. (3.62) and (4.21).

- (ii) The small lightlike non-BPS charge orbit ($\dim_{\mathbb{R}} = 3q + 2$)

$$\mathcal{O}_{\text{lightlike,nBPS}} = \frac{G_5}{(SO(q,1) \times \mathcal{A}_q) \rtimes \mathbb{R}^{(\text{spin}(q+1), \text{spin}(Q_q))}}, \quad (4.24)$$

with

$$S_{\text{max,lightlike,nBPS}} = SO(q,1) \times \mathcal{A}_q, \quad (4.25)$$

$$\mathcal{T}_{\text{lightlike,nBPS}} = \mathbb{R}^{(\text{spin}(q+1), \text{spin}(Q_q))} = \mathcal{T}_{\text{lightlike,BPS}}. \quad (4.26)$$

The related moduli space reads ($\dim_{\mathbb{R}} = 3q$)

$$\begin{aligned} \mathcal{M}_{\text{lightlike,nBPS}} &= \frac{SO(q,1)}{SO(q)} \rtimes \mathbb{R}^{(\text{spin}(q+1), \text{spin}(Q_q))} \\ &= M_{nJ,5,q} \rtimes \mathbb{R}^{(\text{spin}(q+1), \text{spin}(Q_q))}, \end{aligned} \quad (4.27)$$

where $M_{nJ,5,q}$ is the q th element of the generic non-Jordan symmetric sequence (3.45). Thus, by recalling (3.54), the number $\#$ of scalar degrees of freedom on which the ADM mass depends along $\mathcal{O}_{\text{lightlike,nBPS}}$ is

$$\begin{aligned} \#_{\text{light,nBPS}} &\equiv \dim_{\mathbb{R}} M_5 - \dim_{\mathbb{R}} \mathcal{M}_{\text{lightlike,nBPS}} \\ &= 2q + 2 - (\text{spin}(q+1) \cdot \text{spin}(Q_q)) = 2. \end{aligned} \quad (4.28)$$

- (iii) The small *critical* BPS charge orbit ($\dim_{\mathbb{R}} = 2q + 1$)

$$\mathcal{O}_{\text{critical,BPS}} = \frac{G_5}{(G_6 \times \mathcal{A}_q) \rtimes \mathbb{R}^{(\text{spin}(q+1), \text{spin}(Q_q))}}, \quad (4.29)$$

where

$$G_6 = SO(1, q+1) \quad (4.30)$$

is the U -duality group of the corresponding $(1, 0)$, $d = 6$ chiral supergravity theory. Thus,

$$\begin{aligned} S_{\text{max,critical,BPS}} &= G_6 \times \mathcal{A}_q, \\ \mathcal{T}_{\text{critical,BPS}} &= \mathcal{T}_{\text{lightlike,nBPS}} = \mathcal{T}_{\text{lightlike,BPS}}. \end{aligned} \quad (4.31)$$

The related moduli space reads ($\dim_{\mathbb{R}} = 3q + 1$)

$$\begin{aligned} \mathcal{M}_{\text{critical,BPS}} &= \frac{SO(q+1,1)}{SO(q+1)} \rtimes \mathbb{R}^{(\text{spin}(q+1), \text{spin}(Q_q))} \\ &= M_{nJ,5,q+1} \rtimes \mathbb{R}^{(\text{spin}(q+1), \text{spin}(Q_q))}. \end{aligned} \quad (4.32)$$

Thus, by recalling (3.54), the number $\#$ of scalar degrees of freedom on which the ADM mass depends along $\mathcal{O}_{\text{critical,BPS}}$ is

$$\begin{aligned} \#_{\text{crit,BPS}} &\equiv \dim_{\mathbb{R}} M_5 - \dim_{\mathbb{R}} \mathcal{M}_{\text{critical,BPS}} \\ &= 2q + 1 - (\text{spin}(q+1) \cdot \text{spin}(Q_q)) = 1. \end{aligned} \quad (4.33)$$

The unique scalar degree of freedom on which the ADM mass depends can be interpreted as the Kaluza-Klein radius in the $d = 6 \rightarrow d = 5$ reduction. Furthermore, it is worth observing that

$$\mathcal{M}_{\text{critical,BPS}} = M_{(1,0),d=6,J_3^{\mathbb{A}}} \rtimes \mathbb{R}^{(\text{spin}(q+1), \text{spin}(Q_q))}, \quad (4.34)$$

where $M_{(1,0),d=6,J_3^{\mathbb{A}}}$ is the manifold of tensor multiplets' scalars in the corresponding $(1, 0)$, $d = 6$ theory (see e.g. Sec. 5 of [20] for a recent treatment).

It should also be noticed that $\mathcal{O}_{n\text{BPS,large}}$ [given by Eq. (3.59)] and $\mathcal{O}_{\text{critical,BPS}}$ [given by Eq. (4.29)] share the same compact symmetry, or equivalently that $\mathcal{M}_{n\text{BPS,large}}$ [given by Eq. (3.62)] and $\mathcal{M}_{\text{critical,BPS}}$ [given by Eq. (4.32)] share the same stabilizer group (apart from an \mathcal{A}_q commuting factor), but they do not coincide. This is due to the fact that \tilde{H}_5 and $G_6 \times \mathcal{A}_q$ share the same MCS, namely,

$$\begin{aligned} \tilde{h}_5 &\equiv \text{MCS}(\tilde{H}_5) = \text{MCS}(G_6 \times \mathcal{A}_q) \\ &= SO(q+1) \times \mathcal{A}_q. \end{aligned} \quad (4.35)$$

In the case $\mathbb{A} = \mathbb{R}$ ($q = 1$), the following further results hold (see also Tables III and IV):

$$\begin{aligned} \mathcal{M}_{n\text{BPS,large},J_3^{\mathbb{R}}} \rtimes \begin{cases} \mathbb{R}^2 \\ \mathbb{R}^{(2,2)} \end{cases} &= M_{\mathcal{N}=(1,0),d=6,J_3^{\mathbb{R}}} \rtimes \begin{cases} \mathbb{R}^2 \\ \mathbb{R}^{(2,2)} \end{cases} \\ &= \begin{cases} \mathcal{M}_{\text{critical,BPS},J_3^{\mathbb{R}}} \\ \mathcal{M}_{\text{lightlike,nBPS},J_3^{\mathbb{C}}} \end{cases}. \end{aligned} \quad (4.36)$$

Notice that $J_3^{\mathbb{R}}$ is the unique case, among $J_3^{\mathbb{A}}$ in $d = 5$, in which $\mathcal{M}_{n\text{BPS,large}}$ and $\mathcal{M}_{\text{critical,BPS}}$ not only share the same stabilizer, but actually do coincide (up to $\rtimes \mathbb{R}^2$). Moreover,

$\mathcal{M}_{n\text{BPS,large},J_3^{\mathbb{R}}}$ also coincides with $\mathcal{M}_{\text{lightlike},n\text{BPS},J_3^{\mathbb{C}}}$ (up to $\times \mathbb{R}^{(2,2)}$), because the respective charge orbits $\mathcal{O}_{n\text{BPS,large},J_3^{\mathbb{R}}}$ and $\mathcal{O}_{\text{lightlike},n\text{BPS},J_3^{\mathbb{C}}}$ share the same semisimple, namely, nontranslational, part of the stabilizer [apart from a commuting $\mathcal{A}_2 = SO(2)$ factor], i.e. $SO(2, 1)$.

The Jordan symmetric infinite sequence [28–32,35,52] given by Eq. (3.44) needs some extra care (also at the level of large charge orbits), because of the factorization of G_5 . The large and small charge orbits for such a sequence will be treated in Sec. VI. This treatment refines and complete the ones given e.g. in [19,20,23].

A. G_5 -invariant defining constraints

As mentioned above, small charge orbits in *all* symmetric RSG are all characterized by the constraint [recall Eq. (3.48)]

$$\hat{I}_3 = I_3 = 0, \quad (4.37)$$

where $\hat{I}_3 = I_3$ is the unique cubic scalar invariant of the relevant electric representation \mathbf{R}_q of the $d = 5$ U -duality group G_5 (in which the electric charges q_i sit). By recalling definitions (3.32) and (7.2), the “smallness” condition (4.37) can be recast as follows:

$$\hat{I}_3 = 0 \Leftrightarrow Z^3 - \left(\frac{3}{2}\right)^2 ZZ_x Z^x - \left(\frac{3}{2}\right)^{3/2} T_{xyz} Z^x Z^y Z^z = 0, \quad (4.38)$$

$$I_3 = 0 \Leftrightarrow d^{ijk} q_i q_j q_k = 0, \quad (4.39)$$

in the dressed and bare charge bases, respectively.

It is worth noticing here that Eq. (4.38) can be recast as a cubic algebraic equation:

$$Z^3 + \mathbf{p}Z - \mathbf{q} = 0, \quad \mathbf{p} \equiv -\left(\frac{3}{2}\right)^2 Z_x Z^x < 0, \quad (4.40)$$

$$\mathbf{q} \equiv \left(\frac{3}{2}\right)^{3/2} T_{xyz} Z^x Z^y Z^z,$$

with a polynomial discriminant

$$D \equiv \frac{\mathbf{p}^3}{9} + \frac{\mathbf{q}^2}{4} = \frac{3^3}{2^6} [2(T_{xyz} Z^x Z^y Z^z)^2 - (Z_x Z^x)^3]. \quad (4.41)$$

Thus, for $D > 0$ one gets one real and two complex conjugate (unacceptable) roots, whereas for $D < 0$ all roots are real and unequal. In the particular case

$$D = 0 \Leftrightarrow 2(T_{xyz} Z^x Z^y Z^z)^2 = (Z_x Z^x)^3, \quad (4.42)$$

all roots are real, and at least two are equal.

Let us proceed further, by differentiating the functional \hat{I}_3 with respect to the dressed charges

$$Z \equiv \{Z, Z_x\}, \quad (4.43)$$

as well as the function I_3 with respect to the bare charges $\{q_i\}$. One, respectively, obtains

$$\frac{\partial \hat{I}_3}{\partial Z} = \begin{cases} \frac{\partial \hat{I}_3}{\partial Z} = \frac{1}{2} Z^2 - \frac{3}{8} Z_x Z^x \\ \frac{\partial \hat{I}_3}{\partial Z_x} = -\frac{3}{4} Z Z^x - \frac{1}{2} \left(\frac{3}{2}\right)^{3/2} T_{yz}^x Z^y Z^z, \end{cases} \quad (4.44)$$

$$\frac{\partial I_3}{\partial q_i} = \frac{1}{2} d^{ijk} q_j q_k, \quad (4.45)$$

where it should be recalled once again that here we are considering symmetric real special manifolds $\frac{G_5}{H_5}$, where Eqs. (2.16) and (2.17) hold true.

A further differentiation with respect to Z or $\{q_i\}$, respectively, yields

$$\frac{\partial^2 \hat{I}_3}{(\partial Z)^2} = \begin{cases} \frac{\partial^2 \hat{I}_3}{(\partial Z)^2} = Z \\ \frac{\partial^2 \hat{I}_3}{\partial Z \partial Z_x} = -\frac{3}{4} Z^x \\ \frac{\partial^2 \hat{I}_3}{\partial Z_x \partial Z_y} = -\frac{3}{4} Z g^{xy} - \left(\frac{3}{2}\right)^{3/2} T^{xy}_z Z^z = \frac{\partial^2 \hat{I}_3}{\partial Z_x \partial Z_y}, \end{cases} \quad (4.46)$$

$$\frac{\partial^2 I_3}{\partial q_i \partial q_j} = d^{ijk} q_k = \frac{\partial^2 I_3}{\partial q_i \partial q_j}. \quad (4.47)$$

By further differentiating, one then obtains

$$\frac{\partial^3 \hat{I}_3}{(\partial Z)^3} = \begin{cases} \frac{\partial^3 \hat{I}_3}{(\partial Z)^3} = 1 \\ \frac{\partial^3 \hat{I}_3}{(\partial Z)^2 \partial Z_x} = 0 \\ \frac{\partial^3 \hat{I}_3}{\partial Z \partial Z_x \partial Z_y} = -\frac{3}{4} g^{xy} = \frac{\partial^3 \hat{I}_3}{\partial Z \partial Z_x \partial Z_y} \\ \frac{\partial^3 \hat{I}_3}{\partial Z_x \partial Z_y \partial Z_z} = -\left(\frac{3}{2}\right)^{3/2} T^{xyz} = \frac{\partial^3 \hat{I}_3}{\partial Z_x \partial Z_y \partial Z_z}, \end{cases} \quad (4.48)$$

$$\frac{\partial^3 I_3}{\partial q_i \partial q_j \partial q_k} = d^{ijk} = \frac{\partial^3 I_3}{\partial q_i \partial q_j \partial q_k}. \quad (4.49)$$

Starting from the fourth order of differentiation, all derivatives vanish. This is no surprise, because \hat{I}_3 is a homogeneous functional polynomial of degree three in dressed charges Z , and (equivalently) I_3 is a homogeneous polynomial of degree three in bare charges q_i .

At this point, it is possible to classify the various small charge orbits through G_5 -invariant conditions involving \hat{I}_3 and its nonvanishing functional derivatives with respect to Z , or equivalently through G_5 -invariant conditions involving I_3 and its nonvanishing derivatives with respect to q_i 's.

1. Small lightlike orbits

The small lightlike charge orbits are defined by the constraints [recall Eqs. (4.38) and (4.39)]

$$\hat{I}_3 = 0 \Leftrightarrow Z^3 - \left(\frac{3}{2}\right)^2 ZZ_x Z^x - \left(\frac{3}{2}\right)^{3/2} T_{xyz} Z^x Z^y Z^z = 0,$$

$$\frac{\partial \hat{I}_3}{\partial Z} \neq 0 \Leftrightarrow \begin{cases} Z^2 - \frac{3}{4} Z_x Z^x \neq 0; \\ \text{and/or} \\ ZZ^x + \sqrt{\frac{3}{2}} T^x_{yz} Z^y Z^z \neq 0 \text{ (at least for some } x), \end{cases} \quad (4.50)$$

or equivalently,

$$I_3 = 0 \Leftrightarrow d^{ijk} q_i q_j q_k = 0, \quad (4.51)$$

$$\frac{\partial I_3}{\partial q_i} \neq 0 \Leftrightarrow d^{ijk} q_j q_k \neq 0 \text{ (at least for some } i).$$

The sets of constraints (4.50) and (4.51) are both G_5 invariant, but their manifest invariance is different. Indeed, the dressed charge basis Z is covariant with respect to H_5 , and thus the set of constraints (4.50) exhibits a manifest H_5 invariance. Instead, the bare charge basis $\{q_i\}$ is G_5 covariant, and thus the set of constraints (4.50) is manifestly G_5 invariant.

In the dressed charge basis, it is immediate to realize that two classes of small lightlike charge orbits exist.

- (i) A small lightlike charge orbit for which the constraints (4.50) are solved with $Z = 0$:

$$\hat{I}_3|_{Z=0} = 0 \Leftrightarrow T_{xyz} Z^x Z^y Z^z = 0,$$

$$\frac{\partial \hat{I}_3}{\partial Z}|_{Z=0} \neq 0 \Leftrightarrow \begin{cases} Z_x Z^x \neq 0 \\ \text{and/or} \\ T^x_{yz} Z^y Z^z \neq 0 \text{ (at least for some } x). \end{cases} \quad (4.52)$$

Notice that the constraint $Z_x Z^x \neq 0$ is automatically satisfied, because (1) g_{xy} is assumed to be strictly positive definite, and (2) $Z_x \neq 0$ at least for some x (otherwise, since $Z = 0$, one would obtain the trivial limit in which all charges vanish).

- (ii) A small lightlike charge orbit for which the constraints (4.50) are solved with $Z \neq 0$ [also recall Eqs. (4.40), (4.41), and (4.42)]:

$$\hat{I}_3|_{Z \neq 0} = 0 \Leftrightarrow Z^3 - \left(\frac{3}{2}\right)^2 ZZ_x Z^x - \left(\frac{3}{2}\right)^{3/2} T_{xyz} Z^x Z^y Z^z = 0,$$

$$\frac{\partial \hat{I}_3}{\partial Z} \Big|_{Z \neq 0} \neq 0 \Leftrightarrow \begin{cases} Z^2 - \frac{3}{4} Z_x Z^x \neq 0 \\ \text{and/or} \\ ZZ^x + \sqrt{\frac{3}{2}} T^x_{yz} Z^y Z^z \neq 0 \text{ (at least for some } x). \end{cases} \quad (4.53)$$

2. Small critical orbit

The small critical charge orbit is defined by the constraints [recall Eqs. (4.38) and (4.39)]

$$\hat{I}_3 = 0 \Leftrightarrow Z^3 - \left(\frac{3}{2}\right)^2 ZZ_x Z^x - \left(\frac{3}{2}\right)^{3/2} T_{xyz} Z^x Z^y Z^z = 0,$$

$$\frac{\partial \hat{I}_3}{\partial Z} = 0 \Leftrightarrow \begin{cases} Z^2 - \frac{3}{4} Z_x Z^x = 0 \\ ZZ^x + \sqrt{\frac{3}{2}} T^x_{yz} Z^y Z^z = 0, \end{cases} \quad (4.54)$$

or equivalently,

$$I_3 = 0 \Leftrightarrow d^{ijk} q_i q_j q_k = 0;$$

$$\frac{\partial I_3}{\partial q_i} = 0 \Leftrightarrow d^{ijk} q_j q_k = 0. \quad (4.55)$$

As noticed above for the sets of constraints (4.50) and (4.51), the sets of constraints (4.54) and (4.55) are both G_5 invariant: while (4.54) is manifestly invariant only under $H_5 = \text{MCS}(G_5)$, (4.55) is actually manifestly G_5 invariant.

Once again, in the dressed charge basis it is immediate to realize that only one class of small critical charge orbits exists, namely, a small critical charge orbit for which the constraints (4.54) are solved with $Z \neq 0$:

$$\hat{I}_3|_{Z \neq 0} = 0 \Leftrightarrow Z^3 - \left(\frac{3}{2}\right)^2 ZZ_x Z^x - \left(\frac{3}{2}\right)^{3/2} T_{xyz} Z^x Z^y Z^z = 0,$$

$$\frac{\partial \hat{I}_3}{\partial Z} \Big|_{Z \neq 0} = 0 \Leftrightarrow \begin{cases} Z^2 - \frac{3}{4} Z_x Z^x = 0; \\ ZZ^x + \sqrt{\frac{3}{2}} T^x_{yz} Z^y Z^z = 0. \end{cases} \quad (4.56)$$

Notice that, for the same reason the constraint $\frac{\partial \hat{I}_3}{\partial Z}|_{Z=0} \neq 0$ is automatically satisfied for the small lightlike charge orbit whose representative in the dressed charge basis is given by Eq. (4.52), a small critical charge orbit with a representative having $Z = 0$ cannot exist. Indeed, such an orbit should have $Z = 0$ and $Z_x Z^x = 0$. Because of the assumed strictly positive definiteness of g_{xy} , this would be possible only in the trivial limit of the theory in which all charges do vanish. This can be formally stated as follows:

$$\frac{\partial \hat{I}_3}{\partial Z} \Big|_{Z=0} = 0 \Leftrightarrow Z = 0. \quad (4.57)$$

V. J_3^{H} : $\mathcal{N} = 2$ VS $\mathcal{N} = 6$

The rank-3 Euclidean Jordan algebra J_3^{H} ($q = 4$) is related to two different theories, namely, an $\mathcal{N} = 2$ theory

coupled to 14 Abelian vector multiplets and the $\mathcal{N} = 6$ “pure” theory. These two theories share the same bosonic sector [15,19,53], but their fermionic sectors, exploiting the supersymmetric completion of the bosonic one, are different.

