Processing of cosmological perturbations in a cyclic cosmology

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The evolution of the spectrum of cosmological fluctuations from one cycle to the next is studied. It is pointed out that each cycle leads to a reddening of the spectrum. This opens up new ways to generate a scale-invariant spectrum of curvature perturbations. The large increase in the amplitude of the fluctuations quickly leads to a breakdown of the linear theory. More generally, we see that, after including linearized cosmological perturbations, a cyclic universe cannot be truly cyclic.

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I. INTRODUCTION

Recently, there has been renewed interest in cyclic cosmologies. The motivation comes in part from general efforts to construct nonsingular cosmological backgrounds (see e.g. [1] for a recent review), in part from attempts to construct a cyclic cosmology [2] as an extension of the Ekpyrotic universe scenario [3].

A problem which faces most attempts at constructing a cyclic cosmology and which was already pointed out by Tolman [4] is that the background cosmology cannot be cyclic if the fact that entropy is generated in each cycle is taken into account (see, however, [5] for a recent model which partially resolves this problem). However, as discussed in [6], the entropy of cosmological perturbations does not grow as long as the fluctuations remain well described by linear theory. This is due to the fact that each perturbation mode continues to describe a pure state if it starts out describing a pure state.

In this paper we discuss a problem for cyclic cosmology that arises even in the absence of entropy generation: since on super-Hubble scales the curvature fluctuations do not evolve symmetrically—they grow in the contracting phase and are constant in the expanding phase—there is a net growth of the fluctuations from period to period, which destroys the cyclic nature of the cosmology [7].

Since long wavelength modes have a wavelength that in the contracting phase is larger than the Hubble radius for a longer time than short wavelength modes, the spectrum of fluctuations on super-Hubble scales in the contracting phase is redder than the initial spectrum on sub-Hubble scales. Thus, both the amplitude and the slope of the spectrum of fluctuations changes from cycle to cycle—an effect which we call "processing of the spectrum of cosmological fluctuations." The reddening of the spectrum takes place in each contracting phase: the initial sub-Hubble scaling of the spectrum differs from the later super-Hubble scaling.

The amount of reddening depends on the contraction rate of the Universe and hence on the equation of state of the dominant form of matter. As noticed in [8–10], the reddening of the spectrum has the right strength to turn a vacuum spectrum on sub-Hubble scales into a scale-invariant spectrum on super-Hubble scales if the Universe is dominated by cold matter. More specifically, on scales that exit the Hubble radius in a matter-dominated phase, an initial vacuum spectrum on sub-Hubble scales is converted into a scale-invariant one on super-Hubble scales. This observation was used to propose the "matter bounce" alternative to cosmological inflation for creating a scale-invariant spectrum of cosmological fluctuations. Such a matter bounce is naturally realized [11] in the context of the "Lee-Wick" model [12] for scalar field matter, which is a particular case of the more general quintom matter bounce scenario [13]. Modified gravity theories such as the Biswas *et al.* [14] ghost-free higher derivative gravity theory or Hořava-Lifshitz gravity [15] on nonflat spatial sections can also lead to matter bounce scenarios, as studied in [16,17], respectively [18].

In this paper we will compute the change in the amplitude and slope of the spectrum of cosmological perturbations from one cycle to the next. We begin with a short review of the relevant formalism. Then, we compute the change in the spectrum of cosmological perturbations from one cycle to the next.

II. FLUCTUATIONS IN A CYCLIC BACKGROUND COSMOLOGY

We postulate the existence of a cyclic background cosmology. The turnaround between the expanding phase and the contracting phase at large radius could be generated by a spatial curvature term (in the absence of a cosmological constant), for the turnaround between the contracting phase and the expanding phase new ultraviolet (UV) physics, which violates the weak energy condition, is required. In the context of the Einstein field equations, such new UV physics can be modeled by quintom matter [20]. Asymptotically free higher derivative gravity actions such as the ones proposed in [14,21] can also lead to a nonsingular bounce. Finally, in a background with nonvanishing spatial curvature, Hořava-Lifshitz gravity [15] can also lead to a bounce under the conditions on matter spelled out in [17].

In Fig. 1 we present a space-time sketch of a cyclic cosmology. We choose the origin of the time coordinate (vertical axis) to coincide with a bounce point. The equations simplify if we work in terms of conformal time η, which is related to the physical time t via $dt = a(\eta)d\eta$, where $a(\eta)$ is the scale factor. During the time interval between $-\eta_c$ and η_c, the new UV physics that yields the nonsingular bounce is dominant, for other times the effective equations of motion for gravity are assumed to reduce to those of Einstein gravity.

