PARTICLE ORBITS IN FIXED FIELD ALTERNATING GRADIENT ACCELERATORS

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(presented by K. R. Symon)

I. Linearized orbit equations

1. Geometry of the equilibrium orbits

In order to develop a theory of orbit stability applicable to FFAG accelerators generally, it is convenient to characterize a particular accelerator by specifying its equilibrium orbits. We will therefore assume that a set of closed equilibrium orbits lying in the median plane is given. If instead, the magnetic field pattern is specified, the equilibrium orbits must be found by integrating the equations of motion.

The geometrical properties of each orbit, and the relations between orbits, will be periodic in the azimuthal angle θ with period $2\pi/N$. Each orbit is to be specified by its equivalent radius R defined by

$$S = 2\pi R, \tag{1.1}$$

where S is the length of the orbit. In general, R will be slightly larger than the mean radius $\langle r \rangle_{av}$. We define an azimuthal coordinate Θ by the equation

$$s = \Theta R, \tag{1.2}$$

where s is the distance measured along the orbit from some reference point (say at azimuthal angle Θ_0). We shall require that the orbit be perpendicular to the radius from the center of the machine at the reference point, and that the reference points lie along a continuous curve. The parameter Θ will be equal to the azimuthal angle $\theta - \theta_0$ plus a small periodic function with period $2\pi/N$.

Each orbit will now be specified by a periodic parameter $\mu(\Theta, R)$ defined by

$$\mu(\Theta, R) = R/\rho(\Theta, R) \tag{1.3}$$

where ρ is the radius of curvature. Specification of $\mu(\Theta, R)$, together with the requirement that the center of the orbit lie at the origin in the median plane, completely determines the orbit R, provided the reference point $\Theta = 0$ is specified. For our purposes, it will be sufficient to specify the angle $\zeta(R)$ between the radius from the origin and the reference curve $\Theta = 0$ where it crosses the orbit R (figure 1). Choice of the parameter $\mu(\Theta, R)$ is restricted by the requirement that it be periodic in Θ with, period $2\pi/N$ and mean value

$$\langle\mu\rangle_{av} = \frac{1}{2\pi}\int_0^{2\pi} \mu \, d\Theta = \frac{1}{2\pi}\int_0^{s} \frac{ds}{\rho} = 1. \tag{1.4}$$

The function $\mu(\Theta, R)$ is also restricted by the requirement that at the point $\Theta = 0$ the orbit R must be perpendicular to the radius from the origin. This requirement leads to a rather complicated analytical restriction on the function μ. It is sufficient if $\Theta = 0$ is a point of symmetry of the orbit, i.e.,

$$\mu(-\Theta, R) = \mu(\Theta, R) \tag{1.5}$$

We will need also parameters $\eta(\Theta, R)$ and $\varepsilon(\Theta, R)$ relating the perpendicular distance dx between two nearby orbits, and the increment $d\Theta$ in Θ along an orthogonal trajectory to the orbits, to the increment dR in the parameter R (see figure 1) :

$$dx = \eta \, dR \tag{1.6}$$

$$d\Theta = \varepsilon \, dR/R \tag{1.7}$$

* Assisted by the National Science Foundation and the Office of Naval Research.

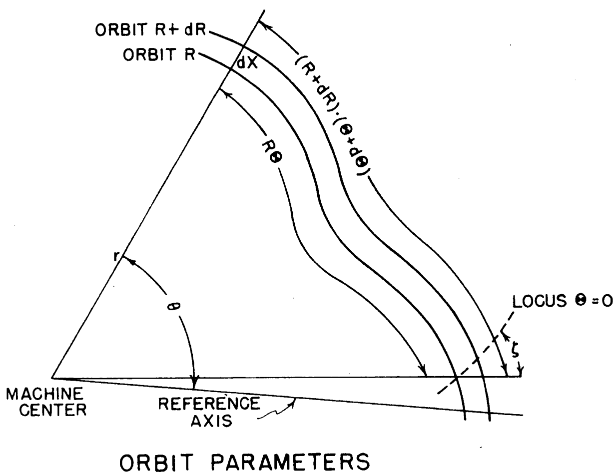

Fig. 1.

It can be shown that η, ε satisfy the differential equations

$$\partial\varepsilon/\partial\Theta = \mu\eta - 1, \tag{1.8}$$

$$\partial\eta/\partial\Theta = -\mu\varepsilon - \int R\, \partial\mu/\partial R \cdot d\Theta, \tag{1.9}$$

where the three constants of integration are to be chosen so that ε and η are periodic functions of Θ (i.e. so that the right hand members of equations (1.8) and (1.9) have zero mean values), and so that

$$[\varepsilon/\eta]_{\theta=0} = \tan\zeta. \tag{1.10}$$

If all equilibrium orbits are geometrically similar, the parameter μ depends only on Θ and not on R. In the interest of simplicity, we will usually restrict our attention to machines of this type. If in addition, ζ is independent of R, then by equations (1.8)-(1.10), the parameters η and ε will be independent of R. In this case, we will say that the quilibrium orbits *scale*; the equilibrium orbits scale if any set of neighboring orbits can be obtained by photographic enlargement or reduction from a set of orbits in the neighborhood of any other orbit.

Let us set

$$\mu = 1 + f g(N\Theta), \tag{1.11}$$

where f is the flutter factor, and the flutter function $g(N\Theta)$ has period 2π in $N\Theta$, zero mean, and is normalized so that its mean square value is $^1/_2$. For example,

$$g(N\Theta) = \cos N\Theta. \tag{1.12}$$

Then an approximate solution of equations (1.8)-(1.9) which is adequate to exhibit the principal features of FFAG orbits is

$$\eta \doteq 1 - f\tan\zeta/N \cdot g_1(N\Theta), \tag{1.13}$$

$$\varepsilon \doteq \tan\zeta, \tag{1.14}$$

where for any function $g(\xi)$, periodic in ξ with zero mean, we define

$$g_1(\xi) = \int g(\xi)\, d\xi, \tag{1.15}$$

where the constant of integration is to be chosen so that $g_1(\xi)$ has zero mean.

2. *Betatron oscillations*

If a particle of momentum p moves in an equilibrium orbit R, then we have by equation (1.3)

$$p c = e H\rho = (e H R)/\mu, \tag{2.1}$$

where H is the magnitude of the magnetic field, so that

$$H(R, \Theta) = pc/eR\, \mu(\Theta, R). \tag{2.2}$$

The magnetic field is thus given in terms of the coordinates R, Θ.

