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Duality, entropy, and ADM mass in supergravity

Bianca L. Cerchiai,[1,2,*] Sergio Ferrara,[3,4,‡] Alessio Marrani,[5,4,§] and Bruno Zumino[1,2,†]

[1]*Lawrence Berkeley National Laboratory, Theory Group, Bldg 50A5104, 1 Cyclotron Rd, Berkeley, California 94720-8162, USA*
[2]*Department of Physics, University of California, Berkeley, California 94720-8162, USA*
[3]*Theory division, CERN, Geneva, Switzerland, CH 1211, Geneva 23, Switzerland*
[4]*INFN-LNF, Via Enrico Fermi 40, I-00044 Frascati, Italy*
[5]*Stanford Institute for Theoretical Physics, Department of Physics, 382 Via Pueblo Mall, Varian Lab, Stanford University, Stanford, California 94305-4060, USA*

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We consider the Bekenstein-Hawking entropy-area formula in four dimensional extended ungauged supergravity and its electric-magnetic duality property. Symmetries of both "large" and "small" extremal black holes are considered, as well as the ADM mass formula for $\mathcal{N} = 4$ and $\mathcal{N} = 8$ supergravity, preserving different fraction of supersymmetry. The interplay between BPS conditions and duality properties is an important aspect of this investigation.

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I. INTRODUCTION

In $d = 4$ extended *ungauged* supergravity theories based on scalar manifolds which are (*at least locally*) *symmetric* spaces

$$M = \frac{G}{H}, \tag{1.1}$$

it is known that the classification of static, spherically symmetric and asymptotically flat extremal black hole (BH) solutions is made in terms of *charge orbits* of the corresponding *classical* electric-magnetic duality group G [1–6] (later called U-duality[1] in string theory).

These orbits correspond to certain values taken by a *duality invariant*[2] combination of the "*dressed*" central charges and matter charges. Denoting such an invariant by $\mathcal{I}$, the set of scalars parametrizing the symmetric manifold M by ϕ, and the set of "*bare*" magnetic and electric charges of the (*dyonic*) BH configuration by the $2n \times 1$ symplectic vector

$$\mathcal{P} \equiv \begin{pmatrix} p^{\Lambda} \\ q_{\Lambda} \end{pmatrix}, \qquad \Lambda = 1, \ldots, n, \tag{1.2}$$

then it holds that

$$\partial_{\phi} \mathcal{I}(\phi, \mathcal{P}) = 0 \Leftrightarrow \mathcal{I} = \mathcal{I}(\mathcal{P}). \tag{1.3}$$

In some cases, the relevant invariant $\mathcal{I}$ is not enough to characterize the orbit, and additional constraints are needed. This is especially the case for the so-called[3] *small* BHs, in which case $\mathcal{I} = 0$ on the corresponding orbit [3,4,9].

An explicit expression for the $E_{7(7)}$-invariant [10] was firstly introduced in supergravity in [11], and then adopted in the study of BH entropy in [12]. The additional U-invariant constraints which specify charge orbits with higher supersymmetry were given in [3]. The corresponding (*large* and *small*) charge orbits for $\mathcal{N} = 8$ and exceptional $\mathcal{N} = 2$ supergravity were determined in [4], whereas the *large* orbits for all other *symmetric* $\mathcal{N} = 2$ supergravities were obtained in [6], and then in [13] for all $\mathcal{N} > 2$-extended theories. Furthermore, the invariant for $\mathcal{N} = 4$ supergravity was earlier discussed in [14,15].

The invariants play an important role in the *attractor mechanism* [16–20], because the Bekenstein-Hawking BH entropy [8], determined by evaluating the *effective black hole potential* ([18–20])

$$V_{\rm BH}(\phi, \mathcal{P}) \equiv -\frac{1}{2} \mathcal{P}^T \mathcal{M}(\phi) \mathcal{P} \tag{1.4}$$

at its critical points, actually coincides with the relevant invariant:

$$\begin{aligned} \frac{S_{\rm BH}}{\pi} &= V_{\rm BH}|_{\partial_\phi V_{\rm BH}=0} = V_{\rm BH}(\phi_H(\mathcal{P}), \mathcal{P}) \\ &= |\mathcal{I}(\mathcal{P})|^{1/2} (\text{or} |\mathcal{I}(\mathcal{P})|). \end{aligned} \tag{1.5}$$

In Eq. (1.4) $\mathcal{M}$ stands for the $2n \times 2n$ real (negative definite) symmetric scalar-dependent symplectic matrix

*BLCerchiai@lbl.gov,
†zumino@thsrv.lbl.gov
‡sergio.ferrara@cern.ch
§marrani@lnf.infn.it

[1]Here U-duality is referred to as the "*continuous*" version, valid for large values of the charges, of the U-duality groups introduced by Hull and Townsend [7].

[2]By *duality invariant*, throughout our treatment we mean that such a combination is G-invariant. Thus, it is actually *independent* on the scalar fields, and it depends only on "*bare*" electric and magnetic (asymptotical) charges (defined in Eq. (1.2)).

[3]Throughout the present treatment, we will, respectively, call *small* or *large* (extremal) BHs those BHs with vanishing or nonvanishing area of the event horizon (and therefore with vanishing or nonvanishing *Bekenstein-Hawking entropy* [8]). For symmetric geometries, they can be G-invariantly characterized, respectively, by $\mathcal{I} = 0$ or by $\mathcal{I} \neq 0$.

$$\mathcal{M}(\phi) \equiv \begin{pmatrix} \mathrm{Im}\mathcal{N}_{\Lambda\Sigma} + \mathrm{Re}\mathcal{N}_{\Lambda\Xi}(\mathrm{Im}\mathcal{N})^{-1|\Xi\Delta}\,\mathrm{Re}\mathcal{N}_{\Delta\Sigma} & -\mathrm{Re}\mathcal{N}_{\Lambda\Xi}(\mathrm{Im}\mathcal{N})^{-1|\Xi\Sigma} \\ -(\mathrm{Im}\mathcal{N})^{-1|\Lambda\Delta}\,\mathrm{Re}\mathcal{N}_{\Xi\Sigma} & (\mathrm{Im}\mathcal{N})^{-1|\Lambda\Sigma} \end{pmatrix}, \tag{1.6}$$

defined in terms of the normalization of the Maxwell and topological terms[4]

$$\mathrm{Im}\,\mathcal{N}_{\Lambda\Sigma}(\phi)F^{\Lambda}F^{\Sigma},\, \mathrm{Re}\mathcal{N}_{\Lambda\Sigma}(\phi)F^{\Lambda}\tilde{F}^{\Sigma} \tag{1.7}$$

of the corresponding supergravity theory (see *e.g.* [21,22] and Refs. therein). Furthermore, in Eq. (1.5) $\phi_H(\mathcal{P})$ denotes the set of charge-dependent, stabilized horizon values of the scalars, solutions of the criticality conditions for V_{BH}:

$$\left.\frac{\partial V_{\mathrm{BH}}(\phi, \mathcal{P})}{\partial\phi}\right|_{\phi=\phi_H(\mathcal{P})} \equiv 0. \tag{1.8}$$

For the case of charge orbits corresponding to *small* BHs, in the case of a *single-center* solution $\mathcal{I}(\mathcal{P}) = 0$, and thus the event horizon area vanishes, and the solution is singular (*i.e.* with vanishing Bekenstein-Hawking entropy). However, the charge orbits with vanishing duality invariant play a role for *multicenter* solutions as well as for elementary BH constituents through which *large* (*i.e.* with nonvanishing Bekenstein-Hawking entropy) BHs are made [23–25].

In the present investigation, we reexamine the duality invariant and the U-invariant classification of charge orbits of $\mathcal{N} = 8$, $d = 4$ supergravity, we give a complete analysis of the $\mathcal{N} = 4$ *large* and *small* charge orbits, and we also derive a diffeomorphism-invariant expression of the $\mathcal{N} = 2$ duality invariant, which is common to all symmetric spaces and which is completely independent on the choice of a symplectic basis.

The paper is organized as follows.

In Sec. II we recall some basic facts about electric-magnetic duality in $\mathcal{N}$-extended supergravity theories, firstly treated in [2]. The treatment follows from the general analysis of [1], and the dictionary between that paper and the present work is given.

In Sec. III we reexamine $\mathcal{N} = 8$, $d = 4$ supergravity and the $E_{7(7)}$-invariant characterization of its charge orbits. This refines, reorganizes and extends the various results of [3–5,9].

In Sec. IV we reconsider *matter coupled* $\mathcal{N} = 4$, $d = 4$ supergravity. The $SL(2,\mathbb{R}) \times SO(6, M)$-invariant characterization of all its BPS and non-BPS charge orbits, firstly obtained in [3,9], is the starting point of the novel results presented in this section.

Sec. V is devoted to the analysis of the $\mathcal{N} = 2$, $d = 4$ case [3]. Beside the generalities on the special Kähler geometry of Abelian vector multiplets' scalar manifold, the results of this section are novel. In particular, a formula for the duality invariant is determined, which is *diffeomorphism-invariant* and holds true for all *symmetric* special Kähler manifolds (see *e.g.* [26] and Refs. therein), regardless of the considered symplectic basis.

Sec. VI, starting from the analysis of [3,9], deals with the issue of the *ADM mass* [27] in $\mathcal{N} = 8$ (Subsec. VI A) and $\mathcal{N} = 4$ (Subsec. VI B), ungauged $d = 4$ supergravities. In general, for all supersymmetric orbits the *ADM mass* has a known explicit expression, depending on the number of supersymmetries preserved by the state which is supported by the considered orbit (saturating the *BPS* [28] bound).

II. ELECTRIC-MAGNETIC DUALITY IN SUPERGRAVITY: BASIC FACTS

The basic requirement for consistent coupling of a non-linear sigma model based on a *symmetric* manifold (1.1) to $\mathcal{N}$-extended, $d = 4$ supergravity (see *e.g.* [21] and Refs. therein) is that the vector field strengths and their duals (through *Legendre transform* with respect the Lagrangian density $\mathcal{L}$)

$$F^{\Lambda}, \qquad G_{\Lambda} \equiv \frac{\delta\mathcal{L}}{\delta F^{\Lambda}}, \tag{2.1}$$

belong to a *symplectic* representation $\mathbf{R}_s$ of the global (*classical*, see Footnote 1) U-duality group G, given by $2n \times 2n$ matrices with block structure

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2n, \mathbb{R}), \tag{2.2}$$

where A, B, C and D are $n \times n$ real matrices. By defining the $2n \times 2n$ *symplectic* metric (each block being $n \times n$)

$$\Omega \equiv \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{2.3}$$

the *finite symplecticity condition* for a $2n \times 2n$ real matrix P

$$P^T\Omega P = \Omega \tag{2.4}$$

yields the following relations to hold for the block components of the matrix defined in Eq. (2.2):

$$A^TC - C^TA = 0; \tag{2.5}$$

$$B^TD - D^TB = 0; \tag{2.6}$$

$$A^TD - C^TB = 1. \tag{2.7}$$

An analogous, equivalent definition of the representation $\mathbf{R}_s$ is the following one: $\mathbf{R}_s$ is real and it contains the

[4]Attention should be paid in order to distinguish between the notations of the number $\mathcal{N}$ of *supercharges* of a supergravity theory and the *kinetic vector matrix* $\mathcal{N}_{\Lambda\Sigma}$ introduced in Eqs. (1.6) and (1.7).

singlet in its 2-fold antisymmetric tensor product

$$(\mathbf{R}_s \times \mathbf{R}_s)_a \ni \mathbf{1}. \tag{2.8}$$

If the basic requirements (2.5), (2.6), and (2.7) or (2.8) are met, the *coset representative* of M in the symplectic representation $\mathbf{R}_s$ is given by the (*scalar-dependent*) $2n \times 2n$ matrix

$$S(\phi) \equiv \begin{pmatrix} A(\phi) & B(\phi) \\ C(\phi) & D(\phi) \end{pmatrix} \in Sp(2n, \mathbb{R}). \tag{2.9}$$

A particular role is played by the two (*scalar-dependent*) complex $n \times n$ matrices f and h, which do satisfy the properties

$$- f^\dagger h + h^\dagger f = i1, \tag{2.10}$$

$$- f^T h + h^T f = 0. \tag{2.11}$$

The constraining relations (2.10) and (2.11) are equivalent to require that

$$S(\phi) = \sqrt{2}\begin{pmatrix} \mathrm{Re} f & -\mathrm{Im} f \\ \mathrm{Re} h & -\mathrm{Im} h \end{pmatrix}, \tag{2.12}$$

or equivalently:

$$f = \frac{1}{\sqrt{2}}(A - iB); \tag{2.13}$$

$$h = \frac{1}{\sqrt{2}}(C - iD). \tag{2.14}$$

In order to make contact with the formalism introduced by Gaillard and Zumino in [1], it is convenient to use another (complex) basis, namely, the one which maps an element $S \in Sp(2n, \mathbb{R})$ into an element $U \in U(n, n) \cap Sp(2n, \mathbb{C})$. The change of basis is exploited through the matrix

$$\mathcal{A} \equiv \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -i1 & i1 \end{pmatrix}, \qquad \mathcal{A}^{-1} = \mathcal{A}^\dagger. \tag{2.15}$$

The (*scalar-dependent*) matrix U is thus defined as follows:

$$\begin{aligned} U(\phi) \equiv \mathcal{A}^{-1} S \mathcal{A} &= \frac{1}{\sqrt{2}}\begin{pmatrix} f + ih & \bar{f} + i\bar{h} \\ f - ih & \bar{f} - i\bar{h} \end{pmatrix} \\ &\in U(n, n) \cap Sp(2n, \mathbb{C}). \end{aligned} \tag{2.16}$$

This is the matrix named S in Eq. (5.1) of [1]. Correspondingly, the $Sp(2n, \mathbb{R})$-covariant vector $(F^\Lambda, G_\Lambda)^T$ is mapped into the vector

$$\begin{aligned} \mathcal{A}^{-1}\begin{pmatrix} F^\Lambda \\ G_\Lambda \end{pmatrix} &= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i1 \\ 1 & -i1 \end{pmatrix}\begin{pmatrix} F^\Lambda \\ G_\Lambda \end{pmatrix} \\ &= \frac{1}{\sqrt{2}}\begin{pmatrix} F^\Lambda + iG_\Lambda \\ F^\Lambda - iG_\Lambda \end{pmatrix}. \end{aligned} \tag{2.17}$$

The kinetic vector matrix $\mathcal{N}_{\Lambda\Sigma}$ appearing in Eqs. (1.6) and (1.7) is given by (in matrix notation)

$$\mathcal{N}(\phi) = hf^{-1} = (f^{-1})^T h^T, \tag{2.18}$$

and it is named $-i\bar{K}$ in [1].

Thus, by introducing the $2n \times 1$ ($n \times n$ *matrix-valued*) complex vector

$$\Xi \equiv \begin{pmatrix} f \\ h \end{pmatrix} \tag{2.19}$$

and recalling the definition (1.6), the matrix $\mathcal{M}$ can be written as

$$\begin{aligned} \mathcal{M}(\phi) &= -i\Omega + 2\Omega\Xi(\Omega\Xi)^\dagger = -i\Omega - 2\Omega\Xi\Xi^\dagger\Omega \\ &= -i\Omega - 2\begin{pmatrix} -h \\ f \end{pmatrix}(h^\dagger, \quad -f^\dagger) \\ &= -i\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + 2\begin{pmatrix} hh^\dagger & -hf^\dagger \\ -fh^\dagger & ff^\dagger \end{pmatrix}. \end{aligned} \tag{2.20}$$

Equations (1.4), (1.6), and (2.20) imply that

$$\begin{aligned} V_{\mathrm{BH}}(\phi, \mathcal{P}) &\equiv -\frac{1}{2}\mathcal{P}^T \mathcal{M}(\phi)\mathcal{P} = \mathrm{Tr}(ZZ^\dagger) = \mathrm{Tr}(Z^\dagger Z) \\ &= \sum_{A>B=1}^{\mathcal{N}} Z_{AB}\bar{Z}^{AB} + Z_I\bar{Z}^I = \frac{1}{2}Z_{AB}\bar{Z}^{AB} + Z_I\bar{Z}^I \\ &= \frac{1}{2}\mathrm{Tr}(ZZ^\dagger) + Z_I\bar{Z}^I = \frac{1}{2}\mathrm{Tr}(Z^\dagger Z) + Z_I\bar{Z}^I, \end{aligned} \tag{2.21}$$

where ($A, B = 1, \ldots, \mathcal{N}$ and $I = 1, \ldots, m$ throughout; recall $\Lambda = 1, \ldots, n$)

$$Z \equiv \mathcal{P}^T \Omega \Xi = qf - ph = (Z_{AB}(\phi, \mathcal{P}), Z_I(\phi \mathcal{P} P)); \tag{2.22}$$

$$\Updownarrow$$

$$Z^\dagger \equiv -\Xi^\dagger \Omega \mathcal{P} = f^\dagger q - h^\dagger p = \begin{pmatrix} \bar{Z}^{AB}(\phi, \mathcal{P}) \\ \bar{Z}^I(\phi, \mathcal{P}) \end{pmatrix}; \tag{2.23}$$

$$Z_{AB}(\phi, \mathcal{P}) \equiv f^\Lambda_{AB} q_\Lambda - h_{AB} \mid \Lambda p^\Lambda; \tag{2.24}$$

$$Z_I(\phi, \mathcal{P}) \equiv \bar{f}^\Lambda_I q_\Lambda - \bar{h}_{I|\Lambda} p^\Lambda. \tag{2.25}$$

Thus, Eq. (2.21) yields the "*BH potential*" $V_{\mathrm{BH}}(\phi, \mathcal{P})$ to be nothing but the sum of the squares of the "*dressed*" charges. It is here worth noticing that $(f^\Lambda_{AB}, \bar{f}^\Lambda_I)$ and $(h_{AB|\Lambda}, \bar{h}_{I|\Lambda})$ are $n \times n$ complex matrices, because it holds that[5] $f^\Lambda_{AB} = f^\Lambda_{[AB]}$, $h_{AB|\Lambda} = h_{[AB]|\Lambda}$ (thus implying $Z_{AB} = Z_{[AB]}$), and

[5] Unless otherwise noted, square brackets denote antisymmetrization with respect to the enclosed indices.

$$n = \frac{\mathcal{N}(\mathcal{N}-1)}{2} + m, \tag{2.26}$$

where $\mathcal{N}$ stands for the number of *spinorial supercharges* (see Footnote 4), and m denotes the number of *matter multiplets* coupled to the supergravity multiplet, except for $\mathcal{N}=6$, $d=4$ *pure* supergravity, for which $m=1$.

