EUROPEAN ORGANIZATION FOR NUCLEAR RESEARCH

CERN ISR-TH/80-36
28th July 1980

WAKEFIELD OF A RELATIVISTIC CURRENT IN A CAVITY

by

T. Weiland, B. Zotter

Geneva, Switzerland

WAKEFIELDS OF A RELATIVISTIC CURRENT IN A CAVITY

T. Weiland, B. Zotter

1. INTRODUCTION

The wakefield of a bunch of charged particles traversing a resonant cavity is of considerable interest for particle accelerators and storage rings, as it permits the calculation of the coupling impedance - and hence the stability - as well as the evaluation of the energy loss of the bunched beam. The only geometry which permits exact analytic calculations of the wakefield is the closed cylindrical cavity, commonly called "pill-box". Several different approaches to calculate the wakefield of a bunch of particles traversing a pill-box cavity have been published in the literature[1-6] but the equivalence of the solutions was not obvious.

Here we compare the solutions obtained by the mode-analysis and in the time-domain amongst each other and also with a recently published numerical method[7] solving the problem for general rotational symmetric cavities.

In general, we find complete agreement for the wakefield of bunches with continuous line-charge densities, and there is no "missing scalar potential" in the mode-analysis as has been assumed before. However, for discontinuous charge-densities such as delta-function pulses (which can be used as Green's function for arbitrary charge-densities) agreement is found only if one disregards divergent terms which are of no consequence for realistic (continuous) charge densities.

Finally, the mode-analysis can be generalized to arbitrary cavities, for which the wakefield is obtained in terms of the loss parameters of each of the resonant modes. The resonant frequencies and loss-parameters can be obtained numerically for certain rotationally symmetric cavities with existing computer programs such as KN7C[8] or SUPERFISH[9]. However, the series for the wakefield converge rather slowly for positions inside the bunch - which is the case of interest for the coupling impedance - and there the obtainable accuracy is quite limited. However for positions well behind the bunch, the series converge faster and thus the energy loss can be evaluated more precisely.

2. PILL-BOX CAVITY

In this chapter the wakefield in a pill-box cavity will be evaluated using the mode concept and the time-domain scheme. Finally these analytical results will be compared with numerical ones.

2.1 Mode analysis

The mode analysis uses the resonant modes of a cavity to compute the wakefield. It is assumed first that the contributions of the free charges, which cannot be taken into account by those modes, vanish. With $z_o > 0$ as the distance between a point charge Q and a test particle behind it, the mode concept gives the wakefield as an infinite sum[1,2] :

$$V_d\ (z_o) = -Q \sum_{\mu} k_{\mu} \cos\ (\omega_{\mu} \cdot z_o/c) \quad . \tag{1}$$

The k_{μ} are the loss parameters defined by :

$$k_{\mu} = \frac{\underline{V}_{\mu} \cdot \underline{V}_{\mu}^{*}}{4 \cdot U_{\mu}} \quad . \tag{2}$$

U_{μ} is the stored energy in the mode μ and $\underline{V}_{\mu}$ is the voltage seen by the point charge. For a pill-box cavity these loss parameters can be given analytically.

The normalized field components are :

$$\left.\begin{aligned} \underline{E}_z^{n,p} &= \frac{j_n}{R} \cdot J_o\left(j_n \frac{r}{R}\right) \cdot \cos\left(\frac{\pi\, p\, z}{g}\right) \exp\ (i\ \omega_{np}\ t) \\ \underline{E}_r^{n,p} &= -\frac{\pi p}{g} \cdot J_1\left(j_n \frac{r}{R}\right) \cdot \sin\left(\frac{\pi\, p\, z}{g}\right) \cdot \exp\ (i\ \omega_{np}\ t) \\ \underline{H}_{\phi}^{n,p} &= i \cdot \omega_{np} \cdot \varepsilon_o \cdot J_1\left(j_n \frac{r}{R}\right) \cdot \cos\left(\frac{\pi\, p\, z}{g}\right) \cdot \exp\ (i\ \omega_{np}\ t) \end{aligned}\right\} \tag{3}$$

where g is the "gap"-length of the cavity of the radius R, j_n is the n-th zero of the Bessel-function $J_o(x)$ and $\frac{\omega_{np}^2}{c^2} = \left(\frac{j_n}{R}\right)^2 + \left(\frac{\pi p}{g}\right)^2$.

Hence the voltage seen by a particle becomes :

$$\underline{V}_{np} = \int_0^g \underline{E}_z \; (r = 0,\; z, t = \frac{z}{c}) \cdot dz$$

$$= \frac{i \cdot \omega_{np} \cdot R}{j_n \cdot c} \left[1 - (-1)^p \exp\left(i \cdot \frac{\omega_{np} \cdot g}{c} \right) \right] \tag{4}$$

and further :

$$\underline{V}_{np} \cdot \underline{V}^{*}_{np} = 2 \cdot \left(\frac{\omega_{np} \cdot R}{j_n \cdot c} \right)^2 \cdot \left[1 - (-1)^p \cdot \cos\left(\omega_{np} \cdot \frac{g}{c} \right) \right] \; . \tag{5}$$

The stored energy is given by :

$$U_{np} = \frac{\mu_o}{2} \int_0^R \int_0^{2\pi} \int_0^g \underline{H}_\phi^{n,p} \cdot \underline{H}_\phi^{*n,p} \cdot dz \cdot r \cdot d\phi \cdot dr$$

$$= \frac{\pi \cdot \varepsilon_o}{4} \cdot \frac{\omega^2_{np}}{c^2} \cdot g \cdot R^2 \cdot J_1^2 \; (j_n) \; . \tag{6}$$

We finally get the loss parameters :

$$k_{np} = \frac{1}{\pi \cdot \varepsilon_o \cdot g} \; \frac{1 - (-1)^p \; \cos\left(\omega_{np} \cdot \frac{g}{c} \right)}{j_n{}^2 \cdot J_1^2 \; (j_n)} \tag{7}$$

The expression for the point charge wakefield becomes :

$$V_d(z_o) = \frac{-2 \cdot Q}{\pi \cdot \varepsilon_o \cdot g} \sum_{n=1}^{\infty} \sum_{p=-\infty}^{+\infty} \frac{1 - (-1)^p \cos\left(\omega_{np} \cdot \frac{g}{c} \right)}{j_n{}^2 \; J_1^2 \; (j_n)} \cdot \cos\left(\omega_{np} \cdot \frac{z_o}{c} \right) \; . \tag{8}$$

(By counting p from $-\infty$ to $+\infty$ rather than from 0 to ∞ we avoid a special factor for p = 0).