Thus, it also follows that the supersymmetry-preserving features of the large and small charge orbits of the relevant irrepr. **15** of $G_5 = SU^*(6)$ are different. The \mathcal{N} -dependent supersymmetry properties of the various orbits are given in Table V (notice they are consistent with the results of [54]). In the large (attractor) cases, these match the results of [20].

VI. $\mathcal{N} = 2, d = 5$ JORDAN SYMMETRIC SEQUENCE

The Jordan symmetric sequence of $\mathcal{N} = 2, d = 5$ supergravity coupled to $n_V = n + 1$ vector multiplets reads ($\dim_{\mathbb{R}} = n + 1, \text{rank} = 2, n \in \mathbb{N} \cup \{0\}$)

$$M_{\mathcal{N}=2,d=5,\text{Jordan,symm}} = SO(1, 1) \times \frac{SO(1, n)}{SO(n)}. \quad (6.1)$$

This sequence is associated with the rank-3 Euclidean reducible Jordan algebra $\mathbb{R} \oplus \Gamma_{1,n}$. In the following treatment, we will determine the large and small orbits of the irrepr. $(\mathbf{1}, \mathbf{1} + \mathbf{n})$ of the U -duality group $SO(1, 1) \times SO(1, n)$.

For the sake of brevity, we will do this only through an analysis in the bare charges’ basis.

Without any loss of generality, one can choose to treat only $d = 5$ extremal (electric) BHs. Indeed, due to the symmetricity of the reducible coset (6.1), the treatment of $d = 5$ extremal (magnetic) black strings is essentially analogous.

Two disconnected geometric structures emerge in the treatment, as follows.

- (i) Timelike two-sheet hyperboloid T_n , with the two disconnected sheets T_n^{\pm} , respectively, related to $q_0 \gtrless 0$:

$$T_n \equiv \frac{SO(1, n)}{SO(n)} \Big|_{q_i^2 > 0} = \begin{matrix} T_n^+ \\ q_0 > 0 \end{matrix} \cup \begin{matrix} T_n^- \\ q_0 < 0 \end{matrix}; \quad (6.2)$$

$$T_n^+ \cap T_n^- = \emptyset.$$

- (ii) Forward/backward light-cone Λ_n of $(n + 1)$ -dimensional Minkowski space with metric η_{IJ} defined by (6.5), with two (forward Λ_n^+ and backward Λ_n^-) cone branches, respectively, related to $q_0 \gtrless 0$:

$$\Lambda_n \equiv \frac{SO(1, n)}{SO(n-1) \times \mathbb{R}^{n-1}} = \Lambda_n^+ \cup \Lambda_n^-; \Lambda_n^+ \cap \Lambda_n^- = 0, \quad (6.3)$$

with “0” here denoting the origin of Λ_n itself.

Because of such structures, as well as the lower ($\mathcal{N} = 2$) supersymmetry, the case study of large and small charge orbits in $\mathcal{N} = 2, d = 5$ Jordan symmetric sequence exhibits some subtleties absent in the $\mathcal{N} = 4, d = 5$ theory analyzed in Sec. VII.

In the bare charges’ basis, the electric cubic invariant of the $(\mathbf{1}, \mathbf{1} + \mathbf{n})$ of $SO(1, 1) \times SO(1, n)$ reads as follows ($I = 0, i$, where $i = 1, \dots, n$, throughout; 0 pertains to the $d = 5$ graviphoton field, which through the dimensional reduction $d = 5 \rightarrow d = 4$ becomes the Maxwell vector field of the axio-dilatonic vector multiplet):

$$I_{3,el} \equiv q_H q_I q_J \eta^{IJ} \equiv q_H q_I^2 = q_H \left(q_0^2 - \sum_{i=1}^n q_i^2 \right), \quad (6.4)$$

where q_H is the electric charge of the dilatonic vector multiplet: it is an $SO(1, n)$ singlet, with $SO(1, 1)$ weight +2. On the other hand, the $SO(1, n)$ vector q_I has $SO(1, 1)$ weight -1 , such that $I_{3,el}$ defined by (6.4) is $SO(1, 1) \times SO(1, n)$ invariant. Notice that the action of the U -duality group does not mix q_H and q_I , and this originates more charge orbits with respect to the irreducible cases. Moreover, $\eta_{IJ} = \eta^{IJ}$ is the Lorentzian metric of $SO(1, n)$:

$$\eta_{IJ} = \eta^{IJ} \equiv \text{diag} \left(+1, \overbrace{-1, \dots, -1}^n \right). \quad (6.5)$$

In $\mathcal{N} = 2, d = 5$ Jordan symmetric sequence, as well as in $\mathcal{N} = 4, d = 5$ theory, the reducibility of the associated rank-3 Jordan algebra gives rise to many subtleties and differences with respect to the theories associated with irreducible Euclidean rank-3 Jordan algebras. In the $\mathcal{N} = 2$ case under consideration, the major difference consists in a higher number of large and small orbits with respect to the magic supergravities.

A. Large orbits

- (i) BPS (3-charge) orbits are defined as follows:

$$q_H > 0, \quad q_0^2 - \sum_{i=1}^n q_i^2 > 0,$$

$$q_0 > 0, \quad \text{or} \quad q_H < 0, \quad q_0^2 - \sum_{i=1}^n q_i^2 > 0,$$

$$q_0 < 0. \quad (6.6)$$

By recalling definition (6.2), the orbit reads ($n \geq 0$)

$$\mathcal{O}_{\text{BPS,large}} = [SO(1, 1)^+ \times T_n^+] \cup [SO(1, 1)^- \times T_n^-], \quad (6.7)$$

with no related moduli space. In particular, for $n = 0$, namely, in the so-called $\mathcal{N} = 2, d = 5$ $SO(1, 1)$ model ($d = 5$ uplift of the $d = 4$ st^2 model), in which only the dilatonic vector multiplet is coupled

to the gravity multiplet, this orbit is actually 2-charge, and it is given by

$$\mathcal{O}_{\text{BPS,large},SO(1,1)} = \{(q_H, q_0) = (+, +), (-, -)\}. \quad (6.8)$$

On the other hand, for $n = 1$, i.e. in the so-called $\mathcal{N} = 2$, $d = 5$ [$SO(1, 1)$]² model ($d = 5$ uplift of the *stu* model), the cubic invariant (6.4) can be rewritten as follows:

$$I_{3,el} \equiv q_H q_I q_J \eta^{IJ} \equiv q_H (q_0^2 - q_1^2) = q_H q_+ q_-, \\ q_{\pm} \equiv q_0 \pm q_1, \quad (6.9)$$

and thus the hyperboloid (6.2) and light-cone (6.3) structures get, respectively, factorized as follows (“+,” “-,” and “0,” respectively, denote strictly positive, strictly negative, and vanishing values):

$$T_1 = SO(1, 1)|_{q_+ q_- > 0} \\ = \begin{matrix} T_1^+ \\ q_0 > 0 \end{matrix} \cup \begin{matrix} T_1^- \\ q_0 < 0 \end{matrix}; T_1^+ \cap T_1^- = \emptyset, \quad (6.10) \\ T_1^+ = \{(q_+, q_-) = (+, +)\}; \\ T_1^- = \{(q_+, q_-) = (-, -)\}.$$

$$\Lambda_1 = SO(1, 1) = \Lambda_1^+ \cup \Lambda_1^-; \Lambda_1^+ \cap \Lambda_1^- = 0, \\ \Lambda_1^+ = \{(q_+, q_-) = (+, 0), (0, +)\}; \quad (6.11) \\ \Lambda_1^- = \{(q_+, q_-) = (-, 0), (0, -)\}.$$

For $n = 1$, orbit (6.7) reads

$$\mathcal{O}_{\text{BPS,3-charge},[SO(1,1)]^2} = \{(q_H, q_+, q_-) \\ = (+, +, +), (-, -, -)\}. \quad (6.12)$$

This is invariant under triality permutation symmetry of q_H , q_+ , and q_- , and it is consistent with the analysis of [34].

- (ii) Non-BPS (3-charge) orbits, with $Z \neq 0$ at the horizon, are defined as follows:

$$q_H > 0, \quad q_0^2 - \sum_{i=1}^n q_i^2 > 0, \\ q_0 < 0, \quad \text{or} \quad q_H < 0, \quad q_0^2 - \sum_{i=1}^n q_i^2 > 0, \\ q_0 > 0. \quad (6.13)$$

By recalling definition (6.2), the orbit reads ($n \geq 0$)

$$\mathcal{O}_{n\text{BPS,large},I} = [SO(1, 1)^+ \times T_n^-] \\ \cup [SO(1, 1)^- \times T_n^+], \quad (6.14)$$

with no related moduli space. In particular, for $n =$

0, this orbit is actually 2-charge, and it is given by

$$\mathcal{O}_{n\text{BPS,large},SO(1,1)} = \{(q_H, q_0) = (+, -), (-, +)\}. \quad (6.15)$$

On the other hand, for $n = 1$, orbit (6.14) reads

$$\mathcal{O}_{n\text{BPS,large},I,[SO(1,1)]^2} = \{(q_H, q_+, q_-) \\ = (+, -, -), (-, +, +)\}. \quad (6.16)$$

The supersymmetry properties of $\mathcal{O}_{\text{BPS,large}}$ and $\mathcal{O}_{n\text{BPS,large},I}$ can be understood by noticing that the flip of the sign of q_H amounts, in the dressed charges’ basis, to the exchange $Z \leftrightarrow \partial_s Z$, where s is the real dilaton scalar field, parametrizing $SO(1, 1)$ of (6.1).

It is worth pointing out that both the $\mathcal{N} = 2$ orbits $\mathcal{O}_{\text{BPS,large}}$ and $\mathcal{O}_{n\text{BPS,large},I}$ [respectively given by (6.7) and (6.14)] uplift to the same $\mathcal{N} = 4$ orbit $\mathcal{O}_{(1/4)\text{-BPS,large},\mathcal{N}=4,d=5}$ given by Eq. (7.4). As mentioned, this is due to the fact that in $\mathcal{N} = 4$, $d = 5$, $q_H > 0 \leftrightarrow q_H < 0$ amounts to exchanging the two gravitinos in the gravity multiplet, i.e. the two (opposite) skew eigenvalues of the skew-traceless central charge matrix \dot{Z}_{AB} ($A, B = 1, \dots, 4$).

Another non-BPS (3-charge) orbit, with $Z \neq 0$ at the horizon, is defined as follows [19]:

$$q_H \geq 0; \quad q_0^2 - \sum_{i=1}^n q_i^2 < 0. \quad (6.17)$$

Thus, the resulting orbit reads (existing only for $n \geq 1$)

$$\mathcal{O}_{n\text{BPS,large},II} = SO(1, 1) \times \frac{SO(1, n)}{SO(1, n-1)}, \quad (6.18)$$

with related moduli space [recall (3.44) and (3.45)]

$$\mathcal{M}_{n\text{BPS,large},II} = \frac{SO(1, n-1)}{SO(n-1)} = \frac{M_{J,5,n-1}}{SO(1, 1)} = M_{nJ,5,n-1} \\ = M_{(1,0),d=6}|_{n-1}, \quad (6.19)$$

where $M_{nJ,5,n-1}$ denotes the $\mathcal{N} = 2$, $d = 5$ non-Jordan symmetric sequence with $n - 1$ vector multiplets [45], and $M_{(1,0),d=6}|_{n-1}$ is the scalar manifold of $(1, 0)$, $d = 6$ supergravity with $n_T = n - 1$ tensor multiplets. Thus, by recalling (6.1), the number \sharp of nonflat scalar degrees of freedom along $\mathcal{O}_{n\text{BPS,large},II}$ is independent of $n > 1$:

$$\sharp_{n\text{BPS,large},II} \equiv \dim_{\mathbb{R}} \mathcal{M}_{\mathcal{N}=2,d=5,\text{Jordan,symm}} \\ - \dim_{\mathbb{R}} \mathcal{M}_{n\text{BPS,large},II} = 2. \quad (6.20)$$

For $n = 1$, orbit (6.18) reads

$$\mathcal{O}_{n\text{BPS,large,II},[SO(1,1)]^2} = \{(q_H, q_+, q_-) = (+, +, -), (+, -, +), (-, +, -), (-, -, +)\}, \quad (6.21)$$

with no corresponding moduli space. Equation (6.21) is equivalent to (6.16) through triality permutation symmetry of q_H , q_+ , and q_- . Thus, consistent with the analysis of [34], the non-BPS large orbit of the $[SO(1, 1)]^2$ model is given, up to permutations of the triplet (q_H, q_+, q_-) , by

$$\begin{aligned} \mathcal{O}_{n\text{BPS,3-charge},[SO(1,1)]^2} &= \{(q_H, q_+, q_-) \\ &= (+, +, -), (+, -, -)\}. \end{aligned} \quad (6.22)$$

B. Small orbits

Let us now consider the small orbits, and compute the criticality and double-criticality conditions on $I_{3,el}$ defined by (6.4):

$$\frac{\partial I_{3,el}}{\partial Q} = \begin{cases} \frac{\partial I_{3,el}}{\partial q_H} = q_I^2 \\ \frac{\partial I_{3,el}}{\partial q_I} = 2q_H q_J \eta^{IJ}, \end{cases} \quad (6.23)$$

$$\frac{\partial^2 I_{3,el}}{\partial Q^2} = \begin{cases} \frac{\partial I_{3,el}}{(\partial q_H)^2} = 0 \\ \frac{\partial I_{3,el}}{\partial q_H \partial q_I} = \frac{\partial I_{3,el}}{\partial q_I \partial q_H} = 2q_J \eta^{IJ} \\ \frac{\partial I_{3,el}}{\partial q_I \partial q_J} = 2q_H, \end{cases} \quad (6.24)$$

where

$$Q \equiv (q_H, q_I) \quad (6.25)$$

is shorthand for the vector of electric charges. As expected from the fact that $I_{3,el}$ is homogeneous of degree three, (6.24) implies that the unique doubly critical orbit is the trivial one with all charges vanishing, because

$$\frac{\partial^2 I_{3,el}}{\partial Q^2} = 0 \Leftrightarrow Q = 0. \quad (6.26)$$

The small orbits of the $(\mathbf{1}, \mathbf{1} + \mathbf{n})$ of the U -duality group $SO(1, 1) \times SO(1, n)$ are listed as follows:

- (1) BPS lightlike ($I_{3,el} = 0$, $\frac{\partial I_{3,el}}{\partial Q} \neq 0$: 2-charge) orbit with vanishing q_H and timelike q_I :

$$q_H = 0, \quad q_0^2 - \sum_{i=1}^n q_i^2 > 0. \quad (6.27)$$

By recalling definition (6.2), the orbit reads ($n \geq 0$)

$$\mathcal{O}_{\text{BPS,small,I}} = SO(1, 1) \times T_n, \quad (6.28)$$

with no corresponding moduli space. In particular, for $n = 0$ this orbit is actually 1-charge, and it is given by

$$\mathcal{O}_{\text{BPS,small,I},SO(1,1)} = \{(q_H, q_0) = (0, +), (0, -)\}. \quad (6.29)$$

On the other hand, for $n = 1$, the orbit (6.28) reads

$$\begin{aligned} \mathcal{O}_{\text{BPS,small,I},[SO(1,1)]^2} &= \{(q_H, q_+, q_-) \\ &= (0, +, +), (0, -, -)\}, \end{aligned} \quad (6.30)$$

with no corresponding moduli space, and thus

$$\#_{\text{BPS,small,I},[SO(1,1)]^2} = 2. \quad (6.31)$$

- (2) Non-BPS lightlike ($I_{3,el} = 0$, $\frac{\partial I_{3,el}}{\partial Q} \neq 0$: 2-charge) orbit with vanishing q_H and spacelike q_I :

$$q_H = 0, \quad q_0^2 - \sum_{i=1}^n q_i^2 < 0. \quad (6.32)$$

It reads (existing only for $n \geq 1$)

$$\mathcal{O}_{n\text{BPS,small,I}} = SO(1, 1) \times \frac{SO(1, n)}{SO(1, n-1)}, \quad (6.33)$$

with corresponding moduli space [recall Eq. (6.19)]

$$\mathcal{M}_{n\text{BPS,small,I}} = \mathcal{M}_{n\text{BPS,large,II}}. \quad (6.34)$$

Thus, by recalling (6.1), the number $\#$ of nonflat scalar degrees of freedom along $\mathcal{O}_{n\text{BPS,small,I}}$ is independent of $n \geq 1$:

$$\begin{aligned} \#_{n\text{BPS,small,I}} &\equiv \dim_{\mathbb{R}} \mathcal{M}_{\mathcal{N}=2, d=5, \text{Jordan, symm}} \\ &- \dim_{\mathbb{R}} \mathcal{M}_{n\text{BPS,small,I}} = 2. \end{aligned} \quad (6.35)$$

For $n = 1$, orbit (6.33) reads

$$\begin{aligned} \mathcal{O}_{n\text{BPS,small,I},[SO(1,1)]^2} &= \{(q_H, q_+, q_-) \\ &= (0, +, -), (0, -, +)\}, \end{aligned} \quad (6.36)$$

with no corresponding moduli space.