The horizontal axis in Fig. 1 denotes comoving distance x. The vertical line corresponds to the wavelength of a cosmological fluctuation mode. This mode crosses the Hubble radius (the dashed curve) at times $\pm\eta_H(k)$.

In the presence of cosmological perturbations without anisotropic stress, the metric in longitudinal gauge takes the form

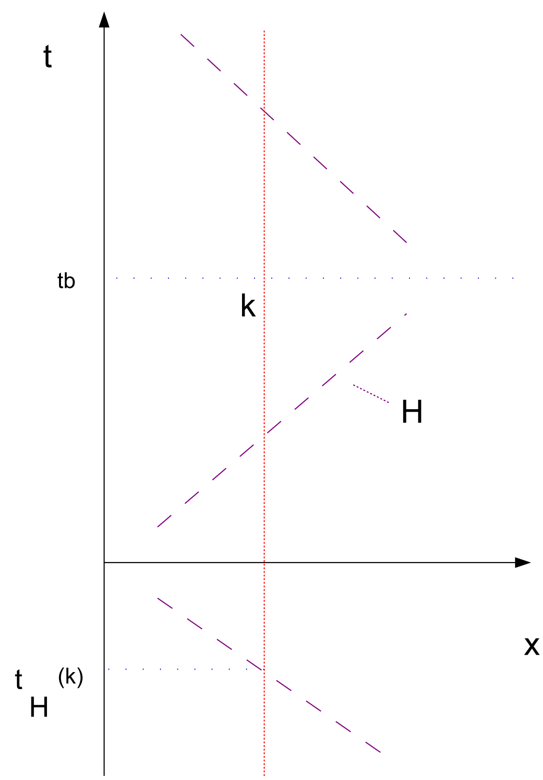

FIG. 1 (color online). Space-time sketch of the cyclic background cosmology. Time is along the vertical axis, comoving length along the horizontal axis. The majenta curve (labeled by H) denotes the Hubble radius $H^{-1}(t)$, the vertical line labeled by k represents the wavelength of a perturbation mode, which exits the Hubble radius at the time $t_H(k)$. The origin of the time axis is chosen to be the nonsingular bounce point (around the bounce the Hubble radius diverges to infinity, an effect not shown on the graph. The time tb is the turnaround from the expanding phase to the contracting phase.

$$ds^2 = a^2(\eta)[(1 + 2\Phi)d\eta^2 - (1 - 2\Phi)d\mathbf{x}^2], \quad (1)$$

where the function $\Phi(x, \eta)$ describes the fluctuations (see e.g. [22] for a review of the theory of cosmological perturbations). We are interested in computing the spectrum of ζ, the function describing the curvature fluctuations in comoving coordinates. ζ is given in terms of Φ via

$$\zeta = \frac{2}{3}(\mathcal{H}\Phi' + \Phi)\frac{1}{1 + w} + \Phi, \quad (2)$$

$\mathcal{H}$ denoting the Hubble expansion rate in conformal time, a prime indicating the derivative with respect of η, and $w = p/\rho$ being the equation of state parameter of matter (p and ρ are pressure and energy density, respectively). The variable ζ in turn is closely related to the variable v [23,24] in terms of which the action for cosmological fluctuations has canonical kinetic term

$$\zeta = \frac{v}{z}. \quad (3)$$

Here, $z(\eta)$ is a function of the background, which for constant equation of state is proportional to the scale factor $a(\eta)$.

At linear order in perturbation theory, the equation of motion for the Fourier mode v_k of v is that of a harmonic oscillator with a mass whose time dependence is given by the background cosmology

$$v_k'' + \left(k^2 - \frac{z''}{z}\right)v_k = 0. \quad (4)$$

This shows that, whereas on sub-Hubble scales v_k is oscillating with approximately constant amplitude, on length scales larger than the Hubble radius (where the k^2 term is negligible) the oscillations of v freeze out and the time dependence of v is given by that of the background. One of the solutions of (4) on super-Hubble scales evolves as $v \sim z \sim a$ and corresponds to a constant value of ζ. The second mode of v corresponds to a decreasing mode of ζ in an expanding Universe. On the other hand, in a contracting phase it corresponds to an increasing mode.