2.2 Time-domain analysis

The electric and magnetic fields induced by a bunch of charged particles traversing a cavity can be derived from the scalar and vector potentials. These potentials can be expressed as infinite series of the products of the eigenmodes of the cavity and of time-dependent factors :

$$\left.\begin{aligned} \phi(\vec{r},t) &= \sum_\mu \phi_\mu(\vec{r}) \cdot r_\mu(t) \quad , \\ \vec{A}(\vec{r},t) &= \sum_\mu \vec{a}_\mu(\vec{r}) \cdot q_\mu(t) \quad . \end{aligned}\right\} \quad (9)$$

The summation extends in general over modes in all 3 spatial directions ($\mu = m,n,p$). For a beam passing along the axis of a rotationnally symmetric cavity however, only azimuthally symmetric fields are excited, and the summation is limited to radial ($1 \leq n < \infty$) and axial ($-\infty < p < \infty$) mode-numbers.

The eigenmodes are normalized solutions of the homogeneous Helmholtz equations :

$$\left.\begin{aligned} \left[\nabla^2 + \frac{\omega_{np}^2}{c^2}\right] \phi_{np} &= 0 \quad , \\ \left[\vec{\nabla}^2 + \frac{\omega_{np}^2}{c^2}\right] \vec{a}_{np} &= 0 \quad , \end{aligned}\right| \quad (10)$$

which fulfill the proper boundary conditions at the cavity walls (assumed to be perfectly conducting for simplicity), and where ω_{np} are the resonant frequencies (x 2π) of the cavity. The time-dependent factors then can be determined from the equations[10] :

$$\left.\begin{aligned} r_{np}(t) &= \frac{c^2}{\varepsilon_o \cdot \omega_{np}^2} \int_V \rho\,(\vec{r}-\vec{v}\cdot t) \cdot \phi_{np}(\vec{r})\, dV \quad , \\ \ddot{q}_{np}(t) + \omega_{np}^2\, \dot{q}_{np}(t) &= \frac{1}{\varepsilon_o} \int_V \vec{J}\,(\vec{r}-\vec{v}\cdot t) \cdot \vec{a}_{np}(\vec{r})\, dV \quad . \end{aligned}\right\} \quad (11)$$

where $\rho(\vec{r})$ is the charge density, and $\vec{J}(\vec{r}) = \rho \cdot \vec{v}$ the current density of the bunch moving with velocity $\vec{v}$. For convenience, we will restrict our considerations to bunches moving with light-velocity along the cavity axis ($\vec{v} = c \cdot \vec{e}_z$). The integration extends generally over the volume of the beam inside the cavity, and reduces to an integral over z for a filamentary beam at the axis (after replacing the volume density ρ by the line-density λ).

The initial conditions for q_{np} will be chosen such that there are no fields in the cavity before the bunch arrives. In order to include bunches of any length, we take $q_{np}(-\infty) = \dot{q}_{np}(-\infty) = 0$.

The electric field can be obtained from the potential with the relation :

$$\vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t} , \tag{12}$$

and hence the axial component on the axis (r=0) of an azimuthally symmetric field ($\partial/\partial\phi = 0$) becomes :

$$E_z(z,t) = -\sum_{n,p} \left[\frac{\partial \phi_{np}}{\partial z} \cdot r_{np}(t) + a_{npz} \cdot \dot{q}_{np}(t)\right] . \tag{13}$$

The wakefield at a distance z_o (or time $t_o = z_o/c$) behind the bunch (center) is defined as the integral over E_z along the z-axis with $ct = z+z_o$ or :

$$V(z_o) = \int_o^g E_z\left(z, \frac{z+z_o}{c}\right) \cdot dz \tag{14}$$

We derive the wakefield from the potentials of a pill-box cavity in the appendix. In this geometry, the eigenmodes and resonant frequencies, are given by closed analytic expressions. For a bunch of the line-charge density $\lambda(z)$ we obtain in general :

$$V(z_o) = -\frac{1}{\pi\varepsilon_o g} \sum_{n=1}^{\infty} \frac{1}{j_n^2 J_1^2(j_n)} \sum_{p=-\infty}^{\infty} \left\{ \int_o^{\infty} dx \cdot \cos \nu_{np} x \cdot \left[2\lambda(x-z_o) + - (-)^p \lambda(x-z_o+g) - (-)^p \lambda(x-z_o-g)\right] + \int_{-g}^{o} dx \cdot \cos\frac{\pi p x}{g} \left[\lambda(x-z_o) - (-)^p \lambda(x-z_o+g)\right] \right\} \tag{15}$$

with $\nu_{np} = \omega_{np}/c$.

For continuous charge distributions, we can interchange the order of integration and summation over p. As shown in the appendix, the wakefield is then given by the much simpler expression

$$V(Z_o) = -\frac{2}{\pi\cdot\varepsilon_o\cdot g}\sum_{n=1}^{\infty}\sum_{p=-\infty}^{\infty}\frac{1-(-)^p\cos\nu_{np}\cdot g}{j_n^2\,J_1^2(j_n)}\int_0^{\infty}\lambda\left(x-z_o\right)\cos\left(\nu_{np}\cdot x\right)dx\,. \tag{16}$$

For discontinuous charge distributions such as the step or delta-function pulse, this equation yields the expressions which are valid after the discontinuity has left the cavity ($z_o > g$). For $z_o < g$, the complete expression Eq.(15) contains a divergent term which is related to the infinite energy loss when such a distribution traverses a (closed) cavity.

For the step function pulse, the infinite sums in Eq.(16) have been summed analytically[4] for $g < z_o < \sqrt{4R^2 + g^2} - g$, i.e. before reflections from the outer-cavity wall arrive at the location where the wakefield is evaluated but after the pulse has left the cavity. If the divergence is ignored, the (different) series yield the same sum also for $z_o < g$ as shown in Ref. 5. If thus appears that Eq.(16) may be used for any distribution, as the divergent term is of no consequence for realistic (continuous) distributions which are always the ultimate aim of the computations.

Eq.(16) could be reduced further by exchanging the order of integration and summation also for the infinite integral. However, this leads to the expressions restricted to $z_o < \sqrt{4R^2 + g^2} - g$ discussed above. We now apply Eq.(16) to a number of typical distributions.

a) <u>Delta function pulse $\lambda(z) = Q\cdot\delta(z)$</u>

$$V_d(z_o) = -\frac{2Q}{\pi\cdot\varepsilon_o\cdot g}\sum_{n,p}\frac{1-(-)^p\cos(\nu_{np}\cdot g)}{j_n^2\,J_1^2\,(j_n)}\cos(\nu_{np}\cdot z_o) \tag{17}$$

