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Electric/magnetic duality for chiral gauge theories with anomaly cancellation

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Abstract: We show that 4D gauge theories with Green-Schwarz anomaly cancellation and possible generalized Chern-Simons terms admit a formulation that is manifestly covariant with respect to electric/magnetic duality transformations. This generalizes previous work on the symplectically covariant formulation of *anomaly-free* gauge theories as they typically occur in extended supergravity, and now also includes general theories with (pseudo-)anomalous gauge interactions as they may occur in global or local $\mathcal{N} = 1$ supersymmetry. This generalization is achieved by relaxing the linear constraint on the embedding tensor so as to allow for a symmetric 3-tensor related to electric and/or magnetic quantum anomalies in these theories. Apart from electric and magnetic gauge fields, the resulting Lagrangians also feature two-form fields and can accommodate various unusual duality frames as they often appear, e.g., in string compactifications with background fluxes.

Keywords: Supergravity Models, Anomalies in Field and String Theories.

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Contents

1. Introduction

In field theories with chiral gauge interactions, the requirement of anomaly-freedom imposes a number of nontrivial constraints on the possible gauge quantum numbers of the chiral fermions. The strongest requirements are obtained if one demands that all anomalous one-loop diagrams due to chiral fermions simply add up to zero.

These constraints on the fermionic spectrum can be somewhat relaxed if some of the anomalous one-loop contributions are instead cancelled by *classical* gauge-variances of certain terms in the tree-level action. The prime example for this is the Green-Schwarz mechanism [1]. In its four-dimensional incarnation, it uses the gauge variance of Peccei-Quinn terms of the form $a\mathcal{F}\wedge\mathcal{F}$, with $a(x)$ being an axionic scalar field and $\mathcal{F}$ some vector

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field strengths, under gauged shift symmetries of the form $a(x) \rightarrow a(x) + c\Lambda(x)$, where $\Lambda(x)$ is the local gauge parameter and c a constant. Gauge variances of this form may cancel mixed Abelian/non-Abelian as well as cubic Abelian gauge anomalies in the quantum effective action. The Abelian gauge bosons that implement the gauged shift symmetries of the axions via Stückelberg-type gauge couplings correspond to the anomalous Abelian gauge groups and gain a mass due to their Stückelberg couplings. If their masses are low enough, these pseudo-anomalous gauge bosons might be observable and could possibly play the rôle of a particular type of Z'-boson. The phenomenology of such Stückelberg Z'-extensions of the Standard Model was studied in various works [2 – 10], which were in part inspired by intersecting brane models in type II orientifolds, where the operation of a 4D Green-Schwarz mechanism is quite generic [11].[1]

In [18 – 20], however, it has recently been pointed out that in these orientifold compactifications, the Green-Schwarz mechanism is often not sufficient to cancel all quantum anomalies.[2] In particular, the cancellation of mixed Abelian anomalies between anomalous and non-anomalous Abelian factors in general needs an additional ingredient, so-called generalized Chern-Simons terms (GCS terms), in the classical action. GCS terms are of the schematic form $A \wedge A \wedge dA$ and $A \wedge A \wedge A \wedge A$, where the vector fields A are not all the same. It is quite obvious that GCS terms are not gauge invariant, and it is precisely this gauge variance that can be used in some cases to cancel possible left-over gauge variances from quantum anomalies and Peccei-Quinn terms. Interestingly, these GCS terms indeed do occur quite generically in the above-mentioned orientifold compactifications [18, 20]. Phenomenologically, they provide extra trilinear (and quartic) couplings between anomalous and non-anomalous gauge bosons, which, given a low Stückelberg mass scale, may lead to Z'-bosons with possibly observable new characteristic signals [18 – 20].

In [26], it is shown how models with all three ingredients (each of which individually breaks gauge symmetry):

(i) anomalous fermionic spectra,

(ii) Peccei-Quinn terms with gauged axionic shift symmetries,

(iii) generalized Chern-Simons terms,

can be compatible with global and local $\mathcal{N} = 1$ supersymmetry. This compatibility is non-trivial, because a violation of gauge symmetries usually also triggers a violation of the on-shell supersymmetry, as is best seen by recalling that in the Wess-Zumino gauge the preserved supersymmetry is a combination of the original superspace supersymmetry and a gauge transformation. Due to the presence of the quantum gauge anomalies, one therefore also has to take into account the corresponding supersymmetry anomalies of the quantum effective action, as they have been determined by Brandt for $\mathcal{N} = 1$ supergravity in [27, 28]. A recent application of the theories studied in [26] to globally supersymmetric models with interesting phenomenology appeared in [29].