If we differentiate equation (2.1) with respect to x, where x is measured perpendicular to the orbit, we have

$$H\, \partial\rho/\partial x + \rho\, (\partial H/\partial x) = c/e \cdot \partial\rho/\partial x. \tag{2.3}$$

The field index is therefore

$$\begin{aligned} n &= -\rho/H \cdot \partial H/\partial x \\ &= \partial\rho/\partial x - \rho \cdot \partial \ln p/\partial x. \end{aligned} \tag{2.4}$$

Making use of equations (1.3), (1.6) and (1.7), we find

$$n = -1/\eta\mu^2\, [k\mu + \varepsilon\, \partial\mu/\partial\Theta + R\, \partial\mu/\partial R], \tag{2.5}$$

where k is a parameter which measures the momentum compaction :

$$k = R(d \ln p)/dR - 1. \tag{2.6}$$

In terms of the mean magnetic field $\bar{H} = pc/eR$, we can write k also as a mean field index :

$$k = R/\bar{H} \cdot d\bar{H}/dR, \tag{2.7}$$

The linearized equations for betatron oscillations about an equilibrium orbit are

$$d^2x/ds^2 - (1-n)/\rho^2 \cdot x = 0, \tag{2.8}$$

$$d^2z/ds^2 - (n/\rho^2)\, z = 0, \tag{2.9}$$

where x and z are the deviations from the equilibrium orbit in the radial and vertical directions. These become by equations (1.2) and (1.3),

$$d^2x/d\Theta^2 + \mu^2 (1-n)\, x = 0, \tag{2.10}$$

$$d^2z/d\Theta^2 - \mu^2 n\, z = 0. \tag{2.11}$$

The character of the betatron oscillations is therefore determined by the functions $\mu^2(\Theta, R)$ and

$$\mu^2 n = -1/\eta - (k\mu + \varepsilon\, \partial\mu/\partial\Theta + R\partial\mu/\partial R). \quad (2.12)$$

By making use of equations (1.8) and (1.9) we can rewrite equation (2.12) in the form

$$\mu^2(1\text{-}n) = (k+1)\mu/\eta - 1/\eta\, \partial^2\eta/\partial\Theta^2. \quad (2.13)$$

If the equilibrium orbits scale, then μ, η and ε are functions only of Θ. Thus μ^2 n will be a function of Θ only, and the betatron oscillations will also scale, provided k is constant. Accelerators with this property will be referred to as accelerators which scale. For accelerators which scale, we have

$$p = p_0 (R/R_0)^{k+1} \quad (2.14)$$

and

$$H = H_0 (R/R_0)^k \mu(\Theta). \quad (2.15)$$

3. *Approximate solution for betatron oscillations*

In this section we develop some approximate formulas which give a useful general picture of the properties of FFAG accelerators. If the betatron wavelengths are long on comparison with the sector length (say at least four sectors), then the smooth approximation equations are applicable[1, 2]. The "smooth" betatron oscillation equations become in this case

$$d^2X/d\Theta^2 - \nu_x^2 X = 0, \quad (3.1)$$

$$d^2Z/d\Theta^2 - \nu_z^2 Z = 0, \quad (3.2)$$

where,

$$\nu_x^2 = \langle\mu^2(1\text{-}n)\rangle_{av} + \langle\{\mu^2(1\text{-}n)\}_1^2\rangle_{av}, \quad (3.3)$$

$$\nu_z^2 = \langle\mu^2 n\rangle_{av} + \langle\{\mu^2 n\}_1^2\rangle_{av}. \quad (3.4)$$

The curly brackets {} indicate that only the oscillatory part of the enclosed function is to be taken; i.e., the mean value is to be subtracted.

The solutions of equations (3.1), (3.2) are

$$X = A\cos\nu_x\Theta + B\sin\nu_x\Theta, \quad (3.5)$$

$$Z = C\cos\nu_z\Theta + D\sin\nu_z\Theta. \quad (3.6)$$

Superposed upon these smooth solutions is a ripple which has the periodicity of the sectors. It is clear that ν_x, ν_z are the numbers of radial and vertical betatron wavelengths around the circumference of the accelerator. The approximate formulas (3.3), (3.4) give ν_x, ν_z within about 10% provided that ν_x, ν_z are *both* less than N/4.

In order to avoid resonance buildup of betatron oscillations, it is necessary to avoid integral and half-integral values for ν_z, ν_z and also to avoid integral values for $\nu_x + \nu_z$. This implies that ν_x, ν_z, must be the same for all orbits, or nearly so, and this is the principal limiting condition on FFAG designs. In accelerators which scale ν_x, ν_z are necessarily the same for all orbits; this is the advantage in designs which scale.

The relation between betatron wavelengths and machine parameters depends upon which term in eq. (2.13) predominates in giving alternating gradient focusing. In a radial sector FFAG accelerator with $\zeta = 0$, and with a large number of sectors (say $N > 10$) η is very nearly unity, and the second term in eq. (2.13) is small except near the edges of the magnets where it gives rise to edge focusing effects. The edge focusing comes from the term $-\eta^{-1}\cdot\varepsilon\,\partial\mu/\partial\Theta$ in eq. (2.12). This term has a non-zero mean value, part of which is included in the μ term in eq.(2-13); thus eq. (3.7) and (3.8) below include most of the mean focusing effect due to edges in radial sector machines. We will call the first term in eq. (2.13) the "μ term" and the second, the "η term". In a spiral sector FFAG accelerator, the alternating gradient focusing comes predominantly from the η term.

It may be noted that the η term includes the term $(R/\eta)(\partial\mu/\partial R)$ which appears when the orbits do not scale. It is not hard to see that in a conventional AG synchrotron this is the dominant alternating gradient term.

Let us first consider a radial sector FFAG accelerator with a large number of sectors, and let us neglect the η term. If $f/N \ll 1$, then $\eta \doteq 1$ according to eq. (1.13), let us write μ in the form given by eq. (1.11). Then eq. (3.3), (3.4) yield, if we substitute from eq. (2.13), with $\eta = 1$,

$$\nu_x^2 = k + 1 + \frac{(k+1)^2 f^2}{N^2}\langle g_1^2\rangle_{av}, \quad (3.7)$$

$$\nu_z^2 = -k + \frac{f^2}{2} + \frac{(k-1)^2 f^2}{N^2}\langle g_1^2\rangle_{av}, \quad (3.8)$$

where we have neglected a small term involving $g^2 - \overline{g^2}$ in eq. (3.8). The betatron oscillation advances in phase by an angle

$$\sigma = 2\pi\nu/N \quad (3.9)$$

per sector. For stability, σ must be less than π, and for the smooth approximation to be valid, σ must be less than about $\pi/2$. If we solve eq. (3.7), (3.8) for k, f in terms of σ_x, σ_z, we obtain

$$k + 1 = \frac{N^2}{8\pi^2}(\sigma_x^2 - \sigma_z^2 - b), \quad (3.10)$$

$$f = \frac{4\pi}{[2\langle g_1^2\rangle_{av}]^{1/2}}\,\frac{[\sigma_x^2 + \sigma_z^2 - b]^{1/2}}{|\sigma_x^2 - \sigma_z^2 - b|}, \quad (3.11)$$

where

$$b = 4\pi^2/N^2\,[1 - f^2 - 4kf^2/N^2 \langle g_1^2\rangle_{av}]. \qquad (3.12)$$

The quantity b is negligible for sufficiently large N.