Equations (2.24) and (2.25) are the basic relation between the (scalar-dependent) "*dressed*" charges Z_{AB} and Z_I and the (scalar-independent) "*bare*" charges $\mathcal{P}$. It is worth remarking that Z_{AB} is the "*central charge matrix function*", whose asymptotical value appears in the right-hand side of the $\mathcal{N}$-extended ($d=4$) supersymmetry algebra, pertaining to the asymptotical Minkowski space-time background:

$$\{Q^A_\alpha, Q^B_\beta\} = \epsilon_{\alpha\beta} Z^{AB}(\phi_\infty, \mathcal{P}), \tag{2.27}$$

where ϕ_∞ denotes the set of values taken by the scalar fields at *radial infinity* ($r \to \infty$) within the considered static, spherically symmetric and asymptotically flat *dyonic extremal* BH background. Notice that the indices A, B of the *central charge matrix* are raised and lowered with the metric of the relevant $\mathcal{R}$*-symmetry* group of the corresponding supersymmetry algebra.

By denoting the *ADM mass* [27] of the considered BH background by $M_{\mathrm{ADM}}(\phi_\infty, \mathcal{P})$, the *BPS bound* [28] implies that

$$M_{\mathrm{ADM}}(\phi_\infty, \mathcal{P}) \geq |\mathbf{Z}_1(\phi_\infty, \mathcal{P})| \geq \ldots \geq |\mathbf{Z}_{[\mathcal{N}/2]}(\phi_\infty, \mathcal{P})|, \tag{2.28}$$

where $\mathbf{Z}_1(\phi, \mathcal{P}), \ldots, \mathbf{Z}_{[\mathcal{N}/2]}(\phi, \mathcal{P})$ denote the set of *skew-eigenvalues* of $Z_{AB}(\phi, \mathcal{P})$, and here square brackets denote the integer part of the enclosed number. If $1 \leq \mathbf{k} \leq [\mathcal{N}/2]$ of the bounds expressed by Eq. (2.28) are saturated, the corresponding extremal BH state is named to be $\frac{\mathbf{k}}{\mathcal{N}}$-BPS. Thus, the minimal fraction of total supersymmetries (pertaining to the asymptotically flat space-time metric) preserved by the extremal BH background within the considered assumptions is $\frac{1}{\mathcal{N}}$ (for $\mathbf{k}=1$), while the maximal one is $\frac{1}{2}$ (for $\mathbf{k} = \frac{\mathcal{N}}{2}$). See Sec. VI for further details.

We end the present Section with some considerations on the issue of duality invariants.

A *duality* invariant $\mathcal{I}$ is a suitable linear combination (in general with complex coefficients) of (ϕ-dependent) H-invariant combinations of $Z_{AB}(\phi, \mathcal{P})$ and $Z_I(\phi, \mathcal{P})$ such that Eq. (1.3) holds, *i.e.* such that $\mathcal{I}$ is invariant under G, and thus ϕ-independent:

$$\mathcal{I} = \mathcal{I}(Z_{AB}(\phi, \mathcal{P}), Z_I(\phi, \mathcal{P})) = \mathcal{I}(\mathcal{P}). \tag{2.29}$$

In presence of *matter coupling*, a charge configuration $\mathcal{P}$ (and thus a certain orbit of the symplectic representation of the U-duality group G, to which $\mathcal{P}$ belongs) is called *supersymmetric iff*, by suitably specifying $\phi = \phi(\mathcal{P})$, it holds that

$$Z_I(\phi(\mathcal{P}), \mathcal{P}) = 0, \quad \forall\ I = 1, \ldots, m. \tag{2.30}$$

Notice that the conditions (2.30) cannot hold *identically* in ϕ, otherwise such conditions would be G-invariant, which generally are *not*. Indeed, in order for the supersymmetry constraints (2.30) to be invariant (or covariant) under G, the following conditions must hold *identically* in ϕ:

$$\partial_\phi Z_I(\phi, \mathcal{P}) = 0, \quad \forall\ \phi \in M. \tag{2.31}$$

Therefore, supersymmetry conditions are *not* generally G-invariant (*i.e.* U-invariant), otherwise extremal BH attractors (which are *large*) supported by supersymmetric charge configurations would not exist.

Nevertheless, in some supergravities it is possible to give U-invariant supersymmetry conditions. In light of previous reasoning, such U-invariant supersymmetric conditions cannot stabilize the scalar fields in terms of charges (by implementing the *attractor mechanism* in the considered framework), because such U-invariant conditions are actually *identities*, *and not equations*, for the set of scalar fields ϕ. Actually, U-invariant supersymmetry conditions can be given for all supersymmetric charge orbits supporting *small* BHs (for which the classical *attractor mechanism* does not hold). This can be seen *e.g.* in $\mathcal{N}=8$ (*pure*) and $\mathcal{N}=4$ (*matter coupled*) $d=4$ supergravities, respectively, treated in Secs. III and IV.

III. $\mathcal{N} = 8$

The scalar manifold of the maximal, namely $\mathcal{N}=8$, supergravity in $d=4$ is the *symmetric* real coset

$$\left(\frac{G}{H}\right)_{\mathcal{N}=8,d=4} = \frac{E_{7(7)}}{SU(8)}, \qquad \dim_{\mathbb{R}} = 70, \tag{3.1}$$

where the usual notation for noncompact forms of exceptional Lie groups is used, with subscripts denoting the difference "*# noncompact generators -# compact generators*". This theory is *pure*, *i.e.* matter coupling is *not* allowed. The *classical* (see Footnote 1) U-duality group is $E_{7(7)}$. Moreover, the $\mathcal{R}$-symmetry group is $SU(8)$ and, due to the absence of *matter multiplets*, it is nothing but the stabilizer of the scalar manifold (3.1) itself.

The Abelian vector field strengths and their *duals*, as well the corresponding *fluxes* (charges), sit in the *fundamental* representation **56** of the global, *classical* U-duality group $E_{7(7)}$. Such a representation determines the embedding of $E_{7(7)}$ into the symplectic group $Sp(56, \mathbb{R})$, which is the largest symmetry acting linearly on charges. The **56** of $E_{7(7)}$ admits an *unique* invariant, which will be denoted by $\mathcal{I}_{4,\mathcal{N}=8}$ throughout. $\mathcal{I}_{4,\mathcal{N}=8}$ is *quartic* in charges, and it was firstly determined in [11].

More precisely, $\mathcal{I}_{4,\mathcal{N}=8}$ is the *unique* combination of $Z_{AB}(\phi, \mathcal{P})$ satisfying

$$\partial_\phi \mathcal{I}_{4,\mathcal{N}=8}(Z_{AB}(\phi, \mathcal{P})) = 0, \quad \forall\ \phi \in \frac{E_{7(7)}}{SU(8)}. \tag{3.2}$$

Equation (3.2) can be computed by using the *Maurer-Cartan Eqs.* of the coset $\frac{E_{7(7)}}{SU(8)}$ (see *e.g.* [29] and Refs. therein):

$$\nabla Z_{AB} = \frac{1}{2} P_{ABCD} \bar{Z}^{CD} \bar{Z}^{CD}, \tag{3.3}$$

or equivalently by performing an infinitesimal $\frac{E_{7(7)}}{SU(8)}$-transformation of the central charge matrix (see *e.g.* [29] and Refs. therein):

$$\delta_{\xi_{ABCD}} Z_{AB} = \frac{1}{2} \xi_{ABCD} \bar{Z}^{CD}, \tag{3.4}$$

where ∇ and P_{ABCD} respectively denote the covariant differential operator and the *Vielbein* 1-form in $\frac{E_{7(7)}}{SU(8)}$, and the infinitesimal $\frac{E_{7(7)}}{SU(8)}$-parameters ξ_{ABCD} satisfy the reality constraint

$$\xi_{ABCD} = \frac{1}{4!} \epsilon_{ABCDEFGH} \bar{\xi}^{EFGH}. \tag{3.5}$$

As first found in [11] and rigorously reobtained in [29], the unique solution of Eq. (3.2) reads:

$$\mathcal{I}_{4,\mathcal{N}=8} = \frac{1}{2^2}[2^2 \operatorname{Tr}((Z_{AC}\bar{Z}^{BC})^2) - (\operatorname{Tr}(Z_{AC}\bar{Z}^{BC}))^2 + 2^5 \operatorname{Re}(Pf(Z_{AB}))], \tag{3.6}$$

where the *Pfaffian* of Z_{AB} is defined as [11]

$$Pf(Z_{AB}) \equiv \frac{1}{2^4 4!} \epsilon^{ABCDEFGH} Z_{AB} Z_{CD} Z_{EF} Z_{GH}, \tag{3.7}$$

and it holds that (see *e.g.* [29])

$$|Pf(Z_{AB})| = |\det(Z_{AB})|^{1/2}. \tag{3.8}$$

In [29] it was indeed shown that, although each of the three terms of the expression (3.6) is $SU(8)$-invariant but *scalar-dependent*, only the combination given by the expression (3.6) is actually $E_{7(7)}$-independent and thus *scalar-independent*, satisfying

$$\delta_{\xi_{ABCD}} \mathcal{I}_{4,\mathcal{N}=8} = 0, \tag{3.9}$$

with Eqs. (3.4) and (3.5) holding true.

It is here worth commenting a bit further about formula (3.6). The first two terms in its right-hand side are actually $U(8)$-invariant, while the third one, namely $2^5 \operatorname{Re}(Pf(Z_{AB}))$, is *only* $SU(8)$-invariant. Such a third term introduces an $SU(8)$-invariant phase φ_Z, defined as (one fourth of) the overall phase of the central charge matrix, when this latter is reduced to a skew-diagonal form in the so-called *normal frame* through an $SU(8)$-transformation:

$$Z_{AB} \overset{SU(8)}{\longrightarrow} Z_{AB,\text{skew-diag.}} \equiv e^{i\varphi_Z/4} \begin{pmatrix} e_1 & & & \\ & e_2 & & \\ & & e_3 & \\ & & & e_4 \end{pmatrix} \otimes \epsilon,$$

$$e_i \in \mathbb{R}^+, \quad \forall\ i = 1, \ldots, 4, \tag{3.10}$$

where the ordering $e_1 \geq e_2 \geq e_3 \geq e_4$ can be performed without any loss of generality, and the 2×2 symplectic metric

$$\epsilon \equiv \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{3.11}$$

has been introduced (notice $\epsilon = \Omega$ for $n = 1$, as defined in Eq. (2.3)). For nonvanishing (in general all different) *skew-eigenvalues* e_i, the symmetry group of $Z_{AB,\text{skew-diag.}}$ is $(USp(2))^4 \sim (SU(2))^4$. Thus, beside the 4 *skew-eigenvalues* e_i and the phase φ_Z, the generic Z_{AB} is described by $51 = \dim_{\mathbb{R}}(\frac{SU(8)}{(SU(2))^4})$ "*generalized angles*". Consistently, the total number of parameters is $4 + 1 + 51 = 56$, which is the real dimension of the *fundamental* representation **56**, defining the embedding of $E_{7(7)}$ into $Sp(56, \mathbb{R})$.

Equivalently, φ_Z can be defined through the *Pfaffian* of Z_{AB} as follows:

$$e^{2i\varphi_Z} \equiv \frac{Pf(Z_{AB})}{Pf(\bar{Z}_{AB})}, \tag{3.12}$$

where clearly $Pf(\bar{Z}_{AB}) = \overline{Pf(Z_{AB})}$, as yielded by the definition (3.7). It is then immediate to compute φ_Z from Eq. (3.6):

$$\cos\varphi_Z(\phi, \mathcal{P}) = \frac{[2^2 \mathcal{I}_{4,\mathcal{N}=8}(\mathcal{P}) - 2^2 \operatorname{Tr}((Z_{AC}\bar{Z}^{BC})^2) + (\operatorname{Tr}(Z_{AB}\bar{Z}^{AC}))^2]}{2^5(\det(Z_{AC}\bar{Z}^{BC}))^{1/4}}. \tag{3.13}$$

Notice that through Eq. (3.13) $(\cos)\varphi_Z$ is determined in terms of the scalar fields ϕ and of the BH charges $\mathcal{P}$, also along the *small* orbits where $\mathcal{I}_{4,\mathcal{N}=8} = 0$. However, Eq. (3.13) is not defined in the cases in which $\det(Z_{AC}\bar{Z}^{BC}) = 0$, *i.e.* when *at least* one of the eigenvalues of the matrix $Z_{AC}\bar{Z}^{BC}$ vanishes. In such cases, φ_Z is actually undetermined.

In $\mathcal{N} = 8$, $d = 4$ supergravity five distinct orbits of the **56** of $E_{7(7)}$ exist, as resulting from the analyses performed in [4,5]. They can be classified in *large* and *small* charge orbits, depending whether they correspond to $\mathcal{I}_{4,\mathcal{N}=8} \neq 0$ or $\mathcal{I}_{4,\mathcal{N}=8} = 0$, respectively.

Only two *large* charge orbits (for which $\mathcal{I}_{4,\mathcal{N}=8} \neq 0$, and the *attractor mechanism* holds) exist in $\mathcal{N} = 8$, $d = 4$ supergravity:

(1) The *large* $\frac{1}{8}$ − BPS orbit [4,5]

$$\mathcal{O}_{(1/8)-\text{BPS,large}} = \frac{E_{7(7)}}{E_{6(2)}}, \qquad \dim_{\mathbb{R}} = 55, \tag{3.14}$$

is defined by the $E_{7(7)}$-invariant constraint

$$\mathcal{I}_{4,\mathcal{N}=8} > 0. \tag{3.15}$$

At the event horizon of the extremal BH, the solution of the $\mathcal{N}=8$, $d=4$ *Attractor Eqs.* yields [3,9,30]

$$e_1 \in \mathbb{R}_0^+, \qquad e_2 = e_3 = e_4 = 0, \tag{3.16}$$

implying $\det(Z_{AB}) = 0 \Leftrightarrow Pf(Z_{AB}) = 0$, and thus φ_Z to be *undetermined*. Thus, at the event horizon, the symmetry of the *skew-diagonalized* central charge matrix $Z_{AB,\text{skew-diag.}}$ defined in Eq. (3.10) gets *enhanced* as follows, revealing the maximal compact symmetry of $\mathcal{O}_{(1/8)-\text{BPS,large}}$:

$$(USp(2))^4 \overset{r\to r_H^+}{\longrightarrow} USp(2) \times SU(6) \sim SU(2) \times SU(6). \tag{3.17}$$

Indeed, $SU(2) \times SU(6)$ is the *maximal compact subgroup* (*mcs*, with symmetric embedding [31]) of $E_{6(2)}$ (stabilizer of $\mathcal{O}_{(1/8)-\text{BPS,large}}$) itself.

(2) The *large* non-BPS ($Z_{AB} \neq 0$) orbit [4,5]

$$\mathcal{O}_{\text{non-BPS},Z_{AB}\neq 0} = \frac{E_{7(7)}}{E_{6(6)}}, \qquad \dim_{\mathbb{R}} = 55, \tag{3.18}$$

is defined by the $E_{7(7)}$-invariant constraint

$$\mathcal{I}_{4,\mathcal{N}=8} < 0. \tag{3.19}$$

At the event horizon of the extremal BH, the solution of the $\mathcal{N}=8$, $d=4$ *Attractor Eqs.* yields [3,9,30]

$$\begin{aligned} &e_1 = e_2 = e_3 = e_4 \in \mathbb{R}_0^+, \\ &\varphi_Z = \pi + 2k\pi, \qquad k \in \mathbb{Z}, \end{aligned} \tag{3.20}$$

so the *skew-eigenvalues* of Z_{AB} at the horizon (see Eq. (3.10)) are complex. Thus, at the event horizon, the symmetry of the *skew-diagonalized* central charge matrix $Z_{AB,skew-diag.}$ defined in Eq. (3.10) gets *enhanced* as follows, revealing the maximal compact symmetry of $\mathcal{O}_{\text{non-BPS},Z_{AB}\neq 0}$:

$$(USp(2))^4 \overset{r\to r_H^+}{\longrightarrow} USp(8). \tag{3.21}$$

Indeed, $USp(8)$ is the *mcs* (with symmetric embedding [31]) of $E_{6(6)}$ (stabilizer of $\mathcal{O}_{\text{non-BPS},Z_{AB}\neq 0}$) itself.