[1] For more details on intersecting brane models, see, e.g. the reviews [12 – 17] and references therein.
[2] See also [21 – 25].

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While in [18, 26] the general interplay of all the above three ingredients is discussed, it should be emphasized that not all three ingredients necessarily need to be present in a gauge invariant theory. This is obvious from the original Stückelberg Z'-models [3 – 10], which do not have GCS terms. However, one can also construct purely classical theories, in which only the last two ingredients (ii) and (iii), i.e. the gauged shift symmetries and the GCS terms, are present and the fermionic spectrum is either absent or non-anomalous. In fact, it was in such a context that GCS terms were first discussed in the literature. More concretely, their possibility was first discovered in extended gauged supergravity theories [30], which are automatically free of quantum anomalies due to the incompatibility of chiral gauge interactions with extended 4D supersymmetry. The ensuing papers [31 – 40] likewise remained focused on – or were inspired by — the structures found in extended supergravity. Recently, axionic gaugings and GCS terms were also considered in the context of global $\mathcal{N} = 1$ supersymmetry in [41]. In all these cases, the absence of quantum anomalies restricts the form of the possible gauged axionic shift symmetries.

Another very important example in this context is the work [42], which combines *classically* gauge invariant local Lagrangians that may also include Peccei-Quinn and GCS terms with the concept of electric/magnetic duality transformations. In four spacetime dimensions, a field theory with n Abelian vector potentials and no charged matter fields admits reparametrizations in the form of electic/magnetic duality transformations. Those transformations that leave the set of field equations and Bianchi identities invariant are the rigid (or global) symmetries of the theory and form the global symmetry group G_{rigid}. In section 3.2, we will discuss how, in general, G_{rigid} is contained in the direct product of the symplectic duality transformations that act on the vector fields and the isometry group of the scalar manifold of the chiral multiplets: $G_{\text{rigid}} \subseteq \text{Sp}(2n, \mathbb{R}) \times \text{Iso}(\mathcal{M}_{\text{scalar}})$.

Note, however, that the Lagrangians that encode the field equations are in general not invariant under such rigid symmetry transformations, as the latter may involve nontrivial mixing of field equations and Bianchi identities. Moreover, the fields before and after a symmetry transformation are, in general, not related by a *local* field transformation.

In order to gauge a rigid symmetry in the standard way (i.e., in order to introduce charges for some of the fields), one needs to be able to go to a symplectic duality frame in which the symmetry leaves the action invariant. This automatically implies that the symmetry is also implemented by *local* field transformations. This would then allow the introduction of minimal couplings and covariant field strengths for the electric vector potentials in the Lagrangian in the usual way. This standard procedure obviously singles out certain duality frames and breaks the original duality covariance.

In [42], it was shown how one can nevertheless reformulate 4D gauge theories in such a way as to maintain, formally, the full duality covariance of the original ungauged theory. In order to do so, the authors consider electric and magnetic gauge potentials $(A_\mu{}^\Lambda, A_{\mu\,\Lambda})$ $(\Lambda = 1, \ldots, n)$ at the same time and combine them into a $2n$-plet, $A_\mu{}^M$ $(M = 1, \ldots, 2n)$ of vector potentials. Introducing then also a set of antisymmetric tensor fields, an intricate system of gauge invariances can be implemented, which ensures that the number of propagating degrees of freedom is the same as before. The coupling of the electric and magnetic vector potentials to charged fields is then encoded in the so-called embedding tensor $\Theta_M{}^\alpha =$

$(\Theta_\Lambda{}^\alpha, \Theta^{\Lambda\,\alpha})$, which enters the covariant derivatives of matter fields, ϕ, schematically,

$$(\partial_\mu - A_\mu{}^M \Theta_M{}^\alpha \delta_\alpha)\phi\,. \tag{1.1}$$

Here, $\alpha = 1, \ldots, \dim(G_{\rm rigid})$ labels the generators of the rigid symmetry group, $G_{\rm rigid}$, acting as $\delta_\alpha\phi$ on the matter fields. In general, the gauge group also acts on the vector fields via $(2n \times 2n)$-matrices,

$$(X_M)_N{}^P \equiv X_{MN}{}^P \equiv \Theta_M{}^\alpha (t_\alpha)_N{}^P\,, \tag{1.2}$$

where the $(t_\alpha)_N{}^P$ are in the fundamental representation of $\mathrm{Sp}(2n,\mathbb{R})$.