We see that the evolution of the curvature fluctuation ζ is asymmetric between the contracting phase and the expanding phase. The dominant mode is constant on super-Hubble scales in the expanding phase whereas it is increasing in the contracting phase. It is this asymmetry that is responsible for the processing of the spectrum of fluctuations in a cyclic background cosmology.

III. PROCESSING OF THE SPECTRUM OF COSMOLOGICAL PERTURBATIONS

In this section, we will study the processing of the spectrum of cosmological perturbations from one cycle to the next. We will consider scales for which the background is expanding (or contracting) as a power of time

$$a(t) \sim t^p. \quad (5)$$

In this case, the solutions of (4) are given by

$$v(\eta) \sim \eta^{\alpha}, \tag{6}$$

with

$$\alpha = \frac{1}{2} \pm \nu, \qquad \nu = \frac{1}{2}\frac{1-3p}{1-p}. \tag{7}$$

We are interested in the range of values $1/3 < p < 1$ for which ν is a negative number. In the contracting phase prior to the bounce at $\eta = 0$, the dominant solution of (4) thus scales as

$$v_k(\eta) \sim \eta^{1/2+\nu} \sim \eta^{((1-2p)/(1-p))}. \tag{8}$$

There are two cases of special interest. First, in a matter-dominated universe $p = 2/3$ and hence the dominant mode of v_k scales as

$$v_k(\eta) \sim \eta^{-1}, \tag{9}$$

whereas in a universe dominated by relativistic radiation $p = 1/2$ and hence

$$v_k(\eta) \sim \text{const.} \tag{10}$$

In fact, from (8) we see that for matter with an equation of state $w > 1/3$ the amplitude of v is decreasing as the bounce is approached, whereas for $w < 1/3$ the amplitude is increasing. We will focus on the more physical second case.

Let us assume initial conditions for fluctuations on sub-Hubble scales at some initial time $-\eta_i$ long before the bounce point

$$v_k(-\eta_i) = z(-\eta_i)A_i^{1/2}k^{-3/2}\left(\frac{k}{k_0}\right)^{(n_i-1)/2}, \tag{11}$$

where A_i is the initial amplitude of the spectrum of ζ, n_i is the initial slope, and k_0 is the pivot scale. Our aim is to calculate the amplitude A_f and slope n_f after one cycle.

Making use of the fact that v_k oscillates until the time $-\eta_H(k)$ when the wavelength exits the Hubble radius and subsequently increases in amplitude as given by (8), we see that at the time $-\eta_c$ immediately before the bounce, the time when the weak energy violating effects that yield the nonsingular bounce start to dominate, the amplitude of v_k is given by

$$v_k(-\eta_c) = \left(\frac{-\eta_H(k)}{|\eta_c|}\right)^{(2p-1)/(1-p)} v_k(-\eta_i). \tag{12}$$

Making use of the Hubble radius crossing condition

$$\eta_H(k) \sim k^{-1}, \tag{13}$$

we obtain the following power spectrum of ζ just before the bounce

$$P_\zeta(k, -\eta_c) \simeq \left(\frac{z(-\eta_i)}{z(-\eta_c)}\right)^2 (\eta_c k)^{-2(2p-1)/(1-p)} A_i \left(\frac{k}{k_0}\right)^{n_i-1} = A_f\left(\frac{k}{k_0}\right)^{n_f-1}, \tag{14}$$

with

$$n_f = n_i - 2\frac{2p-1}{1-p} \tag{15}$$

and

$$A_f = \left(\frac{z(-\eta_i)}{z(-\eta_c)}\right)^2 A_i(\eta_c k_0)^{-2(2p-1)/(1-p)}. \tag{16}$$

The next step of the analysis is to follow v_k through the nonsingular bounce from time $-\eta_c$ to η_c. The first approach to do this would be to match the values of the fluctuations at the 2 times using the Hwang-Vishniac [25] (see also [26]) matching conditions. However, the applicability of this prescription is questionable [27] since the background does not satisfy these matching conditions. In the case of a nonsingular bounce we can, however, follow the evolution of the fluctuations through the bounce explicitly, assuming the validity of the Einstein equations for the fluctuations. Since we are dealing with modes that are in the far infrared (even at the bounce point) compared to the characteristic scale of the bounce, this assumption is a safe one to make (as has been verified explicitly in [16] for the bounce model of [14]). The lesson that has been learned by following fluctuations through the bounce in several models [13,28–30] is that the spectrum of v_k does not change on scales for which the wavelength is long compared to the duration of the bounce. This is clearly satisfied for the case of interest to us.