For $z_o < \sqrt{4R^2+g^2}-g$ these sums can be evaluated analytically, and yield :

$$V_d(z_o) = \frac{Q}{2\cdot\pi\cdot\varepsilon_o\cdot g}\left|\frac{1}{\frac{z_o}{2g}+\left[\frac{z_o}{2g}\right]} - \frac{1}{\frac{z_o}{2g}+\left[\frac{z_o}{2g}\right]+2}\right| \qquad (18)$$

where the square brackets stand for the integer part of the term enclosed. With Q = 1, these expressions could be considered the Green's function for the wakefield of a general distribution $\lambda(z)$, which is obtained formally by

$$V_\lambda(z_o) = \int_{(o)}^{z_o} V_d\,(z_o - z)\;\lambda(z)\;dz \qquad (19)$$

However, this integral diverges at the lower limit and there should be a term included outside the integral which cancels the divergence. We get the correct result by leaving-off the lower limit (or replacing it by $-\infty$ which amounts to the same as $\lambda(-\infty) = 0$), but the same result is obtained without these complications under b.

b) <u>Step-function pulse $\lambda(z) = \lambda_o \cdot s(-z)$</u>

where

$$s(z) = \begin{cases} 0 \text{ for } z < 0 \\ \tfrac{1}{2} \text{ for } z = 0 \\ 1 \text{ for } z > 0 \end{cases} .$$

then

$$V_s(z_o) = -\frac{2\lambda_o}{\pi\,\varepsilon_o\,g}\sum_{n,p}\frac{1-(-)^p\cos\nu_{np}\,g}{j_n^{\,2}\,J_1^{\,2}(j_n)}\;\frac{\sin\nu_{np}\cdot z_o}{\nu_{np}} \qquad (20)$$

Restricting z_o to be smaller than $\sqrt{4R^2+g^2} - g$, one obtains[2)]

$$V_\lambda(z_o) = -\frac{\lambda_o}{2\pi\varepsilon_o}\cdot \ln\left\{1+\frac{1}{\frac{z_o}{2g}+\left[\frac{z_o}{2g}\right]}\right\} \qquad (21)$$

and hence the wakefield for an arbitrary distribution $\lambda(z)$

$$V_\lambda(z_o) = -\frac{1}{2\pi\varepsilon_o}\int_0^{z_o}\frac{d\lambda(z)}{dz}\cdot \ell n\left|1+\frac{1}{\frac{z_o-z}{2g}+\left[\frac{z_o-z}{2g}\right]}\right\}dz \qquad (22)$$

This expression is valid before the arrival of reflections from the cylindrical cavity wall, i.e. for a limited range of z_o (which is here counted from the beginning of the bunch).

c) Parabolic bunch (half length L)

$$\lambda(z) = \begin{cases}\frac{3Q}{4L}\left(1-\frac{z^2}{L^2}\right) & \text{for } |z| < L\\ 0 & \text{for } |z| > L\end{cases}$$

Equation (16) yields

$$V_p = \frac{3Q\,g}{\pi\,\varepsilon_o\,L}\sum_{n,p}\frac{1-(-)^p\cos\nu_{np}\,g}{j_n^2\,\nu_{np}^2\,J_1^2\,(j_n)}\begin{cases}\frac{\sin\nu_{np}\,(z_o+L)}{\nu_{np}\,L}-\cos\nu_{np}\,(z_o+L)-\frac{z_o}{L} & ;\ z_o < L\\ 2\left[\frac{\sin\nu_{np}\,(z_o+L)}{\nu_{np}\,L}-\cos\nu_{np}L\right]\cos\nu_{np}\,z_o & ;\ z_o > L\end{cases}$$

(23)

For $z_o < L < g < \sqrt{4R^2+g^2} - g$ these sums yield

$$V_p(z_o) = \frac{3\,Q\,g}{4\,\pi\,\varepsilon_o\,L^3}\left[(z_o+L) - 2(z_o+g)\,\ell n\left(1+\frac{z_o+L}{g}\right)+\frac{L^2-z_o^2}{2g}\,\ell n\left(1+\frac{2g}{z_o+L}\right)\right]$$

(24)

The same expression is obtained from Eq. (19), which becomes

$$V_p(z_o) = -\frac{3\,Q}{4\,\pi\,\varepsilon_o\,L^3}\int_0^z (L-z)\,\ell n\left(1+\frac{2g}{z_o-z}\right)dz$$

for $z_o < 2g$ (z_o counted from the head of the bunch).

d) Gaussian charge distribution with standard deviation σ

$$\lambda(z) = \frac{Q}{\sigma\sqrt{2\pi}}\, e^{-\frac{z^2}{2\sigma^2}}$$

We find from Eq.(22), after evaluation of the integral (25)

$$V_G(z_o) = -\frac{Q}{\pi\,\varepsilon_o\, g}\, e^{-\frac{z_o^2}{2\sigma^2}} \sum_{n,p} \frac{1-(-)^p \cos \nu_{np}\, g}{j_n^2\, J_1^2\,(j_n)}\, \mathrm{Re}\left\{ w\left(\frac{\nu_{np}\sigma}{\sqrt{2}} - \frac{i z_o}{\sigma\sqrt{2}}\right)\right|$$

where w(z) is the complex error-function, and Re stands for real part. No closed expression is presently known to the authors for this sum, but it has been evaluated numerically and is compared to purely numerical results in the next section.

2.3 Comparison of results obtained by various methods

A comparison between the equations (8) and (15-17) for the δ-function wakefield shows that the time domain and the mode analysis yield the same analytic expression except for the divergent term occurring in the time domain calculations for the case where the point charge is inside the pill-box cavity.

For any realistic charge distribution both methods give exactly the same answer for all positions z_o.

Therefore one can conclude that any contributions to the wakefield due to free charges are correctly obtained in the mode concept and thus there is no missing scalar potential contribution as has been suspected in the past[1,2].

A further comparison was made between the analytic results derived above and numerical results of the computer program BCI[7] which solves the field equations in the time domain directly by a mesh method, including the effects of free charges.

Fig. 1 shows the wakefield in a range of $-4\sigma \leq z_o \leq 36\sigma$ for a Gaussian bunch (σ = 2.5 cm) which has passed a pill-box cavity (R = 5 cm), g = 10 cm). An excellent agreement (better than 10^{-3} can be found for test particles "outside" the bunch ($z_o \geq 4\sigma$). Although a rough mesh was used in BCI (11 x 21 -points), and only 40 modes in the analytic sum, both results can hardly be distinguished in the range $4\sigma \leq z_o \leq 36\sigma$.

"Inside" the bunch ($-4\sigma \leq z_o \leq 4\sigma$) the analytical and numerical results seem to disagree and therefore a second figure is given showing the wakefield in more detail and with increasing precision in both methods. The analytic results (broken lines) approach continuously the numerical results with increasing number of terms in the sum. The numerical results approach the analytical ones from the other side with increasing number of mesh points. The final difference between the most accurate results in Fig. 1b is less than ± 2.5 %.

The reason for this slow convergence of the results "inside" the bunch is the behaviour of the Fourier spectrum of the driving current which is suddenly cut off at z_o for a beam moving with light velocity (see fig.2) (Due to causality a particle at z_o can see fields only from particles in front of itself).