The embedding tensor has to satisfy a quadratic constraint in order to ensure the closure of the gauge algebra inside the algebra of $G_{\rm rigid}$. In [42], this fundamental constraint is supplemented by one additional constraint linear in the embedding tensor, which can be written in terms of the above-mentioned tensor $X_{MN}{}^P$, as[3]

$$X_{(MN}{}^Q \Omega_{P)Q} = 0\,, \tag{1.3}$$

where Ω_{PQ} is the symplectic metric of $\mathrm{Sp}(2n,\mathbb{R})$. This constraint is sometimes called the "representation constraint", as it suppresses a representation of the rigid symmetry group in the tensor $X_{MN}{}^P$. Together with the quadratic constraint, it ensures mutual locality of all physical fields that are present in the action.[4] The full physical meaning of this additional constraint, however, always remained a bit obscure, and was inferred in [42] from identities that are known to be valid in $\mathcal{N}=8$ or $\mathcal{N}=2$ supergravity.

In this paper, we propose a physical interpretation of this representation constraint and recognize it as the condition for the *absence of quantum anomalies*. Quantum anomalies are automatically absent in extended 4D supergravity theories, and so it is no surprise, that the internal consistency of $\mathcal{N}=8$ or $\mathcal{N}=2$ supergravity always hinted at the validity of the constraint (1.3).

We then go one step further and show that if quantum anomalies proportional to a constant, totally symmetric tensor,[5] d_{MNP}, are present, the representation constraint (1.3) has to be relaxed to

$$X_{(MN}{}^Q \Omega_{P)Q} = d_{MNP}\,, \qquad \text{with} \qquad d_{MNP} = \Theta_M{}^\alpha \Theta_N{}^\beta \Theta_P{}^\gamma d_{\alpha\beta\gamma}\,, \tag{1.4}$$

to allow for a gauge invariant quantum effective action. Here $d_{\alpha\beta\gamma}$ is a symmetric tensor that will be defined by the anomalies. We show explicitly how the framework of [42]

[3] This constraint was considered in [42] for general $\mathcal{N}$ and in particular for $\mathcal{N}=1$ gauged supergravity and generalizes an analogous condition originally found in [30]. In the context of rigid $\mathcal{N}=1$ supersymmetry, its electric version already appeared in [41].

[4] A subtlety arises for generators δ_α that have a trivial action on the vector fields, i.e., $(t_\alpha)_M{}^N = 0$. In that case the mutual locality of the corresponding electric/magnetic components of the embedding tensor should be imposed as an independent quadratic constraint.

[5] The tensor d_{MNP} is the one that defines the consistent anomaly in the form given in equation (3.61). As the gauge symmetry in the matter sector is implemented by minimal couplings to the gauge potentials dressed with an embedding tensor, as can be seen from (1.1), the tensor d_{MNP} must be of the form (1.4).

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has to be modified in such a situation and that the resulting gauge variance of the classical Lagrangian precisely gives the negative of the consistent quantum anomaly encoded in d_{MNP}.

Our work can thus be viewed as a generalization of [42] to theories with quantum anomalies or, equivalently, as the covariantization of [18, 26] with respect to electric/magnetic duality transformations, and includes situations in which pseudo-anomalous gauge interactions are mediated by magnetic vector potentials. While already interesting in itself, our results promise to be very useful for the description of flux compactifications with chiral fermionic spectra, as e.g. in intersecting brane models on orientifolds with fluxes, because flux compactifications often give 4D theories which appear naturally in unusual duality frames and contain two-form fields.

The outline of this paper is as follows. In section 2, we briefly recapitulate the results of [26], adapted to the notation of [42]. Section 3 then gives the symplectically covariant framework of [42] in a more general treatment without using the representation constraint (1.3). In section 4 we show how the formalism of [42] has to be modified in order to accommodate quantum anomalies involving the relaxed representation constraint (1.4). We flesh out our results with a simple nontrivial example in section 5 and conclude in section 6.

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2. Anomalies, generalized Chern-Simons terms and gauged shift symmetries in $\mathcal{N} = 1$ supersymmetry

In this section, we summarize the results of [26] which will later motivate our proposed generalization (1.4) of the original constraint (1.3).