- (3) BPS critical ($I_{3,el} = 0$, $\frac{\partial I_{3,el}}{\partial Q} = 0$: 1-charge) orbit with vanishing q_H and lightlike q_I :

$$q_H = 0, \quad q_0^2 - \sum_{i=1}^n q_i^2 = 0. \quad (6.37)$$

By recalling definition (6.3), the orbit reads (existing only for $n \geq 1$)

$$\mathcal{O}_{\text{BPS,small,II}} = \Lambda_n, \quad (6.38)$$

and the corresponding moduli space is ($n \geq 1$)

$$\mathcal{M}_{\text{BPS,small,II}} = SO(1, 1) \times \mathbb{R}^{n-1}. \quad (6.39)$$

Thus, by recalling (6.1), the number $\#$ of nonflat scalar degrees of freedom along $\mathcal{O}_{\text{BPS,small,II}}$ is independent of $n \geq 1$:

$$\begin{aligned} \#_{\text{BPS,small,II}} &\equiv \dim_{\mathbb{R}} \mathcal{M}_{\mathcal{N}=2, d=5, \text{Jordan, symm}} \\ &- \dim_{\mathbb{R}} \mathcal{M}_{\text{BPS,small,II}} = 1. \end{aligned} \quad (6.40)$$

$$\mathcal{O}_{\text{BPS,small,II}, [SO(1,1)]^2} = \{(q_H, q_+, q_-) = (0, 0, +), (0, +, 0), (0, 0, -), (0, -, 0)\}. \quad (6.41)$$

(4) BPS lightlike ($I_{3,el} = 0$, $\frac{\partial I_{3,el}}{\partial Q} \neq 0$: 2-charge) orbit, defined as follows:

$$\begin{aligned} q_H > 0, \quad q_0^2 - \sum_{i=1}^n q_i^2 &= 0, \\ q_0 > 0, \quad \text{or} \quad q_H < 0, \quad q_0^2 - \sum_{i=1}^n q_i^2 &= 0, \\ q_0 < 0. \end{aligned} \quad (6.42)$$

By recalling definition (6.3), the orbit reads ($n \geq 2$)

$$\begin{aligned} \mathcal{O}_{\text{BPS,small,III}} &= [SO(1, 1)^+ \times \Lambda_n^+] \\ &\cup [SO(1, 1)^- \times \Lambda_n^-], \end{aligned} \quad (6.43)$$

$$\mathcal{O}_{\text{BPS,small,III}, [SO(1,1)]^2} = \{(q_H, q_+, q_-) = (+, 0, +), (+, +, 0), (-, 0, -), (-, -, 0)\}, \quad (6.46)$$

with no corresponding moduli space. Equation (6.46) is equivalent to (6.30) through triality permutation symmetry of q_H , q_+ , and q_- . Thus, the BPS 2-charge orbit of the $[SO(1, 1)]^2$ model is given, up to permutations of the triplet (q_H, q_+, q_-) , by

$$\begin{aligned} \mathcal{O}_{\text{BPS,2-charge}, [SO(1,1)]^2} &= \{(q_H, q_+, q_-) \\ &= (+, +, 0), (-, -, 0)\}. \end{aligned} \quad (6.47)$$

(5) Non-BPS lightlike ($I_{3,el} = 0$, $\frac{\partial I_{3,el}}{\partial Q} \neq 0$: 2-charge) orbit, defined as follows:

$$\begin{aligned} q_H < 0, \quad q_0^2 - \sum_{i=1}^n q_i^2 &= 0, \\ q_0 > 0, \quad \text{or} \quad q_H > 0, \quad q_0^2 - \sum_{i=1}^n q_i^2 &= 0, \\ q_0 < 0. \end{aligned} \quad (6.48)$$

Analogously to what holds for symmetric magic RSG [noted below Eq. (4.33)], the unique scalar degree of freedom on which the ADM mass depends can be interpreted as the Kaluza-Klein radius in the $d = 6 \rightarrow d = 5$ reduction. For $n = 1$, orbit (6.38) reads

and the corresponding moduli space is purely translational ($n \geq 2$):

$$\mathcal{M}_{\text{BPS,small,III}} = \mathbb{R}^{n-1} = \mathcal{M}_{\text{BPS,small,II}}. \quad (6.44)$$

Thus, by recalling (6.1), the number $\#$ of nonflat scalar degrees of freedom along $\mathcal{O}_{\text{BPS,small,III}}$ is independent of $n \geq 2$:

$$\begin{aligned} \#_{\text{BPS,small,III}} &\equiv \dim_{\mathbb{R}} \mathcal{M}_{\mathcal{N}=2, d=5, \text{Jordan, symm}} \\ &- \dim_{\mathbb{R}} \mathcal{M}_{\text{BPS,small,III}} = 2. \end{aligned} \quad (6.45)$$

This orbit exists also for $n = 1$, and it reads

By recalling definition (6.3), the orbit reads ($n \geq 2$)

$$\begin{aligned} \mathcal{O}_{n\text{BPS,small,II}} &= [SO(1, 1)^+ \times \Lambda_n^-] \\ &\cup [SO(1, 1)^- \times \Lambda_n^+], \end{aligned} \quad (6.49)$$

with corresponding moduli space ($n \geq 2$)

$$\begin{aligned} \mathcal{M}_{n\text{BPS,small,II}} &= \mathbb{R}^{n-1} = \mathcal{M}_{\text{BPS,small,II}} \\ &= \mathcal{M}_{\text{BPS,small,III}}. \end{aligned} \quad (6.50)$$

Thus, by recalling (6.1), the number $\#$ of nonflat scalar degrees of freedom along $\mathcal{O}_{n\text{BPS,small,II}}$ is independent of $n \geq 2$:

$$\begin{aligned} \#_{n\text{BPS,small,II}} &\equiv \dim_{\mathbb{R}} \mathcal{M}_{\mathcal{N}=2, d=5, \text{Jordan, symm}} \\ &- \dim_{\mathbb{R}} \mathcal{M}_{n\text{BPS,small,II}} = 2. \end{aligned} \quad (6.51)$$

This orbit exists also for $n = 1$, and it reads

$$\mathcal{O}_{n\text{BPS,small,II},[SO(1,1)]^2} = \{(q_H, q_+, q_-) = (+, 0, -), (+, -, 0), (-, 0, +), (-, +, 0)\}, \quad (6.52)$$

with no corresponding moduli space. Equation (6.52) is equivalent to (6.36) through triality permutation symmetry of q_H , q_+ , and q_- . Thus, the non-BPS 2-charge orbit of the $[SO(1, 1)]^2$ model is given, up to permutations of the triplet (q_H, q_+, q_-) , by

$$\begin{aligned} \mathcal{O}_{n\text{BPS,2-charge},[SO(1,1)]^2} &= \{(q_H, q_+, q_-) \\ &= (+, -, 0)\}. \end{aligned} \quad (6.53)$$

- (6) BPS critical ($J_{3,el} = 0$, $\frac{\partial J_{3,el}}{\partial Q} = 0$: 1-charge) orbit with vanishing q_I and nonvanishing q_H :

$$q_H \in \mathbb{R}_0, \quad q_I = 0. \quad (6.54)$$

It exists for every $n \geq 0$, and it reads

$$\mathcal{O}_{\text{BPS,small,IV}} = SO(1, 1), \quad (6.55)$$

with moduli space $[n \geq 1$; recall (3.45)]

$$\mathcal{M}_{\text{BPS,small,IV}} = \frac{SO(1, n)}{SO(n)} = M_{nJ,5,n}. \quad (6.56)$$

Thus, by recalling (6.1), the number \sharp of nonflat scalar degrees of freedom along $\mathcal{O}_{\text{BPS,small,IV}}$ is independent of $n \geq 1$:

$$\begin{aligned} \sharp_{\text{BPS,small,IV}} &\equiv \dim_{\mathbb{R}} \mathcal{M}_{\mathcal{N}=2,d=5,\text{Jordan,symm}} \\ &- \mathcal{M}_{\text{BPS,small,IV}} = 1. \end{aligned} \quad (6.57)$$

Analogously to what holds for symmetric magic RSG [noted below Eq. (4.33)], the unique scalar degree of freedom on which the ADM mass depends can be interpreted as the Kaluza-Klein radius in the $d = 6 \rightarrow d = 5$ reduction. Furthermore, as in the corresponding $\mathcal{N} = 4$, $d = 5$ small orbit [given by Eq. (7.34)], the sign of q_H does not matter here. Orbit (6.55) originates from the $d = 6 \rightarrow d = 5$ reduction of $(1, 0)$ theory with *all* charges switched off. Indeed, q_H is the electric charge of the Kaluza-Klein vector in the reduction $d = 6 \rightarrow d = 5$. In particular, for $n = 0$, this orbit reads

$$\mathcal{O}_{\text{BPS,small,IV},SO(1,1)} = \{(q_H, q_0) = (+, 0), (-, 0)\}, \quad (6.58)$$

with no corresponding moduli space. On the other

hand, for $n = 1$ the orbit (6.55) reads

$$\begin{aligned} \mathcal{O}_{\text{BPS,small,IV},[SO(1,1)]^2} &= \{(q_H, q_+, q_-) \\ &= (+, 0, 0), (-, 0, 0)\}, \end{aligned} \quad (6.59)$$

which is equivalent to (6.41) through triality permutation symmetry of q_H , q_+ , and q_- . Thus, the BPS 1-charge orbit of the $[SO(1, 1)]^2$ model is given, up to permutations of the triplet (q_H, q_+, q_-) , by

$$\begin{aligned} \mathcal{O}_{\text{BPS,1-charge},[SO(1,1)]^2} &= \{(q_H, q_+, q_-) \\ &= (+, 0, 0), (-, 0, 0)\}. \end{aligned} \quad (6.60)$$

Thus, the stratification structure of the $(\mathbf{1}, \mathbf{1} + \mathbf{n})$ -repr. space of the $d = 5$ U -duality group $SO(1, 1) \times SO(1, n)$ can be given through the following two chains of relations, proceeding (left to right) from 1-charge orbits to 2-charge orbits to 3-charge orbits:

$$\mathcal{O}_{\text{BPS,small,II}} \rightarrow \begin{cases} \mathcal{O}_{\text{BPS,small,I}} \rightarrow \begin{cases} \mathcal{O}_{\text{BPS,large}} \\ \mathcal{O}_{n\text{BPS,large,I}} \end{cases} \\ \mathcal{O}_{n\text{BPS,small,I}} \rightarrow \mathcal{O}_{n\text{BPS,large,II}} \\ \mathcal{O}_{\text{BPS,small,III}} \rightarrow \begin{cases} \mathcal{O}_{\text{BPS,large}} \\ \mathcal{O}_{n\text{BPS,large,II}} \end{cases} \\ \mathcal{O}_{n\text{BPS,small,II}} \rightarrow \begin{cases} \mathcal{O}_{n\text{BPS,large,I}} \\ \mathcal{O}_{n\text{BPS,large,II}} \end{cases} \end{cases} \quad (6.61)$$

$$\mathcal{O}_{\text{BPS,small,IV}} \rightarrow \begin{cases} \mathcal{O}_{\text{BPS,small,III}} \rightarrow \begin{cases} \mathcal{O}_{\text{BPS,large}} \\ \mathcal{O}_{n\text{BPS,large,II}} \end{cases} \\ \mathcal{O}_{n\text{BPS,small,II}} \rightarrow \begin{cases} \mathcal{O}_{n\text{BPS,large,I}} \\ \mathcal{O}_{n\text{BPS,large,II}} \end{cases} \end{cases} \quad (6.62)$$

For the $SO(1, 1)$ model ($n = 0$), such a stratification structure simplifies as follows:

$$SO(1, 1): \left. \begin{array}{l} \mathcal{O}_{\text{BPS,small,I}} \\ \mathcal{O}_{\text{BPS,small,IV}} \end{array} \right\} \rightarrow \left\{ \begin{array}{l} \mathcal{O}_{\text{BPS,large}} \\ \mathcal{O}_{n\text{BPS,large}} \end{array} \right. \quad (6.63)$$

On the other hand, for the $[SO(1, 1)]^2$ model ($n = 1$), the stratification structure (6.61) and (6.62) reads

$$[SO(1, 1)]^2: \mathcal{O}_{\text{BPS,1-charge}} \rightarrow \left\{ \begin{array}{l} \mathcal{O}_{\text{BPS,2-charge}} \rightarrow \begin{cases} \mathcal{O}_{\text{BPS,3-charge}} \\ \mathcal{O}_{n\text{BPS,3-charge}} \end{cases} \\ \mathcal{O}_{n\text{BPS,2-charge}} \rightarrow \mathcal{O}_{n\text{BPS,3-charge}} \end{array} \right. \quad (6.64)$$

Thus, summarizing, the $\mathcal{N} = 2$, $d = 5$ Jordan symmetric sequence admits six small charge orbits describing the flux configurations supporting static, spherically symmetric, asymptotically flat small BHs: four $\frac{1}{2}$ -BPS and two non-BPS. Furthermore, there are three large orbits, namely, one $\frac{1}{2}$ -BPS and two non-BPS (with $Z \neq 0$ at the horizon).

VII. $\mathcal{N} = 4$, $d = 5$ SUPERGRAVITY

The scalar manifold of $\mathcal{N} = 4$, $d = 5$ supergravity coupled to $n_V = n \in \mathbb{N} \cup \{0\}$ matter (vector) multiplets reads $[\dim_{\mathbb{R}} = 1 + 5n, \text{rank} = 1 + \min(5, n)]$

$$M_{\mathcal{N}=4,d=5} = SO(1, 1) \times \frac{SO(5, n)}{SO(5) \times SO(n)}. \quad (7.1)$$

This theory is associated with the rank-3 Euclidean reducible Jordan algebra $\mathbb{R} \oplus \Gamma_{5,n}$. In the following treatment, we will determine the large and small orbits of the irrepr. $(\mathbf{1}, \mathbf{5} + \mathbf{n})$ of the U -duality group $SO(1, 1) \times SO(5, n)$.

For the sake of brevity, we will do this only through an analysis in the bare charges' basis.

Without any loss of generality, one can choose to treat only $d = 5$ extremal (electric) BHs. Indeed, due to the symmetricity of the reducible coset (7.1), the treatment of $d = 5$ extremal (magnetic) black strings is essentially analogous.

In the bare charges' basis, the electric cubic invariant of the $(\mathbf{1}, \mathbf{5} + \mathbf{n})$ of $SO(1, 1) \times SO(5, n)$ reads as follows ($I = 1, \dots, 5 + n$ throughout; the indices $1, \dots, 5$, with positive signature, pertain to the five $\mathcal{N} = 4$, $d = 5$ graviphotons):

$$I_{3,el} \equiv q_H q_I q_J \eta^{IJ} \equiv q_H q_I^2, \quad (7.2)$$

where q_H is the electric charge of the 3-form field strength of the 2-form $B_{\mu\nu}$ ($\mu, \nu = 0, 1, \dots, 4$) in the gravity multiplet (see e.g. [55,56]). q_H is an $SO(5, n)$ singlet, with $SO(1, 1)$ weight $+2$. On the other hand, the $SO(5, n)$ vector q_I has $SO(1, 1)$ weight -1 , such that $I_{3,el}$ defined by (7.2) is $SO(1, 1) \times SO(5, n)$ invariant. Notice that the action of the U -duality group does not mix q_H and q_I , and this originates more charge orbits with respect to the irreducible cases. Moreover, $\eta_{IJ} = \eta^{IJ}$ is the pseudo-Euclidean

metric of $SO(5, n)$, with signature $(\overbrace{+, \dots, +}^5, \overbrace{-, \dots, -}^n)$.

A. Large orbits

- (i) $\frac{1}{4}$ -BPS (3-charge) orbit, defined by a timelike q_I vector, with q_H of any sign:

$$q_H \in \mathbb{R}_0, \quad q_I q_J \eta^{IJ} > 0. \quad (7.3)$$

The resulting form of the orbit reads [20] ($n \geq 0$)

$$\mathcal{O}_{(1/4)\text{-BPS,large}} = SO(1, 1) \times \frac{SO(5, n)}{SO(4, n)}, \quad (7.4)$$

with related moduli space

$$\mathcal{M}_{(1/4)\text{-BPS,large}} = \frac{SO(4, n)}{SO(4) \times SO(n)} = \frac{M_{(1,1),d=6}}{SO(1, 1)}, \quad (7.5)$$

where $M_{(1,1),d=6}$ is the scalar manifold of nonchiral half-maximal supergravity in $d = 6$ with n matter (vector) multiplets. The exchange between $q_H > 0$ and $q_H < 0$ amounts to exchanging the two gravitinos in the gravity multiplet, i.e. the two (opposite) skew eigenvalues of the skew-traceless central charge matrix \dot{Z}_{AB} ($A, B = 1, \dots, 4$). Thus, the number $\#$ of nonflat scalar degrees of freedom along $\mathcal{O}_{(1/4)\text{-BPS,large}}$ is (for $n \geq 1$)

$$\begin{aligned} \#_{(1/4)\text{-BPS,large}} &\equiv \dim_{\mathbb{R}} M_{\mathcal{N}=4,d=5} \\ &\quad - \dim_{\mathbb{R}} \mathcal{M}_{(1/4)\text{-BPS,large}} \\ &= n + 1. \end{aligned} \quad (7.6)$$

In $\mathcal{N} > 2$ -extended supergravity theories, in general, $\frac{1}{\mathcal{N}}$ -BPS attractors have a related moduli space [23]. It corresponds to the hypermultiplets' scalar manifold in the supersymmetry reduction $\mathcal{N} > 2 \rightarrow \mathcal{N} = 2$ of the theory under consideration. In this case, it is amusing to observe that $\mathcal{M}_{(1/4)\text{-BPS,large}}$ given by (7.5) is the c map of the vector multiplets' scalar manifold of the $\mathcal{N} = 2$, $d = 4$ Jordan symmetric sequence:

$$\begin{aligned} \mathcal{M}_{(1/4)\text{-BPS,large}} &= c \left(\frac{SU(1, 1)}{U(1)} \right. \\ &\quad \left. \times \frac{SO(2, n-2)}{SO(2) \times SO(n-2)} \right). \end{aligned} \quad (7.7)$$

Thus, $\mathcal{M}_{(1/4)\text{-BPS,large}}$ admits an interpretation either as (1) a scalar manifold of the $\mathcal{N} = 4$, $d = 3$ Jordan symmetric sequence in $d = 3$, or as (2) the hypermultiplets' scalar manifold of the Jordan symmetric sequence in $d = 4, 5$ ($\mathcal{N} = 2$) and 6 [(1, 0)]. In particular, $\mathcal{M}_{(1/4)\text{-BPS,large}}$ parametrizes the $\mathcal{N} = 2$ hyperscalar degrees of freedom in the supersymmetry/Jordan algebra reduction:

$$d = 5: \quad \mathcal{N} = 4 \quad \mathbb{R} \oplus \Gamma_{5,n} \quad \rightarrow \quad \mathcal{N} = 2 \quad \mathbb{R} \oplus \Gamma_{1,n-3}. \quad (7.8)$$

The pure theory (i.e. $n = 0$) limit of orbit (7.4) is actually 2-charge [indeed, $SO(5)$ symmetry can be used to make only one component of the Euclidean vector q_I nonvanishing], and it reads

$$\begin{aligned} \mathcal{O}_{(1/4)\text{-BPS,large},n=0} &= SO(1, 1) \times \frac{SO(5)}{SO(4)} \\ &\equiv SO(1, 1) \times S^4, \end{aligned} \quad (7.9)$$

with no corresponding moduli space, and thus trivially

$$\#_{(1/4)\text{-BPS, large, } n=0} = 1. \quad (7.10)$$

- (ii) Non-BPS (3-charge) orbit with $\dot{Z}_{AB} = 0$ (at the horizon), defined by a spacelike q_I vector, and q_H of any sign:

$$q_H \in \mathbb{R}_0, \quad q_I q_J \eta^{IJ} < 0. \quad (7.11)$$

Notice that both signs of q_H are allowed, due to the fact that the non-BPS $\dot{Z}_{AB} = 0$ attractor equations are quadratic in q_H (see e.g. [20]). The resulting orbit reads ($n \geq 1$, not existing in pure theory) [20]

$$\mathcal{O}_{n\text{BPS, large}} = SO(1, 1) \times \frac{SO(5, n)}{SO(5, n-1)}, \quad (7.12)$$

with related moduli space

$$\begin{aligned} \mathcal{M}_{n\text{BPS, large}} &= \frac{SO(5, n-1)}{SO(5) \times SO(n-1)} \\ &= M_{(2,0), d=6} |_{n-1}, \end{aligned} \quad (7.13)$$

where $M_{(2,0), d=6} |_{n-1}$ is the scalar manifold of (2, 0), $d = 6$ supergravity with $n_T = n - 1$ tensor multiplets. Note that $\mathcal{N} = 4$, $d = 5$ and (2, 0), $d = 6$ supergravities share the same \mathcal{R} symmetry $SO(5) \sim USp(4)$. Thus, the number $\#$ of nonflat scalar degrees of freedom along $\mathcal{O}_{n\text{BPS, large}}$ is independent of $n \geq 2$:

$$\begin{aligned} \#_{n\text{BPS, large}} &\equiv \dim_{\mathbb{R}} M_{\mathcal{N}=4, d=5} - \dim_{\mathbb{R}} \mathcal{M}_{n\text{BPS, large}} \\ &= 6. \end{aligned} \quad (7.14)$$

B. Small orbits

The conditions on $I_{3,el}$ defined by (7.2) are formally the same as the ones holding in $\mathcal{N} = 2$, $d = 5$ Jordan symmetric sequence, and given by Eqs. (6.23) and (6.24). Thus, analogously to the case of $\mathcal{N} = 2$, $d = 5$ Jordan symmetric sequence, and as expected from the fact that $I_{3,el}$ is homogeneous of degree three, (6.24) implies that the unique doubly critical orbit is the trivial one with all charges vanishing [namely, 0-charge orbit; recall Eq. (6.26)].