"Inside" the bunch, the driving current for the wakefield is a function with a large step which leads to a Fourier transform proportional to $1/\omega$ over a large range. "Behind" the bunch the step is small and the Fourier transform of the driving term becomes proportional to $\exp(-\omega^2 \sigma^2 / 2c^2)$.

This problem occurs in both methods. In the analytical expressions the terms with high frequencies do not decay sufficiently fast. In the numerical computations the highest frequency which can be included is given by the size of the largest mesh step[7].

3. GENERAL CAVITIES

As a result of chapter 2, we know that the wakefield is determined by the eigen-modes of the cavity and by the loss-parameters k_μ. A general cavity may then be represented by an LC-network as shown in Fig.3.

The cavity impedance is given by : (26)

$$Z(\omega) = \sum_\mu \frac{-i\,k_\mu}{\omega - \omega_\mu} \;;\; C_\mu = \frac{1}{2k_\mu} \;;\; L_\mu = \frac{2k_\mu}{\omega_\mu^2} \;;\; \omega_\mu^2 = \frac{1}{L_\mu C_\mu} .$$

(C_μ capacitance, L_μ inductance, ω_μ resonant frequency).

This impedance is valid only if the Fourier transform of the driving current has no poles in the ω-plane. However, it can be replaced by a much simpler one giving the same results for the wakefield without any restrictions to the driving term :

$$Z(\omega) = -2\pi \sum_\mu k_\mu\, \delta(\omega - \omega_\mu) \quad . \tag{27}$$

As already mentioned above, the Fourier transform of the bunch current which is cut off at z_o is given by :

$$j_\lambda(\omega, z_o) = \int_{-\infty}^{z_o/c} j_\lambda(t) e^{i\omega t}\, dt \quad . \tag{28}$$

The Fourier transform of the wakefield is simply given by :

$$V_\lambda(\omega, z_o) = j_\lambda(\omega, z_o) \cdot Z(\omega) \tag{29}$$

Using Eq. 27 for the impedance the wakefield as a function of z_o becomes :

$$V_\lambda(z_o) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} j_\lambda(\omega, z_o) \cdot \left[-2\pi \sum_\mu k_\mu\, \delta(\omega - \omega_\mu)\right] e^{-i\omega \frac{z_o}{c}}\, d\omega \quad . \tag{30}$$

Hence we get :

$$V_\lambda(z_o) = -\sum_\mu j_\lambda(\omega_\mu, z_o) \cdot k_\mu \cdot e^{-i\omega_\mu \frac{z_o}{c}} \quad . \qquad (31)$$

To find the wakefield at a position z_o behind a reference point for an arbitrary bunchshape and for an arbitrary cavity one thus only needs the resonant frequencies ω_μ, the loss-parameters k_μ and the Fourier transforms of the bunch (which is cut off at z_o) evaluated at the resonant frequencies.

For realistic cavities only a limited number of resonant frequencies and loss-parameters can be obtained by numerical methods. For z_o inside the bunch a wakefield computation becomes very difficult due to the slow convergence of the sum in Eq.(31). However the series converge much faster for positions (z_o) well behind the bunch and permit a more accurate calculation of the wakefield by this method.

4. CONCLUSIONS

It has been shown that for a pill-box cavity the time-domain calculation and the mode analysis yield the same analytical expression for the wakefield of realistic bunch shapes.

Extrapolating this result to arbitrary cavities yields an expression for the wakefield as a sum over loss-parameters and Fourier transforms. This sum converges very slowly for positions inside the bunch, making it difficult to obtain a precise value for the coupling impedance. However, a good approximation to the wakefield can be obtained after the bunch has passed the cavity, and hence the total energy loss of the bunch passing the cavity can be calculated more accurately.

LITERATURE

1) P.B. Wilson et al, SLAC Pub 1908 ; PEP 240 (1977).

2) K. Bane, P.B. Wilson, IEEE Trans. Nucl. Sci., NS-24, No3 (1977), p.1485-1486.

3) M. Chatard-Moulin et al, Rapport 73-Al, 73-A2, Université de Limoges (1973).

4) A.W. Chao, P.L. Morton, PEP 105 (1975), PEP 119 (1975).

5) B. Zotter, PEP Note 310 (1979).

6) P. Morton, K. Neil, LRL Report 18103 (1968), p.365.

7) T. Weiland, CERN-ISR-TH/80-07 (1980).

8) E. Keil, Nucl. Instr. Meth. 100 (1972), p.419-427.

9) K. Halbach, R.F. Holsinger, Particle Accelerators, Vol 7 (1976), p. 213-222.

10) E. Condon, Journal Appl. Phys. 12, No 129 (1941), p.129-132.

GLOSSARY

$\vec{a}_{np}$	eigenfunctions of the vector potential
c	velocity of light
$\vec{e}_{r,\phi,z}$	unit vector in r, ϕ, z direction
g	length of the pill-box cavity ("gap" length)
i	$\sqrt{-1}$
j_n	n-th root of the Bessel-function J_o
j_λ	current produced by a line charge density λ
k_μ	loss-parameter of the mode μ
L	half length of parabolic bunch
n	radial mode number in a pill-box cavity $n=1,2,3...$
p	longitudinal mode number in a pill-box cavity $p=...-2,-1,0,1,2...$
Q	charge
$q_{np}(t)$	time dependent coefficients of the vector potential
$r_{np}(t)$	time dependent coefficients of the scalar potential
R	radius of the pill-box cavity
U_{np}	stored energy in the mode (n,p)
$\underline{V}_{np}$	voltage seen by a particle due to the mode (n,p)
V	wakefield (= energy gain in volts)
$V_{d,s,G,p,\lambda}$	wakefield for a point charge (delta function), step current, Gaussian bunch, parabolic bunch, arbitrary line charge density $\underline{\lambda}$
z_o	distance from the bunch center or reference point
$Z(\omega)$	impedance
$\lambda(z)$	line-charge density of the driving current
μ	general index for counting resonant modes
ν_{np}	wave number $\nu_{np} = \omega_{np}/c$
σ	standard deviation of a Gaussian bunch
ϕ_{np}	eigenfunctions of the scalar potential
ω_{np}	circular resonant frequencies of a pill-box cavity