By appropriate choice of σ_x, σ_z, k can be made either positive or negative, i.e., in a radial sector FFAG synchrotron, with N large, the high energy orbits may be either on the outside or the inside of the donut. The b-term, which is important when N is small, is positive and therefore favors machines with positive k, i.e., with a given N, /k/ can be larger and f smaller if $k>0$. For maximum momentum compaction, i.e., minimum radial aperture, k, and hence N, should be as large as practicable. If we define a circumference factor C as the ratio between mean and minimum radii of curvature of the equilibrium orbit, then

$$C = |\mu|_{max} = |1 + fg(N\Theta)|_{max}. \qquad (3.13)$$

It is desirable to minimize C, since for a given maximum magnetic field, this yields the smallest accelerator design. It is clear from eq. (3.11), that for a given form of g, the minimum circumference factor is obtained by making σ_z as small, and σ_x as large as possible (or vice versa, if k is to be negative).

Let us assume a rectangular field flutter, with unit mean square:

$$g(\xi) = \begin{cases} \left[\dfrac{1-q}{2q}\right]^{\frac{1}{2}}, -q\pi < \xi < q\pi, & \text{(I)} \\ -\left[\dfrac{q}{2(1-q)}\right]^{\frac{1}{2}}, q\pi < \xi < 2\pi - q\pi, & \text{(II)} \end{cases} \qquad (3.14)$$

$$g(\xi + 2\pi) = g(\xi). \qquad (3.15)$$

When $\xi = N\,\Theta$ lies in regions labeled I, we say that Θ is in a positive half sector; regions labeled II we call negative half sectors. We need to calculate

$$\langle g_1^2\rangle_{av} = \pi^2/\sigma \cdot q(1-q). \qquad (3.16)$$

If now

$$K = f\,[\langle g_1^2\rangle_{av}]^{1/2} \qquad (3.17)$$

is fixed by eq. (3.11), then by eq. (3.13), the circumference factor is

$$C = 1 + \frac{\sqrt{3}K}{\pi q}, \text{ or } \frac{\sqrt{3}K}{\pi(1-q)} - 1, \qquad (3.18)$$

whichever is greater. The minimum value of C occurs when q is chosen so that the two values of the right member of eq. (3.18) are equal. We then have

$$\mu = 1 + fg(N\Theta) = \begin{cases} C, -q\pi < N\Theta < q\pi, & \text{(I)} \\ -C, q\pi < N\Theta < 2\pi - q\pi, & \text{(II)} \end{cases} \qquad (3.19)$$

The radius of curvature, and consequently also the magnetic field, is constant in magnitude along the equilibrium orbit and opposite in sign in the two half sectors. The ratio of half sector lengths is

$$\Gamma = \frac{q}{1-q} = \frac{C+1}{C-1}, \qquad (3.20)$$

and the circumference factor is

$$C = \frac{\Gamma + 1}{\Gamma - 1} = \left[1 - \frac{f^2}{2}\right]^{\frac{1}{2}} \qquad (3.21)$$

If we take $\sigma_z = \pi/6$, $\sigma_x = \pi/2$, $b = 0$, and use the approximate formulas (3.10), (3.11), we obtain $K = 3\sqrt{5}$, $\Gamma = 1.31$, $C = 7.5$, $f = 10.5$, $k = N^2/36$. It will be shown in the next section by a more accurate calculation that the minimum value of C where N is large is about 5.

In a spiral sector FFAG accelerator, ζ is near 90° and the η - term in eq. (2.13) is large. It is then possible to use a much smaller flutter factor, so that the oscillatory part of the μ - term is small. We will again assume that μ is given by eq. (2.11) and will use the approximation (1.13) for η. If we expand $1/\eta$ in a power series in the second term of formula (1.13), we may calculate

$$\langle\mu/\eta\rangle_{av} = 1 - (f^2\tan^2\zeta)/N^2 \langle g_1^2\rangle_{av} - \ldots \qquad (3.22)$$

We will neglect the second and higher order terms, and will neglect also the oscillatory part of μ/η. The η - term can be rewritten in the following way:

$$1/\eta\ \partial^2\eta/\partial\Theta^2 = \partial/\partial\Theta\,(1/\eta\ \partial\eta/\partial\Theta) + (1/\eta\ \partial\eta/\partial\Theta)^2. \qquad (3.23)$$

The first term on the right is large and oscillatory with zero mean, and the second is smaller but has a positive mean value. We neglect the oscillatory part of the second term, and substitute in eq. (3.3) and (3.4), using (2.13) to obtain

$$\nu_x^2 = k + 1, \qquad (3.24)$$

$$\nu_x^2 = -k + f^2/2 + 2\langle(1/\eta\ \partial\eta/\partial\Theta)^2\rangle_{av}. \qquad (3.25)$$

Note that the η - term does not contribute in this approximation to the radial focusing. If we take η as given by formula (1.13), we have

$$\left\langle\left(\frac{1}{\eta}\frac{\partial\eta}{\partial\Theta}\right)^2\right\rangle_{av} = f^2\tan^2\zeta\left\langle\frac{g^2}{(1 - fN^{-1}\tan\zeta\, g_1)^2}\right\rangle_{av}$$

$$= f^2\tan^2\zeta\left[\frac{1}{2} + \frac{2f^2\tan^2\zeta}{N^2}\left\langle g^2 g_1^2\right\rangle_{av} + \ldots\right] \qquad (3.26)$$

We will neglect the second and higher order terms in square brackets and substitute in eq. (3.24), (3.25), to obtain

$$f^2 \tan^2\zeta = (\nu_x^2 + \nu_z^2 - 1), \tag{3.27}$$

where we have also neglected f^2. Note that, to this order of approximation, formulas, (3.24) and (3.27) are independent of the form of the flutter function $g(N\Theta)$; only the circumference factor [eq. (3.13)] depends on $g(N\Theta)$. We can rewrite these formulas in terms of the phase shifts σ per sector :

$$k + 1 = (N^2 \sigma_x^2)/4\pi^2, \tag{3.28}$$

$$f^2 \tan^2\zeta = N^2/4\pi^2 \cdot (\sigma_x^2 - \sigma_z^2) - 1. \tag{3.29}$$

The reference curve $\Theta = 0$, satisfies, in polar coordinates r, θ, the equation

$$1/r \, dr/d\theta = \cot\zeta. \tag{3.30}$$

The radial separation between ridges (points of maximum magnetic field), in units of r is therefore

$$\lambda = \Delta r/r = 2\pi/N \tan\zeta. \tag{3.31}$$

Thus for a given choice of σ_x, σ_z, and N the ratio f/λ is fixed. The maximum allowable gap between the poles of the magnet is proportional to λ; if the field flutter is to be obtained by shaping the poles, without extra forward windings, it can be shown that for f/λ fixed the maximum gap is about $1/4\ \lambda r$ and is obtained for $f \doteq 1/4$. Under these conditions, the field flutter will necessarily be very nearly sinusoidal,

$$g(\xi) = \cos\xi, \tag{3.33}$$

and hence the circumference factor will be

$$C = 1 + f = 1.25.$$

If we take, as above, $\sigma_z = \pi/6$, $\sigma_x = \pi/2$, with $f = 1/4$, we obtain $k + 1 = N^2/16$, $\lambda = 5.95\ N^{-2}\,[1\text{-}14.4N^{-2}]^{-1/2}$, $\tan\zeta = 1.05\ N\,[1\text{-}14.4N^{-2}]^{-1/2}$.