As mentioned above, for such *large* charge orbits, corresponding to a nonvanishing quartic $E_{7(7)}$-invariant $\mathcal{I}_{4,\mathcal{N}=8}$ and thus supporting *large* BHs, the *attractor mechanism* holds. Consequently, the computations of the Bekenstein-Hawking BH entropy can be performed by solving the criticality conditions for the "*BH potential*"

$$V_{\text{BH},\mathcal{N}=8} = \frac{1}{2} Z_{AB}\bar{Z}^{AB}, \tag{3.22}$$

the result being

$$\begin{aligned} \frac{S_{\text{BH},}}{\pi} &= V_{\text{BH},\mathcal{N}=8}|_{\partial V_{\text{BH},\mathcal{N}=8}=0} = V_{\text{BH},\mathcal{N}=8}(\phi_H(\mathcal{P}),\mathcal{P}) \\ &= |\mathcal{I}_{4,\mathcal{N}=8}|^{1/2}, \end{aligned} \tag{3.23}$$

where $\phi_H(\mathcal{P})$ denotes the set of solutions to the *criticality conditions* of $V_{\text{BH},\mathcal{N}=8}$, namely, the *Attractor Eqs.* of $\mathcal{N}=8$, $d=4$ supergravity:

$$\partial_\phi V_{\text{BH},\mathcal{N}=8} = 0, \quad \forall\ \phi \in \frac{E_{7(7)}}{SU(8)}, \tag{3.24}$$

expressing the stabilization of the scalar fields purely in terms of supporting charges $\mathcal{P}$ at the event horizon of the extremal BH. Through Eqs. (3.3) and (3.22), Eqs. (3.24) can be rewritten as follows (notice the strict similarity to Eq. (3.40) further below) [30]:

$$Z_{[AB}Z_{CD]} + \frac{1}{4!}\epsilon_{ABCDEFGH}\bar{Z}^{EF}\bar{Z}^{GH} = 0. \tag{3.25}$$

Actually, the critical potential $V_{\text{BH},\mathcal{N}=8}|_{\partial V_{\text{BH},\mathcal{N}=8}=0}$ exhibits some "*flat*" directions, so not all scalars are stabilized in terms of charges at the event horizon [32,33]. Thus, Eq. (3.23) yields that the *unstabilized* scalars, spanning a related *moduli space* of the considered class of attractor solutions, do not enter in the expression of the BH entropy at all. The *moduli spaces*[6] exhibited by the *Attractor Eqs.* (3.24) and (3.25) are [33]

$$\mathcal{M}_{(1/8)-\text{BPS}B,\text{large}} = \frac{E_{6(2)}}{SU(2)\times SU(6)}, \qquad \dim_{\mathbb{R}} = 40; \tag{3.26}$$

$$\mathcal{M}_{\text{non-BPS},Z_{AB}\neq 0} = \frac{E_{6(6)}}{USp(8)}, \qquad \dim_{\mathbb{R}} = 42. \tag{3.27}$$

As found in [33], the general structure of the *moduli spaces* of attractor solutions in supergravities based on *symmetric* scalar manifolds $\frac{G}{H}$ is

$$\frac{\mathcal{H}_{nc}}{\mathbf{h}}, \tag{3.28}$$

where $\mathcal{H}_{nc}$ is the *noncompact* stabilizer of the charge orbit $\frac{G}{\mathcal{H}_{nc}}$ (apart from eventual $U(1)$ factors, $\mathcal{H}_{nc}$ is a noncompact, real form of H), and $\mathbf{h} = mcs(\mathcal{H}_{nc})$. As justified in [29] and then in [32], $\mathcal{M}_{(1/8)-\text{BPS,large}}$ is a *quaternionic* symmetric manifold. Furthermore, $\mathcal{M}_{\text{non-BPS},Z_{AB}\neq 0}$ given by Eq. (3.27) is nothing but the scalar manifold of $\mathcal{N}=8$, $d=5$ supergravity. The stabilizers of $\mathcal{M}_{(1/8)-\text{BPS,large}}$ and $\mathcal{M}_{\text{non-BPS},Z_{AB}\neq 0}$ exploit the maximal compact symmetry of the corresponding charge orbits; this symmetry becomes

[6] Results obtained by explicit computations within the $\mathcal{N}=2$, $d=4$ *symmetric* so-called *stu* model in [23,34] seem to point out that the *moduli spaces* should be present not only at the event horizon of the considered extremal BH (*i.e.* for $r \to r_H^+$), but also all along the scalar attractor *flow* (*i.e.* $\forall\ r \geq r_H$).

fully manifest through the enhancement of the compact symmetry group of $Z_{AB,\text{skew}-\text{diag}}$ at the event horizon of the extremal BH, respectively, given by Eqs. (3.17) and (3.21).

It is now convenient to denote with λ_i ($i = 1, \ldots, 4$) the four real non-negative eigenvalues of the matrix $Z_{AB}\bar{Z}^{CB} = (ZZ^{\dagger})_A^C$. By recalling Eq. (3.10), one can notice that

$$\lambda_i = e_i^2, \tag{3.29}$$

and one can order them as $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4$, without any loss of generality. The explicit expression of λ_i in terms of $U(8)$-invariants (namely of $\mathrm{Tr}(ZZ^{\dagger})$, $\mathrm{Tr}((ZZ^{\dagger})^2)$, $\mathrm{Tr}((ZZ^{\dagger})^3)$ and $\mathrm{Tr}((ZZ^{\dagger})^4)$, and suitable powers) is given by Eqs. (4.74), (4.75), (4.86) and (4.87) of [9], and it will be used in Sec. VI to determine the *ADM mass* for $\frac{\mathbf{k}}{8}$ − BPS ($\mathbf{k} = 1, 2, 4$) extremal BH states.

Three distinct *small* charge orbits (all with $\mathcal{I}_{4,\mathcal{N}=8} = 0$) exist, and they all are supersymmetric:

(1) The generic *small lightlike* orbit is $\frac{1}{8}$ − BPS, it is defined by the $E_{7(7)}$-invariant constraint

$$\mathcal{I}_{4,\mathcal{N}=8} = 0, \tag{3.30}$$

and it reads [4,5]

$$\mathcal{O}_{(1/8)-\text{BPS,small}} = \frac{E_{7(7)}}{F_{4(4)} \times_s T_{26}}, \qquad \dim_{\mathbb{R}} = 55. \tag{3.31}$$

Generally, it yields four different λ_i's, and in this case Eq. (3.13) reduces to

$$\cos\varphi_Z(\phi, \mathcal{P})|_{\mathcal{I}_{4,\mathcal{N}=8}=0} = -\frac{[2^2\,\mathrm{Tr}((Z_{AC}\bar{Z}^{BC})^2) - (\mathrm{Tr}(Z_{AB}\bar{Z}^{AC}))^2]}{2^5(\det(Z_{AC}\bar{Z}^{BC}))^{1/4}}\Bigg|_{\mathcal{I}_{4,\mathcal{N}=8}=0}. \tag{3.32}$$

In agreement with the results of [4,5], the (maximal compact) symmetry of the *skew-diagonalized* central charge matrix $Z_{AB,\text{skew}-\text{diag}}$ all along the $\frac{1}{8}$ − BPS *small* flow is the generic one: $(SU(2))^4$. The counting of the parameters of $\mathcal{O}_{(1/8)-\text{BPS,small}}$ consistently reads: $55 = 4$ *skew-eigenvalues* $\lambda_i + 1$ phase $\varphi_Z + 51(= \dim_{\mathbb{R}}(\frac{SU(8)}{(SU(2))^4}))$ "*generalized angles*" −1 defining constraint (3.30).

(2) The *small critical* orbit is $\frac{1}{4}$ − BPS. It reads [4,5]

$$\mathcal{O}_{(1/4)-\text{BPS}} = \frac{E_{7(7)}}{(SO(6,5) \times_s T_{32}) \times T_1}, \qquad \dim_{\mathbb{R}} = 45, \tag{3.33}$$

and it is defined by the following differential constraint on $\mathcal{I}_{4,\mathcal{N}=8}$ [3,9]:

$$\frac{\partial \mathcal{I}_{4,\mathcal{N}=8}}{\partial Z_{AB}} = 0, \tag{3.34}$$

which, due to the reality of $\mathcal{I}_{4,\mathcal{N}=8}$, is actually $E_{7(7)}$-invariant. Let us also notice that, due to the homogeneity of $\mathcal{I}_{4,\mathcal{N}=8}$ of degree four in $\mathcal{P}$, Eq. (3.34) implies the constraint (3.30). In particular, along the $\frac{1}{4}$ − BPS orbit it holds that (the labelling does not yield any loss of generality)

$$\lambda_1 = \lambda_2 > \lambda_3 = \lambda_4 \geq 0. \tag{3.35}$$

If $Pf(Z_{AB}) \neq 0$ then

$$\lambda_1 = \lambda_2 > \lambda_3 = \lambda_4 > 0, \tag{3.36}$$

and Eq. (3.13) yields $\varphi_Z = k\pi$, $k \in \mathbb{Z}$, so the *skew-eigenvalues* of Z_{AB} (see Eq. (3.10)) are real and the (maximal) compact symmetry of $Z_{AB,\text{skew}-\text{diag.}}$ is $(USp(4))^2$. On the other hand, if $Pf(Z_{AB}) = 0$ then

$$\lambda_1 = \lambda_2 > \lambda_3 = \lambda_4 = 0, \tag{3.37}$$

and φ_Z is undetermined. In this case, the (maximal compact) symmetry of the *skew-diagonalized* central charge matrix $Z_{AB,\text{skew}-\text{diag.}}$ is $USp(4) \times SU(4) \sim SO(5) \times SO(6)$, which is the *mcs* of the nontranslational part of the stabilizer of $\mathcal{O}_{(1/4)-\text{BPS}}$, expressing the maximal compact symmetry of $\mathcal{O}_{(1/4)-\text{BPS}}$ itself. In agreement with the results of [4,5], the *maximal* (compact) symmetry of the *skew-diagonalized* central charge matrix $Z_{AB,\text{skew}-\text{diag.}}$ along the $\frac{1}{4}$ − BPS *small* flow (fully manifest in the particular solution (3.37)) is $USp(4) \times SU(4)$. The counting of the parameters of $\mathcal{O}_{(1/4)-\text{BPS}}$ consistently reads: $45 = 2$ *skew-eigenvalues* λ_1 and $\lambda_2 + 43(= \dim_{\mathbb{R}}(\frac{SU(8)}{(USp(4))^2}))$ "*generalized angles*".

(3) The *small doubly-critical* orbit is $\frac{1}{2}$ − BPS, and it reads [4,5]

$$\mathcal{O}_{(1/2)-\text{BPS}} = \frac{E_{7(7)}}{E_{6(6)} \times_s T_{27}}, \qquad \dim_{\mathbb{R}} = 28. \tag{3.38}$$

It can be defined in an $E_{7(7)}$-invariant way by performing the following two-step procedure [9]. One starts by considering the requirement that the second derivative of $\mathcal{I}_{4,\mathcal{N}=8}$ (with respect to Z_{AB}) projected along the adjoint representation $\mathbf{Adj}(SU(8)) = \mathbf{63}$ of $SU(8)$ vanishes, yielding [9]

$$\frac{\partial^2 \mathcal{I}_{4,\mathcal{N}=8}}{\partial Z_{AB}\bar{\partial}\bar{Z}^{BC}}\Bigg|_{\mathbf{Adj}(SU(8))} = 0 \Leftrightarrow Z_{AC}\bar{Z}^{BC} = \frac{1}{2^3}\delta_A^B Z_{DE}Z^{DE}. \tag{3.39}$$

This is a mixed rank-2 $SU(8)$-covariant condition. By further differentiating with respect to the scalars ϕ parametrizing $\frac{E_{7(7)}}{SU(8)}$ and using the Maurer-Cartan Eqs. (3.3), one obtains another $SU(8)$-covariant relation [notice the strict similarity to the $\mathcal{N} = 8$,

$d = 4$ *Attractor* Eqs. (3.25)] [9]:

$$Z_{[AB}Z_{CD]} - \frac{1}{4!}\epsilon_{ABCDEFGH}\bar{Z}^{EF}\bar{Z}^{GH} = 0. \quad (3.40)$$

Actually, Eq. (3.40) form with Eq. (3.39) an $E_{7(7)}$-invariant set of differential conditions defining $\mathcal{O}_{(1/2)-\text{BPS}}$. Indeed, as noticed in [9], Eq. (3.40) can be rewritten as

$$\frac{\partial^2 \mathcal{I}_{4,\mathcal{N}=8}}{\partial Z_{[AB}\partial Z_{CD]}} - \frac{1}{4!}\epsilon^{ABCDEFGH}\frac{\partial^2 \mathcal{I}_{4,\mathcal{N}=8}}{\partial \bar{Z}^{[EF}\partial \bar{Z}^{GH]}} = 0. \quad (3.41)$$

Thus, by using the notation $Z_{\mathbf{56}} \equiv (\mathcal{Z}, \mathcal{Z}^T) = (Z_{AB}, \bar{Z}^{AB}\bar{Z}^{AB})$ (recall Eqs. (2.22) and (2.23), Eqs. (3.39), (3.40), and (3.41) can be rewritten in the manifestly $E_{7(7)}$-invariant fashion

$$\left.\frac{\partial^2 \mathcal{I}_{4,\mathcal{N}=8}}{\partial Z_{\mathbf{56}}\partial Z_{\mathbf{56}}}\right|_{\mathbf{Adj}(E_{7(7)})} = 0, \quad (3.42)$$

where $\mathbf{Adj}(E_{7(7)}) = \mathbf{133}$ is the adjoint representation of $E_{7(7)}$. Notice that $\frac{\partial^2 \mathcal{I}_{4,\mathcal{N}=8}}{\partial Z_{\mathbf{56}}\partial Z_{\mathbf{56}}}$ is a rank-2 symmetric true-tensor $E_{7(7)}$-tensor, thus sitting in the symmetric product representation $(\mathbf{56}\times\mathbf{56})_s = \mathbf{1596}$ of $E_{7(7)}$, which in turns enjoys the following branching with respect to $E_{7(7)}$ [9,31]:

$$(\mathbf{56}\times\mathbf{56})_s = \mathbf{1596} \rightarrow \mathbf{1463} + \underset{\mathbf{Adj}(E_{7(7)})}{\mathbf{133}}. \quad (3.43)$$

It is here worth remarking that the constraints (3.39), (3.40), and (3.41) (or equivalently ((3.42))) imply the constraint (3.34), because in fact they are stronger constraints.

Along the $\frac{1}{2}$ − BPS orbit it holds that

$$\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4. \quad (3.44)$$

Furthermore, it can be shown that $\varphi_Z = 2k\pi$, $k \in \mathbb{Z}$, so the *skew-eigenvalues* of Z_{AB} (see Eq. (3.10)) are real. In agreement with the results of [4,5], the (maximal compact) symmetry of the *skew-diagonalized* central charge matrix $Z_{AB,\text{skew-diag.}}$ all along the $\frac{1}{2}$ − BPS *small* flow is $USp(8)$, which is the *mcs* of the nontranslational part of the stabilizer of $\mathcal{O}_{(1/2)-\text{BPS}}$, expressing the maximal compact symmetry of $\mathcal{O}_{(1/2)-\text{BPS}}$ itself. The counting of the parameters of $\mathcal{O}_{(1/2)-\text{BPS}}$ consistently reads: $28 = 1$ *skew-eigenvalue* $\lambda_1 + 27(= \dim_{\mathbb{R}}(\frac{SU(8)}{USp(8)}))$ "*generalized angles*".

Interestingly, $USp(8)$ also is the *enhanced* compact symmetry of $Z_{AB,\text{skew-diag.}}$ at the event horizon of the *large* non-BPS $Z_{AB} \neq 0$ *attractor* scalar flow (see Eq. (3.21) above). Indeed, the charge orbits $\mathcal{O}_{\text{non-BPS},Z_{AB}\neq 0}$ and $\mathcal{O}_{(1/2)-\text{BPS}}$ (respectively given by Eqs. (3.18) and (3.38)) coincide, up to the translational factor T_{27} in the stabilizer, and thus they have the same maximal compact symmetry.

As given by the analysis of [3], the classification of *large* and *small* orbits of the **56** of $E_{7(7)}$ can be performed also considering the symplectic basis composed by the fluxes q_Λ ($\Lambda = 1, \ldots, 56$). In general, the symplectic basis of charges is useful in order to determine, through constraints imposed on the relevant U-invariant, the number and typology of orbits of the relevant representation of the U-duality group. On the other hand, using the manifestly H-covariant basis of central charges and matter charges one can achieve a symplectic-invariant characterization of charge orbits, and also study the related supersymmetry-preserving features.

Finally, it is worth pointing out once again that there is a crucial difference among the various constraints defining the two *large* and the three *small* charge orbits of $\mathcal{N} = 8$, $d = 4$ supergravity listed above:

(i) The *large* charge orbits $\mathcal{O}_{(1/8)-\text{BPS,large}}$ and $\mathcal{O}_{\text{non-BPS},Z_{AB}\neq 0}$, respectively, given by Eqs. (3.18) and (3.38), are in order defined by the $E_{7(7)}$-invariant conditions $\mathcal{I}_{4,\mathcal{N}=8} > 0$ and $\mathcal{I}_{4,\mathcal{N}=8} < 0$. Because of their $E_{7(7)}$-invariance, these conditions are *identities* for the scalar fields ϕ spanning $\frac{E_{7(7)}}{SU(8)}$. However, the *classical attractor mechanism* does hold for *large extremal* BHs, and the scalars ϕ are stabilized purely in terms of charges $\mathcal{P}$ at the event horizon ($r \rightarrow r_H^+$) through the only two independent solutions (3.16) and (3.20) to the $\mathcal{N} = 8$, $d = 4$ *Attractor Eqs.* (3.24) and (3.25).

(ii) The *small* charge orbits $\mathcal{O}_{(1/8)-\text{BPS,small}}$, $\mathcal{O}_{(1/4)-\text{BPS}}$ and $\mathcal{O}_{(1/2)-\text{BPS}}$, respectively, given by Eqs. (3.31), (3.33), and (3.38), are in order defined by the $E_{7(7)}$-invariant conditions (3.30), (3.34), and (3.42). Because of their $E_{7(7)}$-invariance, these conditions are *identities* for the scalars ϕ, which thus are *not* stabilized along such orbits. Indeed, the *classical attractor mechanism* does *not* hold for *small* BHs.

IV. $\mathcal{N} = 4$

In $\mathcal{N} = 4$, $d = 4$ supergravity, unlike the $\mathcal{N} = 8$ case, matter (*vector*) multiplets appear (see *e.g.* [35,36]). By denoting their number with M, the related scalar manifold is the *symmetric* coset

$$\left(\frac{G}{H}\right)_{\mathcal{N}=4,d=4} = \frac{SL(2,\mathbb{R})}{U(1)} \times \frac{SO(6,M)}{SO(6)\times SO(M)}, \quad \dim_{\mathbb{R}} = 6M + 2. \quad (4.1)$$

The Abelian vector field strengths and their *duals*, as well the corresponding *fluxes* (charges), sit in the bi-fundamental $(\mathbf{2}, \mathbf{6}+\mathbf{M})$ representation of the global, *classical* (see Footnote 1) U-duality group $SL(2,\mathbb{R}) \times SO(6,M)$ [37]. Such a representation determines the embedding of $SL(2,\mathbb{R}) \times SO(6,M)$ into the symplectic group $Sp(12 + 2M, \mathbb{R})$. The representation $(\mathbf{2}, \mathbf{6}+\mathbf{M})$ is endowed with a natural symplectic metric

$$\mathbf{\Omega} \equiv \epsilon_{\alpha\beta}\eta_{\Lambda\Sigma}, \tag{4.2}$$

where $\epsilon_{\alpha\beta}$ (α, $\beta = 1, 2$) is the (inverse of the) $SL(2,\mathbb{R})$ skew-symmetric metric defined in Eq. (3.11), and $\eta_{\Lambda\Sigma}$ (Λ, $\Sigma = 1, \ldots, 6+M = n$; recall Eq. (2.26)) is the Lorentzian metric of $SO(6, M)$. Moreover, the $\mathcal{R}$-symmetry group is $U(4)$.

Furthermore, $(\mathbf{2}, \mathbf{6}+\mathbf{M})$ admits an *unique* invariant, which will be denoted by $\mathcal{I}_{4,\mathcal{N}=4}$ throughout. $\mathcal{I}_{4,\mathcal{N}=4}$ is *quartic* in charges, and it was firstly determined in [14,19,38].