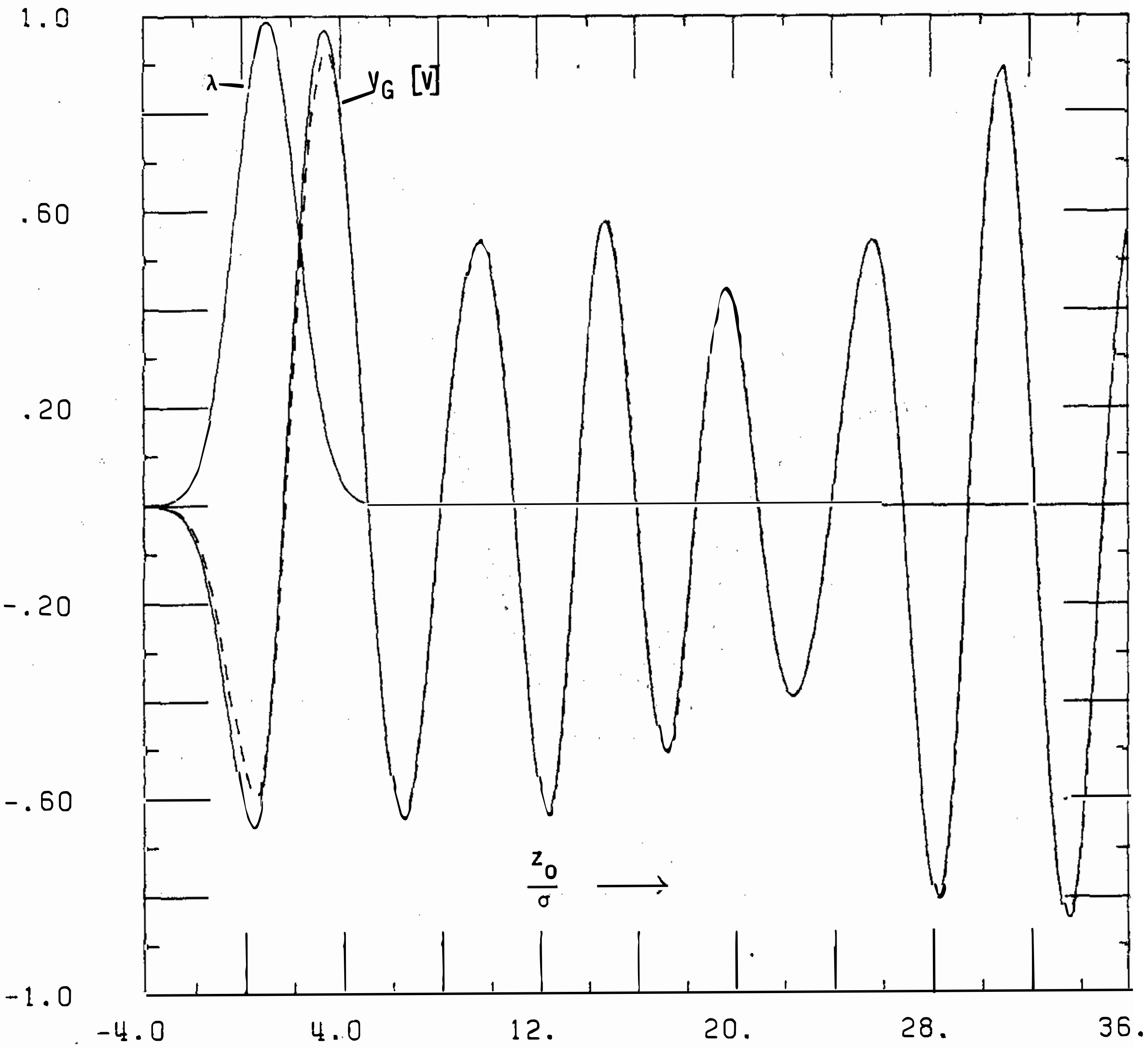


Figure 1a : The wakefield of a Gaussian bunch (σ = 2.5 cm) due to a pill box cavity (R = 5 cm, g = 10 cm)

$-4\sigma \leq z_0 \leq 36\sigma$

--- mode-analysis results (40 modes)

—— results of BCI (11 x 21 mesh)

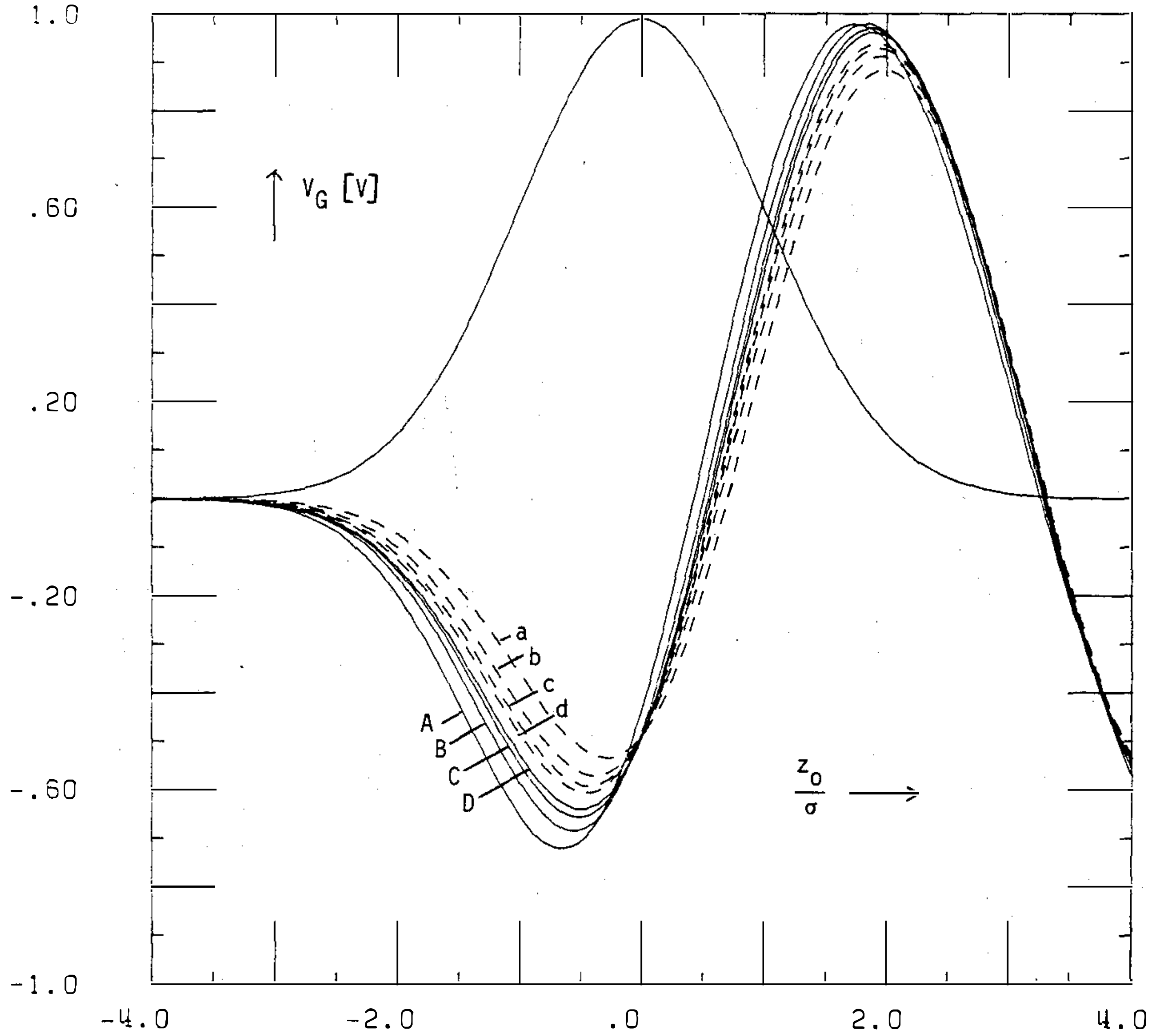


Figure 1b : The wakefield of a Gaussian bunch (σ = 2.5 cm) due to a pill-box cavity (R = 5 cm, g = 10 cm)