4. *Linear stability for radial sectors*

In order to get more accurate relations between the parameters, we return to the betatron oscillation equations (2.10), (2.11). Making use of eq. (2.12), (1.13) and (1.14), with $\zeta = 0$, we rewrite eq. (2.10), (2.11) for the case of a rectangular field flutter of the form (3.19) :

$$d^2x/d\Theta^2 \pm kCx = 0, \tag{4.1}$$

$$d^2z/d\Theta^2 \mp kCz = 0, \tag{4.2}$$

where the upper signs apply in positive half sectors, and the lower in negative half-sectors. The term $\varepsilon\, \partial\mu/\partial\Theta$ in eq. (2.12) gives rise to terms in eq. (2.10), (2.11) which represent the focusing which occurs at the sector edges, which we will here neglect. These approximations are valid only when $N \gg f$, and we have accordingly also neglected 1 in comparison with n. When N is small, edge effects and higher order terms in η must be taken into account. The oscillatory terms in η will give rise to effects resulting from the fact that neighboring equilibrium orbits are not everywhere equidistant. For small N, edge effects turn out to increase the vertical focusing and decrease the radial focusing, so that considerably smaller values of the flutter factor f may be used if $k > 0$, without losing vertical stability.

Let $N\Theta_0 = -q\pi$, $N\Theta_1 = q\pi$, $N\Theta_2 = (2\text{-}q)\pi$. Then the solutions of eq. (4.1) within the positive and negative halfsectors separately yield the following matrix relations between x and $x' = dx/d\Theta$ at the points Θ_0, Θ_1, Θ_2:

$$\begin{pmatrix} x_1 \\ x_1' \end{pmatrix} = M_+ \begin{pmatrix} x_0 \\ x_0' \end{pmatrix}, \quad \begin{pmatrix} x_2 \\ x_2' \end{pmatrix} = M_- \begin{pmatrix} x_1 \\ x_1' \end{pmatrix}, \tag{4.3}$$

where

$$M_+ = \begin{pmatrix} \cos\psi_+ & (KC)^{-1/2}\sin\psi_+ \\ -(KC)^{1/2}\sin\psi_+ & \cos\psi_+ \end{pmatrix}, \quad M_- = \begin{pmatrix} \cosh\psi_- & (KC)^{-1/2}\sinh\psi_- \\ (KC)^{1/2}\sinh\psi_- & \cosh\psi_- \end{pmatrix}, \tag{4.4}$$

$$\psi_+ = \frac{2\pi q}{N}(KC)^{1/2}, \quad \psi_- = \frac{2\pi(1\text{-}q)}{N}(KC)^{1/2} \tag{4.5}$$

We thus obtain

$$\begin{pmatrix} x_2 \\ x'_2 \end{pmatrix} = M \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}, \tag{4.6}$$

with

$$M = M_- M_+ = \begin{pmatrix} \cos\psi_+\cosh\psi_- - \sin\psi_+\sinh\psi_-, & (KC)^{-1/2}(\cos\psi_+\sinh\psi_- - \sin\psi_+\cosh\psi_-) \\ (KC)^{1/2}(\cos\psi_+\sinh\psi_- + \sin\psi_+\cosh\psi_-), & \cos\psi_+\cosh\psi_- + \sin\psi_+\sinh\psi_- \end{pmatrix} \tag{4.7}$$

We can now calculate

$$\cos \sigma_x = \tfrac{1}{2} \text{Trace}(M) = \cos \psi_+ \cosh \psi_- \tag{4.8}$$

and in the same way,

$$\cos \sigma_z = \cos \psi_- - \cosh \psi_+ . \tag{4.9}$$

In terms of the local field index

$$n = k/C, \tag{4.10}$$

within the magnets (we take n as positive here), and the ratio Γ of sector lengths [eq. (3.20)], we may rewrite ψ_+ and ψ_- :

$$\psi_+ = \frac{2\pi}{N} \frac{\Gamma}{\Gamma - 1} n^{1/2}, \; \psi_- = \frac{2\pi}{N} \frac{1}{\Gamma - 1} n^{1/2}. \tag{4.11}$$

Formulas (4.5), (4.8), (4.9) and (4.11) have been written for $k > 0$. However they may also be used for $k < 0$, in which case it is convenient to regard C as negative.

The smallest circumference factor is obtained by choosing σ_x as large as possible and σ_z as small as possible (or vice versa). If we choose $\sigma_x = 3\pi/4$, $\sigma_z = \pi/6$, we calculate from eq. (4.8), (4.9), $\psi_+ = 1.32$, $\psi_- = 1.93$. From eq. (4.11), (3.21) we have

$$\Gamma = \psi_+/\psi_- = 1.46, \; C = 5.35 \tag{4.12}$$

The theoretical minimum value of C is 4.45 for $\sigma_x = \pi$, $\sigma_z = 0$. In order to keep the amplitude of betatron oscillations within reasonable bounds, the above choices of σ_x, σ_z run about as close to the stability limits as it is safe to go. (For the choice $\sigma_x = \pi/2$, $\sigma_z = \pi/6$, these more exact formulas give $\Gamma = 1.29$, $C = 7.9$, which may be compared with the approximate values 1.31, 7.5 obtained in the preceding section.)

5. *Linear stability for spiral sectors*

For spiral sector accelerators, the circumference factor is close to unity, and minimizing C is no longer a major consideration. The ridge separation λ is, however, rather small, and if the gap between magnet poles is to be kept as large as possible, it appears that the field flutter in the median plane must be at least approximately sinusoidal. We will therefore assume a field in the median plane of the form.

$$B_{z0} = B_0 (r/r_0)^k [1 - f \sin (1/w \cdot \ln (r/r_0) - N\theta)], \tag{5.1}$$

where r, θ are polar coordinates in the median plane. The argument of the sine function is made logarithmic rather than linear in r in order to make the magnetic field (and hence the particle orbits) scale. The constant w is related to the spiral angle and the ridge separation (eq. 3.31) by

$$1/w = N \tan \zeta = 2\pi/\lambda. \tag{5.2}$$

The linearized equations for the betatron oscillations in the field (5.1) can be obtained from the general analysis of the first two sections, but it is perhaps more illuminating to derive them directly.

If one undertakes to write the linear terms in the differential equations characterizing the departure of the particle from a reference circle of radius $r_1 = p/eB_0(r_0/r_1)^k$ one obtains substantially the following, where $x \equiv (r - r_1)/r_1$ and $y \equiv z/r_1$.

$$x'' + [1 + k + f/w \cdot \cos N\theta]\, x \doteq f \sin N\theta \tag{5.3}$$

$$y'' - [k + f/w \cdot \cos N\theta]\, y \doteq 0. \tag{5.4}$$

These equations suggest alternate gradient focusing of the type characterized by the Mathieu differential equation, but the presence of the forcing term on the right hand side of the equation for the x-motion indicates that a forced oscillation will be expected and will be given approximately by

$$x = -\frac{f}{N^2 - (k+1)} \sin N\theta. \tag{5.5}$$

Because of the presence of this forced motion one realizes that not only will the nonlinear terms in the differential equations be large but that a noticeable influence upon the betatron oscillation wavelength can result.