More precisely, $\mathcal{I}_{4,\mathcal{N}=4}$ is the *unique* combination of "*dressed*" charges $Z_{AB} = Z_{[AB]}(\phi, \mathcal{P})$ (*central charge matrix*, A, $B = 1, \ldots, 4$) and $Z_I(\phi, \mathcal{P})$ (*matter charges*, $I = 1, \ldots, M$) satisfying

$$\partial_\phi \mathcal{I}_{4,\mathcal{N}=4}(Z_{AB}(\phi,\mathcal{P}), Z_I(\phi,\mathcal{P})) = 0, \quad \forall\ \phi \in \left(\frac{G}{H}\right)_{\mathcal{N}=4,d=4}. \tag{4.3}$$

Equation (4.3) can be computed by using the *Maurer-Cartan* Eqs. of the coset $\frac{SL(2,\mathbb{R})}{U(1)} \times \frac{SO(6,M)}{SO(6)\times SO(M)}$ (see *e.g.* [29], and Refs. therein):

$$\nabla Z_{AB} = \frac{1}{2} P \epsilon_{ABCD} \bar{Z}^{CD} + P_{ABI}\bar{Z}^I; \tag{4.4}$$

$$\nabla Z_I = \frac{1}{2} P_{ABI}\bar{Z}^{AB} + P\eta_{IJ}\bar{Z}^J, \tag{4.5}$$

or equivalently by performing an infinitesimal $\frac{SL(2,\mathbb{R})}{U(1)} \times \frac{SO(6,M)}{SO(6)\times SO(M)}$-transformation of the central charge matrix and of matter charges (see *e.g.* [29], and Refs. therein):

$$\delta_{(\xi,\xi_{AB|I})} Z_{AB} = \frac{1}{2}\xi\epsilon_{ABCD}\bar{Z}^{CD} + \xi_{AB|I}Z^I; \tag{4.6}$$

$$\delta_{(\xi,\xi_{AB|I})} Z_I = \bar{\xi}\eta_{IJ}\bar{Z}^J + \frac{1}{2}\xi_{AB|I}\bar{Z}^{AB}, \tag{4.7}$$

where ∇ stands for the covariant differential operator in $\frac{SL(2,\mathbb{R})}{U(1)} \times \frac{SO(6,M)}{SO(6)\times SO(M)}$. P and P_{ABI} respectively are the *Vielbein* 1-forms of $\frac{SL(2,\mathbb{R})}{U(1)}$ and $\frac{SO(6,M)}{SO(6)\times SO(M)}$, with P_{ABI} satisfying the reality condition:

$$P_{ABI} = \frac{1}{2}\eta_{IJ}\epsilon_{ABCD}\bar{P}^{CDJ}. \tag{4.8}$$

Moreover, ξ is the infinitesimal $\frac{SL(2,\mathbb{R})}{U(1)}$-parameter and $\xi_{AB|I}$ are the infinitesimal $\frac{SO(6,M)}{SO(6)\times SO(M)}$-parameters, satisfying the reality condition

$$\bar{\xi}^{AB|I} = \frac{1}{2}\eta^{IJ}\epsilon^{ABCD}\xi_{CD|J}. \tag{4.9}$$

As found in [14,19,38] and rigorously reobtained in [29], in terms of Z_{AB} and Z_I the unique solution of Eq. (4.3) reads:

$$\mathcal{I}_{4,\mathcal{N}=4} = \mathcal{S}_1^2 - |\mathcal{S}_2|^2, \tag{4.10}$$

where one can identify $\mathcal{S}_1 \equiv L_0$, $\mathcal{S}_2 = L_1 + iL_2$, with $L \equiv (L_0, L_1, L_2)$ being an $SL(2,\mathbb{R}) \sim SO(1,2)$-vector with square norm

$$L^2 = L_0^2 - L_1^2 - L_2^2 = \mathcal{S}_1^2 - |\mathcal{S}_2|^2. \tag{4.11}$$

$\mathcal{S}_1$ and $\mathcal{S}_2$ are defined as [29]

$$\mathcal{S}_1 \equiv \frac{1}{2}Z_{AB}\bar{Z}^{AB} - Z_I\bar{Z}^I \in \mathbb{R}; \tag{4.12}$$

$$\mathcal{S}_2 \equiv \frac{1}{4}\epsilon^{ABCD}Z_{AB}Z_{CD} - \bar{Z}_I\bar{Z}_I \in \mathbb{C}. \tag{4.13}$$

In [29] it was indeed shown that $\mathcal{I}_{4,\mathcal{N}=4}$ given by Eq. (4.10) is the unique combination of $SO(6, M)$-invariant and scalar-dependent quantities, which is actually *also* $SL(2,\mathbb{R})$-independent and thus *scalar-independent*, satisfying

$$\delta_\xi \mathcal{I}_{4,\mathcal{N}=4} = 0; \tag{4.14}$$

$$\delta_{\xi_{AB|I}} \mathcal{I}_{4,\mathcal{N}=4} = 0, \tag{4.15}$$

with Eqs. (4.6), (4.7), and (4.9) holding true.

On the other hand, the expression of $\mathcal{I}_{4,\mathcal{N}=4}$ in terms of the "*bare*" charges $\mathcal{P}$ reads [14,15,18,19]

$$\begin{aligned}\mathcal{I}_{4,\mathcal{N}=4} &= p^2q^2 - (p\cdot q)^2 \\ &= \frac{1}{2}(p_\Lambda q_\Sigma - p_\Sigma q_\Lambda)(p_\Xi q_\Omega - p_\Omega q_\Xi)\eta^{\Lambda\Xi}\eta^{\Sigma\Omega} \\ &= \frac{1}{2}T^{(a)}_{\Lambda\Sigma}T^{(a)|\Lambda\Sigma},\end{aligned} \tag{4.16}$$

where

$$p^2 \equiv p\cdot p \equiv p_\Lambda p_\Sigma\eta^{\Lambda\Sigma}, \qquad q^2 \equiv q\cdot q \equiv q_\Lambda q_\Sigma\eta^{\Lambda\Sigma}, \qquad p\cdot q \equiv p_\Lambda q_\Sigma\eta^{\Lambda\Sigma}, \tag{4.17}$$

and the tensor

$$T^{(a)}_{\Lambda\Sigma} \equiv p_\Lambda q_\Sigma - p_\Sigma q_\Lambda = T^{(a)}_{[\Lambda\Sigma]} \tag{4.18}$$

has been introduced (the upperscript "(a)" stands for "*antisymmetric*").

The classification of charge orbits, in particular, the BPS ones, was performed in [3,9]. By performing a suitable $U(1)\times SO(6)(\sim U(4))$-transformation, the *central charge matrix* Z_{AB} can be *skew-diagonalized* in the *normal frame* (recall definition (3.11)):

$$Z_{AB}U(4) \longrightarrow Z_{AB,\text{skew-diag}} \equiv \begin{pmatrix} z_1 & \\ & z_2 \end{pmatrix} \otimes \epsilon,\ z_1, z_2 \in \mathbb{R}^+, \tag{4.19}$$

where the ordering $z_1 \geq z_2$ does not imply any loss of generality. Furthermore, by performing a suitable

$SO(M)$-transformation, the vector Z_I of *matter charges* can be reduced to have only two nonvanishing entries, one real positive and the other one complex, say (without loss of generality, with the subscript "*red.*" standing for "*reduced*")

$$Z_I \overset{SO(M)}{\longrightarrow} Z_{I,\text{red}} \equiv (\rho_1 e^{i\theta}, \rho_2, 0, \ldots, 0), \qquad \rho_1, \rho_2 \in \mathbb{R}^+, \qquad \theta \in \mathbb{R}. \tag{4.20}$$

For nonvanishing (in general different) *skew-eigenvalues* z_1 and z_2, the symmetry group of $Z_{AB,\text{skew-diag.}}$ is $(USp(2))^2 \sim (SU(2))^2$. Analogously, for nonvanishing (in general different) ρ_1 and ρ_2 (and nonvanishing phase θ) the symmetry group of $Z_{I,\text{red}}$ is $SO(M-2)$. Thus, beside z_1, z_2, ρ_1, ρ_2 and θ the generic Z_{AB} and Z_I are described by $7 + 2M = \dim_{\mathbb{R}}\left(\frac{U(4)\times SO(M)}{(SU(2))^2\times SO(M-2)}\right)$ "*generalized angles*". Consistently, the total number of parameters is $2 + 2 + 1 + 7 + 2M = 12 + 2M$, which is the real dimension of the bi-fundamental representation $(\mathbf{2}, \mathbf{6} + \mathbf{M})$, defining the embedding of $SL(2,\mathbb{R}) \times SO(6, M)$ into $Sp(12 + 2M, \mathbb{R})$.

In $\mathcal{N} = 4$, $d = 4$ *matter coupled* supergravity three distinct *large* charge orbits of the $(\mathbf{2}, \mathbf{6} + \mathbf{M})$ of $SL(2,\mathbb{R}) \times SO(6, M)$ (for which $\mathcal{I}_{4,\mathcal{N}=4} \neq 0$, and the *attractor mechanism* holds) exist, as resulting from the analysis performed in[7] [13]:

(1) The *large* $\frac{1}{4}$ − BPS orbit

$$\mathcal{O}_{(1/4)-\text{BPS,large}} = SL(2,\mathbb{R}) \times \frac{SO(6, M)}{SO(4, M) \times SO(2)}, \qquad \dim_{\mathbb{R}} = 11 + 2M, \tag{4.21}$$

is defined by the $SL(2,\mathbb{R}) \times SO(6, M)$-invariant constraint

$$\mathcal{I}_{4,\mathcal{N}=4} > 0. \tag{4.22}$$

Thus, the corresponding horizon solution of the $\mathcal{N} = 4$, $d = 4$ *Attractor Eqs.* yields [3,9,13]

$$z_1 \in \mathbb{R}_0^+, \qquad z_2 = 0, \qquad \rho_1 = \rho_2 = 0, \theta \quad \text{undetermined}; \tag{4.23}$$

$$\mathcal{S}_1 = z_1^2 > 0, \qquad \mathcal{S}_2 = 0. \tag{4.24}$$

Therefore, at the event horizon, the symmetry group of $Z_{AB,\text{skew-diag.}}$ defined in Eq. (4.19) does not get enhanced, while the symmetry group of $Z_{i,\text{red}}$ defined in Eq. (4.20) gets *enhanced* as follows:

$$SO(M-2) \overset{r\to r_H^+}{\longrightarrow} SO(M). \tag{4.25}$$

As a consequence, the horizon attractor solution exploits the maximal compact symmetry $SU(2) \times SU(2) \times SO(M) \times SO(2)$, which is the *mcs* [31] of the stabilizer of $\mathcal{O}_{(1/4)-\text{BPS,large}}$ itself.

(2) The *large* non-BPS $Z_{AB} = 0$ orbit (existing for $M \geq 2$) [13]

$$\mathcal{O}_{\text{non-BPS},Z_{AB}=0,\text{large}} = SL(2,\mathbb{R}) \times \frac{SO(6, M)}{SO(6, M-2) \times SO(2)}, \qquad \dim_{\mathbb{R}} = 11 + 2M, \tag{4.26}$$

is defined by the $SL(2,\mathbb{R}) \times SO(6, M)$-invariant constraint

$$\mathcal{I}_{4,\mathcal{N}=4} > 0. \tag{4.27}$$

Thus, the corresponding attractor solution of the $\mathcal{N} = 4$, $d = 4$ *Attractor Eqs.* yields (for $M \geq 2$) [3,9,13]

$$z_1 = z_2 = 0, \qquad \rho_1^2 e^{2i\theta} + \rho_2^2 = 0 \Leftrightarrow \rho_1 = \rho_2 \in \mathbb{R}_0^+, \qquad \theta = \frac{\pi}{2} + k\pi, \qquad k \in \mathbb{Z}; \tag{4.28}$$

$$\mathcal{S}_1 = -2\rho_1^2 < 0, \qquad \mathcal{S}_2 = 0. \tag{4.29}$$

Therefore, at the event horizon, the symmetry group of $Z_{AB,\text{skew-diag.}}$ defined in Eq. (4.19) gets enhanced as follows:

$$(SU(2))^2 \overset{r\to r_H^+}{\longrightarrow} SU(4), \tag{4.30}$$

and the symmetry group of $Z_{i,\text{red}}$ defined in Eq. (4.20) does not get *enhanced*. Consequently, the horizon attractor solution exploits the maximal compact symmetry $SU(4) \times SO(M-2) \times SO(2)$, which is the *mcs* [31] of the stabilizer of $\mathcal{O}_{\text{non-BPS},Z_{AB}=0,\text{large}}$ itself.

(3) The *large* non-BPS $Z_{AB} \neq 0$ orbit (existing for $M \geq 1$) [13]

$$\mathcal{O}_{\text{non-BPS},Z_{AB}\neq 0,\text{large}} = SL(2,\mathbb{R}) \times \frac{SO(6, M)}{SO(5, M-1) \times SO(1,1)}, \qquad \dim_{\mathbb{R}} = 11 + 2M, \tag{4.31}$$

is defined by the $SL(2,\mathbb{R}) \times SO(6, M)$-invariant constraint

$$\mathcal{I}_{4,\mathcal{N}=4} < 0. \tag{4.32}$$

At the event horizon of the extremal BH, the solution of the $\mathcal{N} = 4$, $d = 4$ *Attractor Eqs.* yields (for $M \geq 1$) [3,9,13]

[7] Consistent with the analysis of [13], Eqs. (4.21), (4.26), and (4.31), fix a slightly misleading notation for the *large* charge orbits of $\mathcal{N} = 4$, $d = 4$ *matter coupled* supergravity, as given by Table 1 of [39].

$$z_1 = z_2 = \frac{\rho_1}{\sqrt{2}} \in \mathbb{R}_0^+, \qquad \rho_2 = 0, \qquad \theta = \frac{\pi}{2} + k\pi, \qquad k \in \mathbb{Z}; \tag{4.33}$$

$$\mathcal{S}_1 = 0, \qquad \mathcal{S}_2 = 3z_1^2 > 0. \tag{4.34}$$

Thus, at the event horizon, the symmetry group of $Z_{AB,\mathrm{skew-diag.}}$ defined in Eq. (4.19) gets enhanced as follows:

$$(SU(2))^2 \stackrel{r \to r_H^+}{\longrightarrow} USp(4), \tag{4.35}$$

and the symmetry group of $Z_{i,\mathrm{red}}$ defined in Eq. (4.20) gets also *enhanced* as

$$SO(M-2) \stackrel{r \to r_H^+}{\longrightarrow} SO(M-1). \tag{4.36}$$

As a consequence, the horizon attractor solution exploits the maximal compact symmetry $USp(4) \times SO(M-1)$ which, due to the isomorphism $USp(4) \sim SO(5)$, is the *mcs* [31] of the stabilizer of $\mathcal{O}_{\mathrm{non-BPS}, Z_{AB} \neq 0, \mathrm{large}}$ itself.

As mentioned above, for such *large* charge orbits, corresponding to a nonvanishing quartic $SL(2, \mathbb{R}) \times SO(6, M)$-invariant $I_{4,\mathcal{N}=4}$ and thus supporting *large* BHs, the *attractor mechanism* holds. Consequently, the computations of the Bekenstein-Hawking BH entropy can be performed by solving the criticality conditions for the "*BH potential*"

$$V_{\mathrm{BH},\mathcal{N}=4} = \frac{1}{2} Z_{AB}\bar{Z}^{AB} + Z_I\bar{Z}^I, \tag{4.37}$$

the result being

$$\frac{S_{\mathrm{BH},}}{\pi} = V_{\mathrm{BH},\mathcal{N}=4}|_{\partial V_{\mathrm{BH},\mathcal{N}=4}=0} = V_{\mathrm{BH},\mathcal{N}=4}(\phi_H(\mathcal{P}), \mathcal{P}) = |I_{4,\mathcal{N}=4}|^{1/2}, \tag{4.38}$$

where $\phi_H(\mathcal{P})$ denotes the set of solutions to the *criticality conditions* of $V_{\mathrm{BH},\mathcal{N}=4}$, namely, the *Attractor Eqs.* of $\mathcal{N} = 4$, $d = 4$ *matter coupled* supergravity:

$$\partial_\phi V_{\mathrm{BH},\mathcal{N}=4} = 0, \quad \forall\ \phi \in \frac{SL(2,\mathbb{R})}{U(1)} \times \frac{SO(6,M)}{SO(6) \times SO(M)}, \tag{4.39}$$

expressing the stabilization of the scalar fields purely in terms of supporting charges $\mathcal{P}$ at the event horizon of the extremal BH. Through Eqs. (4.4), (4.5), and (4.37), Equations (4.39) can be rewritten as follows [13]:

$$\left(\bar{Z}^{AB} + \frac{1}{2}\epsilon^{ABCD}Z_{CD}\right)Z^I = 0; \qquad Z^IZ^J\delta_{IJ} + \frac{1}{4}\epsilon_{ABCD}\bar{Z}^{AB}\bar{Z}^{CD} = 0. \tag{4.40}$$

Actually, the critical potential $V_{\mathrm{BH},\mathcal{N}=4}|_{\partial V_{\mathrm{BH},\mathcal{N}=4}=0}$ exhibits some "*flat*" directions, so not all scalars are stabilized in terms of charges at the event horizon [39]. Thus, Eq. (4.38) yields that the *unstabilized* scalars, spanning a related *moduli space* of the considered class of attractor solutions, do not enter in the expression of the BH entropy at all. The *moduli spaces* exhibited by the *Attractor Eqs.* (4.39) and (4.40) are [39]

$$\mathcal{M}_{(1/4)-\mathrm{BPS,large}} = \frac{SO(4,M)}{SU(2) \times SU(2) \times SO(M)}, \qquad \dim_{\mathbb{R}} = 4M; \tag{4.41}$$

$$\mathcal{M}_{\mathrm{non-BPS},Z_{AB}=0,\mathrm{large}} = \frac{SO(6,M-2)}{SU(4) \times SO(M-2)}, \qquad \dim_{\mathbb{R}} = 6(M-2); \tag{4.42}$$

$$\mathcal{M}_{\mathrm{non-BPS},Z_{AB}\neq 0,\mathrm{large}} = SO(1,1) \times \frac{SO(5,M-1)}{USp(4) \times SO(M-1)}, \qquad \dim_{\mathbb{R}} = 5(M-1) + 1. \tag{4.43}$$

As justified in [29] and then in [39], $\mathcal{M}_{(1/4)-\mathrm{BPS,large}}$ is a *quaternionic* symmetric manifold. Furthermore, $\mathcal{M}_{\mathrm{non-BPS},Z_{AB}\neq 0,\mathrm{large}}$ given by Eq. (4.43) is nothing but the scalar manifold of $\mathcal{N} = 4$, $d = 5$ *matter coupled* supergravity. The stabilizers of $\mathcal{M}_{(1/4)-\mathrm{BPS,large}}$, $\mathcal{M}_{\mathrm{non-BPS},Z_{AB}=0,\mathrm{large}}$ and $\mathcal{M}_{\mathrm{non-BPS},Z_{AB}\neq 0,\mathrm{large}}$ exploit the maximal compact symmetry of the corresponding charge orbits; this symmetry becomes fully manifest through the enhancement of the compact symmetry group of $Z_{AB,\mathrm{skew-diag.}}$ and $Z_{I,\mathrm{red}}$ at the event horizon of the extremal BH, respectively, given by Eqs. (4.25), (4.30), (4.35), and (4.36).