$- 4\sigma \leq z_0 \leq 4\sigma$

--- mode-analysis results for
a : 10 modes, b : 40, c : 160, d : 640

—— BCI-results for different meshes
A : 6 x 11, B : 11 x 21, C : 21 x 41, D : 41 x 81

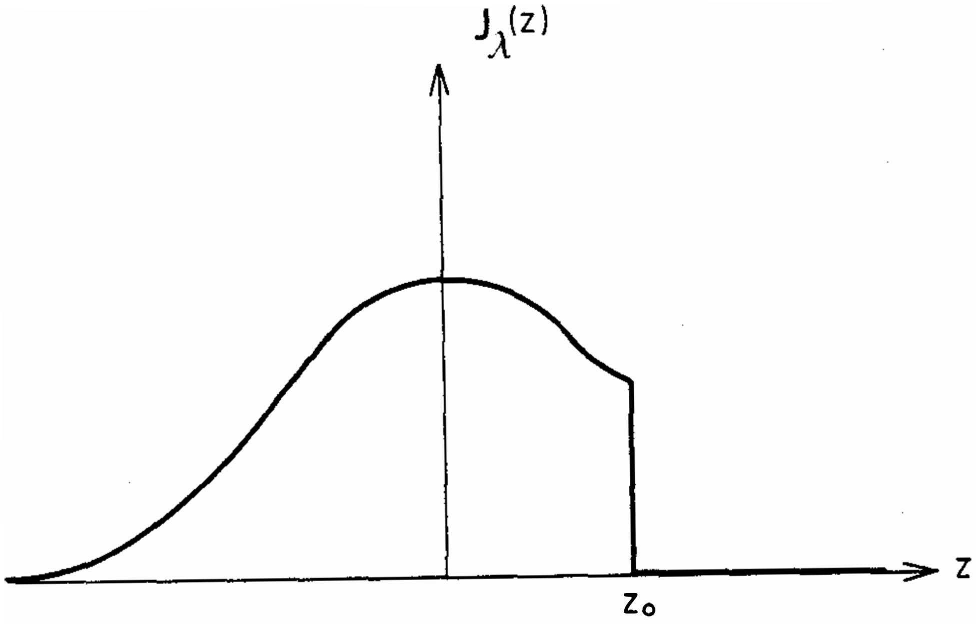


Figure 2 : The driving current seen by a particle at z_0.

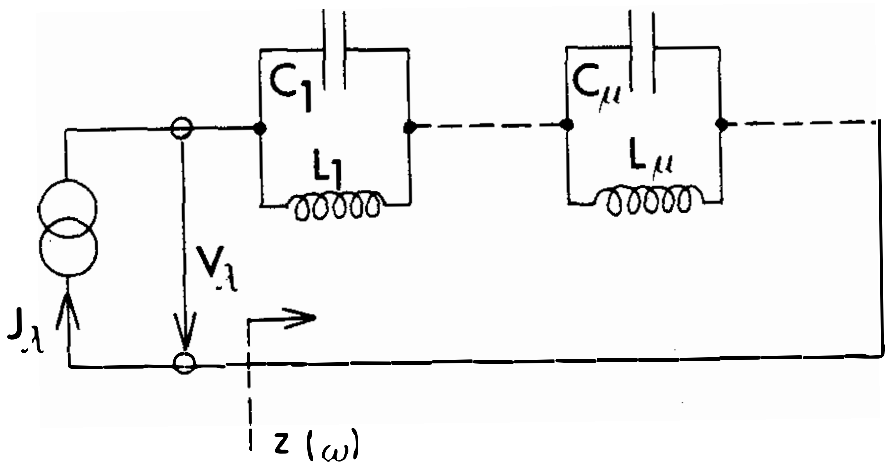


Figure 3 : An LC-network representing the cavity.

APPENDIX

WAKEFIELD IN A PILL-BOX CAVITY

1) Normalized eigenmodes in a cavity of ("gap"-) length g and radius R

$$\phi_{np}(\vec{r}) = \sqrt{\frac{2}{\pi g}}\ \frac{J_o\ (j_n r/R)}{R\ J_1(j_n)}\ \sin\frac{\pi p z}{g}$$

$$\vec{a}_{np}(\vec{r}) = \frac{1}{\sqrt{\pi g}}\ \frac{c}{R\ \omega_{np}\ J_1(j_n)} \left\{ \begin{array}{l} \frac{\pi p}{g}\ J_1\left(\frac{j_n r}{R}\right) \sin\frac{\pi p z}{g} \cdot \vec{e}_r \\ 0 \cdot \vec{e}_\phi \\ \frac{j_n}{R}\ J_o\left(\frac{j_n r}{R}\right) \cos\frac{\pi p z}{f} \cdot \vec{e}_z \end{array} \right\} \qquad \text{(A1)}$$

where $1 \leq n < \infty,\ -\infty < p < \infty.$ (A2)

The resonant frequencies are given by

$$\omega_{np} = c\left[\left(\frac{j_n}{R}\right)^2 + \left(\frac{\pi p}{g}\right)^2\right]^{1/2}$$

2) Time-dependent factors for a bunch with line-density $\lambda(z)$, moving along the z-axis with light velocity v = c.

$$r_{np}(t) = \frac{c^2}{\varepsilon_o\ \omega_{np}^2} \int_0^g \lambda(z-ct)\ \phi_{np}(z,r=0)\,dz$$

$$\ddot{q}_{np}(t) + \omega_{np}^{\ 2}\ q_{np}(t) = \frac{c}{\varepsilon_o} \int_0^g \lambda(z-ct)\ a_{npz}(z,r=0)\,dz = F(t) \qquad \text{(A3)}$$

with the boundary conditions $q_{np}(-\infty) = \dot{q}_{np}(-\infty) = 0$. The general solution for q_{np} thus is

$$q_{np}(t) = \frac{1}{\omega_{np}} \int_{-\infty}^{t} F(\tau)\ \sin\omega_{np}(t-\tau)\,d\tau \qquad \text{(A4a)}$$

and for its derivative (which we need for the calculation of the electric field rather than q_{np})

$$\dot{q}_{np}(t) = \int_{-\infty}^{t} F(\tau)\ \cos\omega_{np}(t-\tau)\,d\tau \qquad \text{(A4b)}$$