It is appropriate, therefore, to perform an expansion about a more suitable reference curve by writing

$$v = x - \frac{f}{N^2 - (k+1)} \sin N\theta \tag{5.6}$$

In this way one obtains equations of which the most significant terms appear below :

$$v'' + \left[k + 1 - \tfrac{1}{2} \frac{f^2/w^2}{N^2 - (k+1)} + \frac{f}{w} \cos N\theta + \tfrac{1}{2} \frac{f^2/w^2}{N^2 - (k+1)} \cos 2N\theta \right] v = 0 \tag{5.7}$$

$$y'' - \left[k - \tfrac{1}{2} \frac{f^2/w^2}{N^2 - (k+1)} + \frac{f}{w} \cos N\theta + \tfrac{1}{2} \frac{f^2/w^2}{N^2 - (k+1)} \cos 2N\theta \right] y = 0 \tag{5.8}$$

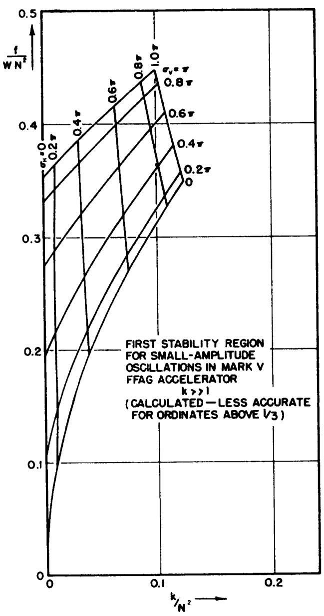

Fig. 2.

Each of these equations is of the form

$$d^2z/dr^2 + [A + B \cos 2\tau + C \cos 4\tau]\, z = 0.$$

Tables of the characteristic exponent (σ/π) of the extended Mathieu equation (5.9) have been computed on the ILLIAC, using a variational method[3]. Values of A are tabulated for a range of values of σ, B, C covering the significant portion of the first stability region. Results for the Mathieu equation (C = 0) are included. So far as we are aware there are at present no published tables of characteristic exponents for the Mathieu equation within the stability region.

In fig. 2 we plot a stability diagram for a spiral sector FFAG accelerator with $k \gg 1$ computed from the above formulas and the tabulated solutions. If $k \gg 1$, the coefficients A, B, C, depend only on k/N^2 and $f/w\,N^2$. We accordingly plot curves of constant σ_x and σ_z vs k/N^2 and f/WN^2. If we take $\sigma_z = \pi/6$, $\sigma_x = \pi/2$, with $f = 1/4$, we obtain $k = .057N^2$, $f/WN^2 = .25$, $\lambda = 6.3N^{-2}$, which may be compared with the approximate values $k = .062\,N^2$, $f/WN^2 = .265$, $\lambda = 5.95\,N^{-2}$ obtained in Section 3.

II. Non-linear effects in FFAG orbits

6. *General description of non-linear effects*

The preceding analysis of betatron oscillations has been based on an expansion of the equations of motion in powers of the displacement from the equilibrium orbit, keeping only the linear terms. The small amplitude betatron oscillations in x and z are then found to satisfy linear differential equations with coefficients periodic in the independent variable Θ.

In a perfectly constructed accelerator, the only periodicity would be that associated with the N identical sectors around the machine, and the period of the coefficients would be $2\pi/N$. In an actual accelerator, there will be imperfections, so that the coefficients will be strictly periodic with the period 2π in Θ, and approximately periodic with period $2\pi/N$. Associated with the period $2\pi/N$ is the requirement that σ_x and σ_z must not be integral or half integral multiples of 2π; in practice it appears that σ should be less than π since otherwise the tolerances on magnet construction and alignment become very severe. Associated with the period 2π is the requirement that ν_x and ν_z must not be integral or half-integral if imperfection resonances are to be avoided, and, in addition, if imperfections can couple the x- and z- motions, $\nu_x + \nu_z$ must not be an integer.

The study of the effects of non-linear terms in the equations of motion has not advanced nearly as far as the study of the linearized equations. Approximate analytic methods of treating non-linear equations with periodic coefficients have been developed by J. Moser[4] and P. A. Sturrock[5]. Their results can be summarized as follows. If the coefficients in the equations have period 2π in Θ, and if ν_x, ν_z are the numbers of betatron oscillations in one period 2π, then imperfection resonances can occur when

$$n_x \nu_x + n_z \nu_z = \text{any integer, for} \qquad (6.1)$$

$$n_x, n_z = 0, 1, 2, \ldots\ldots$$

Let

$$n_x + n_z = q. \qquad (6.2)$$

Then if q = 1 or q = 2, the motion is unstable even in linear approximation (this is the rule stated in the preceding paragraph). If q = 3, then in general, the effects of quadratic terms in the differential equations are such as to make the motion unstable even at very small amplitudes. If q = 4, then the effects of cubic terms may be to render the motion unstable, depending on the form of the cubic (and linear) terms. If $q > 4$, then, in general, the motion is stable for sufficiently small amplitudes of betatron oscillation. In any case, if $q \geqslant 4$, and if the equations of motion are non-linear, then there will be in general a limiting amplitude of betatron oscillations beyond which the oscillations are unstable in the sense that they leave the donut. Numerical studies carried out on the ILLIAC at the University of Illinois seem to confirm these conclusions.

If we apply the above criteria to the sector periodicity $2\pi/N$, then we must replace ν_x, ν_z in eq. (6.1) by $\sigma_x/2\pi$, $\sigma_z/2\pi$, the number of betatron oscillations per sector. We then conclude that values of σ_x or σ_z near $2\pi/3$ are to be avoided as well as values such that $\sigma_x + 2\sigma_z$ or $\sigma_z + 2\,\sigma_x$ is nearly 2π. We call these resonances with the periodicity of the structure itself "sector resonances". We have indeed found in numerical studies that the limiting amplitude for betatron oscillations in spiral sector machines become very small when σ approaches $2\pi/3$.

It should be pointed out that non-linear terms in the equations for the radial sector accelerator are not very large, being not greater in order of magnitude than non-linear terms which arise in some conventional alternating gradient accelerators which have been contemplated. However, the non-linear terms which arise when the sectors spiral are much larger and play a very important role in determining the character of the betatron oscillations. Numerical studies indicate that although the motion in spiral sector synchrotrons exhibits marked non-linear effects, the amplitude limits are large enough to accommodate reasonable betatron oscillations provided σ is not close to $2\pi/3$. (Say $\sigma_x < .6\pi$).

7. *Characteristics of particle motion in spiral sector structures*

The digital computer studies have been carried out with the aid of the Electronic Digital Computer of the Graduate College of the University of Illinois (ILLIAC). A large fraction of the computations pertained to structures for which the parameters fell in the range suitable for the spiral-sector model, which is under development at the University of Illinois, but the majority of the orbit characteristics revealed in this way appear to be common to large-scale spiral-sector machines, including cyclotrons of the type currently being studied by groups in other laboratories.