In a generic low energy effective field theory, the kinetic and the theta angle terms of vector fields, ${A_\mu}^\Lambda$, appear with scalar field dependent coefficients,[6]

$$\mathcal{L}_{\text{g.k.}} = \frac{1}{4} e \mathcal{I}_{\Lambda\Sigma}(z,\bar{z}) {\mathcal{F}_{\mu\nu}}^\Lambda \mathcal{F}^{\mu\nu\Sigma} - \frac{1}{8} \mathcal{R}_{\Lambda\Sigma}(z,\bar{z}) \varepsilon^{\mu\nu\rho\sigma} {\mathcal{F}_{\mu\nu}}^\Lambda {\mathcal{F}_{\rho\sigma}}^\Sigma \,. \tag{2.1}$$

Here, ${\mathcal{F}_{\mu\nu}}^\Lambda \equiv 2\partial_{[\mu} {A_{\nu]}}^\Lambda + {X_{\Sigma\Omega}}^\Lambda {A_\mu}^\Sigma {A_\nu}^\Omega$ denotes the non-Abelian field strengths with ${X_{\Sigma\Omega}}^\Lambda = {X_{[\Sigma\Omega]}}^\Lambda$ being the structure constants of the gauge group. We use the metric signature $(-+++)$ and work with real $\varepsilon_{0123} = 1$. As usual, e denotes the vierbein determinant. The second term in (2.1) is often referred to as the Peccei-Quinn term, and the functions $\mathcal{I}_{\Lambda\Sigma}(z,\bar{z})$ and $\mathcal{R}_{\Lambda\Sigma}(z,\bar{z})$ depend nontrivially on the scalar fields, z^i, of the theory. One can combine these functions to a complex function $\mathcal{N}_{\Lambda\Sigma}(z,\bar{z}) = \mathcal{R}_{\Lambda\Sigma}(z,\bar{z}) + \mathrm{i}\mathcal{I}_{\Lambda\Sigma}(z,\bar{z})$. In a supersymmetric context, $\mathcal{N}_{\Lambda\Sigma}(z,\bar{z})$ has to satisfy certain conditions, depending on the amount of supersymmetry. In $\mathcal{N} = 1$ global and local supersymmetry, which will be the

[6] To compare notations between this paper, ref. [26] and ref. [42], note that the vector fields were denoted as ${W_\mu}^A$ in [26], and are here and in [42] denoted as ${A_\mu}^\Lambda$ (upper greek letters are electric indices). In [26], the kinetic matrix for the vector multiplets is, as in most of the $\mathcal{N} = 1$ literature, denoted as f_{AB}, which corresponds to $-\mathrm{i}\mathcal{N}^*_{\Lambda\Sigma}$ in this paper. The structure constants ${f_{AB}}^C$ of [26] correspond to the ${X_{\Lambda\Sigma}}^\Omega = {f_{\Lambda\Sigma}}^\Omega$ here, and the axionic shift tensors $C_{AB,C}$ of [26] are now called $X_{\Lambda\Sigma\Omega} = X_{\Lambda(\Sigma\Omega)} = C_{\Sigma\Omega,\Lambda}$. To compare formulae of [42] to those here and in [26], the Levi-Civita symbol $\varepsilon^{\mu\nu\rho\sigma}$ appears in covariant equations with opposite sign (but $\varepsilon_{0123} = 1$ is valid in both cases due to another orientation of the spacetime directions).

subject of the remainder of this section, $\mathcal{N}_{\Lambda\Sigma} = \mathcal{N}_{\Lambda\Sigma}(\bar{z})$ simply has to be antiholomorphic in the complex scalars of the chiral multiplets.

If, under a gauge transformation with gauge parameter $\Lambda^\Omega(x)$, acting on the field strengths as $\delta(\Lambda)\mathcal{F}^\Lambda_{\mu\nu} = \Lambda^\Xi \mathcal{F}^\Omega_{\mu\nu} X_{\Omega\Xi}{}^\Lambda$, some of the z^i transform nontrivially, this may induce a corresponding gauge transformation of $\mathcal{N}_{\Lambda\Sigma}(\bar{z})$. In case this transformation is of the form of a symmetric product of two adjoint representations of the gauge group,

$$\delta(\Lambda)\mathcal{N}_{\Lambda\Sigma} = \Lambda^\Omega \delta_\Omega \mathcal{N}_{\Lambda\Sigma}\,, \qquad \delta_\Omega \mathcal{N}_{\Lambda\Sigma} = X_{\Omega\Lambda}{}^\Gamma \mathcal{N}_{\Sigma\Gamma} + X_{\Omega\Sigma}{}^\Gamma \mathcal{N}_{\Lambda\Gamma}\,, \tag{2.2}$$

the kinetic term (2.1) is obviously gauge invariant. This is what was assumed in the action of general matter-coupled supergravity in [43].[7]