3) Time-dependant factors for the pill-box cavity

$$\left.\begin{aligned} r_{np}(t) &= \sqrt{\frac{2}{\pi g}}\ \frac{c^2}{\varepsilon_o\ \omega_{np}\ R\ J_1(j_n)} \int_0^g \lambda(z-ct)\ \sin\frac{\pi p z}{g}\, dz \\ \dot{q}_{np}(t) &= \frac{1}{\sqrt{\pi g}}\ \frac{j_n\ c^2}{\varepsilon_o\ \omega_{np}\ R^2\ J_1(j_n)} \int_{-\infty}^{t} d\tau\ \cos\omega(t-\tau) \int_0^g dz\ \lambda(z-c\tau)\ \cos\frac{\pi p z}{g} \end{aligned}\right\} \quad (A5)$$

The double integral in the second equation can be removed by changing the order of integration, but first we substitude $z-c\tau = u$, $c\tau = v$ to get (see Figure A1)

$$\int_{-\infty}^{t} dt \int_0^g dz = \int_{-\infty}^{ct} dv \int_{-v}^{g-v} du = \int_{-ct}^{g-ct} du \int_{-u}^{ct} dv + \int_{g-ct}^{\infty} du \int_{-u}^{g-u} dv$$

We further need the integral

$$\int \cos\ \nu(v-ct)\ \cos\alpha(v+u)\,dv = \frac{1}{\nu^2-\alpha^2}\Big[\sin\nu(v-ct)\ \cos\alpha(v+u) + \\ -\ \alpha\ \cos\nu(v-ct)\ \sin\alpha(v+u)\Big]$$

With $\alpha = \frac{\pi p}{g}$, $\nu^2 = \alpha^2 + \frac{j_n^2}{R^2}$, and $\nu^2 - \alpha^2 = \frac{j_n^2}{R^2}$ we get

$$\int_{-u}^{ct} dv = \frac{R^2}{j_n^2}\Big[\nu\ \sin\nu(u+ct) - \alpha\ \sin\alpha(u+ct)\Big]$$

$$\int_{-u}^{g-u} dv = \frac{R^2}{j_n^2}\Big[\nu\ \sin\nu(u+ct) - (-)^p\ \nu\sin\nu(u+ct-g)\Big]$$

and hence

$$\dot{q}_{np}(t) = \frac{1}{\sqrt{\pi g}}\ \frac{c}{\varepsilon_o\ \omega_{np}\ j_n\ J_1(j_n)}\Bigg\{\nu\int_0^{\infty}\Big[\lambda(u-ct) - (-)^p\lambda(u+ct-g)\Big]\sin\nu u\cdot du + \\ -\ \frac{\pi p}{g}\int^{g} \lambda(u-ct)\cdot\sin\frac{\pi p u}{g}\cdot du\Bigg\} \quad (A6)$$

Figure A1

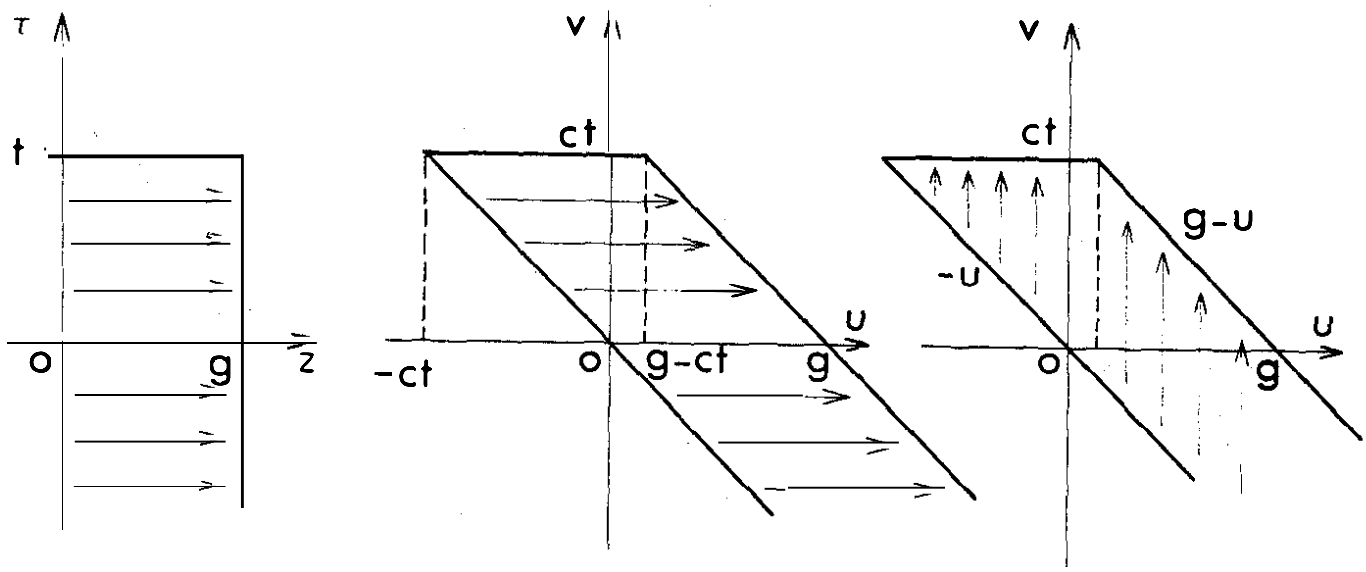
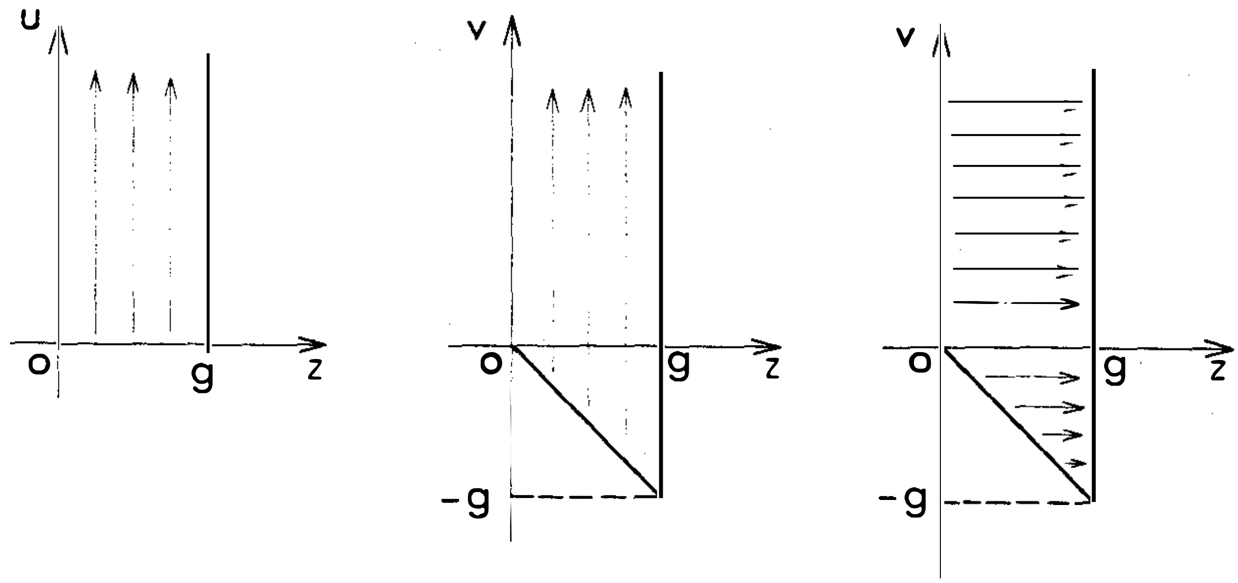