The computational studies for spiral-sector machines have so far involved integration of differential equations describing the particle-trajectories, although attention is being directed towards the formulation of transformations (suitable for rapid computation of particle-motion through successive sectors) akin to those employed earlier as part of an analogous study of non-linear alternate-gradient structures similar in form to the Courant-Livingston-Snyder design.

The differential equations have involved (i) a set of exact equations covering motion in the median plane and (ii) a set of approximate, but Hamiltonian, equations describing both radial and exial motion in a magnetic field of the form necessarily associated with that prescribed in the median plane. The present programs have confined attention to fields with a sinusoidal dependence upon azimuth angle, but active programming has been begun on others free of this restriction. The utility of structures possessing poles which do not lead to pure sinusoidal fields is under study. The analytic work for a two-part computational program has been completed, involving (i) solution of the magnetostatic problem in the space between such poles, employing only two position variables

$$\xi \equiv \frac{1}{2\pi}\left[\frac{ln\,(1+x)}{w} - N\theta\right] \text{ and } \eta \equiv \frac{\sqrt{1+(wN)^2}}{2\pi w}\,\frac{y}{1+x}$$

when use is made of the scaling property of the structure, and (ii) solving the differential equations for trajectories in this field, which will, in effect, be stored in the computer memory.

The results of computations pertaining to motion with one degree of freedom are appropriately and conveniently represented by means of phase plots, depicting on invariant curves the position and associated momentum of a particle as it progresses through successive " sectors " (periods of the structure) from one homologous point to another. Such studies provide information concerning the location of "fixed-points", corresponding to an equilibrium orbit; the phase-change of the betatron oscillation per sector (σ); the displacement associated with trajectory directions different from that of the equilibrium orbit; and the extent of the region within which stable motion is possible. The characteristics of small-amplitude motion found in this way agree well, for sinusoidal fields, with the predictions of the analytic theory. At large amplitudes, unstable fixed-points—representing unstable equilibrium orbits—make their appearance. These fixed-points are usually 3 or 4 in number, corresponding to an unstable periodic solution 3 or 4 sectors in wavelength, although other cases have also been observed.

Associated with the unstable fixed-points one finds a separatrix, constituting an effective stability limit, which in the majority of cases the ILLIAC results depict as a sharp boundary and outside of which it is frequently possible to draw the initial portions of what appear to be invariant curves for unstable motion. Fig. 3 shows a number of invariant curves, on a phase plot of this nature, for parameters not far from those which would be suitable for a model. In this case the phase change per sector is close to $\sigma_x = .571\pi$ for small-amplitude motion; σ_x does not change greatly with increasing amplitude and it is noteworthy that ultimately 7 unstable fixed-points ($\sigma_x = 4\pi/7 \doteq .5714\pi$ make their appearance. In this example a rather large permissible amplitude of stable motion is found ($|\Delta\, r|$ approximately 0.08 or 0.09 times the radius, at $N\theta = 0$, mod 2π). The existence of this relatively large region of stability is connected with the fact that

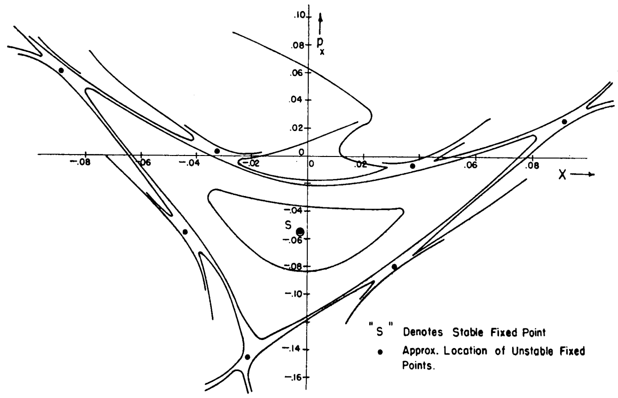

Fig. 3.

σ_{x0} differs materially from the value $2\pi/3$, for which a prominent non-linear sector-resonance makes its presence felt.

When axial motion is also permitted, there is in general coupling between this motion and that occurring in the radial direction. For small-amplitude oscillations about the equilibrium orbit, however, the motion is virtually decoupled. Limits of axial stability can be readily examined for special cases such as that in which the radial motion is introduced with initial conditions characteristic of the stable equilibrium orbit. For the structure with the parameters to which fig. 3 pertains, one finds in this way an axial amplitude limit of slightly over 0.014r—this limit applies to locations such that $N\theta = 0 \pmod{2\pi}$, near the center of an axially defocusing region, and has associated with it amplitude limits which become almost twice as large at intermediate points.

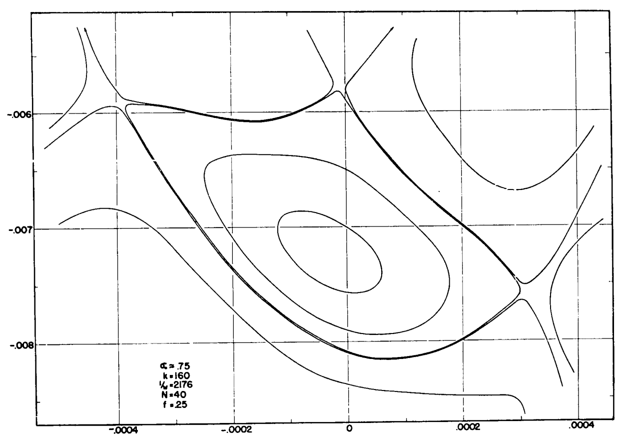

Fig. 4.

Similarly constructed phase plots for other values of machine parameters are shown in fig. 4 and the following figures. We are indebted to N. Vogt-Nilsen for supplying these plots from his studies of orbit stability.

Coupled axial and radial motion is more difficult to study systematically. By examining the behavior of the axial motion for various amplitudes of radial oscillation, however, some progress has already been made in the examination of the importance of various resonances involving the two frequencies which characterize the small-amplitude motion.

When the machine as-a-whole is considered, as it must because the presence of unavoidable misalignments makes the basic period strictly not one sector but one complete revolution, numerous additional resonances become possible. The effect of some of these has been examined with the ILLIAC, and further active investigation of this question is planned.

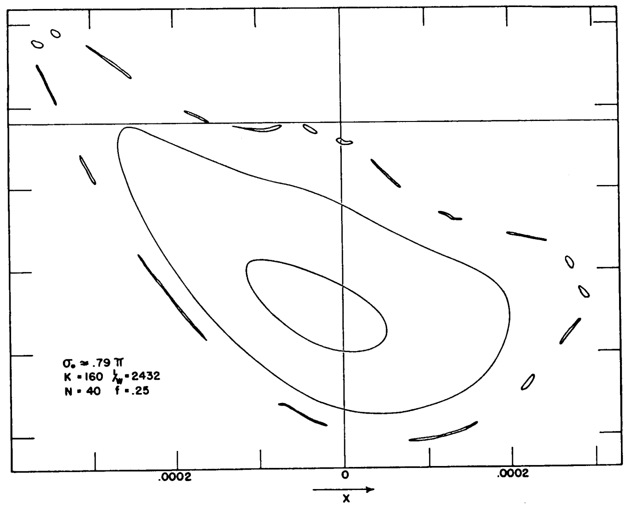

Fig. 5.