The small orbits of the $(\mathbf{1}, \mathbf{5} + \mathbf{n})$ of the U -duality group $SO(1, 1) \times SO(5, n)$ can be listed as follows:

- (1) Lightlike ($I_{3,el} = 0$, $\frac{\partial I_{3,el}}{\partial Q} \neq 0$: 2-charge) orbit with vanishing q_H and timelike q_I :

$$q_H = 0, \quad q_I^2 > 0. \quad (7.15)$$

This orbit is $\frac{1}{2}$ -BPS [14]. It reads ($n \geq 0$)

$$\mathcal{O}_{(1/2)\text{-BPS, small, I}} = SO(1, 1) \times \frac{SO(5, n)}{SO(4, n)}, \quad (7.16)$$

with corresponding moduli space [recall Eq. (7.5)]

$$\mathcal{M}_{(1/2)\text{-BPS, small, I}} = \mathcal{M}_{(1/4)\text{-BPS, large}}. \quad (7.17)$$

Thus, the number $\#$ of nonflat scalar degrees of freedom along $\mathcal{O}_{(1/2)\text{-BPS, small, I}}$ is (for $n \geq 1$)

$$\begin{aligned} \#_{(1/2)\text{-BPS, small, I}} &\equiv \dim_{\mathbb{R}} M_{\mathcal{N}=4, d=5} \\ &\quad - \dim_{\mathbb{R}} \mathcal{M}_{(1/2)\text{-BPS, small, I}} \\ &= n + 1. \end{aligned} \quad (7.18)$$

The pure theory (i.e. $n = 0$) limit of orbit (7.16) is actually 1-charge, and it reads

$$\mathcal{O}_{(1/2)\text{-BPS, small, I, } n=0} = SO(1, 1) \times S^4, \quad (7.19)$$

with no related moduli space, and thus

$$\#_{(1/2)\text{-BPS, small, I, } n=0} = 1. \quad (7.20)$$

- (2) Lightlike ($I_{3,el} = 0$, $\frac{\partial I_{3,el}}{\partial Q} \neq 0$: 2-charge) orbit with vanishing q_H and spacelike q_I :

$$q_H = 0, \quad q_I^2 < 0. \quad (7.21)$$

This orbit is non-BPS. It reads ($n \geq 1$, not existing in pure theory)

$$\mathcal{O}_{n\text{BPS, small}} = SO(1, 1) \times \frac{SO(5, n)}{SO(5, n-1)}, \quad (7.22)$$

with corresponding moduli space [recall Eq. (7.13)]

$$\mathcal{M}_{n\text{BPS, small}} = \mathcal{M}_{n\text{BPS, large}}. \quad (7.23)$$

Thus, the number $\#$ of nonflat scalar degrees of freedom along $\mathcal{O}_{n\text{BPS, small}}$ is independent of $n \geq 1$:

$$\begin{aligned} \#_{n\text{BPS, small}} &\equiv \dim_{\mathbb{R}} M_{\mathcal{N}=4, d=5} - \dim_{\mathbb{R}} \mathcal{M}_{n\text{BPS, small}} \\ &= 6. \end{aligned} \quad (7.24)$$

- (3) Critical ($I_{3,el} = 0$, $\frac{\partial I_{3,el}}{\partial Q} = 0$: 1-charge) orbit with vanishing q_H and lightlike q_I :

$$q_H = 0; \quad q_I^2 = 0. \quad (7.25)$$

This orbit is $\frac{1}{2}$ -BPS [14]. It reads ($n \geq 1$, not existing in pure theory)

$$\mathcal{O}_{(1/2)\text{-BPS, small, II}} = \frac{SO(5, n)}{SO(4, n-1) \times \mathbb{R}^{4, n-1}}, \quad (7.26)$$

with corresponding moduli space [recall Eq. (7.17)]

$$\begin{aligned} \mathcal{M}_{(1/2)\text{-BPS, small, II}} &= SO(1, 1) \times \mathcal{M}_{(1/2)\text{-BPS, small, I}} |_{n \rightarrow n-1} \times \mathbb{R}^{4, n-1} \\ &= SO(1, 1) \times \mathcal{M}_{(1/4)\text{-BPS, large}} |_{n \rightarrow n-1} \times \mathbb{R}^{4, n-1}. \end{aligned} \quad (7.27)$$

Thus, the number \sharp of nonflat scalar degrees of freedom along $\mathcal{O}_{(1/2)\text{-BPS,small,II}}$ is independent of $n \geq 1$:

$$\begin{aligned} \sharp_{(1/2)\text{-BPS,small,II}} &\equiv \dim_{\mathbb{R}} M_{\mathcal{N}=4,d=5} \\ &\quad - \dim_{\mathbb{R}} \mathcal{M}_{(1/2)\text{-BPS,small,II}} = 1. \end{aligned} \quad (7.28)$$

Analogously to what holds for symmetric magic RSG [noted below Eq. (4.33)] and for $N = 2$, $d = 5$ Jordan symmetric sequence treated in Sec. VI, the unique scalar degree of freedom on which the ADM mass depends can be interpreted as the Kaluza-Klein radius in the $d = 6 \rightarrow d = 5$ reduction.

- (4) Lightlike ($I_{3,el} = 0$, $\frac{\partial I_{3,el}}{\partial Q} \neq 0$: 2-charge) orbit with nonvanishing q_H and lightlike q_I :

$$q_H \in \mathbb{R}_0; \quad q_I^2 = 0. \quad (7.29)$$

This orbit is $\frac{1}{4}$ -BPS. It reads ($n \geq 1$)

$$\begin{aligned} \mathcal{O}_{(1/4)\text{-BPS,small}} &= SO(1, 1) \\ &\quad \times \frac{SO(5, n)}{SO(4, n-1) \rtimes \mathbb{R}^{4, n-1}}, \end{aligned} \quad (7.30)$$

with corresponding moduli space [recall Eq. (7.27)]

$$\begin{aligned} \mathcal{M}_{(1/4)\text{-BPS,small}} &= \mathcal{M}_{(1/2)\text{-BPS,small,I}}|_{n \rightarrow n-1} \rtimes \mathbb{R}^{4, n-1} \\ &= \mathcal{M}_{(1/4)\text{-BPS,large}}|_{n \rightarrow n-1} \rtimes \mathbb{R}^{4, n-1}. \end{aligned} \quad (7.31)$$

Thus, the number \sharp of nonflat scalar degrees of freedom along $\mathcal{O}_{(1/2)\text{-BPS,small,II}}$ is independent of $n \geq 1$:

$$\begin{aligned} \sharp_{(1/4)\text{-BPS,small}} &\equiv \dim_{\mathbb{R}} M_{\mathcal{N}=4,d=5} \\ &\quad - \dim_{\mathbb{R}} \mathcal{M}_{(1/4)\text{-BPS,small}} = 2. \end{aligned} \quad (7.32)$$

Critical ($I_{3,el} = 0$, $\frac{\partial I_{3,el}}{\partial Q} = 0$: 1-charge) orbit with vanishing q_I and nonvanishing q_H :

$$q_H \in \mathbb{R}_0, \quad q_I = 0. \quad (7.33)$$

This orbit is $\frac{1}{2}$ -BPS [14]. It reads (independent of $n \geq 0$)

$$\mathcal{O}_{(1/2)\text{-BPS,small,III}} = SO(1, 1), \quad (7.34)$$

with moduli space

$$\mathcal{M}_{(1/2)\text{-BPS,small,III}} = \frac{SO(5, n)}{SO(5) \times SO(n)}. \quad (7.35)$$

Thus, the number \sharp of nonflat scalar degrees of freedom along $\mathcal{O}_{(1/2)\text{-BPS,small,III}}$ is independent of $n \geq 0$:

$$\begin{aligned} \sharp_{(1/2)\text{-BPS,small,III}} &\equiv \dim_{\mathbb{R}} M_{\mathcal{N}=4,d=5} \\ &\quad - \mathcal{M}_{(1/2)\text{-BPS,small,III}} = 1. \end{aligned} \quad (7.36)$$

Notice that $\mathcal{O}_{(1/2)\text{-BPS,small,III}}$ can also be seen as the “ $n = 0$ formal limit” of $\mathcal{O}_{(1/4)\text{-BPS,small}}$ given by Eq. (7.30). Indeed, the $n = 0$ limit of (7.29) is given by (7.33) itself. Furthermore, analogously to what holds for symmetric magic RSG [noted below Eq. (4.33)] and for $N = 2$, $d = 5$ Jordan symmetric sequence treated in Sec. VI, the unique scalar degree of freedom on which the ADM mass depends can be interpreted as the Kaluza-Klein radius in the $d = 6 \rightarrow d = 5$ reduction. Orbit (7.34) is originated by the $d = 6 \rightarrow d = 5$ reduction of (2, 0) theory with *all* charges switched off. Indeed, q_H is the electric charge of the Kaluza-Klein vector in the reduction $d = 6 \rightarrow d = 5$. Notice that in the pure theory (i.e. $n = 0$) $\mathcal{M}_{(1/2)\text{-BPS,small,III}}$ vanishes, and thus

$$\sharp_{(1/2)\text{-BPS,small,III},n=0} = 1. \quad (7.37)$$

Thus, the stratification structure of the $(\mathbf{1}, \mathbf{5} + \mathbf{n})$ -repr. space of the $d = 5$ U -duality group $SO(1, 1) \times SO(5, n)$ can be given through the two chains of relations, proceeding (left to right) from 1-charge orbits to 2-charge and then 3-charge orbits:

$$\mathcal{O}_{(1/2)\text{-BPS,small,II}} \rightarrow \begin{cases} \mathcal{O}_{(1/2)\text{-BPS,small,I}} \rightarrow \mathcal{O}_{(1/4)\text{-BPS,large}} \\ \mathcal{O}_{n\text{BPS,small}} \rightarrow \mathcal{O}_{n\text{BPS,large}} \\ \mathcal{O}_{(1/4)\text{-BPS,small}} \rightarrow \begin{cases} \mathcal{O}_{(1/4)\text{-BPS,large}} \\ \mathcal{O}_{n\text{BPS,large}} \end{cases} \end{cases} \quad (7.38)$$

$$\mathcal{O}_{(1/2)\text{-BPS,small,III}} \rightarrow \mathcal{O}_{(1/4)\text{-BPS,small}} \rightarrow \begin{cases} \mathcal{O}_{(1/4)\text{-BPS,large}} \\ \mathcal{O}_{n\text{BPS,large}} \end{cases} \quad (7.39)$$

For pure $\mathcal{N} = 4$, $d = 5$ supergravity, such a stratification structure simplifies as follows:

$$\left. \begin{array}{l} \text{1-charge} \\ \mathcal{O}_{(1/2)\text{-BPS,small,I},n=0} \\ \mathcal{O}_{(1/2)\text{-BPS,small,III}} \end{array} \right\} \rightarrow \begin{array}{l} \text{2-charge} \\ \mathcal{O}_{(1/4)\text{-BPS,large},n=0} \end{array} \quad (7.40)$$

Thus, summarizing, $\mathcal{N} = 4$, $d = 5$ supergravity theory admits five small charge orbits describing the flux configurations supporting static, spherically symmetric, asymptotically flat small BHs: one $\frac{1}{4}$ -BPS, three $\frac{1}{2}$ -BPS, and one non-BPS. There are two large orbits, namely, one $\frac{1}{4}$ -BPS and one non-BPS (with $\dot{Z}_{AB} = 0$ at the horizon).

The relations among the charge orbits of $\mathcal{N} = 4$, $d = 5$ supergravity and the charge orbits of $\mathcal{N} = 2$, $d = 5$ Jordan symmetric sequence can be determined through the supersymmetry reduction

TABLE VI. Large (rank = 3) and small (rank = 1 and 2) charge orbits of the repr. $(\mathbf{1}, \mathbf{5} + \mathbf{n})$ and $(\mathbf{1}, \mathbf{1} + \mathbf{n})$ of the $d = 5$ U -duality groups $SO(1, 1) \times SO(5, n)$ and $SO(1, 1) \times SO(1, n)$ of $\mathcal{N} = 4$ supergravity (based on $\mathbb{R} \oplus \Gamma_{5,n}$) and $\mathcal{N} = 2$ Jordan symmetric sequence (based on $\mathbb{R} \oplus \Gamma_{1,n}$), respectively. The rank r of the orbit is defined as the minimal number of charges defining a representative solution. “ $\updownarrow *$ ” denotes the fact that the orbits are related through a flip of the sign of q_H . The disconnected timelike hyperboloid T_n and light-cone Λ_n structures are defined by (6.2) and (6.3), respectively. The symbol $\#$, defined in (4.3), denotes the number of nonflat scalar degrees of freedom supported by the charge orbit.

r	$\mathcal{N} = 4: \mathbb{R} \oplus \Gamma_{5,n}$	$\mathcal{N} = 2: \mathbb{R} \oplus \Gamma_{1,n}$
3	$\mathcal{O}_{(1/4)\text{-BPS,large}}$ $SO(1, 1) \times \frac{SO(5,n)}{SO(4,n)}$ $\# = n + 1$	$\mathcal{O}_{\text{BPS,large}}$ $[SO(1, 1)^+ \times T_n^+] \cup [SO(1, 1)^- \times T_n^-]$ $\# = n + 1$ $\updownarrow *$ $\mathcal{O}_{n\text{BPS,large,I}}$ $[SO(1, 1)^+ \times T_n^-] \cup [SO(1, 1)^- \times T_n^+]$ $\# = n + 1$
3	$\mathcal{O}_{n\text{BPS,large}}$ $SO(1, 1) \times \frac{SO(5,n)}{SO(5,n-1)}$ $\# = 6$	$\mathcal{O}_{n\text{BPS,large,II}}$ $SO(1, 1) \times \frac{SO(1,n)}{SO(1,n-1)}$ $\# = 2$
2	$\mathcal{O}_{(1/2)\text{-BPS,small,I}}$ $SO(1, 1) \times \frac{SO(5,n)}{SO(4,n)}$ $\# = n + 1$	$\mathcal{O}_{\text{BPS,small,I}}$ $SO(1, 1) \times T_n$ $\# = n + 1$
2	$\mathcal{O}_{(1/4)\text{-BPS,small}}$ $SO(1, 1) \times \frac{SO(5,n)}{SO(4,n-1) \rtimes \mathbb{R}^{4,n-1}}$ $\# = 2$	$\mathcal{O}_{\text{BPS,small,III}}$ $[SO(1, 1)^+ \times \Lambda_n^+] \cup [SO(1, 1)^- \times \Lambda_n^-]$ $\# = 2$ $\updownarrow *$ $\mathcal{O}_{n\text{BPS,small,II}}$ $[SO(1, 1)^+ \times \Lambda_n^-] \cup [SO(1, 1)^- \times \Lambda_n^+]$ $\# = 2$
2	$\mathcal{O}_{n\text{BPS,small}}$ $SO(1, 1) \times \frac{SO(5,n)}{SO(5,n-1)}$ $\# = 6$	$\mathcal{O}_{n\text{BPS,small,I}}$ $SO(1, 1) \times \frac{SO(1,n)}{SO(1,n-1)}$ $\# = 2$
1	$\mathcal{O}_{(1/2)\text{-BPS,small,II}}$ $\frac{SO(5,n)}{SO(4,n-1) \rtimes \mathbb{R}^{4,n-1}}$ $\# = 1$	$\mathcal{O}_{\text{BPS,small,II}}$ Λ_n $\# = 1$
1	$\mathcal{O}_{(1/2)\text{-BPS,small,III}}$ $SO(1, 1)$ $\# = 1$	$\mathcal{O}_{\text{BPS,small,IV}}$ $SO(1, 1)$ $\# = 1$

$$d = 5: \begin{matrix} \mathcal{N} = 4 \\ \mathbb{R} \oplus \Gamma_{5,n} \end{matrix} \rightarrow \begin{matrix} \mathcal{N} = 2 \\ \mathbb{R} \oplus \Gamma_{1,n} \end{matrix}, \quad (7.41)$$

yielding the results summarized in Table VI.

Finally, it is worth summarizing the results obtained about the number $\#$ of nonflat scalar degrees of freedom, within the symmetric RSG studied in previous sections. For the magic supergravities, it holds that

$$J_3^{\mathbb{A}}: \begin{cases} \text{“large” (rank = 3):} & \begin{cases} \text{BPS: } \# = 3q + 2 \\ n\text{BPS: } \# = q + 2 \end{cases} \\ \text{“small” (rank = 3):} & \begin{cases} \text{BPS: } \# = q + 2 \\ n\text{BPS: } \# = 2 \end{cases} \\ \text{rank = 3:} & \text{BPS: } \# = 1, \end{cases} \quad (7.42)$$

whereas for $\mathcal{N} = 4$ supergravity and $\mathcal{N} = 2$ Jordan sym-

metric sequence the results are reported in Table VI. As pointed out above, in the symmetric RSG’s under consideration the unique scalar degree of freedom on which the ADM mass depends along the 1-charge $\frac{1}{2}$ -BPS (maximally symmetric) charge orbits can be interpreted as the Kaluza-Klein radius in the $d = 6 \rightarrow d = 5$ reduction.