Let us now analyze the *small* charge orbits of the $(\mathbf{2}, \mathbf{6}+\mathbf{M})$ of $SL(2,\mathbb{R}) \times SO(6,M)$, associated to $I_{4,\mathcal{N}=4} = 0$, for which the *attractor mechanism* does not hold. The analysis performed below completes the one given in [3,9].

While in $\mathcal{N} = 8$, $d = 4$ supergravity all three *small* charge orbits are BPS (with various degrees of supersymmetry-preservation), in the considered $\mathcal{N} = 4$, $d = 4$ theory there are five *small* charge orbits, two of them being $\frac{1}{2}-$ BPS one $\frac{1}{4}$-BPS, and the other two non-BPS (one with $Z_{AB} = 0$ and the other with $Z_{AB} \neq 0$). Such an abundance of different charge orbits can be traced back to the *factorized* nature of the U-duality group $SL(2,\mathbb{R}) \times SO(6,M)$. Furthermore, it should be remarked that in $\mathcal{N} = 4$, $d = 4$ supergravity the $\frac{1}{(\mathcal{N}=)4}$-BPS charge orbit exists only in its *large* version, differently from the $d = 4$ maximal theory, in which both *large* and *small* $\frac{1}{(\mathcal{N}=)8}-$ BPS charge orbits exist.

It is now convenient to denote with α_1 and α_2 the two real non-negative eigenvalues of the matrix $Z_{AB}\bar{Z}^{CB} = (ZZ^\dagger)_A^C$. By recalling Eq. (4.19), one can notice that

$(i = 1, 2)$

$$\alpha_i = z_i^2. \tag{4.44}$$

and one can order them as $\alpha_1 \geq \alpha_2$, without any loss of generality. The explicit expression of α_i in terms of $U(4) \times SO(M)$-invariants (namely of $\mathrm{Tr}(ZZ^\dagger)$, $\mathrm{Tr}((ZZ^\dagger)^2)$, and suitable powers) is given by Eqs. (5.108) and (5.109) of [9].

Firstly, let us observe that from Eqs. (4.16) and (4.11) the $SL(2,\mathbb{R}) \times SO(6,M)$-invariant "*degeneracy*" condition can be written in the "*dressed*" ($\mathcal{R}$-symmetry- and $SO(M)$- covariant) and "*bare*" (symplectic-, *i.e.* $Sp(12+2M,\mathbb{R})$- covariant) charges' bases, respectively, as follows:

$$\mathcal{I}_{4,\mathcal{N}=4} = 0 \Leftrightarrow \mathcal{S}_1^2 = |\mathcal{S}_2|^2 \Leftrightarrow p^2q^2 = (p\cdot q)^2 \geq 0. \tag{4.45}$$

Then, in order to determine the number and typology of *small* orbits, it is convenient to start differentiating $\mathcal{I}_{4,\mathcal{N}=4}$ in the symplectic "*bare*" charges' basis $\mathcal{P} \equiv (p^\Lambda, q_\Lambda)^T$ (recall definition (1.2)). Equations (4.16) and (4.18) yield the constraints defining the *small critical* orbits to read

$$\frac{\partial \mathcal{I}_{4,\mathcal{N}=4}}{\partial p_\Lambda} = 2[q^2p^\Lambda - (q\cdot p)q^\Lambda] = 2T^{(a)|\Lambda\Sigma}q_\Sigma = 0; \tag{4.46}$$

$$\frac{\partial \mathcal{I}_{4,\mathcal{N}=4}}{\partial q_\Lambda} = 2[p^2q^\Lambda - (q\cdot p)p^\Lambda] = -2T^{(a)|\Lambda\Sigma}p_\Sigma = 0. \tag{4.47}$$

Because of the definition (4.18), or equivalently to the homogeneity (of degree four) in charges of $\mathcal{I}_{4,\mathcal{N}=4}$, it is worth noticing that the "*criticality*" constraints (4.46) and (4.47) imply the "*degeneracy*" condition (4.45).

Beside the trivial one ($p_\Lambda = 0 = q_\Lambda \forall \Lambda$), all the solutions to the "*criticality*" constraints (4.46) and (4.47) list as follows:

$$A]\left\{\begin{array}{l} T^{(a)}_{\Lambda\Sigma} = 0; \\ \left\{\begin{array}{ll} p^2q^2 = (p\cdot q)^2 > 0: & \left\{\begin{array}{l} A.1]p^2>0,\, q^2>0; \\ aut \\ A.2]p^2<0,\, q^2<0; \end{array}\right. \\ A.3]p^2q^2 = (p\cdot q)^2 = 0: & p^2=0,\, q^2=0; \end{array}\right. \end{array}\right. \tag{4.48}$$

$$B]\left\{\begin{array}{l} T^{(a)}_{\Lambda\Sigma} \neq 0; \\ p^2 = q^2 = p\cdot q = 0 \Leftrightarrow T^{(0)} = 0 \end{array}\right. . \tag{4.49}$$

Notice that each set (**A.1**, **A.2**, **A.3** and **B**) of constraints is $SL(2,\mathbb{R}) \times SO(6,M)$-invariant, but formulated in terms of the symplectic charge basis $\mathcal{P}$.

The solutions (4.48) and (4.49) can be rewritten by noticing that $\frac{\partial^2 \mathcal{I}_{4,\mathcal{N}=4}}{\partial\mathcal{P}\partial\mathcal{P}}$, *i.e.* the tensor of second derivatives of $\mathcal{I}_{4,\mathcal{N}=4}$ with respect to $\mathcal{P}$, sits in the symmetric product representation $((\mathbf{2},\mathbf{6}+\mathbf{M})\times(\mathbf{2},\mathbf{6}+\mathbf{M}))_s$ of the U-duality group $SL(2,\mathbb{R})\times SO(6,M)$, which decomposes as follows [9]:

$$\begin{aligned}((\mathbf{2},\mathbf{6}+\mathbf{M})&\times(\mathbf{2},\mathbf{6}+\mathbf{M}))_s \\ &\xrightarrow{SL(2,\mathbb{R})\times SO(6,M)} \underset{T^{(0)}}{(\mathbf{3},\mathbf{1})} + \underset{T^{(\mathrm{tr}-s)}_{\Lambda\Sigma}}{(\mathbf{3},\,\mathbf{TrSym}(SO(6,M)))} \\ &\quad + \underset{T^{(a)}_{\Lambda\Sigma}}{(\mathbf{1},\mathbf{Adj}(SO(6,M)))}.\end{aligned} \tag{4.50}$$

The antisymmetric tensor

$$T^{(a)}_{\Lambda\Sigma} \equiv \left.\frac{\partial^2\mathcal{I}_{4,\mathcal{N}=4}}{\partial\mathcal{P}\partial\mathcal{P}}\right|_{(\mathbf{1},\mathbf{Adj}(SO(6,M)))} \tag{4.51}$$

was already introduced in Eq. (4.18). **TrSym** and **Adj** respectively denote the *traceless symmetric* and *adjoint* representations, and [9]

$$\begin{aligned}T^{(\mathrm{tr}-s)}_{\Lambda\Sigma} &\equiv \left.\frac{\partial^2\mathcal{I}_{4,\mathcal{N}=4}}{\partial\mathcal{P}\partial\mathcal{P}}\right|_{(\mathbf{3},\mathbf{TrSym}(SO(6,M)))} \\ &\equiv \left(q_\Lambda q_\Sigma - \frac{q^2}{6+M}\eta_{\Lambda\Sigma},\, p_\Lambda p_\Sigma - \frac{p^2}{6+M}\eta_{\Lambda\Sigma},\, \frac{1}{2}\right. \\ &\quad \left.\times(q_\Lambda p_\Sigma + q_\Sigma p_\Lambda) - \frac{q\cdot p}{6+M}\eta_{\Lambda\Sigma}\right);\end{aligned} \tag{4.52}$$

$$\begin{aligned}T^{(0)} &\equiv \left.\frac{\partial^2\mathcal{I}_{4,\mathcal{N}=4}}{\partial\mathcal{P}\partial\mathcal{P}}\right|_{(\mathbf{3},\mathbf{1})} \equiv \mathrm{Tr}_{SO(6,M)}(T^{(s)}_{\Lambda\Sigma}) \\ &\equiv \mathrm{Tr}_{SO(6,M)}\left(\left.\frac{\partial^2\mathcal{I}_{4,\mathcal{N}=4}}{\partial\mathcal{P}\partial\mathcal{P}}\right|_{(\mathbf{3},\mathbf{Sym}(SO(6,M)))}\right) \\ &= (q^2, p^2, q\cdot p) = \begin{pmatrix} q^2 & q\cdot p \\ q\cdot p & p^2\end{pmatrix}.\end{aligned} \tag{4.53}$$

The definition (4.53) of $T^{(0)}$ implies that (recall Eq. (4.16))

$$\mathcal{I}_{4,\mathcal{N}=4} = \det(T^{(0)}) = \det\left(\left.\frac{\partial^2\mathcal{I}_{4,\mathcal{N}=4}}{\partial\mathcal{P}\partial\mathcal{P}}\right|_{(\mathbf{3},\mathbf{1})}\right), \tag{4.54}$$

in turn yielding another, equivalent $SL(2,\mathbb{R})\times SO(6,M)$-invariant characterization of the "*degeneracy*" condition (4.45):

$$\det(T^{(0)}) = \det\left(\left.\frac{\partial^2\mathcal{I}_{4,\mathcal{N}=4}}{\partial\mathcal{P}\partial\mathcal{P}}\right|_{(\mathbf{3},\mathbf{1})}\right) = 0. \tag{4.55}$$

Thus, Eqs. (4.48) and (4.49) can be recast as follows:

$$A]\left\{\begin{array}{l} T^{(a)}_{\Lambda\Sigma} = 0; \\ \det(T^{(0)}) = 0, \quad \left\{\begin{array}{l} A.1]\,\mathrm{Tr}(T^{(0)}) > 0; \\ aut \\ A.2]\,\mathrm{Tr}(T^{(0)}) < 0; \\ aut \\ A.3]\,Tr(T^{(0)}) = 0 \Leftrightarrow T^{(0)} = 0. \end{array}\right. \end{array}\right. \tag{4.56}$$

$$B]\begin{cases} T^{(a)}_{\Lambda\Sigma} \neq 0; \\ \mathrm{T}^{(0)} = 0. \end{cases} \tag{4.57}$$

As mentioned above, each set (**A.1**, **A.2**, **A.3** and **B**) of constraints is $SL(2,\mathbb{R}) \times SO(6,M)$-invariant, but formulated in terms of the symplectic charge basis $\mathcal{P}$.

It is interesting to point out that, differently from $\mathcal{N} = 8$, $d = 4$ supergravity treated in Sec. III, in $\mathcal{N} = 4$, $d = 4$ supergravity there are no *small doubly-critical* (or with higher degree of criticality) charge orbits *independent* from the *small critical* ones. This can be easily seen by noticing that the solutions (4.56) and (4.57) to the "*criticality*" constraints (4.46) and (4.47) can actually be rewritten in a *doubly-critical* fashion, *i.e.* through $\frac{\partial^2 I_{4,\mathcal{N}=4}}{\partial\mathcal{P}\partial\mathcal{P}}$ and related projections (according to decomposition (4.50)). For completeness' sake, we report here the second-order derivatives of $I_{4,\mathcal{N}=4}$ with respect to the "*bare*" *symplectic* charges:

$$\frac{\partial^2 I_{4,\mathcal{N}=4}}{\partial p_\Sigma \partial p_\Lambda} = 2(q^2\eta^{\Lambda\Sigma} - q^\Lambda q^\Sigma); \tag{4.58}$$

$$\frac{\partial^2 I_{4,\mathcal{N}=4}}{\partial q_\Sigma \partial q_\Lambda} = 2(p^2\eta^{\Lambda\Sigma} - p^\Lambda p^\Sigma); \tag{4.59}$$

$$\frac{\partial^2 I_{4,\mathcal{N}=4}}{\partial q_\Sigma \partial p_\Lambda} = 4T^{(a)|\Lambda\Sigma}. \tag{4.60}$$

In order to determine the *small* orbits of the bi-fundamental representation $(\mathbf{2}, \mathbf{6}+\mathbf{M})$ of the U-duality group $SL(2,\mathbb{R}) \times SO(6,M)$ and to study their supersymmetry-preserving properties, it is now convenient to switch to the basis of "*dressed*" charges (recall Eqs. (2.22) and (2.23))

$$\mathcal{U} \equiv (Z, \bar{Z})^T = (Z_{AB,} Z^I, \bar{Z}_{AB}, \bar{Z}^I)^T. \tag{4.61}$$

From the analysis of [9], one obtains the following equivalence:

$$T^{(a)}_{\Lambda\Sigma} \equiv \frac{\partial^2 I_{4,\mathcal{N}=4}}{\partial\mathcal{P}\partial\mathcal{P}}\bigg|_{\mathbf{Adj}(SO(6,M))} = 0 \Leftrightarrow \frac{\partial^2 I_{4,\mathcal{N}=4}}{\partial\mathcal{U}\partial\mathcal{U}}\bigg|_{\mathbf{Adj}(SO(6,M))} = 0. \tag{4.62}$$

The $SL(2,\mathbb{R}) \times SO(6,M)$-invariant constraint (4.62) is common to the *small critical* charge orbits determined by the solutions **A.1**, **A.2** and **A.3** of Eqs. (4.56). It also implies that $\alpha_1 = \alpha_2$ [9]. Then, the further $SL(2,\mathbb{R}) \times SO(6,M)$-invariant constraints $\mathrm{Tr}(T^{(0)})0$ can equivalently be rewritten as (recall definition (4.12))

$$\mathrm{Tr}\,(T^{(0)})0 \Leftrightarrow \mathcal{S}_1 0. \tag{4.63}$$

Therefore, one can characterize the *small critical* orbits **A.1**, **A.2**, and**A.3** of Eqs. (4.48) and (4.56) as follows:

$$A]\begin{cases} \frac{\partial^2 I_{4,\mathcal{N}=4}}{\partial\mathcal{U}\partial\mathcal{U}}|_{\mathbf{Adj}(SO(6,M))} = 0; \\ \mathcal{S}_1^2 = |\mathcal{S}_2|^2, \quad \begin{cases} A.1]\mathcal{S}_1 > 0; \\ aut \\ A.2]\mathcal{S}_1 < 0; \\ aut \\ A.3]\mathcal{S}_1 = 0 \Leftrightarrow \mathcal{S}_2 = 0. \end{cases} \end{cases} \tag{4.64}$$

Notice that each set (**A.1**, **A.2**, **A.3** and **B**) of constraints is $SL(2,\mathbb{R}) \times SO(6,M)$-invariant but, differently from Eqs. (4.48) and (4.56), it is also independent from the symplectic basis eventually considered.

On the other hand, the $SL(2,\mathbb{R}) \times SO(6,M)$-invariant constraints (4.49) and (4.57) defining the *small critical* orbit **B** can be recast in a form which (differently from Eqs. (4.49) and (4.57)) is independent from the symplectic basis eventually considered, as follows:

$$B]\begin{cases} \frac{\partial^2 I_{4,\mathcal{N}=4}}{\partial\mathcal{U}\partial\mathcal{U}}|_{\mathbf{Adj}(SO(6,M))} \neq 0; \\ \mathcal{S}_1^2 = |\mathcal{S}_2|^2 = 0. \end{cases} \tag{4.65}$$

Thus, five distinct *small* charge orbits (all with $I_{4,\mathcal{N}=4} = 0$) exist:

(1) The *critical* orbit **A.1** is defined by the $SL(2,\mathbb{R}) \times SO(6,M)$-invariant constraints (4.48) (or (4.56), or (4.64)). Such constraints are solved by the following flow solution (exhibiting maximal symmetry):

$$z_1 = z_2 \in \mathbb{R}_0^+, \qquad \rho_1 = \rho_2 = 0, \theta \quad \text{undetermined.} \tag{4.66}$$

Thus, from the reasoning performed at the end of Sec. II and the analysis of [9], the considered *small critical* orbit is $\frac{1}{2}$ − BPS. Along the corresponding *small critical* $\frac{1}{2}$ − BPS flow, the (maximal compact) symmetry of the *skew-diagonalized* central charge matrix $Z_{AB,\text{skew-diag}}$ defined in Eq. (4.19) is $USp(4)$, whereas the one of $Z_{I,\text{red}}$ defined in Eq. (4.20) is $SO(M)$. Therefore, the resulting maximal compact symmetry of the *critical* orbit **A.1** is $USp(4) \times SO(M)$.

(2) The *critical* orbit **A.2** is defined by the $SL(2,\mathbb{R}) \times SO(6,M)$-invariant constraints (4.48) (or (4.56), or (4.64)). Such constraints are solved by the following flow solution, existing for $M \geq 1$ (and exhibiting maximal symmetry)

$$z_1 = z_2 = 0, \qquad \rho_1 \in \mathbb{R}_0^+, \qquad \rho_2 = 0. \tag{4.67}$$

Thus, the considered *small critical* orbit is non-BPS $Z_{AB} = 0$. Along the corresponding *small critical* non-BPS $Z_{AB} = 0$ flow, the (maximal compact) symmetry of the *skew-diagonalized* central charge matrix $Z_{AB,\text{skew-diag}}$ defined in Eq. (4.19) is $SU(4)$, whereas the one of $Z_{I,\text{red}}$ defined in Eq. (4.20) is $SO(M-1)$. Therefore, the resulting maximal compact symmetry of the *critical* orbit **A.2** is $SU(4) \times SO(M-1)$.

(3) The *critical* orbit **A.3** is defined by the $SL(2,\mathbb{R}) \times SO(6,M)$-invariant constraints (4.48) (or (4.56), or (4.64)). Such constraints are solved by the following flow solution, existing for $M \geq 1$ (and exhibiting maximal symmetry)

$$z_1 = z_2 = \frac{\rho_2}{\sqrt{2}} \in \mathbb{R}_0^+, \rho_1 = 0, \theta \text{undetermined}. \tag{4.68}$$

This *small critical* orbit is $\frac{1}{2}$-BPS. Along the corresponding *small critical* non-BPS $Z_{AB} \neq 0$ flow, the (maximal compact) symmetry of the *skew-diagonalized* central charge matrix $Z_{AB,\text{skew-diag.}}$ defined in Eq. (4.19) is $USp(4)$, whereas the one of $Z_{I,\text{red}}$ defined in Eq. (4.20) is $SO(M-1)$. Therefore, the resulting maximal compact symmetry of the *critical* orbit **A.3** is $USp(4) \times SO(M-1)$.