8. *Application of Walkinshaw's equation to the $2\sigma_y = \sigma_x$ resonance*

A method of analysis which appears to account for the behavior of the axial motion, in the presence of appreciable radial oscillation, has been developed by Walkinshaw[6]. The differential equation characterizing the axial motion is trated as *linear*, but contains a coefficient which involves the radial motion. As is well-known, the forced radial motion enhances the A-G focusing which appears in the axial equation—now, however, the additional effect of the free radial betatron oscillations is also included in the axial equation. The super-position of the comparatively-long-wavelength radial oscillations on the forced motion in effect modulates the smooth-approximation coefficient in the axial equation, to yield a Mathieu equation with a coefficient *having the period of the radial motion.* Under "resonant" conditions, which will be seen to include the case of interest here, this equation may have unstable solutions and, in such cases, the characteristic exponent of the solution appears to compare reasonably in magnitude with the lapserate characterizing the exponential growth of the ILLIAC solutions of the "Feckless Five" equations.

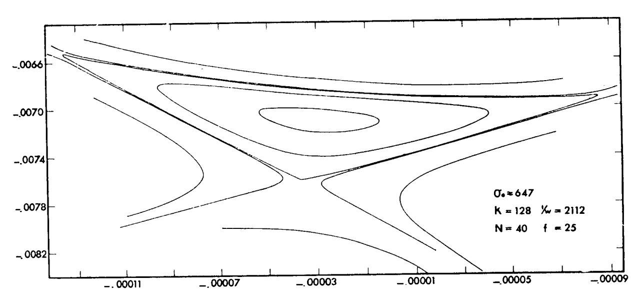

Fig. 6.

Walkinshaw's analysis pertains to differential equations which, in the MURA notation (f. ex., LJL(MURA) - 5), are taken to be of the form

$$x'' + (k + 1)\,x = -f \sin (x/w - N\theta), \qquad (8.1)$$

$$y'' + [-k - (f/w) \cos (x/w - N\theta)\,y = 0 \qquad (8.2)$$

(cf. LJL MURA Notes 6-22 Oct. 1955, Sect. 6, for $y/w \ll 1$). A solution for the radial motion, representing a free oscillation of amplitude A superposed on the forced motion, is taken of the form

$$x = A \cos (\nu_x\theta + \varepsilon) - (f/\Omega^2) \sin \textstyle\int \Omega d\theta, \qquad (8.3)$$

where

$$\Omega \cong N + A\,(\nu_x/w) \sin (\nu_x\theta + \varepsilon) \text{ and } \nu_x \doteq (k+1)^{1/2} \quad (8.4)$$

This solution is substituted into the axial equation to yield, after some approximation (and a shift of the origin of θ which we introduce for convenience),

$$y'' + \left[-k + \frac{f^2}{w^2N^2}\left(1 + \frac{2A\,\nu_x}{wN} \cos \nu_x\theta\right)\right] y = 0. \qquad (8.5)$$

It is noted that, when $A = 0$, this equation reduces to that given by the smooth approximation—we accordingly write

$$y'' + \left[\nu_y^2 + \frac{2Af^2\,\nu_x}{w^3N^3} \cos \nu_x\theta\right] y = 0, \qquad (8.6)$$

to obtain an equation of the Mathieu type with a coefficient of period $2\pi/\nu_x$ in θ. By the transformation $\nu_x\theta = 2\,t$, we have the standard form

$$d^2y/dt^2 + \left[(2\,\nu_y/\nu_x)^2 + \frac{8\,f^2}{w^3N^3}\frac{A}{\nu_x} \cos 2\,t\right] y = 0 \qquad (8.7)$$

with a coefficient of period π in the independent variable t.

A solution of the Mathieu equation

$$d^2y/dt^2 + [a + b \cos 2t]\,y = 0, \qquad (8.8)$$

for b small but not zero, will exhibit instability when the coefficient a is equal or close to the square of an integer. In the present application stop-bands may thus be expected at operating points such that $2\,\nu_y/\nu_x = m$, the broad band of instability at $2\,\nu_y/\nu_x = 1$ (or z $\sigma_y/\sigma_x = 1$) being of chief interest in connection with the work presented here. It appears, moreover, possible to employ the Mathieu equation to account semi-quantitatively for (i) the range of b, and hence of the amplitude of free radial oscillation, which may be permitted when the oscillation frequencies depart by a specified amount from the resonant condition, and (ii) the lapse rate found to characterize the growth of the axial motion when the radial oscillations exceed this limit.

The numerical application of the Mathieu equation to specific problems of stability or instability may be accomplished by reference to ILLIAC solutions for the stability boundaries or for the characteristic exponent characterizing the solution.

(i) A useful estimate of the expected restrictions on the radial motion may be obtained, however, by appeal to the fact that near $a = 1$, $b = 0$ the stability boundaries can be represented rather well by the condition

$$|b| \doteq 2\,|a - 1|. \qquad (8.9)$$

We find in this way the following estimate for the limiting amplitude :

$$A_1 = \frac{w^3N^3}{4f^2}\,\nu_x\,|(2\,\nu_y/\nu_x)^2 - 1|$$

$$\cong \frac{w^3N^3}{2f^2}\,|2\nu_y - \nu_x|\ \left(\text{for } \frac{2\nu_y}{\nu_x} - 1 \ll 1\right). \qquad (8.10)$$

It may be noted that this result, although expressed in terms of ν_x and ν_y, concerns an inherent *sector resonance* which arises when $2\,\sigma_y/\sigma_x = 1$. This resonance is particularly interesting in that it does not appear to fall under the general criteria outlined in Section 6.

(ii) An estimate of the lapse rate characterizing unstable solutions near $a = 1$, $b = 0$ may, moreover, be made by taking

$$\mu \doteq \frac{\pi}{4}\sqrt{b^2 - 4(a-1)^2} \text{ nepers for } \Delta t = \pi \quad \text{when } (|b| > 2|a-1|)$$

$$= \frac{\pi}{4}\frac{\nu_x}{N}\sqrt{b^2 - 4(a-1)^2} \text{ nepers per sector}$$

$$= \frac{\pi/4}{N}\sqrt{\left(\frac{8F^2A}{w^3N^3}\right)^2 - 4\left[(2\,\nu_y)^2 - \nu_x^2\right]^2/\nu_x^2} \text{ nepers per sector}$$

$$= \frac{0.68}{N}\sqrt{\left(\frac{4f^2A}{w^3N^3}\right)^2 - \left[(2\,\nu_y)^2 - \nu_x^2\right]^2/\nu_x^2} \text{ decades per sector} \qquad (8.11)$$

A convenient alternative form for this last result is

$$\mu = \frac{2\pi\,F^2}{w^3\,N^4}\sqrt{A^2 - A_1^2} \text{ nepers/sector}$$

$$\frac{2.73\,F^2}{w^3\,N^4}\sqrt{A^2 - A_1^2} \text{ decades/sector.} \qquad (8.12)$$