ACKNOWLEDGMENTS

S.F. and A.M. would like to thank L. Borsten and M. Trigiante for enlightening discussions. B.L.C. and A.M. would like to thank the CTP of the University of California, Berkeley, where part of this work was done, for kind hospitality and a stimulating environment. A.M. would also like to acknowledge the warm hospitality and inspiring environment of the Department of Physics, Theory Unit Group at CERN, Geneva. The work of

B.L.C. has been supported in part by the European Commission under the FP7-PEOPLE-IRG-2008 Grant No. PIRG04-GA-2008-239412 “*String Theory and Noncommutative Geometry*” (*STRING*). The work of S.F. has been supported by the ERC Advanced Grant No. 226455, “*Supersymmetry, Quantum Gravity and Gauge Fields*” (*SUPERFIELDS*), and also in part by INFN—Frascati National Laboratories, and by DOE Grant No. DE-FG03-91ER40662, Task C. The work of A.M. has been supported by INFN as a visitor at Stanford University. The work of B.Z. has been supported in part by the Director, Office of Science, Office of High Energy and Nuclear Physics, Division of High Energy Physics of the U.S. Department of Energy under Contract No. DE-AC02-05CH11231, and in part by NSF Grant No. 10996-13607-44 PHHXM.

APPENDIX A: RESOLUTION OF G_5 -INVARIANT CONSTRAINTS

In this appendix, we explicitly solve the G_5 -invariant defining constraints of small charge orbits in magic symmetric RSG, both in the bare (Appendix A 1) and dressed (Appendix A 2) charge bases.

1. Bare charge basis

Let us start by noticing that for each of the four magic symmetric RSG’s a unique maximal symmetric embedding into G_5 exists containing a factor $SO(1, 1)$. It reads [recall Eq. (4.30)] [57]

$$G_5 \underset{\max}{\supseteq} G_6 \times \mathcal{A}_q \times SO(1, 1), \quad (\text{A1})$$

where the group \mathcal{A}_q has been defined in Table II. Notice that, in the cases $q = 4$ and 2 , $G_6 \times SO(1, 1)$ is not embedded maximally (also considering nonsymmetric embeddings [58]) into G_5 itself.

When removing \mathcal{A}_q in the cases $q = 4$ and 2 (and thus losing the maximality), the embedding (A1) has a nice interpretation in terms of truncation of the magic supergravity to theories belonging to the Jordan symmetric sequence (3.44) [19]:

$$\begin{aligned} J_3^{\text{O}} \underset{\max}{\supseteq} \mathbb{R} \oplus J_2^{\text{O}}: E_{6(-26)} \underset{\max}{\supseteq} SO(1, 1) \times SO(1, 9), \\ J_3^{\text{H}} \underset{\max}{\supseteq} \mathbb{R} \oplus J_2^{\text{H}}: SU^*(6) \supseteq SO(1, 1) \times SO(1, 5), \\ J_3^{\text{C}} \underset{\max}{\supseteq} \mathbb{R} \oplus J_2^{\text{C}}: SL(3, \mathbb{C}) \supseteq SO(1, 1) \times SO(1, 3), \\ J_3^{\text{R}} \underset{\max}{\supseteq} \mathbb{R} \oplus J_2^{\text{R}}: SL(3, \mathbb{R}) \underset{\max}{\supseteq} SO(1, 1) \times SO(1, 2), \end{aligned} \quad (\text{A2})$$

where it should be recalled that ($q = 8, 4, 2, 1$; see e.g. [46])

$$J_2^{\text{A}} \sim \Gamma_{1,q+1}. \quad (\text{A3})$$

a. $\mathcal{O}_{\text{lightlike,BPS}}$

In order to solve the small lightlike G_5 -invariant defining constraints (4.51) in bare charges in a way consistent with an orbit representative having $Z \neq 0$, let us further embed the MCS of the group in the right-hand side of Eq. (A1), thus obtaining

$$G_5 \underset{\max}{\supseteq} G_6 \times \mathcal{A}_q \times SO(1, 1) \underset{\max}{\supseteq}^{MCS} SO(q+1) \times \mathcal{A}_q. \quad (\text{A4})$$

Thus, under the “branching” (A4) the irrepr. \mathbf{R}_Q of G_5 in which the electric charges q_i sit decomposes as follows:

$$\begin{aligned} \mathbf{R}_Q &\rightarrow (\mathbf{1}, \mathbf{1})_{+4} + (\mathbf{q} + \mathbf{2}, \mathbf{1})_{-2} \\ &\quad + (\mathbf{Spin}(q+2), \mathbf{Spin}(Q_q))_{+1} \\ &\rightarrow (\mathbf{1}, \mathbf{1})_{\text{I}} + (\mathbf{q} + \mathbf{1}, \mathbf{1}) + (\mathbf{1}, \mathbf{1})_{\text{II}} \\ &\quad + (\mathbf{Spin}(q+1), \mathbf{Spin}(Q_q)). \end{aligned} \quad (\text{A5})$$

This in turn entails the branching

$$q_i \rightarrow (q_{(\mathbf{1}, \mathbf{1})_{\text{I}}}, q_{(\mathbf{1}, \mathbf{1})_{\text{II}}}, q_{(\mathbf{q} + \mathbf{1}, \mathbf{1})}, q_{(\mathbf{Spin}(q+1), \mathbf{Spin}(Q_q))}). \quad (\text{A6})$$

In the first and second lines of (A5) subscripts denote the weight with respect to $SO(1, 1)$, whereas in the third and fourth lines they just discriminate between the two singlets of $SO(q+1) \times \mathcal{A}_q$. Also recall that, as given in Table II, \mathcal{A}_q and Q_q are absent for $q = 8$ and $q = 1$.

Therefore, with respect to $SO(q+1) \times \mathcal{A}_q$, one obtains:

- (i) two singlets [note that $(\mathbf{1}, \mathbf{1})_{\text{I}}$ is a singlet of $SO(q+1, 1) \times \mathcal{A}_q$, as well];
- (ii) one vector $(\mathbf{q} + \mathbf{1}, \mathbf{1})$;
- (iii) a (double) spinor $(\mathbf{Spin}(q+1), \mathbf{Spin}(Q_q))$.

The representation decomposition (A5) yields that d^{ijk} , the rank-3 completely symmetric G_5 -invariant tensor [namely, the unique singlet in the tensor product $(\mathbf{R}_Q)^3$], decomposes in such a way that $(\mathbf{1}, \mathbf{1})_{\text{II}}$ and $(\mathbf{q} + \mathbf{1}, \mathbf{1})$ have the same couplings inside $(\mathbf{R}_Q)^3$.

Details concerning the various magic symmetric RSG’s are given further below.

The position which solves [with maximal—compact—symmetry $SO(q+1) \times \mathcal{A}_q$] the small lightlike G_5 -invariant defining constraints (4.51) in bare charges (and in a way consistent with an orbit representative having $Z \neq 0$) reads as follows:

$$\begin{aligned} q_{(\mathbf{1}, \mathbf{1})_{\text{I}}} &= 0, & q_{(\mathbf{q} + \mathbf{1}, \mathbf{1})} &= 0, \\ q_{(\mathbf{Spin}(q+1), \mathbf{Spin}(Q_q))} &= 0, & q_{(\mathbf{1}, \mathbf{1})_{\text{II}}} &\neq 0. \end{aligned} \quad (\text{A7})$$

Since $SO(q+1) \times \mathcal{A}_q$ is the unique group maximally (and symmetrically) embedded into $G_6 \times \mathcal{A}_q \times SO(1, 1)$ which has $SO(q+1) \times \mathcal{A}_q$ as (in this case, improper) MCS, it follows that $SO(q+1) \times \mathcal{A}_q$ is also the maximal

semisimple symmetry of $\mathcal{O}_{\text{lightlike,BPS}}$, which is thus given by Eq. (4.14).

The origin of the non-semi-simple Abelian (namely, translational) factor $\mathbb{R}^{(\text{spin}(q+1), \text{spin}(Q_q))}$ in the stabilizer of $\mathcal{O}_{\text{lightlike,BPS}}$ will be explained through the procedure of suitable Inönü-Wigner contraction performed in Appendix B 1.

b. $\mathcal{O}_{\text{critical,BPS}}$

Equation (A4) and subsequent ones are also relevant for the resolution of the small critical G_5 -invariant defining constraints (4.55) in bare charges in a way consistent with an orbit representative having $Z \neq 0$ (which is the unique possible case; see treatment above). In this case, the position which solves (with maximal—noncompact—symmetry $G_6 \times \mathcal{A}_q$) the constraints (4.55) in bare charges reads as follows:

$$\begin{aligned} q_{(\mathbf{1}, \mathbf{1})_{\text{II}}} &= 0, & q_{(\mathbf{q}+1, \mathbf{1})} &= 0, \\ q_{(\text{Spin}(q+1), \text{Spin}(Q_q))} &= 0, & q_{(\mathbf{1}, \mathbf{1})_{\text{I}}} &\neq 0. \end{aligned} \quad (\text{A8})$$

At least for the relevant values $q = 8, 4, 2, 1$, it holds that $\text{spin}(q+2) = \text{spin}(q+1)$ [recall definition (4.17)]. Therefore, since

$$q_{(\mathbf{1}, \mathbf{1})_{\text{II}}} = 0, \quad q_{(\mathbf{q}+1, \mathbf{1})} = 0 \Leftrightarrow q_{(\mathbf{q}+2, \mathbf{1})} = 0, \quad (\text{A9})$$

it follows that the position (A8) exhibits maximal—noncompact—symmetry $G_6 \times \mathcal{A}_q$, which is then the maximal semisimple symmetry of $\mathcal{O}_{\text{critical,BPS}}$, which is thus given by Eq. (4.29).

The origin of $\mathbb{R}^{(\text{spin}(q+2), \text{spin}(Q_q))}$ in the stabilizer of $\mathcal{O}_{\text{critical,BPS}}$ will be explained through the procedure of suitable $SO(1, 1)$ (three-)grading performed in Appendix B 2.

c. $\mathcal{O}_{\text{lightlike}, n\text{BPS}}$

In order to solve the small *lightlike* G_5 -invariant defining constraints (4.51) in bare charges in a way consistent with an orbit representative having $Z = 0$, the embedding (A1) has to be further elaborated as follows:

$$\begin{aligned} G_5 \supseteq_{\text{max}} G_6 \times \mathcal{A}_q \times SO(1, 1) \supseteq_{\text{max}} SO(q, 1) \times \mathcal{A}_q \\ \times SO(1, 1) \supseteq_{\text{MCS}} SO(q) \times \mathcal{A}_q. \end{aligned} \quad (\text{A10})$$

Thus, under the branching (A10) the irrepr. \mathbf{R}_Q decomposes as follows:

$$\begin{aligned} \mathbf{R}_Q &\rightarrow (\mathbf{1}, \mathbf{1})_{+4} + (\mathbf{q} + \mathbf{2}, \mathbf{1})_{-2} \\ &\quad + (\mathbf{Spin}(q+2), \mathbf{Spin}(Q_q))_{+1} \\ &\rightarrow (\mathbf{1}, \mathbf{1})_{+4} + (\mathbf{q} + \mathbf{1}, \mathbf{1})_{-2} + (\mathbf{1}, \mathbf{1})_{-2} \\ &\quad + (\mathbf{Spin}(q+1), \mathbf{Spin}(Q_q))_{+1} \\ &\rightarrow (\mathbf{1}, \mathbf{1})_{\text{I}} + (\mathbf{q}, \mathbf{1}) + (\mathbf{1}, \mathbf{1})_{\text{III}} + (\mathbf{1}, \mathbf{1})_{\text{II}} \\ &\quad + (\mathbf{Spin}'(q), \mathbf{Spin}(Q_q)) + (\mathbf{Spin}''(q), \mathbf{Spin}(Q_q)), \end{aligned} \quad (\text{A11})$$

where, besides the obvious irrepr. decompositions determining the last two lines of (A11), one should recall that

$$\begin{aligned} (\mathbf{Spin}(q+1), \mathbf{Spin}(Q_q)) &\rightarrow (\mathbf{Spin}'(q), \mathbf{Spin}(Q_q)) \\ &\quad + (\mathbf{Spin}''(q), \mathbf{Spin}(Q_q)), \end{aligned} \quad (\text{A12})$$

where the primes discriminate between the two spinor irreprs. of $SO(q) \times \mathcal{A}_q$. The branching of electric charges corresponding to (A11) reads

$$\begin{aligned} q_i \rightarrow (q_{(\mathbf{1}, \mathbf{1})_{\text{I}}}, q_{(\mathbf{1}, \mathbf{1})_{\text{II}}}, q_{(\mathbf{1}, \mathbf{1})_{\text{III}}}, q_{(\mathbf{q}, \mathbf{1})}, q_{(\mathbf{Spin}'(q), \mathbf{Spin}(Q_q))}, \\ q_{(\mathbf{Spin}''(q), \mathbf{Spin}(Q_q))}). \end{aligned} \quad (\text{A13})$$

In the first four lines of (A11) subscripts denote the weight with respect to $SO(1, 1)$, whereas in the fifth and sixth lines they just discriminate between the three singlets of $SO(q) \times \mathcal{A}_q$.

Therefore, with respect to $SO(q) \times \mathcal{A}_q$, one obtains

- (i) three singlets [notice that $(\mathbf{1}, \mathbf{1})_{\text{I}}$ is also a singlet of $SO(q, 1) \times \mathcal{A}_q$ and of $G_6 \times \mathcal{A}_q$, and that $(\mathbf{1}, \mathbf{1})_{\text{II}}$ is a singlet of $SO(q, 1) \times \mathcal{A}_q$, as well];
- (ii) a vector $(\mathbf{q}, \mathbf{1})$;
- (iii) two (double) spinors $(\mathbf{Spin}'(q), \mathbf{Spin}(Q_q))$ and $(\mathbf{Spin}''(q), \mathbf{Spin}(Q_q))$.

As a feature peculiar to (A11), the vector $(\mathbf{q}, \mathbf{1})$ and the two (double) spinors $(\mathbf{Spin}'(q), \mathbf{Spin}(Q_q))$ and $(\mathbf{Spin}''(q), \mathbf{Spin}(Q_q))$ do exhibit a “triatlity symmetry,” realized differently depending on $q = 8, 4, 2, 1$, as given in Appendix A 1.

The representation decomposition (A11) yields that d^{ijk} decomposes in such a way that the manifest “triatlity” exhibited by the branching of \mathbf{R}_Q is removed, and the two (double) spinors are put on a different footing with respect to the vector. As a consequence,

- (i) $(\mathbf{1}, \mathbf{1})_{\text{II}}$, $(\mathbf{1}, \mathbf{1})_{\text{III}}$, and $(\mathbf{q}, \mathbf{1})$;
- (ii) $(\mathbf{Spin}'(q), \mathbf{Spin}(Q_q))$ and $(\mathbf{Spin}''(q), \mathbf{Spin}(Q_q))$

separately have the same couplings inside $(\mathbf{R}_Q)^3$.

The position which solves [with maximal—compact—symmetry $SO(q) \times \mathcal{A}_q$] the small lightlike G_5 -invariant defining constraints (4.51) in bare charges (and in a way consistent with an orbit representative having $Z = 0$) reads as follows:

$$\begin{aligned}
q_{(\mathbf{q},1)} = 0, \quad q_{(\text{Spin}'(q), \text{Spin}(Q_q))} = 0, \\
q_{(\text{Spin}''(q), \text{Spin}(Q_q))} = 0,
\end{aligned} \tag{A14}$$

with the three singlets $q_{(\mathbf{1},1)_I}$, $q_{(\mathbf{1},1)_{II}}$, and $q_{(\mathbf{1},1)_{III}}$ constrained by

$$q_{(\mathbf{1},1)_I} \left[\begin{array}{c} d_{(\mathbf{1},1)_I(\mathbf{1},1)_{II}(\mathbf{1},1)_{III}} q_{(\mathbf{1},1)_{II}}^2 \\ + 2d_{(\mathbf{1},1)_I(\mathbf{1},1)_{II}(\mathbf{1},1)_{III}} q_{(\mathbf{1},1)_{II}} q_{(\mathbf{1},1)_{III}} \\ + d_{(\mathbf{1},1)_I(\mathbf{1},1)_{III}(\mathbf{1},1)_{III}} q_{(\mathbf{1},1)_{III}}^2 \end{array} \right] = 0. \tag{A15}$$

Notice that in (A14) the charges related to the vector and to the two (double) spinors are on equal footing, thus exhibiting a triality symmetry, as already mentioned above.

Notice that $SO(q, 1) \times \mathcal{A}_q$ is the unique group which is maximally [if one considers also the factor $SO(1, 1)$] and symmetrically embedded into $G_6 \times \mathcal{A}_q \times SO(1, 1)$, and also which has $SO(q) \times \mathcal{A}_q$ as MCS. Therefore, it follows that $SO(q, 1) \times \mathcal{A}_q$ is also the maximal semisimple symmetry of $\mathcal{O}_{\text{lightlike}, n\text{BPS}}$, which is thus given by Eq. (4.24).

As mentioned above, the origin of $\mathbb{R}^{(\text{spin}(q+1), \text{spin}(Q_q))}$ in the stabilizer of $\mathcal{O}_{\text{lightlike}, n\text{BPS}}$ will be explained through the procedure of suitable İnönü-Wigner contraction performed in Appendix B 1.

d. Details

We now give some details of the treatment of symmetric magic RSG.

We start by giving the explicit form of Eqs. (A4) and (A5) for all $q = 8, 4, 2, 1$ classifying symmetric magic RSG.