(4) The *critical* orbit **B** is defined by the $SL(2,\mathbb{R}) \times SO(6,M)$-invariant constraints (4.49) (or (4.57), or (4.65)). Such constraints are solved by the following flow solution, existing for $M \geq 2$ (and exhibiting maximal symmetry)

$$z_1 \in \mathbb{R}_0^+, \qquad z_2 = 0, \qquad \rho_1 = \rho_2 = \frac{z_1}{\sqrt{2}}; \tag{4.69}$$

$$\theta = \frac{\pi}{2} + k\pi, \qquad k \in \mathbb{Z}. \tag{4.70}$$

This *small critical* orbit is $\frac{1}{4}$-BPS. Along the corresponding *small critical* non-BPS $Z_{AB} \neq 0$ flow, the (maximal compact) symmetry of the *skew-diagonalized* central charge matrix $Z_{AB,\text{skew-diag.}}$ defined in Eq. (4.19) is $(SU(2))^2$, whereas the one of $Z_{I,\text{red}}$ defined in Eq. (4.20) is $SO(M-2)$. Therefore, the resulting maximal compact symmetry of the *critical* orbit **B** is $(SU(2))^2 \times SO(M-2)$.

(5) The generic *small lightlike* case is defined by the $SL(2,\mathbb{R}) \times SO(6,M)$-invariant constraints (4.45) (or (4.55)). In this case, it is more convenient to consider the symplectic basis of "*bare*" charges $\mathcal{P}$ and, in order to determine the maximal compact symmetry of the flow solution(s), one can consider the saturation of the bound (4.45), namely:

$$p^2 q^2 = (p \times q)^2 = 0. \tag{4.71}$$

This is in general solved by $p^2 = 0$, $p \cdot q = 0$ and $q^2 \neq 0$ (or equivalently by $q^2 = 0$, $p \cdot q = 0$ and $p^2 \neq 0$). It is easy to realize that the maximal compact symmetry of the flow solution is $SO(4) \times SO(M-1)$ in the case $q^2 > 0$, and $SO(5) \times SO(M-2)$ in the case $q^2 < 0$. In the first case the solution exists for $M \geq 1$, whereas in the second case the solution exists for $M \geq 2$. Thus, one actually gets two generic *small lightlike* orbits, both non-BPS $Z_{AB} \neq 0$, with maximal compact symmetry, respectively, given by $SO(4) \times SO(M-1)$ and $SO(5) \times SO(M-2)$.

Mutatis mutandis, the same considerations made at the end of Sect. III for $\mathcal{N} = 8$, $d = 4$ supergravity also hold for $\mathcal{N} = 4$, $d = 4$ *matter coupled* supergravity.

Notice that in *pure* $\mathcal{N} = 4$, $d = 4$ supergravity only the *small* $\frac{1}{2}$-BPS orbit **A.1** and the *large* $\frac{1}{4}$-BPS orbit exist. Indeed, the non-BPS $Z_{AB} \neq 0$ and non-BPS $Z_{AB} = 0$ *large* orbits and the *small* orbits **A.2**, **A.3**, and **B** cannot be realized, and the *small lightlike* orbit(s) of point 5 above coincide with *small* orbit **A.1**.

Finally, it is worth noticing that the $U(1)$ (stabilizer of the factor $\frac{SL(2,\mathbb{R})}{U(1)}$ of the scalar manifold (4.1)) is broken both in *large* and *small* charge orbits, because both the *central charge matrix* Z_{AB} and the *matter charges* Z_I are charged with respect to it.

V. $\mathcal{N} = 2$

In $\mathcal{N} = 2$, $d = 4$ supergravity one can repeat the analysis of [1,40] (see also [41]), by using the properties of *special Kähler geometry* (SKG, see *e.g.* [22], and Refs. therein). Indeed, in SKG one can define an $Sp(2n,\mathbb{R})$ matrix over the scalar manifold (as in Eq. (2.9)), as well complex matrices f and h (as in Eqs. (2.10), (2.11), (2.12), (2.13), and (2.14)), without the need for the manifold to be necessarily a (*n at least locally*) symmetric space (see *e.g.* [13,21]).

The basic identities of SKG applied to the (covariantly holomorphic) $\mathcal{N} = 2$, $d = 4$ *central charge* section

$$Z \equiv e^{K/2}(X^\Lambda q_\Lambda - F_\Lambda p^\Lambda) \tag{5.1}$$

of the $U(1)$ Kähler-Hodge bundle (with Kähler weights $(1,-1)$) read as follows [20] ($i, \bar{j} = 1, \ldots, n-1$, with $n-1$ denoting the number of Abelian vector multiplets coupled to the supergravity one)

$$\bar{D}_{\bar{i}} Z = 0; \tag{5.2}$$

$$D_i D_j Z = i C_{ijk} g^{k\bar{k}} \bar{D}_{\bar{k}} \bar{Z}; \tag{5.3}$$

$$\bar{D}_{\bar{j}} D_i Z = g_{i\bar{j}} Z, \tag{5.4}$$

where (X^Λ, F_Λ) are the holomorphic symplectic sections of the $U(1)$ Kähler-Hodge bundle (with Kähler weights (2, 0)), and K denotes the Kähler potential of the Abelian vector multiplets' scalar manifold, with metric $g_{i\bar{j}} = \bar{\partial}_{\bar{j}} \partial_i K$. C_{ijk} is the rank-3 symmetric and covariantly holomorphic *C-tensor* of SKG (see *e.g.* [22], and Refs. therein):

$$\bar{D}_{\bar{l}} C_{ijk} = 0; \tag{5.5}$$

$$D_{[l}C_{i]jk} = 0. \tag{5.6}$$

Thus, in $\mathcal{N} = 2$, $d = 4$ supergravity coupled to $n - 1$ Abelian vector multiplets, the *"BH potential"* is given by [18,19]

$$V_{\mathrm{BH}}(\phi, \mathcal{P}) = Z\bar{Z} + g^{i\bar{j}}(D_i Z)\bar{D}_{\bar{j}}\bar{Z}, \tag{5.7}$$

and the *Attractor Eqs.* read [20]

$$\partial_i V_{\mathrm{BH}} = 0 \Leftrightarrow 2\bar{Z}D_i Z + iC_{ijk}g^{j\bar{j}}g^{k\bar{k}}(\bar{D}_{\bar{j}}\bar{Z})\bar{D}_{\bar{k}}\bar{Z} = 0. \tag{5.8}$$

(1) The ($\frac{1}{2}$ − BPS) supersymmetric solution to *Attractor Eqs.* (5.8) is determined by

$$(D_i Z)_{(1/2)-\mathrm{BPS}} = 0, \quad \forall\ i, \tag{5.9}$$

and therefore Eq. (5.7) yields

$$V_{\mathrm{BH},(1/2)-\mathrm{BPS}} = |Z|^2_{(1/2)-\mathrm{BPS}}, \tag{5.10}$$

and the corresponding Hessian matrix of V_{BH} has block components given by [20]

$$(D_i\partial_j V_{\mathrm{BH}})_{(1/2)-\mathrm{BPS}} = (\partial_i\partial_j V\partial_{\mathrm{BH}})_{(1/2)-\mathrm{BPS}} = 0; \tag{5.11}$$

$$(\partial_i\bar{\partial}_{\bar{j}} V_{\mathrm{BH}})_{(1/2)-\mathrm{BPS}} = 2g_{i\bar{j},(1/2)-\mathrm{BPS}}|Z|^2_{(1/2)-\mathrm{BPS}}, \tag{5.12}$$

showing that there are no "*flat*" *directions* for such the ($\frac{1}{2}$ − BPS class of solutions to *Attractor Eqs.* (5.8) [33].

(2) Nonsupersymmetric (non-BPS) solutions to *Attractor Eqs.* (5.8) have $D_i Z \neq 0$ (*at least*) for some $i \in \{1, \dots, n-1\}$. Generally, such solutions fall into two class [6], and they exhibit "*flat*" *directions* of V_{BH} itself [33]. The non-BPS, $Z = 0$ class is defined by the following constraints:

$$D_i Z = \partial_i Z \neq 0, \quad \text{for some } i, Z = 0, \tag{5.13}$$

thus yielding (from Eqs. (5.8))

$$[C_{ijk}g^{j\bar{j}}g^{k\bar{k}}(\bar{\partial}_{\bar{j}}\bar{Z})\bar{\partial}_{\bar{k}}\bar{Z}]_{\mathrm{non-BPS},Z=0} = 0. \tag{5.14}$$

Thus, Eqs. (5.7) and (5.13) yield

$$\begin{aligned} V_{\mathrm{BH,non-BPS},Z=0} &= [g^{i\bar{j}}(D_i Z)\bar{D}_{\bar{j}}\bar{Z}]_{\mathrm{non-BPS},Z=0} \\ &= [g^{i\bar{j}}(\partial_i Z)\bar{\partial}_{\bar{j}}\bar{Z}]_{\mathrm{non-BPS},Z=0}. \end{aligned} \tag{5.15}$$

(3) The non-BPS, $Z \neq 0$ class is defined by the following constraints:

$$D_i Z \neq 0, \quad \text{for some } i, Z \neq 0. \tag{5.16}$$

It is worth remarking that Eqs. (5.8) and the non-BPS $Z \neq 0$ defining constraints (5.16) imply the following relations to hold at the non-BPS $Z \neq 0$ critical points of V_{BH} [13]:

$$\begin{aligned} [g^{i\bar{j}}(D_i Z)\bar{D}_{\bar{j}}\bar{Z}]_{\mathrm{non-BPS},Z\neq 0} &= -\frac{i}{2} \\ &\quad \times \left[\frac{N_3(\bar{Z})}{\bar{Z}}\right]_{\mathrm{non-BPS},Z\neq 0} \\ &= \frac{i}{2}\left[\frac{\bar{N}_3(Z)}{Z}\right]_{\mathrm{non-BPS},Z\neq 0}, \end{aligned} \tag{5.17}$$

where the cubic form $N_3(\bar{Z})$ is defined as [13]

$$N_3(\bar{Z}) \equiv C_{ijk}\bar{Z}^i\bar{Z}^j\bar{Z}^k \Leftrightarrow \bar{N}_3(Z) \equiv \bar{C}_{\bar{i}\bar{j}\bar{k}}Z^{\bar{i}}Z^{\bar{j}}Z^{\bar{k}}. \tag{5.18}$$

For an arbitrary SKG, it is in general hard to compute

$$\frac{S_{\mathrm{BH}}}{\pi} = V_{\mathrm{BH}}|_{\partial_\phi V_{\mathrm{BH}}=0} = V_{\mathrm{BH}}(\phi_H(\mathcal{P}), \mathcal{P}), \tag{5.19}$$

where $\phi_H(\mathcal{P})$ are the horizon scalar configurations solving the *Attractor Eqs.* (5.8). However, the situation dramatically simplifies for *symmetric* SK manifolds

$$\frac{G_4}{H_4}, \tag{5.20}$$

in which case a classification, analogous to the one available for $\mathcal{N} > 2$-extended, $d = 4$ supergravities (see *e.g.* [13] and Refs. therein; see also Secs. III and IV) can be performed [6].

In the treatment below, we are going to give a remarkable general *topological* formula for $V_{\mathrm{BH}}(\phi_H(\mathcal{P}), \mathcal{P})$ for *symmetric* SKG, which is manifestly *invariant* under diffeomorphisms of the SK scalar manifold, and which holds *for any* choice of symplectic basis of *"bare"* charges $\mathcal{P}$ and of *special coordinates* (see *e.g.* [22] and Refs. therein) of the SK manifold itself. Indeed, such a formula by no means does refer to *special coordinates*, which may not even exist for certain parametrizations of $\frac{G_4}{H_4}$ itself.

It should be pointed out that a general formula for the G_4-invariant $\mathcal{I}_{4,\mathcal{N}=2}$ is known for the so-called d-SK homogeneous *symmetric* manifolds [26], and it reads ($a = 1, \dots, n-1$) [4]:

$$\begin{aligned} \mathcal{I}_{4,\mathcal{N}=2}(\mathcal{P}) = -(p^0q_0 + p^aq_a)^2 + 4[q_0\mathcal{I}_{3,\mathcal{N}=2}(p) \\ - p^0\mathcal{I}_{3,\mathcal{N}=2}(q) \\ + \{\mathcal{I}_{3,\mathcal{N}=2}(p), \mathcal{I}_{3,\mathcal{N}=2}(q)\}], \end{aligned} \tag{5.21}$$

where

$$\mathcal{I}_{3,\mathcal{N}=2}(p) \equiv \frac{1}{3!}d_{abc}p^ap^bp^c; \tag{5.22}$$

$$\mathcal{I}_{3,\mathcal{N}=2}(q) \equiv \frac{1}{3!} d^{abc} q_a q_b q_c; \tag{5.23}$$

$$\{\mathcal{I}_{3,\mathcal{N}=2}(p), \mathcal{I}_{3,\mathcal{N}=2}(q)\} \equiv \frac{\partial \mathcal{I}_{3,\mathcal{N}=2}(p)}{\partial p^a} \frac{\partial \mathcal{I}_{3,\mathcal{N}=2}(q)}{\partial q_a}, \tag{5.24}$$

in which the constant (number) rank-3 symmetric tensor d_{abc} has been introduced (and d^{abc} is its suitably defined completely contravariant form). However, such a formula holds for a particular symplectic basis (namely the one inherited from the $\mathcal{N} = 2$, $d = 5$ theory, *i.e.* the one of *special coordinates*), in which the holomorphic prepotential $F(X)$ of SKG can be written as

$$F(X) \equiv \frac{1}{3!} d_{abc} \frac{X^a X^b X^c}{X^0}. \tag{5.25}$$

In such a symplectic basis, the manifest symmetry is the $d = 5$ U-duality G_5, under which G_4 branches as $G_4 \longrightarrow G_5 \times SO(1, 1)$. Indeed, $\mathcal{I}_{3,\mathcal{N}=2}(p)$ and $\mathcal{I}_{3,\mathcal{N}=2}(q)$ are nothing but, respectively, the *magnetic* and *electric* invariants (both *cubic* in $\mathcal{P}$) of the relevant symplectic representations of G_5.

Equation (5.21) excludes the so-called *quadratic* (or *minimally coupled* [42]) sequence of symmetric SK manifolds (particular *complex Grassmannians*)

$$\frac{SU(1, n-1)}{SU(n-1) \times U(1)}, \qquad n \in \mathbb{N} \tag{5.26}$$

(not upliftable to $d = 5$), for which $F(X)$ is given by (in the symplectic basis exhibiting the maximal noncompact symmetry $SU(1, n-1)$)

$$F(X) = -\frac{i}{2}\left[(X^0)^2 - \sum_{i=1}^{n-1}(X^i)^2\right], \tag{5.27}$$

and the invariant of the symplectic representation of $G_4 = SU(1, n-1)$ reads as follows (notice it is *quadratic* in $\mathcal{P}$) [29]:

$$\begin{aligned}\mathcal{I}_{2,\mathcal{N}=2}(\mathcal{P}) &= (p^0)^2 + q_0^2 - \sum_{i=1}^{n-1}((p^i)^2 + q_i^2) \\ &= |Z|^2 - g^{i\bar{j}}(D_i Z)\bar{D}_{\bar{j}}\bar{Z}.\end{aligned} \tag{5.28}$$

Because of the *quadratic* nature of the G_4-invariant $\mathcal{I}_{2,\mathcal{N}=2}(\mathcal{P})$ given by Eq. (5.28), the *quadratic* sequence of symmetric SK manifolds (5.26) exhibits only one *small* charge orbit, namely, the *lightlike* one, beside the two *large* charge orbits determined in [6].

The symmetric SK manifolds whose geometry is determined by the holomorphic prepotential function (5.25) and the *minimally coupled* ones determined by Eq. (5.27) are *all* the possible symmetric SK manifolds. After [43], from the geometric perspective of SKG, symmetric SK manifolds can be characterized in the following way.

In SKG the Riemann tensor obeys to the following constraint (see *e.g.* [22] and Refs. therein):

$$R_{i\bar{j}k\bar{l}} = -g_{i\bar{j}}g_{k\bar{l}} - g_{i\bar{l}}g_{k\bar{j}} + C_{ikm}\bar{C}_{\bar{l}\bar{j}\bar{n}}g^{m\bar{n}}. \tag{5.29}$$

The requirement that the manifold to be symmetric demands the Riemann to be covariantly constant:

$$D_m R_{i\bar{j}k\bar{l}} = 0. \tag{5.30}$$

Because of the SKG constraint (5.29) and to covariant holomorphicity of the C-tensor (expressed by Eq. (5.5)), Eq. (5.30) generally implies (for nonvanishing C_{ijk})

$$D_l C_{ijk} = D_{(l}C_{i)jk} = 0, \tag{5.31}$$

where in the last step Eq. (5.6) was used. Thus, in a SK symmetric space both the Riemann tensor and the C-tensor are covariantly constant. Equation (5.31) implies the following relation [6]

$$C_{j(lm}C_{pq)k}\bar{C}_{\bar{i}\bar{j}\bar{k}}g^{j\bar{j}}g^{k\bar{k}} = \frac{4}{3}C_{(lmp}g_{q)\bar{i}}, \tag{5.32}$$

which is nothing but the *"dressed"* form of the analogous relation holding for the d-tensor itself [43,44]

$$d_{j(lm}d_{pq)k}d^{ijk} = \frac{4}{3}d_{(lmp}\delta^i_{q)}. \tag{5.33}$$

The quadratic sequence of symmetric manifolds (5.26) whose SKG is determined by the prepotential (5.27) has

$$C_{ijk} = 0, \tag{5.34}$$

whereas the remaining symmetric SK manifolds, whose prepotential in the special coordinates is given by Eq. (5.25) (with d_{abc} constrained by Eq. (5.33)), correspond to

$$C_{abc} = e^K d_{abc}. \tag{5.35}$$

By using Eqs. (5.31) and (5.32), as well as the SKG identities (5.2), (5.3), and (5.4) (which, for symmetric SKG, are equivalent to the *Maurer-Cartan Eqs.*, as Eqs. (3.3), (4.4), and (4.5) for $\mathcal{N} = 8$ and $\mathcal{N} = 4$, $d = 4$ supergravities, respectively; see *e.g.* [21,29]), one can prove that the following *quartic* expression is a *duality* invariant for all symmetric SK manifolds:

$$\begin{aligned}\mathcal{I}_{4,\mathcal{N}=2,\mathrm{symm}}(\phi, \mathcal{P}) = (Z\bar{Z} - Z_i\bar{Z}^i)^2 \\ + \frac{2}{3}i(ZN_3(\bar{Z}) - \bar{Z}\bar{N}_3(Z)) \\ - g^{i\bar{i}}C_{ijk}\bar{C}_{\bar{i}\bar{l}\bar{m}}\bar{Z}^j\bar{Z}^k Z^{\bar{l}} Z^{\bar{m}},\end{aligned} \tag{5.36}$$

where the *matter charges* have been renoted as $Z_i \equiv D_i Z$, $Z^{\bar{i}} = g^{j\bar{i}}Z_j$, and definition (5.18) was recalled.