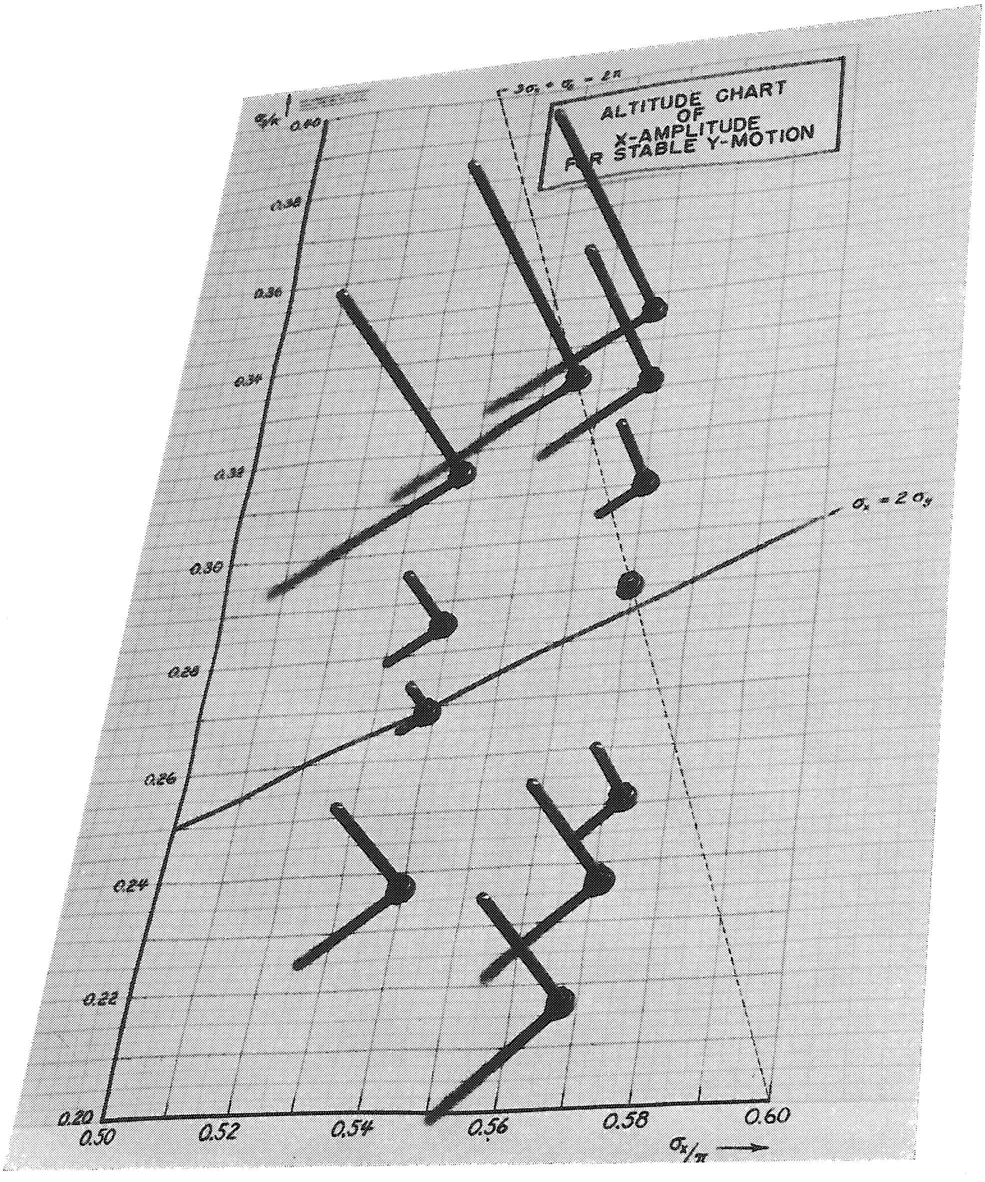
ALTITUDE CHART OF X-AMPLITUDE FOR STABLE Y-MOTION
$3\sigma_x + \sigma_y = 2\pi$
$\sigma_x = 2\sigma_y$
σ_y/π
0.40
0.38
0.36
0.34
0.32
0.30
0.28
0.26
0.24
0.22
0.20
0.50
0.52
0.54
0.56
0.58
0.60
σ_x/π

Fig. 7.

Results obtained with the ILLIAC, for 5-sector machines with model-like parameters such that $0.5\pi < \sigma_{x0} < 0.6\pi$ and $0.2\pi < \sigma_{y0} < 0.4\pi$, appear fairly close to these estimates. In all the ILLIAC runs the radial amplitudes were measured, however, near the center of a focusing region, at $N\theta = 0$ (Mod. 2π), where the amplitudes of the non-sinusoidal A-G oscillations can exceed those corresponding to the smooth approximation representation of the motion. By way of example we present here the results for an accelerator for which

$$k = 0.6436 \qquad 1/w = 20.82 \qquad f = 1/4 \qquad N = 5:$$

In this case the oscillation frequencies are such that

$$\left.\begin{array}{l}\sigma_{x0} = 0.5388\pi \\ \sigma_{y0} = 0.2855\pi\end{array}\right\} \text{ or } \left\{\begin{array}{l}\nu_{x0} = 1.347 \\ \nu_{y0} = 0.714\end{array}\right.$$

and the limiting amplitude for x appeared to be some 0.0075 units to the left of the stable fixed point ($N\theta = 0$, mod. 2π). For these machine parameters the equation for A_1 yields

$$A_1 = \frac{500}{(20.82)^3}\, 1.347\, [(1.06)^2 - 1]$$

$= 0.0092$, the observed limiting amplitude at $N\theta = 0$ (mod 2π) thus being within 20% of this estimate. With respect to the lapse rate, we continue this example by consideration of the case $A = 0.0225$. Then $\sqrt{A^2 - A_1^2} = 0.02035$, and one expects

$$\mu = \frac{0.171\,(20.82)^3}{625}\,(0.02035)$$

$$= 0.050 \text{ decades/sector},$$

in close agreement with the value 0.055 decades/sector found from the ILLIAC work. (For this case the coefficients in the Mathieu equation are $a = 1.12$, $b = 0.604$, for which an independent extrapolation of coarse tables extending to $a \leqslant 1$ suggests $\mu = 0.107$ nepers/sector = 0.046 decades/sector.) In fig. 7, we plot the amplitude of radial motion for which the vertical motion becomes unstable (represented by the lengths of the rods) at various points in the σ_x, σ_z – plane.

Growth of the axial motion, similar in appearance to that reported here, has also been observed in the neighborhood of the $2\,\sigma_x + 2\,\sigma_y = 2\pi$ and $\sigma_x + 2\sigma_y = 2\pi$ resonances. It appears that these sum resonances may be connected with the presence of terms in the y-equation which involve $u^2y \cos N\theta$ and $uy \sin N\theta$, where u represents the radial oscillation about the scalloped equilibrium orbit.

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