(i) $q = 8$ ($J_3^{\mathbb{O}}$):

$$\begin{aligned}
E_{6(-26)} \underset{\text{max}}{\supseteq} SO(9, 1) \times SO(1, 1) \underset{\text{MCS}}{\supseteq} SO(9), \\
\mathbf{27} \rightarrow \mathbf{1}_{+4} + \mathbf{10}_{-2} + \mathbf{16}_{+1} \rightarrow \mathbf{1}_I + \mathbf{9} + \mathbf{1}_{II} + \mathbf{16}.
\end{aligned} \tag{A16}$$

(ii) $q = 4$ ($J_3^{\mathbb{H}}$) [$SO(5, 1) \sim SU^*(4)$, $SO(5) \sim USp(4)$]:

$$\begin{aligned}
SU^*(6) \underset{\text{max}}{\supseteq} SO(5, 1) \times SO(3) \times SO(1, 1) \\
\underset{\text{MCS}}{\supseteq} SO(5) \times SO(3), \\
\mathbf{15} \rightarrow (\mathbf{1}, \mathbf{1})_{+4} + (\mathbf{6}, \mathbf{1})_{-2} + (\mathbf{4}, \mathbf{2})_{+1} \\
\rightarrow (\mathbf{1}, \mathbf{1})_I + (\mathbf{5}, \mathbf{1}) + (\mathbf{1}, \mathbf{1})_{II} + (\mathbf{4}, \mathbf{2}).
\end{aligned} \tag{A17}$$

(iii) $q = 2$ ($J_3^{\mathbb{C}}$) [$SL(2, \mathbb{C}) \sim SO(3, 1)$, $GL(1, \mathbb{C}) \sim SO(2) \times SO(1, 1)$]:

$$\begin{aligned}
SL(3, \mathbb{C}) \underset{\text{max}}{\supseteq} SL(2, \mathbb{C}) \times SL(1, \mathbb{C}) \times GL(1, \mathbb{C}) \\
\underset{\text{MCS}}{\supseteq} SO(3) \times SO(2),
\end{aligned}$$

$$\begin{aligned}
\mathbf{9} \rightarrow (\mathbf{1}_0)_{+4} + (\mathbf{3}_0 + \mathbf{1}_0)_{-2} + (\mathbf{2}_3 + \bar{\mathbf{2}}_{-3})_{+1} \\
\rightarrow (\mathbf{1}_0)_I + \mathbf{3}_0 + (\mathbf{1}_0)_{II} + \mathbf{2}_3 + \bar{\mathbf{2}}_{-3},
\end{aligned} \tag{A18}$$

where the first subscript in the second step and the subscript in the last step denote charges with respect to (w.r.t.) $SO(2) \sim U(1)$, and the second subscript in the second step denotes weights w.r.t. $SO(1, 1)$. In order to derive (A18), the decompositions of the irreprs. of $SL(3, \mathbb{C})$ under $SL(2, \mathbb{C}) \times SL(1, \mathbb{C}) \times GL(1, \mathbb{C}) \sim SL(2, \mathbb{C}) \times SO(2) \times SO(1, 1)$ have been recalled [the charges and weights w.r.t. $SO(2)$ and $SO(1, 1)$ are given]:

$$\mathbf{3} \rightarrow (\mathbf{2}, \mathbf{1}, -1) + (\mathbf{1}, -2, \mathbf{2}), \tag{A19}$$

$$\bar{\mathbf{3}} \rightarrow (\bar{\mathbf{2}}, -1, -1) + (\mathbf{1}, \mathbf{2}, \mathbf{2}), \tag{A20}$$

$$\mathbf{3}' \rightarrow (\mathbf{2}, -1, \mathbf{1}) + (\mathbf{1}, \mathbf{2}, -2), \tag{A21}$$

$$\bar{\mathbf{3}}' \rightarrow (\bar{\mathbf{2}}, \mathbf{1}, \mathbf{1}) + (\mathbf{1}, -2, -2). \tag{A22}$$

Thus, through (A19) and (A20), the irrepre.

$$\mathbf{R}_{q=2} = \mathbf{9} \equiv \mathbf{3} \times \bar{\mathbf{3}} \tag{A23}$$

branches as given by (A18).

(iv) $q = 1$ ($J_3^{\mathbb{R}}$) [$SL(2, \mathbb{R}) \sim SO(2, 1)$]:

$$\begin{aligned}
SL(3, \mathbb{R}) \underset{\text{max}}{\supseteq} SO(2, 1) \times SO(1, 1) \underset{\text{MCS}}{\supseteq} SO(2), \\
\mathbf{6}' \rightarrow \mathbf{1}_{+4} + \mathbf{3}_{-2} + \mathbf{2}_{+1} \rightarrow \mathbf{1}_I + \mathbf{2} + \mathbf{1}_{II} + \mathbf{2},
\end{aligned} \tag{A24}$$

where the normalizations and conventions of Table 58 of [58] have been adopted.

Next, we write down Eqs. (A10) and (A11) for all $q = 8, 4, 2, 1$ classifying symmetric magic RSG.

(i) $q = 8$ ($J_3^{\mathbb{O}}$):

$$\begin{aligned}
E_{6(-26)} \underset{\text{max}}{\supseteq} SO(9, 1) \times SO(1, 1) \underset{\text{max}}{\supseteq} SO(8, 1) \\
\times SO(1, 1) \underset{\text{MCS}}{\supseteq} SO(8), \\
\mathbf{27} \rightarrow \mathbf{1}_{+4} + \mathbf{10}_{-2} + \mathbf{16}_{+1} \rightarrow \mathbf{1}_{+4} + \mathbf{9}_{-2} + \mathbf{1}_{-2} \\
+ \mathbf{16}_{+1} \rightarrow \mathbf{1}_I + \mathbf{8}_v + \mathbf{1}_{III} + \mathbf{1}_{II} + \mathbf{8}_s + \mathbf{8}_c.
\end{aligned} \tag{A25}$$

The triality in irreprs. of $SO(q)$ is implemented here through the triality of $(\mathbf{8}_v, \mathbf{8}_s, \mathbf{8}_c)$ of $SO(8)$.

(ii) $q = 4 (J_3^{\mathbb{H}})$:

$$\begin{aligned}
SU^*(6) &\underset{\max}{\supseteq} SO(5, 1) \times SO(3) \times SO(1, 1) \\
&\underset{\max}{\supseteq} SO(4, 1) \times SO(3) \times SO(1, 1) \\
&\underset{\text{MCS}}{\supseteq} SO(4) \times SO(3) \sim SU(2) \\
&\quad \times SU(2) \times SU(2), \tag{A26}
\end{aligned}$$

$$\begin{aligned}
\mathbf{15} &\rightarrow (\mathbf{1}, \mathbf{1})_{+4} + (\mathbf{6}, \mathbf{1})_{-2} + (\mathbf{4}, \mathbf{2})_{+1} \\
&\rightarrow (\mathbf{1}, \mathbf{1})_{+4} + (\mathbf{5}, \mathbf{1})_{-2} + (\mathbf{1}, \mathbf{1})_{-2} + (\mathbf{4}, \mathbf{2})_{+1} \\
&\rightarrow (\mathbf{1}, \mathbf{1}, \mathbf{1})_{\text{I}} + (\mathbf{2}, \mathbf{2}, \mathbf{1}) + (\mathbf{1}, \mathbf{1}, \mathbf{1})_{\text{III}} + (\mathbf{1}, \mathbf{1}, \mathbf{1})_{\text{II}} \\
&\quad + (\mathbf{1}, \mathbf{2}, \mathbf{2}) + (\mathbf{2}, \mathbf{1}, \mathbf{2}). \tag{A27}
\end{aligned}$$

Thus, the triality in irreprs. of $SO(q) \times \mathcal{A}_q$ is implemented for $q = 4$ through the triality of $((\mathbf{2}, \mathbf{2}, \mathbf{1}), (\mathbf{2}, \mathbf{1}, \mathbf{2}), (\mathbf{1}, \mathbf{2}, \mathbf{2}))$ of $SU(2) \times SU(2) \times SU(2)$.

(iii) $q = 2 (J_3^{\mathbb{C}})$:

$$\begin{aligned}
SL(3, \mathbb{C}) &\underset{\max}{\supseteq} SL(2, \mathbb{C}) \times SL(1, \mathbb{C}) \times GL(1, \mathbb{C}) \\
&\underset{\max}{\supseteq} SO(2, 1) \times SO(2) \times SO(1, 1) \\
&\underset{\text{MCS}}{\supseteq} SO(2) \times SO(2), \tag{A28}
\end{aligned}$$

$$\begin{aligned}
\mathbf{9} &\rightarrow (\mathbf{1}_0)_{+4} + (\mathbf{3}_0 + \mathbf{1}_0)_{-2} + (\mathbf{2}_3 + \bar{\mathbf{2}}_{-3})_{+1} \\
&\rightarrow (\mathbf{1}_0)_{+4} + (\mathbf{3}_0)_{-2} + (\mathbf{1}_0)_{-2} + (\mathbf{2}_3)_{+1} \\
&\quad + (\bar{\mathbf{2}}_{-3})_{+1} \\
&\rightarrow (\mathbf{1}_0)_{\text{I}} + \mathbf{2}_0 + (\mathbf{1}_0)_{\text{III}} + (\mathbf{1}_0)_{\text{II}} + \mathbf{2}_3 + \mathbf{2}_{-3}. \tag{A29}
\end{aligned}$$

Thus, the triality in irreprs. of $SO(q) \times \mathcal{A}_q$ is implemented for $q = 2$ through the triality of $(\mathbf{2}_0, \mathbf{2}_3, \bar{\mathbf{2}}_{-3})$ of $SO(2) \times SO(2)$ [notice the different charges w.r.t. $\mathcal{A}_{q=2} = SO(2) \sim U(1)$].

(iv) $q = 1 (J_3^{\mathbb{R}})$:

$$\begin{aligned}
SL(3, \mathbb{R}) &\underset{\max}{\supseteq} SO(2, 1) \times SO(1, 1) \\
&\underset{\max}{\supseteq} SO(1, 1) \times SO(1, 1) \underset{\text{MCS}}{\supseteq} \mathbf{1}, \\
\mathbf{6}' &\rightarrow \mathbf{1}_{+4} + \mathbf{3}_{-2} + \mathbf{2}_{+1} \rightarrow \mathbf{1}_{+4} + \mathbf{2}_{-2} \\
&\quad + \mathbf{1}_{-2} + \mathbf{2}_{+1} \\
&\rightarrow \mathbf{1}_{\text{I}} + \mathbf{1}_{\text{II}} + \mathbf{1}_{\text{III}} + \mathbf{1}_{\text{IV}} + \mathbf{1}_{\text{V}} + \mathbf{1}_{\text{VI}}, \tag{A30}
\end{aligned}$$

where in the second line “1” denotes the identity element. Notice that there is no compact symmetry in $\mathcal{O}_{\text{lightlike}, n\text{BPS}, J_3^{\mathbb{R}}, d=5}$, as also given by the fact that $\mathcal{M}_{\text{lightlike}, n\text{BPS}, J_3^{\mathbb{R}}, d=5} = SO(1, 1) \rtimes \mathbb{R}^2$ (see Table III).

Thus, the triality of irreprs. of $SO(q)$ in this case trivially degenerates into a “sexuality” [six singlets in the last line of (A30)].

2. Dressed charge basis

Concerning the resolution of the G_5 -invariant (sets of) constraints in the basis of dressed charges, one should notice that for each of the four magic symmetric RSG’s a unique noncompact, real form \tilde{H}_5 of the compact group $H_5 \equiv \text{MCS}(G_5)$ exists with maximal symmetric embedding into G_5 (see e.g. [57]; also recall Sec. III D and Table I):

$$G_5 \underset{\max}{\supseteq} \tilde{H}_5. \tag{A31}$$

a. $\mathcal{O}_{\text{lightlike}, \text{BPS}}$

In order to solve the small *lightlike* G_5 -invariant defining constraints (4.51) in dressed charges in a way consistent with an orbit representative with $Z \neq 0$, let us further embed

$$\tilde{h}_5 \equiv \text{MCS}(\tilde{H}_5) = SO(q+1) \times \mathcal{A}_q, \tag{A32}$$

thus obtaining

$$G_5(\underset{\max}{\supseteq} \tilde{H}_5) \underset{\text{MCS}}{\supseteq} SO(q+1) \times \mathcal{A}_q, \tag{A33}$$

where the brackets denote the auxiliary nature of the embedding. Thus, under the branching (A33) \mathbf{R}_Q decomposes as follows:

$$\begin{aligned}
\mathbf{R}_Q(\rightarrow \mathbf{1} + \hat{\mathbf{R}}) &\rightarrow (\mathbf{1}, \mathbf{1})_{\text{I}} + (\mathbf{q} + \mathbf{1}, \mathbf{1}) \\
&\quad + (\mathbf{Spin}(q+1), \mathbf{Spin}(Q_q)) + (\mathbf{1}, \mathbf{1})_{\text{II}}, \tag{A34}
\end{aligned}$$

where $\hat{\mathbf{R}}$ is an irrepr. of \tilde{H}_5 used as an intermediate step. Equation (A34) corresponds to the branching

$$Z \equiv (Z, Z_x) \rightarrow (Z, Z_{(\mathbf{1}, \mathbf{1})_{\text{II}}}, Z_{(\mathbf{q} + \mathbf{1}, \mathbf{1})}, Z_{(\mathbf{Spin}(q+1), \mathbf{Spin}(Q_q))}), \tag{A35}$$

where

$$Z_{(\mathbf{1}, \mathbf{1})_{\text{I}}} \equiv Z \tag{A36}$$

throughout. Therefore, with respect to $SO(q+1) \times \mathcal{A}_q$, one obtains

- (i) two singlets;
- (ii) one vector $(\mathbf{q} + \mathbf{1}, \mathbf{1})$;
- (iii) one (double) spinor $(\mathbf{Spin}(q+1), \mathbf{Spin}(Q_q))$.

The position which solves [with maximal—compact—symmetry $SO(q+1) \times \mathcal{A}_q$] the small lightlike G_5 -invariant defining constraints (4.51) in dressed charges (and in a way consistent with an orbit representative having $Z \neq 0$) reads as follows:

$$Z_{(\mathbf{q}+1, \mathbf{1})} = 0, \quad Z_{(\mathbf{Spin}(q+1), \mathbf{Spin}(Q_q))} = 0, \quad (\text{A37}) \quad (\text{iv}) \quad q = 1 (J_3^{\mathbb{R}}):$$

with Z and $Z_{(\mathbf{1}, \mathbf{1})_{\text{II}}}$ constrained by

$$Z^3 - \left(\frac{3}{2}\right)^2 ZZ_{(\mathbf{1}, \mathbf{1})_{\text{II}}}^2 - \left(\frac{3}{2}\right)^{3/2} T_{(\mathbf{1}, \mathbf{1})_{\text{II}}(\mathbf{1}, \mathbf{1})_{\text{II}}(\mathbf{1}, \mathbf{1})_{\text{II}}} Z_{(\mathbf{1}, \mathbf{1})_{\text{II}}}^3 = 0. \quad (\text{A38})$$

Notice that $SO(q+1) \times \mathcal{A}_q$ is the unique group which is maximally (and symmetrically) embedded into \tilde{H}_5 and which has $SO(q+1) \times \mathcal{A}_q$ as (in this case, improper) MCS [actually, $SO(q+1) \times \mathcal{A}_q = \text{MCS}(\tilde{H}_5)$]. Therefore, it follows that $SO(q+1) \times \mathcal{A}_q$ is also the maximal semisimple symmetry of $\mathcal{O}_{\text{lightlike, BPS}}$, which is thus given by Eq. (4.14).

The explicit form of Eqs. (A33) and (A34) for all $q = 8, 4, 2, 1$ classifying symmetric magic RSG is given below.

(i) $q = 8 (J_3^{\mathbb{O}})$:

$$E_{6(-26)} \left(\underset{\text{max}}{\supseteq} F_{4(-20)} \underset{\text{max}}{\supseteq}^{\text{MCS}} SO(9) \right), \quad (\text{A39})$$

$$27(\rightarrow \mathbf{1} + \mathbf{26}) \rightarrow \mathbf{1}_I + \mathbf{9} + \mathbf{16} + \mathbf{1}_{\text{II}}.$$

(ii) $q = 4 (J_3^{\mathbb{H}})$:

$$SU^*(6) \left(\underset{\text{max}}{\supseteq} USp(4, 2) \underset{\text{max}}{\supseteq}^{\text{MCS}} USp(4) \times USp(2) \right)$$

$$\sim SO(5) \times SO(3),$$

$$15(\rightarrow \mathbf{1} + \mathbf{14}) \rightarrow (\mathbf{1}, \mathbf{1})_I + (\mathbf{5}, \mathbf{1})$$

$$+ (\mathbf{4}, \mathbf{2}) + (\mathbf{1}, \mathbf{1})_{\text{II}}. \quad (\text{A40})$$

(iii) $q = 2 (J_3^{\mathbb{C}})$:

$$SL(3, \mathbb{C}) \left(\underset{\text{max}}{\supseteq} SU(2, 1) \underset{\text{max}}{\supseteq}^{\text{MCS}} SU(2) \times U(1) \right)$$

$$\sim SO(3) \times SO(2),$$

$$9(\rightarrow \mathbf{1} + \mathbf{8}) \rightarrow (\mathbf{1}_0)_I + \mathbf{2}_{-3} + \mathbf{2}_3$$

$$+ \mathbf{3}_0 + (\mathbf{1}_0)_{\text{II}}. \quad (\text{A41})$$

$$SL(3, \mathbb{R}) \left(\underset{\text{max}}{\supseteq} SO(2, 1) \underset{\text{max}}{\supseteq}^{\text{MCS}} SO(2) \right),$$

$$6'(\rightarrow \mathbf{1} + \mathbf{5}) \rightarrow \mathbf{1}_I + \mathbf{2} + \mathbf{2} + \mathbf{1}_{\text{II}}. \quad (\text{A42})$$

As mentioned in the resolution in the basis of bare (electric) charges q_i , the origin of $\mathbb{R}^{(\text{spin}(q+1), \text{spin}(Q_q))}$ in the stabilizer of $\mathcal{O}_{\text{lightlike, BPS}}$ will be explained through the procedure of suitable İnönü-Wigner contraction performed in Appendix B 1.

b. $\mathcal{O}_{\text{lightlike, } n\text{BPS}}$

In order to solve the small lightlike G_5 -invariant defining constraints (4.51) in dressed charges in a way consistent with an orbit representative having $Z = 0$, the embedding (A31) has to be further elaborated as follows:

$$G_5 \left(\underset{\text{max}}{\supseteq} \tilde{H}_5 \right) \underset{\text{max}}{\supseteq} \hat{h}_5 \underset{\text{max}}{\supseteq}^{\text{MCS}} SO(q) \times \mathcal{A}_q, \quad (\text{A43})$$

where

$$\hat{h}_5 = SO(q, 1) \times \mathcal{A}_q \quad (\text{A44})$$

is the unique noncompact form of \tilde{h}_5 [defined by (A32)] to be embedded maximally and symmetrically into \tilde{H}_5 (see e.g. [57]).

Thus, under the branching (A43), \mathbf{R}_Q decomposes as follows:

$$\mathbf{R}_Q(\rightarrow \mathbf{1} + \hat{\mathbf{R}}) \rightarrow (\mathbf{1}, \mathbf{1})_I + (\mathbf{q} + \mathbf{1}, \mathbf{1})$$

$$+ (\mathbf{Spin}(q+1), \mathbf{Spin}(Q_q)) + (\mathbf{1}, \mathbf{1})_{\text{II}}$$

$$\rightarrow (\mathbf{1}, \mathbf{1})_I + (\mathbf{q}, \mathbf{1}) + (\mathbf{1}, \mathbf{1})_{\text{III}}$$

$$+ (\mathbf{Spin}'(q), \mathbf{Spin}(Q_q))$$

$$+ (\mathbf{Spin}''(q), \mathbf{Spin}(Q_q)) + (\mathbf{1}, \mathbf{1})_{\text{II}}. \quad (\text{A45})$$

Equation (A45) corresponds to the branching [recall Eq. (A36)]

$$Z \equiv (Z, Z_x) \rightarrow (Z, Z_{(\mathbf{1}, \mathbf{1})_{\text{II}}}, Z_{(\mathbf{1}, \mathbf{1})_{\text{III}}}, Z_{(\mathbf{q}, \mathbf{1})}, Z_{(\mathbf{Spin}'(q), \mathbf{Spin}(Q_q))}, Z_{(\mathbf{Spin}''(q), \mathbf{Spin}(Q_q))}). \quad (\text{A46})$$

Therefore, with respect to $SO(q) \times \mathcal{A}_q$, besides Z , one obtains

- (i) two singlets [note that $(\mathbf{1}, \mathbf{1})_{\text{II}}$ is a singlet of $SO(q, 1) \times \mathcal{A}_q$, as well];
- (ii) one vector $(\mathbf{q}, \mathbf{1})$;
- (iii) two (double) spinors $(\mathbf{Spin}'(q), \mathbf{Spin}(Q_q))$ and $(\mathbf{Spin}''(q), \mathbf{Spin}(Q_q))$.