As claimed above, $\mathcal{I}_{4,\mathcal{N}=2,\mathrm{symm}}$ given by Eq. (5.36) is ϕ-dependent *only apparently*, *i.e.* it is *topological*, merely charge-dependent:

$$\frac{\partial \mathcal{I}_{4,\mathcal{N}=2,\text{symm}}(\phi, \mathcal{P})}{\partial \phi} = 0 \Leftrightarrow \mathcal{I}_{4,\mathcal{N}=2,\text{symm}} = \mathcal{I}_{4,\mathcal{N}=2,\text{symm}}(\mathcal{P}). \quad (5.37)$$

Thus, by recalling Eq. (1.5), the general entropy-area formula [8] for extremal BHs in $\mathcal{N} = 2$, $d = 4$ supergravity coupled to Abelian vector multiplets whose scalar manifold is a symmetric (SK) space reads as follows:

$$\frac{S_{\text{BH}}}{\pi} = V_{\text{BH}}|_{\partial_\phi V_{\text{BH}}=0} = V_{\text{BH}}(\phi_H(\mathcal{P}), \mathcal{P}) = |\mathcal{I}_{4,\mathcal{N}=2,\text{symm}}(\mathcal{P})|^{1/2}. \quad (5.38)$$

Let us briefly analyze Eq. (5.36).

As for the case of $\mathcal{N} = 8$, $d = 4$ supergravity treated in Sec. III, one can introduce a phase ϑ as follows (recall definitions (5.18)):

$$e^{2i\vartheta} \equiv -\frac{ZN_3(\bar{Z})}{\bar{Z}\bar{N}_3(Z)} = \frac{iZC_{ijk}\bar{Z}^i\bar{Z}^j\bar{Z}^k}{-i\bar{C}_{\bar{l}\bar{m}\bar{n}}Z^{\bar{l}}Z^{\bar{m}}Z^{\bar{n}}}. \quad (5.39)$$

Thus, ϑ is the phase of the quantity $iZN_3(\bar{Z})$: $\vartheta \equiv \vartheta_{iZN_3(\bar{Z})}$. It is then immediate to compute ϑ from Eq. (5.36):

$$\cos\vartheta(\phi, \mathcal{P}) = \frac{3[\mathcal{I}_{4,\mathcal{N}=2,\text{symm}}(\mathcal{P}) - (Z\bar{Z} - Z_i\bar{Z}^i)^2 + g^{i\bar{i}}C_{ijk}\bar{C}_{\bar{i}\bar{l}\bar{m}}\bar{Z}^j\bar{Z}^kZ^{\bar{l}}Z^{\bar{m}}]}{2^2|ZN_3(\bar{Z})|}. \quad (5.40)$$

Notice that through Eq. (5.40) (cos)ϑ is determined in terms of the scalar fields ϕ and of the BH charges $\mathcal{P}$, also along the *small* orbits where $\mathcal{I}_{4,\mathcal{N}=2,\text{symm}} = 0$. However, Eq. (5.40) is not defined in the cases in which $ZN_3(\bar{Z}) = 0$. In such cases, ϑ is actually undetermined. It should be clearly pointed out that the phase ϑ has nothing to do with the phase of the $U(1)$ bundle over the SK-Hodge vector multiplets' scalar manifold (see *e.g.* [22] and Refs. therein).

(1) For $\frac{1}{2}-$ BPS attractors (defined by the constraints (5.9)), Eq. (5.36) yields

$$\mathcal{I}_{4,\mathcal{N}=2,\text{symm}}|_{(1/2)-\text{BPS}} = (Z\bar{Z})^2_{(1/2)-\text{BPS}} = |Z|^4_{(1/2)-\text{BPS}}, \quad (5.41)$$

as in turn also implied by Eqs. (1.5) and (5.10) (or equivalently (5.38)). Notice that Eqs. (5.10) and (5.41) are *general*, *i.e.* they hold for any SKG, regardless the symmetric nature of the SK vector multiplets' scalar manifold. Furthermore, the constraints (5.9) imply that at the event horizon of $\frac{1}{2}-$ BPS extremal BHs it holds

$$[N_3(\bar{Z})]_{(1/2)-\text{BPS}} = 0 \Rightarrow \vartheta_{(1/2)-\text{BPS}} \text{ undetermined}. \quad (5.42)$$

(2) For non-BPS $Z = 0$ attractors (defined by the constraints (5.13) which, through Eqs. (5.8), imply Eq. (5.14)), Eq. (5.36) yields

$$\mathcal{I}_{4,\mathcal{N}=2,\text{symm}}|_{\text{non-BPS},Z=0} = (Z_i\bar{Z}^i)^2_{\text{non-BPS},Z=0} = [g^{i\bar{j}}(\partial_i Z)\bar{\partial}_{\bar{j}}\bar{Z}]^2_{\text{non-BPS},Z=0}. \quad (5.43)$$

Notice that Eqs. (5.15) and (5.43) are *general*, *i.e.* they hold for any SKG, regardless the symmetric nature of the SK vector multiplets' scalar manifold. Furthermore, the constraints (5.9) imply that at the event horizon of non-BPS $Z = 0$ extremal BHs it holds

$$Z_{\text{non-BPS},Z=0} = 0 \Rightarrow \vartheta_{\text{non-BPS},Z=0} \text{ undetermined}. \quad (5.44)$$

(3) For non-BPS $Z \neq 0$ attractors (defined by the constraints (5.16) as well as by Eqs. (5.8)), Eqs. (5.17) and (5.36) yield

$$\mathcal{I}_{4,\mathcal{N}=2,\text{symm}}|_{\text{non-BPS},Z\neq0} = -16|Z|^4_{\text{non-BPS},Z\neq0}, \quad (5.45)$$

thus implying, through Eq. (5.7) [6,13,30,45]

$$Z_i\bar{Z}^i|_{\text{non-BPS},Z\neq0} = 3|Z|^2_{\text{non-BPS},Z\neq0} \Leftrightarrow V_{\text{BH},\text{non-BPS},Z\neq0} = 4|Z|^2_{\text{non-BPS},Z\neq0}. \quad (5.46)$$

By plugging Eqs. (5.8), (5.16), (5.17), and (5.45) into Eq. (5.40), it follows that at the event horizon of non-BPS $Z \neq 0$ extremal BHs it holds that

$$\vartheta_{\text{non-BPS},Z\neq0} = \pi + 2k\pi, \qquad k \in \mathbb{Z}. \quad (5.47)$$

It should be remarked that, differently from the results (5.10), (5.11), (5.12), (5.41), and (5.42) (holding for $\frac{1}{2}-$ BPS attractors) and from the results (5.14), (5.15), (5.43), and (5.44) (holding for non-BPS $Z = 0$ attractors), Eqs. (5.45), (5.46), and (5.47) are not *general*: *i.e.* they hold at the event horizon of extremal non-BPS $Z \neq 0$ BHs for symmetric SK manifolds, but they do not hold true for generic SKG. However, when going *beyond* the symmetric SK case (and thus encompassing both homogeneous nonsymmetric [26,46] and nonhomogeneous SK spaces), one can compute both $V_{BH,non-BPS,Z\neq0}$ and $\mathcal{I}_{4,\mathcal{N}=2,symm}|_{non-BPS,Z\neq0}$, and express the deviation from the symmetric case considered above in terms of the complex quantity [13]

$$\Delta \equiv -\frac{3}{4}\frac{E_{i\bar{j}\bar{k}\bar{l}\bar{m}}\bar{Z}^i Z^{\bar{j}} Z^{\bar{k}} Z^{\bar{l}} Z^{\bar{m}}}{\bar{N}_3(Z)}, \quad (5.48)$$

where the tensor $E_{i\bar{j}\bar{k}\bar{l}\bar{m}}$ was firstly introduced in [26] (see also [13]). The results of straightforward computations read as follows:

$$V_{BH,non-BPS,Z\neq0} = 4|Z|^2_{non-BPS,Z\neq0} + \Delta_{non-BPS,Z\neq0}; \quad (5.49)$$

$$\mathcal{I}_{4,\mathcal{N}=2,symm}|_{non-BPS,Z\neq0} = \left[-16|Z|^4 + \Delta^2 - \frac{8}{3}\Delta|Z|^2\right]_{non-BPS,Z\neq0}. \quad (5.50)$$

Notice that, as yielded *e.g.* by Eq. (5.49), Δ is real at the non-BPS $Z \neq 0$ critical points of V_{BH}. For symmetric SK manifolds $E_{i\bar{j}\bar{k}\bar{l}\bar{m}} = 0$ globally, and thus Eqs. (5.49) and (5.50) respectively reduce to Eqs. (5.45) and (5.46). On the other hand, the results (5.45) and (5.46) hold also for those nonsymmetry SK spaces ($E_{i\bar{j}\bar{k}\bar{l}\bar{m}} \neq 0$) such that

$$\Delta_{non-BPS,Z\neq0} = 0 \Leftrightarrow (E_{i\bar{j}\bar{k}\bar{l}\bar{m}}\bar{Z}^i Z^{\bar{j}} Z^{\bar{k}} Z^{\bar{l}} Z^{\bar{m}})_{non-BPS,Z\neq0}, \quad (5.51)$$

where in the implication "$\Rightarrow$" the assumption $[\bar{N}_3(Z)]_{non-BPS,Z\neq0} \neq 0$ was made. The condition (5.51) might explain some results obtained for generic (d)-SKGs in some particular supporting BH charge configurations in [45] (see also the treatment in [13,39]).

Consistently, for the quadratic *minimally coupled* sequence (5.26), for which Eq. (5.34) holds, Eq. (5.36) formally reduces to

$$\mathcal{I}_{4,\mathcal{N}=2,symm}|_{C_{ijk=0}} = (Z\bar{Z} - Z_i\bar{Z}^i)^2; \qquad |\mathcal{I}_{4,\mathcal{N}=2,symm}|_{C_{ijk=0}}|^{1/2} = |\mathcal{I}_{2,\mathcal{N}=2}|, \quad (5.52)$$

where $\mathcal{I}_{2,\mathcal{N}=2}$ is given by Eq. (5.28).

Remarkably, Eq. (5.36) turns out to be directly related to the quantity $-h$ given by Eq. (2.31) of [26] (see also the treatment of [47]). This is seen by noticing that Eq. (4.42) of [26] coincides with Eq. (5.21) (along with definitions (5.22), (5.23), and (5.24)). Note that the mapping of quaternionic coordinates $(A^\Lambda, B_\Lambda)^T$ into the charges $\mathcal{P}^T = (p^\Lambda, q_\Lambda)^T$ (in *special coordinates*) is related to the $d = 3$ attractor flows (see *e.g.* [48–50]).

For *symmetric* SK manifolds, *small* charge orbits of the symplectic representation of G_4 are known to exist since [4,5].

(i) *Small lightlike* charge orbits are defined by the G_4-invariant constraint

$$\mathcal{I}_{4,\mathcal{N}=2,symm} = 0; \quad (5.53)$$

$$\Updownarrow$$

$$(Z\bar{Z} - Z_i\bar{Z}^i)^2 + \frac{2}{3}i(ZN_3(\bar{Z}) - \bar{Z}\bar{N}_3(Z)) = g^{i\bar{i}}C_{ijk}\bar{C}_{\bar{i}\bar{l}\bar{m}}\bar{Z}^j\bar{Z}^k Z^{\bar{l}} Z^{\bar{m}}. \quad (5.54)$$

In this case, Eq. (5.40) reduces to

$$\cos\vartheta(\phi,\mathcal{P})|_{\mathcal{I}_{4,\mathcal{N}=2,symm}=0} = -\frac{3[(Z\bar{Z} - Z_i\bar{Z}^i)^2 - g^{i\bar{i}}C_{ijk}\bar{C}_{\bar{i}\bar{l}\bar{m}}\bar{Z}^j\bar{Z}^k Z^{\bar{l}} Z^{\bar{m}}]}{2^2|ZN_3(\bar{Z})|}\Bigg|_{\mathcal{I}_{4,\mathcal{N}=2,symm}=0}. \quad (5.55)$$

(ii) Beside the constraint (5.53) and (5.54), *small critical* charge orbits are defined by the following G_4-invariant set of first order differential constraints, as well:

$$\frac{\partial\mathcal{I}_{4,\mathcal{N}=2,symm}}{\partial Z} = 0 = \frac{\partial\mathcal{I}_{4,\mathcal{N}=2,symm}}{\partial Z_i}. \quad (5.56)$$

(iii) Beside the constraints (5.53), (5.54), and (5.56), *small doubly-critical* charge orbits are also defined by the following set of second-order differential constraints, as well:

$$\mathcal{D}_{i\bar{j}}\mathcal{I}_{4,\mathcal{N}=2,symm} = 0 = \mathcal{D}_i\mathcal{I}_{4,\mathcal{N}=2,symm}, \quad (5.57)$$

where the second-order differential operators $\mathcal{D}_{i\bar{j}}$ and $\mathcal{D}_i$ have been introduced:

$$\mathcal{D}_{i\bar{j}} \equiv R_{i\bar{j}k}{}^{l}\frac{\partial}{\partial Z_k}\frac{\partial}{\partial\bar{Z}^l}; \quad (5.58)$$

$$\mathcal{D}_i \equiv C_{ijk}\frac{\partial}{\partial Z_j}\frac{\partial}{\partial Z_k}. \quad (5.59)$$

Notice that, through the definitions (5.58) and (5.59), the constraints (5.57) are G_4-invariant, because they are equivalent to the following constraint:

$$\frac{\partial^2\mathcal{I}_{4,\mathcal{N}=2,symm}}{\partial Z_{\mathbf{sympl}(G_4)}\partial Z_{\mathbf{sympl}(G_4)}}\Bigg|_{\mathbf{Adj}(G_4)} = 0, \quad (5.60)$$

where

$$Z_{\mathbf{sympl}(G_4)} \equiv (Z, \bar{Z}_{\bar{i}}, \bar{Z}, Z_i)^T, \tag{5.61}$$

and the change of charge basis between the manifestly H_4-covariant (in "*flat*" *local* coordinates) basis $Z_{\mathbf{sympl}(G_4)}$ and the manifestly $Sp(2n, \mathbb{R})$-covariant basis $\mathcal{P}$ (defined by Eq. (1.2)) is expressed by the fundamental *identities* of the SKG (see *e.g.* [22,51] and Refs. therein). Indeed, by considering the *Cartan decomposition* of the Lie algebra of G_4:

$$\mathfrak{g}_4 = \mathfrak{h}_{44} + \mathfrak{k}_4, \tag{5.62}$$

and switching to "*flat*" *local* coordinates in the scalar manifold (here denoted by capital Latin indices), it holds that $\mathcal{D}_I$ ("*flat*" version of the operator defined in Eq. (5.59)) is $\mathfrak{k}_4$-valued. Furthermore, in symmetric manifolds $R_{I\bar{J}K}{}^L$ is a twoform (in the first two "*flat*" *local* indices) which is Lie algebra-valued in $\mathfrak{h}_4$, and thus $\mathcal{D}_{I\bar{J}}$ (*"flat"* version of the operator defined in Eq. (5.58)) turns out to be $\mathfrak{h}_4$-valued. Notice that Eq. (5.60), G_4-invariantly defining the *small doubly-critical* charge orbit(s) of the $\mathcal{N} = 2$, $d = 4$ vector multiplets' symmetric SK scalar manifolds, is the analogue of Eq. (3.42), which defines in an $E_{7(7)}$-invariant way the *small doubly-critical* charge orbit of $\mathcal{N} = 8$, $d = 4$ *pure* supergravity. It should be also recalled that in $\mathcal{N} = 4$, $d = 4$ *matter coupled* supergravity *smalldoubly-critical* (or *higher-order-critical*) charge orbits (independent from the *small critical* ones) are absent. As treated in Sec. IV, all *small critical* charge orbits of the $\mathcal{N} = 4$ theory actually are *doubly-critical*, and the analogues of Eqs. (3.42) and (5.60) are given, through Eq. (4.50) and definitions (4.51) and (4.53), by the rich case study exhibited by Eqs. [(4.48), (4.49), (4.56), and (4.57)].

The classification of *small* charge orbits of the relevant symplectic representation of G_4 for $\mathcal{N} = 2$, $d = 4$ supergravity coupled to Abelian vector multiplets whose scalar manifold $\frac{G_4}{H_4}$ is (SK) symmetric, performed in accordance to their *order of criticality* (*lightlike*, *critical*, *doubly-critical*), will be given elsewhere.

VI. ADM MASS FOR BPS EXTREMAL BLACK HOLE STATES

For BPS BH states in $d = 4$ ungauged[8] supergravity theories, the *ADM mass* [27] $M_{\text{ADM}}(\phi_\infty, \mathcal{P})$ is defined as the largest (of the absolute values) of the *skew-eigenvalues* of the (spatially asymptotically) *central charge matrix* $Z_{AB}(\phi_\infty, \mathcal{P})$ which saturate the *BPS bound* (2.28). The *skew-diagonalization* of Z_{AB} is made by performing a suitable transformation of the $\mathcal{R}$-*symmetry*, and thus by going to the so-called *normal frame*. In such a frame, the *skew-eigenvalues* of Z_{AB} can be taken to be real and positive (up to an eventual overall *phase*). By saturating the *BPS bound* (2.28), it therefore holds that

$$M_{\text{ADM}}(\phi_\infty, \mathcal{P}) = |\mathbf{Z}_1(\phi_\infty, \mathcal{P})| \geq \ldots \geq |\mathbf{Z}_{[\mathcal{N}/2]}(\phi_\infty, \mathcal{P})|, \tag{6.1}$$

where $\mathbf{Z}_1(\phi, \mathcal{P}), \ldots, \mathbf{Z}_{[\mathcal{N}/2]}(\phi, \mathcal{P})$ denote the set of *skew-eigenvalues* of $Z_{AB}(\phi, \mathcal{P})$, and square brackets denote the integer part of the enclosed number. As mentioned at the end of Sec. II, if $1 \leq \mathbf{k} \leq [\mathcal{N}/2]$ of the bounds expressed by Eq. (2.28) are saturated, the corresponding extremal BH state is named to be $\frac{\mathbf{k}}{\mathcal{N}} -$ BPS. Thus, the minimal fraction of total supersymmetries (pertaining to the asymptotically flat space-time metric) preserved by the extremal BH background within the considered assumptions is $\frac{1}{\mathcal{N}}$ (for $\mathbf{k} = 1$), while the maximal one is $\frac{1}{2}$ (for $\mathbf{k} = \frac{\mathcal{N}}{2}$).