In the expanding phase, the amplitude of ζ_k is constant on super-Hubble scales. After the wavelength reenters the Hubble radius, the amplitude of v_k does not change until the mode leaves the Hubble radius once more in the contracting phase of the following cycle. To obtain the change in the spectrum of cosmological perturbations, we need to evaluate the spectrum of fluctuations at the time η_i, the mirror image of the initial time. The spectrum at η_i will be identical to the spectrum at the time corresponding to $-\eta_i$ in the next cycle.

To compute this spectrum, note that the amplitude of ζ_k is decreasing between the time $\eta_H(k)$ when the mode reenters the Hubble radius and the time η_i since it is v_k, which has constant amplitude during this time interval. Thus,

$$P_\zeta(k, \eta_i) = \left(\frac{z(\eta_H(k))}{z(\eta_c)}\right)^2 P_\zeta(k, \eta_c). \tag{17}$$

Making use of the fact that z scales as $\eta^{p/(1-p)}$ and of the relations (14) and (16) we find

$$P_\zeta(k,\eta_i) = \left(\frac{\eta_H(k)}{\eta_c}\right)^{2p/(1-p)} (\eta_c k_0)^{-2((2p-1)/(1-p))} A_i \left(\frac{k}{k_0}\right)^{n_f - 1} = A_F \left(\frac{k}{k_0}\right)^{n_F - 1}, \quad (18)$$

with

$$n_F = n_i - 2\frac{3p-1}{1-p} \quad (19)$$

and

$$A_F = A_i (\eta_c k_0)^{-2(3p-1)/(1-p)}. \quad (20)$$

The results (19) and (20) give the change in the slope and in the amplitude of the spectrum of cosmological perturbations from one cycle to the next. Each cycle leads to a reddening of the spectrum and to an increase in its amplitude. The slope changes by $-2(3p-1)/(1-p)$. The increase in amplitude soon leads to a breakdown of the validity of the perturbative analysis, with consequences for the multiverse discussed in [31].

As mentioned in the Introduction, if $p = 2/3$, then an initial vacuum spectrum ($n = 3$) on sub-Hubble scales is transformed into a scale-invariant spectrum after the bounce. If we want an initial Poisson spectrum ($n = 4$) to be transformed into a scale-invariant spectrum after one complete cycle in the second expanding phase, the background needs to satisfy $p = 7/13$, i.e. an equation of state close to that of radiation. This is a simple application of the processing of the spectrum of cosmological perturbations, which we have discussed here.

IV. CONCLUSIONS AND DISCUSSION

We have studied the evolution of the linear cosmological fluctuations from one cycle to the next in a cosmology with a periodic background. Because of the asymmetry in the evolution of fluctuations in the contracting and expanding phases, there is a large increase in the amplitude of the perturbations from one bounce to the next. In addition, there is a characteristic reddening of the shape of the spectrum.

The immediate implication of our analysis is that the evolution of the Universe in a cosmology with a cyclic background is not cyclic (as already stressed in [31]).

The characteristic processing of the slope of the spectrum of perturbations thatwe discuss in this paper in principle opens up new avenues of generating a scale-invariant spectrum of perturbations during a specific cycle. For example, a matter-dominated background with a spectrum with an initial steep blue index $n_s = 5$ will yield a scale-invariant spectrum after two bounces, in the same way that an initial vacuum spectrum with $n_s = 3$ yields a scale-invariant spectrum after one bounce [9]. The processing of fluctuations also opens up new ways of generating a scale-invariant spectrum of curvature perturbations starting from thermal inhomogeneities [32]. Note that a characteristic feature of such scenarios are large non-Gaussianities with a particular shape, as worked out in [33].

However, the increase in the amplitude of the fluctuations quickly leads to a breakdown of the linear analysis. The nonlinear evolution will then lead to entropy production and to the usual problems for cyclic background cosmologies first discussed in [4].

We should note that the cyclic version [2] of the Ekpyrotic universe scenario does not suffer from the problems discussed here since the scale factor of ‘‘our’’ Universe in the Ekpyrotic model [3] is not cyclic. The only cyclicity in the Ekpyrotic scenario is in the evolution of the distance between the two boundary branes. The scale factor of our Universe is monotonically increasing. Thus, perturbations produced at a fixed physical scale in one ‘‘cycle’’ are redshifted by the time the next cycle arrives.

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