The position which solves [with maximal—compact—symmetry $SO(q) \times \mathcal{A}_q$] the small lightlike G_5 -invariant

defining constraints (4.50) in dressed charges (and in a way consistent with an orbit representative having $Z = 0$) reads as follows:

$$Z \equiv Z_{(\mathbf{1}, \mathbf{1})_I} = 0, \quad Z_{(\mathbf{q}, \mathbf{1})} = 0, \quad (\text{A47})$$

$$q_{(\mathbf{Spin}'(q), \mathbf{Spin}(Q_q))} = 0, \quad q_{(\mathbf{Spin}''(q), \mathbf{Spin}(Q_q))} = 0,$$

with the two singlets $Z_{(\mathbf{1}, \mathbf{1})_{\text{II}}}$ and $Z_{(\mathbf{1}, \mathbf{1})_{\text{III}}}$ constrained by

$$T_{(\mathbf{1},\mathbf{1})_{\text{II}}(\mathbf{1},\mathbf{1})_{\text{II}}(\mathbf{1},\mathbf{1})_{\text{II}}} Z_{(\mathbf{1},\mathbf{1})_{\text{II}}}^2 + 3T_{(\mathbf{1},\mathbf{1})_{\text{II}}(\mathbf{1},\mathbf{1})_{\text{III}}(\mathbf{1},\mathbf{1})_{\text{III}}} Z_{(\mathbf{1},\mathbf{1})_{\text{III}}}^2 = 0. \quad (\text{A48})$$

Besides $SO(q+1) \times \mathcal{A}_q$, the only other group which is maximally (and symmetrically) embedded into \tilde{H}_5 and which has $SO(q) \times \mathcal{A}_q$ as MCS is $SO(q,1) \times \mathcal{A}_q$. Therefore, $SO(q,1) \times \mathcal{A}_q$ is also the maximal semisimple symmetry of $\mathcal{O}_{\text{lightlike,BPS}}$, which is thus given by Eq. (4.24).

The explicit form of Eqs. (A43)–(A45) for all $q = 8, 4, 2, 1$ classifying symmetric magic RSG is given below.

(i) $q = 8$ (J_3^{O}):

$$\begin{aligned} E_{6(-26)}(\underset{\text{max}}{\supseteq} F_{4(-20)}) \underset{\text{max}}{\supseteq} SO(8,1) \underset{\text{max}}{\supseteq}^{MCS} SO(8), \\ \mathbf{27}(\rightarrow \mathbf{1} + \mathbf{26}) \rightarrow \mathbf{1}_I + \mathbf{9} + \mathbf{16} + \mathbf{1}_{\text{II}} \\ \rightarrow \mathbf{1}_I + \mathbf{8}_v + \mathbf{1}_{\text{III}} + \mathbf{1}_{\text{II}} + \mathbf{8}_s + \mathbf{8}_c. \end{aligned} \quad (\text{A49})$$

(ii) $q = 4$ (J_3^{H}) [$USp(2,2) \sim SO(5,1)$, $USp(2) \sim SU(2)$]:

$$\begin{aligned} SU^*(6)(\underset{\text{max}}{\supseteq} USp(4,2)) \underset{\text{max}}{\supseteq} USp(2,2) \times USp(2) \\ \underset{\text{max}}{\supseteq}^{MCS} USp(2) \times USp(2) \\ \times USp(2), \end{aligned} \quad (\text{A50})$$

$$\begin{aligned} \mathbf{15}(\rightarrow \mathbf{1} + \mathbf{14}) \rightarrow (\mathbf{1}, \mathbf{1})_I + (\mathbf{5}, \mathbf{1}) + (\mathbf{4}, \mathbf{2}) + (\mathbf{1}, \mathbf{1})_{\text{II}} \\ \rightarrow (\mathbf{1}, \mathbf{1}, \mathbf{1})_I + (\mathbf{1}, \mathbf{1}, \mathbf{1})_{\text{III}} \\ + (\mathbf{2}, \mathbf{2}, \mathbf{1}) + (\mathbf{2}, \mathbf{1}, \mathbf{2}) \\ + (\mathbf{1}, \mathbf{2}, \mathbf{2}) + (\mathbf{1}, \mathbf{1}, \mathbf{1})_{\text{II}}. \end{aligned} \quad (\text{A51})$$

(iii) $q = 2$ (J_3^{C}):

$$\begin{aligned} SL(3, \mathbb{C})(\underset{\text{max}}{\supseteq} SU(2,1)) \underset{\text{max}}{\supseteq} SU(1,1) \times U(1) \\ \underset{\text{max}}{\supseteq}^{MCS} U(1) \times U(1), \\ \mathbf{9}(\rightarrow \mathbf{1} + \mathbf{8}) \rightarrow (\mathbf{1}_0)_I + \mathbf{2}_3 + \mathbf{2}_{-3} + \mathbf{3}_0 \\ + (\mathbf{1}_0)_{\text{II}} \\ \rightarrow (\mathbf{1}_0)_I + \mathbf{2}_0 + \mathbf{2}_3 + \mathbf{2}_{-3} \\ + (\mathbf{1}_0)_{\text{III}} + (\mathbf{1}_0)_{\text{II}}. \end{aligned} \quad (\text{A52})$$

(iv) $q = 1$ (J_3^{R}):

$$\begin{aligned} SL(3, \mathbb{R})(\underset{\text{max}}{\supseteq} SO(2,1)) \underset{\text{max}}{\supseteq} SO(1,1) \underset{\text{max}}{\supseteq}^{MCS} \mathbf{1}, \\ \mathbf{6}'(\rightarrow \mathbf{1} + \mathbf{5}) \rightarrow \mathbf{1}_I + \mathbf{2} + \mathbf{1}_{\text{II}} + \mathbf{2} \\ \rightarrow \mathbf{1}_I + \mathbf{1}_{\text{II}} + \mathbf{1}_{\text{III}} + \mathbf{1}_{\text{IV}} \\ + \mathbf{1}_V + \mathbf{1}_{\text{VI}}, \end{aligned} \quad (\text{A53})$$

where “1” denotes the identity element.

The origin of $\mathbb{R}^{(\text{spin}(q+1), \text{spin}(Q_q))}$ in the stabilizer of $\mathcal{O}_{\text{lightlike,BPS}}$ will be explained through the procedure of suitable İnönü-Wigner contraction performed in Appendix B 1.

APPENDIX B: EQUIVALENT DERIVATIONS

In this appendix, we determine the general form of small charge orbits of symmetric magic RSG [see Eqs. (4.14), (4.24), and (4.29)] through suitable group theoretical procedures, namely,

- (i) İnönü-Wigner contractions, for small lightlike orbits (Appendix B 1),
- (ii) $SO(1,1)$ three-grading, for small critical orbits (Appendix B 2).

Such procedures will clarify the origin of the non-semisimple Abelian (namely, translational) factor [recall Eq. (4.1), definitions (4.17) and (4.18), and see Eq. (B41) below]

$$\mathcal{T} = \mathbb{R}^{(\text{spin}(q+1), \text{spin}(Q_q))} \quad (\text{B1})$$

in all three classes (lightlike BPS, lightlike non-BPS, and critical BPS) of small orbits (for each relevant $q = 8, 4, 2, 1$).

1. İnönü-Wigner contractions

a. $\mathcal{O}_{\text{lightlike,BPS}}$

In order to deal with $\mathcal{O}_{\text{lightlike,BPS}}$, we start from the group embedding (A33). This determines the following decompositions of irreprs. (Adj and Fund, respectively, denoting the adjoint and fundamental irrepr.):

$$\mathbf{A} \mathbf{d}j(G_5) \rightarrow \mathbf{A} \mathbf{d}j(\tilde{H}_5) + \mathbf{F} \mathbf{u} \mathbf{n} \mathbf{d}(\tilde{H}_5), \quad (\text{B2})$$

and further

$$\begin{aligned} \mathbf{A} \mathbf{d}j(\tilde{H}_5) \rightarrow (\mathbf{A} \mathbf{d}j(SO(q+1)), \mathbf{1}) + (\mathbf{1}, \mathbf{A} \mathbf{d}j(\mathcal{A}_q)) \\ + (\mathbf{S} \mathbf{p} \mathbf{i} \mathbf{n}(q+1), \mathbf{S} \mathbf{p} \mathbf{i} \mathbf{n}(Q_q))_{\text{I}}, \end{aligned} \quad (\text{B3})$$

$$\begin{aligned} \mathbf{F} \mathbf{u} \mathbf{n} \mathbf{d}(\tilde{H}_5) \rightarrow (\mathbf{1}, \mathbf{1}) + (\mathbf{q} + \mathbf{1}, \mathbf{1}) \\ + (\mathbf{S} \mathbf{p} \mathbf{i} \mathbf{n}(q+1), \mathbf{S} \mathbf{p} \mathbf{i} \mathbf{n}(Q_q))_{\text{II}}, \end{aligned} \quad (\text{B4})$$

where trivially $\mathbf{A} \mathbf{d}j(SO(q+1)) = \frac{\mathfrak{q}(q+1)}{2}$. Equations (B2)–(B4) thus imply

$$\begin{aligned}
\mathbf{Adj}(G_5) &\rightarrow (\mathbf{Spin}(q+1), \mathbf{Spin}(Q_q))_I \\
&+ (\mathbf{Adj}(SO(q+1)), \mathbf{1}) + (\mathbf{1}, \mathbf{Adj}(\mathcal{A}_q)) \\
&+ (\mathbf{1}, \mathbf{1}) + (\mathbf{q} + \mathbf{1}, \mathbf{1}) \\
&+ (\mathbf{Spin}(q+1), \mathbf{Spin}(Q_q))_{II}. \tag{B5}
\end{aligned}$$

The decomposition of the branching (B3) yields

$$\begin{aligned}
\underbrace{\mathbf{Adj}(\tilde{H}_5)}_{\mathfrak{g}_{\tilde{H}_5}} &\rightarrow \underbrace{(\mathbf{Adj}(SO(q+1)), \mathbf{1}) + (\mathbf{1}, \mathbf{Adj}(\mathcal{A}_q))}_{\mathfrak{h}_{\tilde{H}_5}} \\
&+ \underbrace{(\mathbf{Spin}(q+1), \mathbf{Spin}(Q_q))_I}_{\mathfrak{k}_{\tilde{H}_5}}. \tag{B6}
\end{aligned}$$

The coset [recall Eq. (3.62)]

$$\frac{\tilde{H}_5}{\text{MCS}(\tilde{H}_5)} = \frac{\tilde{H}_5}{SO(q+1) \times \mathcal{A}_q} = \mathcal{M}_{n\text{BPS}, \text{large}} \tag{B7}$$

is symmetric, with real dimension, Euclidean signature and character, respectively (see e.g. [57,59]; here ‘‘c’’ and ‘‘nc,’’ respectively, stand for ‘‘compact’’ and ‘‘non-compact’’):

$$\begin{aligned}
\dim_{\mathbb{R}} &= 2q, & (c, nc) &= (0, 2q), \\
\chi &\equiv c - nc = -2q. \tag{B8}
\end{aligned}$$

By definition, the symmetricity of $\mathcal{M}_{n\text{BPS}, \text{large}}$ implies that

$$\begin{aligned}
[\mathfrak{h}_{\tilde{H}_5}, \mathfrak{h}_{\tilde{H}_5}] &= \mathfrak{h}_{\tilde{H}_5}, & [\mathfrak{h}_{\tilde{H}_5}, \mathfrak{k}_{\tilde{H}_5}] &= \mathfrak{k}_{\tilde{H}_5}, \\
[\mathfrak{k}_{\tilde{H}_5}, \mathfrak{k}_{\tilde{H}_5}] &= \mathfrak{h}_{\tilde{H}_5}. \tag{B9}
\end{aligned}$$

The ‘‘decoupling’’ of $\mathfrak{h}_{\tilde{H}_5}$, with subsequent transformation of the irrepr. $(\mathbf{Spin}(q+1), \mathbf{Spin}(Q_q))_I$ of $SO(q+1) \times \mathcal{A}_q$ into the non-semisimple, Abelian (namely, translational) part of the stabilizer of $\mathcal{O}_{\text{lightlike}, \text{BPS}}$, is achieved by performing a uniform rescaling of the generators of $\mathfrak{k}_{\tilde{H}_5}$:

$$\mathfrak{k}_{\tilde{H}_5} \rightarrow \lambda \mathfrak{k}_{\tilde{H}_5}, \quad \lambda \in \mathbb{R}_0^+, \tag{B10}$$

and then by letting $\lambda \rightarrow \infty$. This amounts to performing an IW contraction [50,51] on $\mathfrak{k}_{\tilde{H}_5}$. Thus [recall Eqs. (4.14) and (4.16)]

$$\begin{aligned}
\text{IW}\left(\mathcal{O}_{n\text{BPS}, \text{large}} = \frac{G_5}{\tilde{H}_5}\right)^{(A.33)} &\rightarrow \mathcal{O}_{\text{lightlike}, \text{BPS}} \\
&= \frac{G_5}{(SO(q+1) \times \mathcal{A}_q) \rtimes \mathbb{R}^{(\text{spin}(q+1), \text{spin}(Q_q))}}, \tag{B11}
\end{aligned}$$

$$\mathcal{T}_{\text{lightlike}, \text{BPS}} \equiv \mathbb{R}^{(\text{spin}(q+1), \text{spin}(Q_q))}. \tag{B12}$$

Thus, $\mathcal{T}_{\text{lightlike}, \text{BPS}}$ given by (B12) is the $\mathfrak{k}_{\tilde{H}_5}$ part of the decomposition (B6) of the Lie algebra $\mathfrak{g}_{\tilde{H}_5}$ of \tilde{H}_5 with respect to $\text{MCS}(\tilde{H}_5) = SO(q+1) \times \mathcal{A}_q$, which then

gets ‘‘decoupled’’ from $\mathfrak{g}_{\tilde{H}_5}$ and Abelianized through the IW contraction procedure (B10) and (B11).

b. $\mathcal{O}_{\text{lightlike}, n\text{BPS}}$

On the other hand, the treatment of $\mathcal{O}_{\text{lightlike}, n\text{BPS}}$ requires one to start from the embedding (A43) [actually, without the last step involving $SO(q) \times \mathcal{A}_q = \text{MCS}(\hat{h}_5)$; recall Eq. (A44)]:

$$G_5 \supseteq_{\text{max}} \tilde{H}_5 \supseteq_{\text{max}} \hat{h}_5 = SO(q, 1) \times \mathcal{A}_q. \tag{B13}$$

The subsequent decompositions of $\mathbf{Adj}(G_5)$, $\mathbf{Adj}(\tilde{H}_5)$, and $\mathbf{Fund}(\tilde{H}_5)$ are given by Eqs. (B2)–(B4), respectively, thus yielding the same decomposition as in (B5). Consequently, the decomposition of the branching (B3) yields the same result as in (B6).

The coset [recall Eq. (3.62)]

$$\frac{\tilde{H}_5}{\hat{h}_5} = \frac{\tilde{H}_5}{SO(q, 1) \times \mathcal{A}_q} \tag{B14}$$

is symmetric, with real dimension, Euclidean signature and character, respectively:

$$\dim_{\mathbb{R}} = 2q, \quad (c, nc) = (q, q), \quad \chi \equiv c - nc = 0. \tag{B15}$$

By definition, the symmetricity of $\frac{\tilde{H}_5}{\hat{h}_5}$ implies the same relations as in (B9).

Thus, the decoupling of $\mathfrak{h}_{\tilde{H}_5}$, with subsequent transformation of the irrepr. $(\mathbf{Spin}(q+1), \mathbf{Spin}(Q_q))_I$ of $SO(q, 1) \times \mathcal{A}_q$ into the non-semi-simple, Abelian (namely, translational) part of the stabilizer of $\mathcal{O}_{\text{lightlike}, n\text{BPS}}$, is achieved by performing a uniform rescaling of the generators of $\mathfrak{k}_{\tilde{H}_5}$ as given by Eq. (B10), and then by letting $\lambda \rightarrow \infty$. This amounts to performing an IW contraction [50,51] on $\mathfrak{k}_{\tilde{H}_5}$. Therefore, one obtains [recall Eqs. (4.24) and (4.26)]

$$\begin{aligned}
\text{IW}\left(\mathcal{O}_{n\text{BPS}, \text{large}}\right)^{(A.43)} &\rightarrow \mathcal{O}_{\text{lightlike}, n\text{BPS}} \\
&= \frac{G_5}{(SO(q+1) \times \mathcal{A}_q) \rtimes \mathbb{R}^{(\text{spin}(q+1), \text{spin}(Q_q))}}, \tag{B16}
\end{aligned}$$

$$\mathcal{T}_{\text{lightlike}, n\text{BPS}} = \mathcal{T}_{\text{lightlike}, \text{BPS}} = \mathbb{R}^{(\text{spin}(q+1), \text{spin}(Q_q))}. \tag{B17}$$

Thus, $\mathcal{T}_{\text{lightlike}, n\text{BPS}}$ given by (B17) is the $\mathfrak{k}_{\tilde{H}_5}$ part of the decomposition (B6) of the Lie algebra $\mathfrak{g}_{\tilde{H}_5}$ of \tilde{H}_5 with respect to $\hat{h}_5 = SO(q, 1) \times \mathcal{A}_q$, which then gets decoupled from $\mathfrak{g}_{\tilde{H}_5}$ and Abelianized through the IW contraction procedure [see Eqs. (B10) and (B16)].