Figure A2



4) For the electric field (Eq.13) the contributions from the scalar potential (first Eq.A5) and from the second integral in the term of the vector-potential (Eq.A6) cancel, and we get simply

$$E_z(z,t) = -\frac{c}{\pi \varepsilon_o g R^2} \sum_{n,p} \frac{\cos \alpha_p z}{\omega_{np} J_1^2(j_n)} \int_0^\infty \left[\lambda(u-ct) - (-)^p \lambda(u-ct+g)\right] \sin \nu_{np} u \, du \tag{A7}$$

where $\alpha_p = \pi p/g$ and $\nu_{np} = \frac{\omega_{np}}{c}$. For a delta-function, pulse $\lambda(z) = Q\delta(z)$ this yields :

and the double-integral in Eq.(A10) becomes (A11)

$$\frac{\nu R^2}{j_n^2}\left|\left\{\int_0^\infty dv\ \left[\cos\nu v-(-)^p\ \cos\nu(v+g)\right]+\int_{-g}^{0} dv\ \left[\cos\alpha v-(-)^p\ \cos\nu(v+g)\right]\right\}\left[\lambda(v-z_o)+\right.\right.$$
$$\left.-(-)^p\lambda(v-z_o+g)\right]$$

We can combine terms with cos $\nu(v+g)$ and substitude $v+g=u$ to get for the term in the curly brackets

$$\int_0^\infty du\ \cos\nu u\ \left[2\ \lambda(u-z_o)\ -\ (-)^p\lambda(u-z_o+g)\ -\ (-)^p\lambda(u-z_o-g)\right]\ +$$
$$+\int_{-g}^{0} du\ \cos\alpha u\ \left[\lambda(u-z_o)-(-)^p\ \lambda(u-z_o+g)\right]$$

which yields Eq.(15) for the wakefield.

6) Final expressions for the wakefield

By substituting $u\pm g=z$ in the first integral and splitting off integrals from 0 to ∞ one gets for the expression (A12)

$$2\ \left[1-(-)^p\ \cos\nu g\right]\int_0^\infty\lambda(z-z_o)\ \cos\nu z\ dz-(-)^p\int_0^g dz\ \cos\nu z\ \ \left[\lambda(z-g-z_o)-\lambda(-z-g-z_o\right]$$

For smooth distributions, one can change the order of integration and summation over p. We then need the sums of three infinite series for $0\lesssim z\lesssim g$

$$\sum_{-\infty}^{+\infty}\cos\frac{\pi pz}{g}\ =\ \delta(z),\ \sum_{-\infty}^{\infty}(-)^p\cos\frac{\pi pz}{g}=\sum_{-\infty}^{\infty}(-)^p\cos\frac{z}{g}\sqrt{\pi^2p^2+\beta^2}=\ \delta(z-g)$$

(for any β) to show that the terms in both finite integrals in the expression for the wakefield cancel, and we thus obtain Eq.(16).

$$E_z(z,t) = -\frac{Q_c}{\pi\,\varepsilon_o\,g\,R^2}\sum_{n,p}\frac{\cos\alpha_p z}{\omega_{np}\,J_1{}^2(j_n)}\begin{cases}\sin\omega_{np}t \;;\quad 0 < ct < g \\ \sin\omega_{np}t - (-)^p\sin\omega_{np}(t-g/c) \;;\quad ct > g\end{cases} \quad (A8)$$

while for a gaussian distribution $\lambda(z) = \frac{Q}{\sigma\sqrt{2\pi}}\,e^{-z^2/2\sigma^2}$ one finds

$$E_z(z,t) = -\frac{Q}{2\pi\,\varepsilon_o\,g\,R^2}\sum_{n,p}\frac{\cos\frac{\pi pz}{g}}{\nu_{np}\,J_1{}^2(j_n)}\left\{\left[e^{-\frac{c^2t^2}{2\sigma^2}}\,\mathrm{Jm}\left|w\left(\frac{\nu_{np}}{\sqrt{2}} - i\frac{ct}{\sigma\sqrt{2}}\right)\right| + - (-)^p\,e^{-\frac{(g-ct)^2}{2\sigma^2}}\,\mathrm{Jm}\left|w\left(\frac{\nu_{np}\,\sigma}{\sqrt{2}} + i\,\frac{g-ct}{\sqrt{2}}\right)\right|\right]\right\} \quad (A9)$$

where Jm w(z) is the imaginary part of the complex error function.

5) <u>The wakefield at position z_o behind the (center of the) bunch</u> is found with Eqs (14) and (A7)

$$V(z_o) = -\frac{c}{\pi\,\varepsilon_o\,g\,R^2}\sum_{n,p}\frac{1}{\omega_{np}\,J_1{}^2(j_n)}\int_o^g dz\,\cos\frac{\pi pz}{g}\int_o^\infty du\,\sin\nu_{np}u\left[\lambda(u-z+z_o) - (-)^p\,\lambda(u-z-z_o+g)\right] \quad (A10)$$

Again, we can perform one of the integrations if we substitute u-z = v and interchange the order of integration (see Fig.A2)

$$\int_o^g dz\int_o^\infty du = \int_o^g dz\int_{-z}^\infty dv = \int_{-o}^\infty dv\int_o^g dz + \int_{-g}^o dv\int_{-v}^g dz$$

We now need the integral

$$\int\cos\alpha z\cdot\sin\nu(v+z)dz = \frac{1}{\alpha^2-\nu^2}\left[\alpha\sin\alpha z\cdot\sin\nu(v+z) + \nu\cdot\cos\alpha z\cdot\cos\nu(v+z)\right]$$

With $\alpha = \frac{\pi p}{g}$, $\nu^2 = \alpha^2 + \frac{j_n{}^2}{R^2}$ as before we get

$$\int_o^g dz = \frac{\nu\,R^2}{j_n{}^2}\left[\cos\nu v - (-)^p\cos\nu(v+g)\right]$$

$$\int_v^g dz = \frac{\nu\,R^2}{j_n{}^2}\left[\cos\nu v - (-)^p\cos\nu(v+g)\right]$$

26