The ADM mass and its symmetries are different, depending on $\mathbf{k}$.

A. $\mathcal{N} = 8$

In $\mathcal{N} = 8$, $d = 4$ supergravity (treated in Sec. III), the $E_{7(7)}$ U-duality symmetry only allows the cases [3] $\mathbf{k} = 1$, 2, 4. By recalling the review given in Sec. III, the maximal compact symmetries of the supporting charge orbits, respectively, read [3,4,13,30,32,33]

$$\mathbf{k} = 1\text{: } SU(2) \times SU(6); \tag{6.2}$$

$$\mathbf{k} = 2\text{: } USp(4) \times SU(4); \tag{6.3}$$

$$\mathbf{k} = 4\text{: } USp(8), \tag{6.4}$$

and they hold all along the respective scalar flows. While cases $\mathbf{k} = 2$ and 4 are *small* (thus not enjoying the *attractor mechanism*), case $\mathbf{k} = 1$ can be either *large* or *small*.

In the *large* $\mathbf{k} = 1$ case, the *attractor mechanism* makes the maximal compact symmetry $SU(2) \times SU(6)$ of the supporting charge orbit $\mathcal{O}_{(1/8)-\text{BPS,large}}$ fully manifest as a symmetry of the central charge matrix Z_{AB} through the *symmetry enhancement* (3.17) at the event horizon of the considered extremal BH.

Furthermore, the $\frac{1}{4} -$ BPS saturation of the $\mathcal{N} = 8$ BPS bound (all along the $\frac{1}{4} -$ BPS scalar flow) has the following peculiar structure [recall Eq. (3.35)] [3]

$$|\mathbf{Z}_1(\phi, \mathcal{P})| = |\mathbf{Z}_2(\phi, \mathcal{P})| > |\mathbf{Z}_3(\phi, \mathcal{P})| = |\mathbf{Z}_4(\phi, \mathcal{P})|, \tag{6.5}$$

where it should be recalled that in Sec. III the notation $e_i \equiv |\mathbf{Z}_i|$ $(i = 1, \ldots, 4)$ was used.

[8] In the present paper only ungauged supergravities are treated. It is here worth remarking that the definition of the *ADM mass* for (eventually rotating) asymptotically nonflat black holes in *gauged* supergravities is a fairly subtle issue, addressed by various studies in literature (see *e.g.* [52,53], and Refs. therein).

As done in Sec. III, let us denote with λ_i ($i = 1, \ldots, 4$) the four real non-negative eigenvalues of the 8×8 Hermitian matrix $Z_{AB}\bar{Z}^{CB} = (ZZ^\dagger)_A^C \equiv A_A^C$. Their relation with the absolute values of the complex *skew-eigenvalues* e_i of Z_{AB} is given by Eq. (3.29). As mentioned, the ordering $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4$ does not imply any loss of generality. After [9] (see, in particular, Eqs. (4.74), (4.75), (4.86), and (4.87) therein), the explicit expression of λ_i in terms of $U(8)$-invariants (namely of $\mathrm{Tr}A$, $\mathrm{Tr}(A^2)$, $\mathrm{Tr}(A^3)$, and $\mathrm{Tr}(A^4)$, and suitable powers) is known, and it can be thus be used in order to compute the ADM mass of $\frac{\mathbf{k}}{8}-$ BPS extremal BH states of $\mathcal{N} = 8$, $d = 4$ supergravity.

The λ_i's are solution of the (square root of) *characteristic equation* [9]

$$\sqrt{\det(A - \lambda \mathbb{I})} = \prod_{i=1}^{4}(\lambda - \lambda_i) = \lambda^4 + a\lambda^3 + b\lambda^2 + c\lambda + d = 0, \quad (6.6)$$

where [9]

$$a \equiv -\frac{1}{2}\,\mathrm{Tr}A = -(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4); \quad (6.7)$$

$$b \equiv \frac{1}{4}\left[\frac{1}{2}(\mathrm{Tr}A)^2 - \mathrm{Tr}(A^2)\right] = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_1\lambda_4 + \lambda_2\lambda_3 + \lambda_2\lambda_4 + \lambda_3\lambda_4; \quad (6.8)$$

$$c \equiv -\frac{1}{6}\left[\frac{1}{8}(\mathrm{Tr}A)^3 + \mathrm{Tr}(A^3) - \frac{3}{4}\,\mathrm{Tr}(A^2)\,\mathrm{Tr}A\right] = -(\lambda_1\lambda_2\lambda_3 + \lambda_1\lambda_2\lambda_4 + \lambda_1\lambda_3\lambda_4 + \lambda_2\lambda_3\lambda_4); \quad (6.9)$$

$$d \equiv \frac{1}{4}\left[\begin{array}{c}\frac{1}{96}(\mathrm{Tr}A)^4 + \frac{1}{8}\,\mathrm{Tr}^2(A^2) + \frac{1}{3}\,\mathrm{Tr}(A^3)\,\mathrm{Tr}A + \\ -\frac{1}{2}\,\mathrm{Tr}(A^4) - \frac{1}{8}\,\mathrm{Tr}(A^2)\,\mathrm{Tr}^2 A\end{array}\right] = \sqrt{\det A} = \lambda_1\lambda_2\lambda_3\lambda_4. \quad (6.10)$$

The system (6.7), (6.8), (6.9), and (6.10) can be inverted, yielding

$$\lambda_{1,2} = -\frac{a}{4} + \frac{s}{2} \pm \frac{1}{2} \times \sqrt{\frac{a^2}{2} - \frac{4b}{3} - \frac{(a^3 - 4ab + 8c)}{4s} - \frac{u}{3w} - \frac{w}{3}}; \quad (6.11)$$

$$\lambda_{3,4} = -\frac{a}{4} - \frac{s}{2} \pm \frac{1}{2} \times \sqrt{\frac{a^2}{2} - \frac{4b}{3} + \frac{(a^3 - 4ab + 8c)}{4s} - \frac{u}{3w} - \frac{w}{3}}, \quad (6.12)$$

where

$$u \equiv b^2 + 12d - 3ac; \quad (6.13)$$

$$v \equiv 2b^3 + 27c^2 - 72bd - 9abc + 27a^2 d; \quad (6.14)$$

$$w \equiv \left(\frac{v + \sqrt{v^2 - 4u^3}}{2}\right)^{1/3}; \quad (6.15)$$

$$s \equiv \sqrt{\frac{a^2}{4} - \frac{2b}{3} + \frac{u}{3w} + \frac{w}{3}}. \quad (6.16)$$

Notice that the positivity of quantities under square root in Eqs. (6.11), (6.12), (6.15), and (6.16) always holds. Furthermore, Eq. (6.6) is at most of fourth order (for $\mathbf{k} = 1$), of second-order for $\mathbf{k} = 2$, and of first order for $\mathbf{k} = 1$.

(1) $\mathbf{k} = 1$ ($\frac{1}{8}-$ BPS, either *large* or *small*). The $\frac{1}{8}-$ BPS extremal BH square ADM mass is

$$M^2_{\mathrm{ADM},(1/8)-\mathrm{BPS}}(\phi_\infty, \mathcal{P}) = \lambda_1(\phi_\infty, \mathcal{P}), \quad (6.17)$$

where λ_1 ($> \lambda_2 > \lambda_3 > \lambda_4$, since $a < 0$ and $s > 0$) is given by Eq. (6.11). In the *large* $\mathbf{k} = 1$ case $\lambda_2 = \lambda_3 = \lambda_4 = 0$ at the event horizon of the extremal BH, as given by Eq. (3.16).

(2) $\mathbf{k} = 2$ ($\frac{1}{4}-$ BPS, *small*). As given by Eq. (3.35), the eigenvalues are equal *in pairs*. By suitably renaming the two noncoinciding λ's, one gets

$$\lambda_{1,2} = \frac{1}{8}\,\mathrm{Tr}A \pm \frac{1}{2}\sqrt{\frac{1}{2}\,\mathrm{Tr}(A^2) - \frac{1}{16}(\mathrm{Tr}A)^2}. \quad (6.18)$$

As mentioned above, the maximal (compact) symmetry is manifest when λ_2 (in the renaming of Eq. (6.18)) vanishes (see treatment in Sec. III). Equation (3.35) implies [9]

$$c = \frac{1}{2}a\left(b - \frac{1}{4}a^2\right); \quad (6.19)$$

$$d = \frac{1}{4}\left(b - \frac{1}{4}a^2\right)^2. \quad (6.20)$$

In [9] Eqs. (6.19) and (6.20) were shown to be consequences of the criticality constraints (3.34). Thus, the $\frac{1}{4}$-BPS extremal BH square ADM mass is

$$M^2_{\mathrm{ADM},(1/4)-\mathrm{BPS}}(\phi_\infty, \mathcal{P}) = \lambda_1(\phi_\infty, \mathcal{P}), \quad (6.21)$$

where λ_1 ($> \lambda_2$) is given by Eq. (6.18):

$$M^2_{\mathrm{ADM},(1/4)-\mathrm{BPS}}(\phi_\infty, \mathcal{P}) = \frac{1}{8}\,\mathrm{Tr}A(\phi_\infty, \mathcal{P}) + \frac{1}{2}\sqrt{\frac{1}{2}\,\mathrm{Tr}(A^2)(\phi_\infty, \mathcal{P}) - \frac{1}{16}(\mathrm{Tr}A(\phi_\infty, \mathcal{P}))^2}. \quad (6.22)$$

(3) $\mathbf{k} = 4$ ($\frac{1}{2}-$ BPS, *small*). This case can be obtained from the $\frac{1}{4}$-BPS considered at point 2 by further

putting $\lambda_1 = \lambda_2$ in Eq. (6.18). Thus, *all* eigenvalues of the Hermitian 8×8 matrix A are equal:

$$A_A^C = \frac{1}{8}(\mathrm{Tr}A)\delta_A^C, \tag{6.23}$$

which implies

$$\mathrm{Tr}\,(A^2) = \frac{1}{8}(\mathrm{Tr}A)^2. \tag{6.24}$$

Therefore, $\frac{1}{2}$ − BPS extremal BH square ADM mass is given by

$$M^2_{\mathrm{ADM},(1/2)-\mathrm{BPS}}(\phi_\infty, \mathcal{P}) = \frac{1}{8}\mathrm{Tr}A(\phi_\infty, \mathcal{P}) = \frac{1}{16}Z_{AB}(\phi_\infty, \mathcal{P})\bar{Z}^{AB}(\phi_\infty, \mathcal{P}). \tag{6.25}$$

B. $\mathcal{N} = 4$

In $\mathcal{N} = 4$, $d = 4$ supergravity (treated in Sec. IV), the $SL(2, \mathbb{R}) \times SO(6, M)$ U-duality symmetry only allows the cases [3] $\mathbf{k} = 1, 2$. By recalling the treatment of Sec. IV, the respective maximal compact symmetries read [3,4,13,39]

$$\mathbf{k} = 1: (SU(2))^2 \times SO(M) \times SO(2); \tag{6.26}$$

$$\mathbf{k} = 2: USp(4) \times SO(M), \tag{6.27}$$

and they hold all along the respective scalar flows. While case $\mathbf{k} = 1$ is *large*, case $\mathbf{k} = 2$ is *small* (thus not enjoying the *attractor mechanism*).

In the *large* $\mathbf{k} = 1$ case, the *attractor mechanism* makes the maximal compact symmetry $(SU(2))^2 \times SO(M) \times SO(2)$ of the supporting charge orbit $\mathcal{O}_{(1/4)-\mathrm{BPS,large}}$ fully manifest as a symmetry of the central charge matrix Z_{AB} through the *symmetry enhancement* (recall Eq. (4.25))

$$(SU(2))^2 \times SO(M-2) \times SO(2) \xrightarrow{r \to r_H^+} (SU(2))^2 \times SO(M) \times SO(2) \tag{6.28}$$

at the event horizon of the considered extremal BH.

As done in Sec. IV and in the treatment of case $\mathcal{N} = 8$, $d = 4$ above, let us denote with λ_1 and λ_2 the two real non-negative eigenvalues of the 4×4 Hermitian matrix $Z_{AB}\bar{Z}^{CB} = (ZZ^\dagger)_A^C \equiv A_A^C$. Their relation with the absolute values of the complex *skew-eigenvalues* e_i of Z_{AB} is given by Eq. (3.29). As mentioned, the ordering $\lambda_1 \geq \lambda_2$ does not imply any loss of generality. After [9], the explicit expression of λ_1 and λ_2 in terms of $(U(4) \times SO(M))$-invariants (namely of $\mathrm{Tr}A$, $\mathrm{Tr}(A^2)$ and $(\mathrm{Tr}A)^2$) is known, and it can be thus be used in order to compute the ADM mass of $\frac{\mathbf{k}}{4}$ − BPS extremal BH states of $\mathcal{N} = 4$, $d = 4$ supergravity.

Indeed, λ_1 and λ_2 are solutions of the (square root of) *characteristic equation* [9]

$$\sqrt{\det(A - \lambda\mathbb{I})} = \prod_{i=1}^{2}(\lambda - \lambda_i) = \lambda^2 - \frac{1}{2}(\mathrm{Tr}A)\lambda + (\det A)^{1/2} = 0, \tag{6.29}$$

whose solution reads

$$\lambda_{1,2} = \frac{1}{2}\left(\frac{1}{2}\,\mathrm{Tr}A \pm \sqrt{\mathrm{Tr}(A^2) - \frac{1}{4}(\mathrm{Tr}A)^2}\right). \tag{6.30}$$

Notice that the positivity of quantities under square root in Eq. (6.30) always holds. Furthermore, Eq. (6.29) is at most of second-order (for $\mathbf{k} = 1$) and of first order for $\mathbf{k} = 2$.

(1) $\mathbf{k} = 1$ ($\frac{1}{4}$ − BPS *large*). The $\frac{1}{4}$-BPS extremal BH square ADM mass is

$$M^2_{\mathrm{ADM},(1/4)\text{-}\mathrm{BPS}}(\phi_\infty, \mathcal{P}) = \lambda_1(\phi_\infty, \mathcal{P}) = \frac{1}{2}\left(\frac{1}{2}\,\mathrm{Tr}A(\phi_\infty, \mathcal{P}) + \sqrt{\mathrm{Tr}(A^2)(\phi_\infty, \mathcal{P}) - \frac{1}{4}(\mathrm{Tr}A(\phi_\infty, \mathcal{P}))^2}\right), \tag{6.31}$$

where $\lambda_1 > \lambda_2$. Notice that $\lambda_2 = 0$ at the event horizon of the extremal BH, as given by Eq. (4.23).

(2) $\mathbf{k} = 2$ ($\frac{1}{2}$ − BPS, *small*). This case can be obtained from the $\frac{1}{4}$ − BPS considered at point 1 by further putting $\lambda_1 = \lambda_2$ in Eq. (6.30). Thus, *all* eigenvalues of the Hermitian 4×4 matrix A are equal:

$$A_A^C = \frac{1}{4}(\mathrm{Tr}A)\delta_A^C, \tag{6.32}$$

which implies

$$\mathrm{Tr}\,(A^2) = \frac{1}{4}(\mathrm{Tr}A)^2. \tag{6.33}$$

Thus, the $\frac{1}{2}$ − BPS extremal BH square ADM mass is

$$M^2_{\mathrm{ADM},(1/2)-\mathrm{BPS}}(\phi_\infty, \mathcal{P}) = \lambda_1(\phi_\infty, \mathcal{P}) = \lambda_2(\phi_\infty, \mathcal{P}) = \frac{1}{4}\,\mathrm{Tr}A(\phi_\infty, \mathcal{P}). \tag{6.34}$$

It should be here remarked that the $\mathcal{R}$-*symmetry* of the $\frac{\mathbf{k}}{\mathcal{N}}$ − BPS extremal BH states, *i.e.* the compact symmetry of the solution in the *normal frame* (determining the automorphism group of the supersymmetry algebra in the *rest frame*) gets broken as follows:

$$\mathcal{R} \to USp(2\mathbf{k}) \times \ldots . \tag{6.35}$$

This is precisely the symmetry of the $\frac{\mathbf{k}}{\mathcal{N}}$ − BPS saturated massive multiplets of the $\mathcal{N}$-extended, $d = 4$ Poincaré supersymmetry algebra [54].

We end this section by finally commenting about the ADM mass for non-BPS extremal BH states.

In non-BPS cases, ADM mass of extremal BH states is not directly related to the *skew-eigenvalues* of the *central charge matrix* Z_{AB}. For some non-BPS extremal BHs a *fake supergravity (first order) formalism* [55] can be consistently formulated in terms of a *fake superpotential* $\mathcal{W}(\phi, \mathcal{P})$ [56–59] such that (also recall Eq. (1.5))

$$\begin{aligned} &\mathcal{W}^2_{\text{non-BPS}}(\phi, \mathcal{P})|_{((\partial \mathcal{W})/(\partial\phi))=0} \\ &\equiv \mathcal{W}^2_{\text{non-BPS}}(\phi_{H,\text{non-BPS}}(\mathcal{P}), \mathcal{P}) \\ &= V_{\text{BH}}(\phi, \mathcal{P})|_{((\partial V_{\text{BH}})/(\partial\phi))=0} \\ &\equiv V_{\text{BH}}(\phi_{H,\text{non-BPS}}(\mathcal{P}), \mathcal{P}) = \frac{S_{\text{BH,non-BPS}}(\mathcal{P})}{\pi}, \end{aligned} \tag{6.36}$$

with $\mathcal{W}_{\text{non-BPS}}$ varying, dependently on whether $Z_{AB} = 0$ or not. In such frameworks, the general expression of the non-BPS ADM mass reads as follows [56–58]

$$M_{\text{ADM,non-BPS}}(\phi_\infty, \mathcal{P}) = \mathcal{W}_{\text{non-BPS}}(\phi_\infty, \mathcal{P}). \tag{6.37}$$

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