Note that the IW contraction does not change the dimension of the starting orbit. Indeed $\mathcal{O}_{\text{lightlike}, \text{BPS}}$, obtained through the IW contraction of $\mathcal{O}_{n\text{BPS}, \text{large}}$ along the branch-

ing (A33), has the same real dimension of $\mathcal{O}_{n\text{BPS,large}}$ itself. Analogously, also $\mathcal{O}_{\text{lightlike},n\text{BPS}}$, obtained through the IW contraction of $\mathcal{O}_{n\text{BPS,large}}$ along the branching (A43), has the same real dimension of $\mathcal{O}_{n\text{BPS,large}}$ itself.

c. Details

Below, besides (B2)–(B4), we write down the relevant formulas of the derivations given above, namely, Eqs. (B7),

$$\frac{\tilde{H}_5}{\text{MCS}(\tilde{H}_5)} = \frac{\tilde{H}_5}{SO(q+1) \times \mathcal{A}_q} \Big|_{q=8} = \mathcal{M}_{n\text{BPS,large},J_3^0,d=5} = \frac{F_{4(-20)}}{SO(9)}, \quad \dim_{\mathbb{R}} = 16, \quad (\text{c, nc}) = (0, 16),$$

$$\chi = -16, \quad \text{IW}\left(\mathcal{O}_{n\text{BPS,large},J_3^0} = \frac{E_{6(-26)}}{F_{4(-20)}}\right) \xrightarrow{(A.33)} \mathcal{O}_{\text{lightlike,BPS},J_3^0} = \frac{E_{6(-26)}}{SO(9) \rtimes \mathbb{R}^{16}},$$
(B19)

$$\frac{\tilde{H}_5}{\hat{h}_5} = \frac{\tilde{H}_5}{SO(q, 1) \times \mathcal{A}_q} \Big|_{q=8} = \frac{F_{4(-20)}}{SO(8, 1)}, \quad \dim_{\mathbb{R}} = 16, \quad (\text{c, nc}) = (8, 8), \quad \chi = 0,$$

$$\text{IW}(\mathcal{O}_{n\text{BPS,large},J_3^0}) \xrightarrow{(A.43)} \mathcal{O}_{\text{lightlike},n\text{BPS},J_3^0} = \frac{E_{6(-26)}}{SO(8, 1) \rtimes \mathbb{R}^{16}}.$$
(B20)

(ii) $q = 4$ (J_3^{H}):

$$\mathbf{35} \rightarrow \mathbf{14} + \mathbf{21}, \quad \mathbf{21} \rightarrow (\mathbf{4}, \mathbf{2})_{\text{I}} + (\mathbf{10}, \mathbf{1}) + (\mathbf{1}, \mathbf{3}), \quad \mathbf{14} \rightarrow (\mathbf{1}, \mathbf{1}) + (\mathbf{5}, \mathbf{1}) + (\mathbf{4}, \mathbf{2})_{\text{II}},$$
(B21)

$$\frac{\tilde{H}_5}{\text{MCS}(\tilde{H}_5)} = \frac{\tilde{H}_5}{SO(q+1) \times \mathcal{A}_q} \Big|_{q=4} = \mathcal{M}_{n\text{BPS,large},J_3^{\text{H}},d=5} = \frac{USp(4, 2)}{USp(4) \times USp(2)}, \quad \dim_{\mathbb{R}} = 8, \quad (\text{c, nc}) = (0, 8),$$

$$\chi = -8, \quad \text{IW}\left(\mathcal{O}_{n\text{BPS,large},J_3^{\text{H}}} = \frac{SU^*(6)}{USp(4, 2)}\right) \xrightarrow{(A.33)} \mathcal{O}_{\text{lightlike,BPS},J_3^{\text{H}}} = \frac{SU^*(6)}{(SO(5) \times SO(3)) \rtimes \mathbb{R}^{(4,2)}},$$
(B22)

$$\frac{\tilde{H}_5}{\hat{h}_5} = \frac{\tilde{H}_5}{SO(q, 1) \times \mathcal{A}_q} \Big|_{q=4} = \frac{USp(4, 2)}{USp(2, 2) \times USp(2)}, \quad \dim_{\mathbb{R}} = 8, \quad (\text{c, nc}) = (4, 4), \quad \chi = 0,$$

$$\text{IW}(\mathcal{O}_{n\text{BPS,large},J_3^{\text{H}}}) \xrightarrow{(A.43)} \mathcal{O}_{\text{lightlike},n\text{BPS},J_3^{\text{H}}} = \frac{SU^*(6)}{(SO(4, 1) \times SO(3)) \rtimes \mathbb{R}^{(4,2)}}.$$
(B23)

(iii) $q = 2$ (J_3^{C}). Notice that in this case Eq. (B2) gets modified into

$$\mathbf{Adj}(G_5) \rightarrow \mathbf{Adj}(\tilde{H}_5) + \mathbf{Adj}(\tilde{H}_5), \quad \mathbf{16} \rightarrow \mathbf{8} + \mathbf{8}, \quad \mathbf{8} \rightarrow \mathbf{3}_0 + \mathbf{1}_0 + \mathbf{2}_3 + \mathbf{2}_{-3}.$$
(B24)

Everything fits also because for $q = 2$ it holds that

$$(\mathbf{q} + \mathbf{1}, \mathbf{1}) = (\mathbf{Adj}(SO(q+1)), \mathbf{1}) = \mathbf{3}_0, \quad (\mathbf{1}, \mathbf{Adj}(\mathcal{A}_q)) = (\mathbf{1}, \mathbf{1}) = \mathbf{1}_0.$$
(B25)

$$\frac{\tilde{H}_5}{\text{MCS}(\tilde{H}_5)} = \frac{\tilde{H}_5}{SO(q+1) \times \mathcal{A}_q} \Big|_{q=2} = \mathcal{M}_{n\text{BPS,large},J_3^{\text{C}},d=5} = \frac{SU(2, 1)}{SU(2) \times U(1)}, \quad \dim_{\mathbb{R}} = 4, \quad (\text{c, nc}) = (0, 4),$$

$$\chi = -4, \quad \text{IW}\left(\mathcal{O}_{n\text{BPS,large},J_3^{\text{C}}} = \frac{SL(3, \mathbb{C})}{SU(2, 1)}\right) \xrightarrow{(A.33)} \mathcal{O}_{\text{lightlike,BPS},J_3^{\text{C}}} = \frac{SL(3, \mathbb{C})}{(SO(3) \times SO(2)) \rtimes \mathbb{R}^{(2,2)}}.$$
(B26)

$$\frac{\tilde{H}_5}{\hat{h}_5} = \frac{\tilde{H}_5}{SO(q, 1) \times \mathcal{A}_q} \Big|_{q=2} = \frac{SU(2, 1)}{SU(1, 1) \times U(1)}, \quad \dim_{\mathbb{R}} = 4, \quad (\text{c, nc}) = (2, 2), \quad \chi = 0, \quad (\text{B27})$$

$$\text{IW}(\mathcal{O}_{n\text{BPS, large, } J_3^{\mathbb{C}}} \xrightarrow{(\text{A.43})} \mathcal{O}_{\text{lightlike, } n\text{BPS, } J_3^{\mathbb{C}}} = \frac{SL(3, \mathbb{C})}{(SO(2, 1) \times SO(2)) \rtimes \mathbb{R}^{(2,2)}}.$$

(iv) $q = 1$ ($J_3^{\mathbb{R}}$). Notice that in this case Eq. (B2) gets modified into

$$\mathbf{Adj}(G_5) \rightarrow \mathbf{Adj}(\tilde{H}_5) + \mathbf{Spin}_{s=2}(\tilde{H}_5), \quad \mathbf{8} \rightarrow \mathbf{3} + \mathbf{5}, \quad \mathbf{3} \rightarrow \mathbf{1}_{\text{II}} + \mathbf{2}_{\text{I}}, \quad \mathbf{5} \rightarrow \mathbf{1}_{\text{I}} + \mathbf{2}_{\text{III}} + \mathbf{2}_{\text{II}}. \quad (\text{B28})$$

Everything fits also because for $q = 1$ it holds that

$$(\mathbf{q} + \mathbf{1}, \mathbf{1}) = (\mathbf{Adj}(SO(q + 1)), \mathbf{1}) = \mathbf{2}, \quad (\mathbf{1}, \mathbf{Adj}(\mathcal{A}_q)) = (\mathbf{1}, \mathbf{1}) = \mathbf{1}. \quad (\text{B29})$$

$$\frac{\tilde{H}_5}{\text{MCS}(\tilde{H}_5)} = \frac{\tilde{H}_5}{SO(q + 1) \times \mathcal{A}_q} \Big|_{q=1} = \mathcal{M}_{n\text{BPS, large, } J_3^{\mathbb{R}}, d=5} = \frac{SO(2, 1)}{SO(2)} \sim \frac{SU(1, 1)}{U(1)}, \quad \dim_{\mathbb{R}} = 2, \quad (\text{c, nc}) = (0, 2),$$

$$\chi = -2, \quad \text{IW}(\mathcal{O}_{n\text{BPS, large, } J_3^{\mathbb{R}}} = \frac{SL(3, \mathbb{R})}{SO(2, 1)}) \xrightarrow{(\text{A.33})} \mathcal{O}_{\text{lightlike, BPS, } J_3^{\mathbb{R}}} = \frac{SL(3, \mathbb{R})}{SO(2) \rtimes \mathbb{R}^2}. \quad (\text{B30})$$

$$\frac{\tilde{H}_5}{\hat{h}_5} = \frac{\tilde{H}_5}{SO(q, 1) \times \mathcal{A}_q} \Big|_{q=1} = \frac{SO(2, 1)}{SO(1, 1)}, \quad \dim_{\mathbb{R}} = 2, \quad (\text{c, nc}) = (1, 1), \quad \chi = 0, \quad (\text{B31})$$

$$\text{IW}(\mathcal{O}_{n\text{BPS, large, } J_3^{\mathbb{R}}} \xrightarrow{(\text{A.43})} \mathcal{O}_{\text{lightlike, } n\text{BPS, } J_3^{\mathbb{R}}} = \frac{SL(3, \mathbb{R})}{(SO(1, 1)) \rtimes \mathbb{R}^2}.$$

2. $SO(1, 1)$ three-grading and $\mathcal{O}_{\text{critical, BPS}}$

In order to deal with $\mathcal{O}_{\text{critical, BPS}}$, we start from the group embedding (A1). As pointed out above, this is the unique maximal embedding (*at least* among the symmetric ones; see e.g. [57]) into G_5 to exhibit a commuting factor $SO(1, 1)$.

Therefore, the Lie algebra \mathfrak{g}_{G_5} of G_5 admits a three-grading with respect to the Lie algebra \mathbb{R} of $SO(1, 1)$ as follows:

$$\mathfrak{g}_{G_5} = \mathcal{W}^{+3} \oplus_s \mathcal{W}^0 \oplus_s \mathcal{W}^{-3}, \quad (\text{B32})$$

where as above the subscripts denote the weights with respect to $SO(1, 1)$ itself. At the level of branching of $\mathbf{Adj}(G_5)$, the $SO(1, 1)$ three-grading reads as follows:

$$\begin{aligned} \mathbf{Adj}(G_5) \rightarrow & (\mathbf{1}, \mathbf{1})_0 + (\mathbf{Adj}(G_6), \mathbf{1})_0 + (\mathbf{1}, \mathbf{Adj}(\mathcal{A}_q))_0 \\ & + (\mathbf{Spin}(q + 2), \mathbf{Spin}(Q_q))_{-3} \\ & + (\mathbf{Spin}'(q + 2), \mathbf{Spin}(Q_q))_{+3}. \end{aligned} \quad (\text{B33})$$

Thus, the decomposition (B33) yields the following identification of the graded terms in (B32):

$$\begin{aligned} & (\mathbf{1}, \mathbf{1})_0 + (\mathbf{Adj}(G_6), \mathbf{1})_0 + (\mathbf{1}, \mathbf{Adj}(\mathcal{A}_q))_0; \\ \mathcal{W}^0 \equiv & \downarrow \text{exp} \quad \downarrow \text{exp} \quad \downarrow \text{exp} \quad (\text{B34}) \\ & SO(1, 1) \quad G_6 \quad \mathcal{A}_q \end{aligned}$$

$$\mathcal{W}^{+3} \equiv (\mathbf{Spin}'(q + 2), \mathbf{Spin}(Q_q))_{+3}, \quad (\text{B35})$$

$$\mathcal{W}^{-3} \equiv (\mathbf{Spin}(q + 2), \mathbf{Spin}(Q_q))_{-3}, \quad (\text{B36})$$

with “exp” denoting the exponential mapping.

Thus, $\mathcal{O}_{\text{critical, BPS}}$ is obtained by cosetting G_5 with the +3-graded (or equivalently -3-graded) extension of $\mathcal{W}^0 - (\mathbf{1}, \mathbf{1})_0$, namely,

$$\mathcal{O}_{\text{critical, BPS}} = \frac{G_5}{\mathcal{N}_{+3(-3)}}, \quad (\text{B37})$$

where

$$\begin{aligned} \mathcal{N}_{+3} & \equiv \text{exp}[(\mathcal{W}^0 - (\mathbf{1}, \mathbf{1})_0) \oplus_s \mathcal{W}^{+3}] \\ & = \text{exp}[(\mathbf{Adj}(G_6), \mathbf{1})_0 + (\mathbf{1}, \mathbf{Adj}(\mathcal{A}_q))_0 \\ & \quad \oplus_s (\mathbf{Spin}'(q + 2), \mathbf{Spin}(Q_q))_{+3}] \\ & = (G_6 \times \mathcal{A}_q) \rtimes_{\mathbb{R}}^{(\text{spin}(q+2), \text{spin}(Q_q))}, \end{aligned} \quad (\text{B38})$$

$$\begin{aligned} \mathcal{N}_{-3} & \equiv \text{exp}[(\mathcal{W}^0 - (\mathbf{1}, \mathbf{1})_0) \oplus_s \mathcal{W}^{-3}] \\ & = \text{exp}[(\mathbf{Adj}(G_6), \mathbf{1})_0 + (\mathbf{1}, \mathbf{Adj}(\mathcal{A}_q))_0 \\ & \quad \oplus_s (\mathbf{Spin}(q + 2), \mathbf{Spin}(Q_q))_{-3}] \\ & = (G_6 \times \mathcal{A}_q) \rtimes_{\mathbb{R}}^{(\text{spin}(q+2), \text{spin}(Q_q))}. \end{aligned} \quad (\text{B39})$$

Thus, it holds that Eqs. (B37) and (B38) [or equivalently

Eqs. (B37) and (B39)] are consistent with the general form of $\mathcal{O}_{\text{critical,BPS}}$ given by Eq. (4.29).

Therefore, in the stabilizer of $\mathcal{O}_{\text{critical,BPS}}$, the factor

$$\mathcal{T}_{\text{critical,BPS}} = \mathbb{R}^{\langle \text{spin}(q+2), \text{spin}(Q_q) \rangle} = \mathbb{R}^{\langle \text{spin}(q+1), \text{spin}(Q_q) \rangle} \quad (\text{B40})$$

is given by the exponential mapping of the Abelian subalgebra of \mathfrak{g}_{G_5} contained into the +3-graded (or equivalently -3-graded) extension of $\mathcal{W}^0 - (\mathbf{1}, \mathbf{1})_0$ through the $SO(1, 1)$ three-grading (B32), corresponding to the irrepr. $(\mathbf{Spin}'(q+2), \mathbf{Spin}(Q_q))_{+3}$ [or equivalently $(\mathbf{Spin}(q+2), \mathbf{Spin}(Q_q))_{-3}$] of $G_6 \times \mathcal{A}_q (\times SO(1, 1))$.

The results obtained in Appendixes B 1 and B 2 (and reported in Tables III and IV) allow one to conclude that all small charge orbits of symmetric magic RSG (classified by $q = 8, 4, 2, 1$) share the same non-semi-simple, Abelian (namely, translational) part of the stabilizer. Namely, Eqs. (B17) and (B40) yield

$$\begin{aligned} \mathcal{T}_{\text{lightlike,BPS}} &= \mathcal{T}_{\text{lightlike,nBPS}} = \mathcal{T}_{\text{critical,BPS}} \\ &= \mathbb{R}^{\langle \text{spin}(q+1), \text{spin}(Q_q) \rangle}. \end{aligned} \quad (\text{B41})$$

Details

Below, we write down Eqs. (B33)–(B36) for all $q = 8, 4, 2, 1$ classifying symmetric magic RSG.

(i) $q = 8 (J_3^{\mathbb{Q}})$:

$$78 \rightarrow \overbrace{\mathbf{1}_0}^{\mathcal{W}^0} + \overbrace{\mathbf{45}_0}^{\mathcal{W}^0} + \overbrace{\mathbf{16}_{-3}}^{\mathcal{W}^{-3}} + \overbrace{\mathbf{16}'_{+3}}^{\mathcal{W}^{+3}}. \quad (\text{B42})$$

(ii) $q = 4 (J_3^{\mathbb{H}})$:

$$\begin{aligned} 35 \rightarrow & \overbrace{(\mathbf{1}, \mathbf{1})_0 + (\mathbf{15}, \mathbf{1})_0 + (\mathbf{1}, \mathbf{3})_0}^{\mathcal{W}^0} + \overbrace{(\mathbf{4}, \mathbf{2})_{-3}}^{\mathcal{W}^{-3}} \\ & + \overbrace{(\mathbf{4}, \mathbf{2})_{+3}}^{\mathcal{W}^{+3}}. \end{aligned} \quad (\text{B43})$$

(iii) $q = 2 (J_3^{\mathbb{C}})$. In this case it should be recalled that

$$\begin{aligned} \mathbf{Adj}(SL(3, \mathbb{C})) &= \mathbf{16} \\ &\equiv \mathbf{3} \times \mathbf{3}' + \bar{\mathbf{3}} \times \bar{\mathbf{3}}' - 2 \text{ singlets}. \end{aligned} \quad (\text{B44})$$

Thus, by recalling Eqs. (A19)–(A22), one can compute that under $SL(3, \mathbb{C}) \supset_{\text{max}} SL(2, \mathbb{C}) \times SL(1, \mathbb{C}) \times GL(1, \mathbb{C})$,

$$\begin{aligned} \mathbf{3} \times \mathbf{3}' &\rightarrow (\mathbf{3}_0)_0 + (\mathbf{1}_0)_0 + (\mathbf{2}_3)_{-3} + (\bar{\mathbf{2}}_{-3})_3 + (\mathbf{1}_0)_0, \\ \bar{\mathbf{3}} \times \bar{\mathbf{3}}' &\rightarrow (\mathbf{3}_0)_0 + (\mathbf{1}_0)_0 + (\bar{\mathbf{2}}_{-3})_{-3} + (\mathbf{2}_3)_3 + (\mathbf{1}_0)_0. \end{aligned} \quad (\text{B45})$$

Therefore,

$$\begin{aligned} \mathbf{Adj}(SL(3, \mathbb{C})) &= \mathbf{16} \rightarrow \overbrace{2(\mathbf{3}_0)_0 + 2(\mathbf{1}_0)_0}^{\mathcal{W}^0} + \overbrace{(\mathbf{2}_3)_{-3} + (\bar{\mathbf{2}}_{-3})_{-3}}^{\mathcal{W}^{-3}} \\ &+ \overbrace{(\mathbf{2}_3)_{+3} + (\bar{\mathbf{2}}_{-3})_{+3}}^{\mathcal{W}^{+3}}. \end{aligned} \quad (\text{B46})$$

(iv) $q = 1 (J_3^{\mathbb{R}})$:

$$8 \rightarrow \overbrace{\mathbf{1}_0}^{\mathcal{W}^0} + \overbrace{\mathbf{3}_0}^{\mathcal{W}^0} + \overbrace{\mathbf{2}_{-3}}^{\mathcal{W}^{-3}} + \overbrace{\mathbf{2}_{+3}}^{\mathcal{W}^{+3}}. \quad (\text{B47})$$

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