If, however, one takes into account also other terms in the (quantum) effective action, a more general transformation rule for $\mathcal{N}_{\Lambda\Sigma}(\bar{z})$ may be allowed:

$$\delta_\Omega \mathcal{N}_{\Lambda\Sigma} = -X_{\Omega\Lambda\Sigma} + X_{\Omega\Lambda}{}^\Gamma \mathcal{N}_{\Sigma\Gamma} + X_{\Omega\Sigma}{}^\Gamma \mathcal{N}_{\Lambda\Gamma}\,. \tag{2.3}$$

Here, $X_{\Omega\Lambda\Sigma}$ is a constant real tensor symmetric in the last two indices, which can be recognized as a natural generalization in the context of symplectic duality transformations [41, 26]. Closure of the gauge algebra requires the constraint

$$X_{\Omega\Lambda\Sigma} X_{\Gamma\Xi}{}^\Omega + 2X_{\Sigma[\Xi}{}^\Omega X_{\Gamma]\Lambda\Omega} + 2X_{\Lambda[\Xi}{}^\Omega X_{\Gamma]\Sigma\Omega} = 0\,. \tag{2.4}$$

If $X_{\Omega\Lambda\Sigma}$ is non-zero, this leads to a non-gauge invariance of the Peccei-Quinn term in $\mathcal{L}_{\text{g.k.}}$:

$$\delta(\Lambda)\mathcal{L}_{\text{g.k.}} = \frac{1}{8}\varepsilon^{\mu\nu\rho\sigma}\, X_{\Omega\Lambda\Sigma}\Lambda^\Omega \mathcal{F}_{\mu\nu}{}^\Lambda \mathcal{F}_{\rho\sigma}{}^\Sigma\,. \tag{2.5}$$

For rigid parameters, $\Lambda^\Omega = \text{const.}$, this is just a total derivative, but for local gauge parameters, $\Lambda^\Omega(x)$, it is obviously not.

In order to understand how this broken invariance can be restored, it is convenient to split the coefficients $X_{\Omega\Lambda\Sigma}$ into a sum,

$$X_{\Omega\Lambda\Sigma} = X^{(\text{s})}_{\Omega\Lambda\Sigma} + X^{(\text{m})}_{\Omega\Lambda\Sigma}\,, \qquad X^{(\text{s})}_{\Omega\Lambda\Sigma} = X_{(\Omega\Lambda\Sigma)}\,, \qquad X^{(\text{m})}_{(\Omega\Lambda\Sigma)} = 0\,, \tag{2.6}$$

where $X^{(\text{s})}_{\Omega\Lambda\Sigma}$ is completely symmetric, and $X^{(\text{m})}_{\Omega\Lambda\Sigma}$ denotes the part of mixed symmetry. Terms of the form (2.5) may then in principle be cancelled by the following two mechanisms, or a combination thereof:

(i) As was first realized in a similar context in $\mathcal{N} = 2$ supergravity in [30] (see also the systematic analysis [31]), the gauge variation due to a non-vanishing mixed part, $X^{(\text{m})}_{\Omega\Lambda\Sigma} \neq 0$, may be cancelled by adding a generalized Chern-Simons term (GCS term) that contains a cubic and a quartic part in the vector fields,

$$\mathcal{L}_{\text{GCS}} = \frac{1}{3} X^{(\text{CS})}_{\Omega\Lambda\Sigma}\, \varepsilon^{\mu\nu\rho\sigma} \left(A_\mu{}^\Omega A_\nu{}^\Lambda \partial_\rho A^\Sigma_\sigma + \frac{3}{8} X_{\Gamma\Xi}{}^\Sigma A_\mu{}^\Omega A_\nu{}^\Lambda A_\rho{}^\Gamma A_\sigma{}^\Xi \right)\,. \tag{2.7}$$

[7] This construction of general matter-couplings has been reviewed in [44]. There, the possibility (2.3) was already mentioned, but the extra terms necessary for its consistency were not considered.

This term depends on a constant tensor $X^{(\mathrm{CS})}_{\Omega\Lambda\Sigma}$, which has the same mixed symmetry structure as $X^{(\mathrm{m})}_{\Omega\Lambda\Sigma}$. The cancellation occurs provided the tensors $X^{(\mathrm{m})}_{\Omega\Lambda\Sigma}$ and $X^{(\mathrm{CS})}_{\Omega\Lambda\Sigma}$ are, in fact, the same. It was first shown in [41] that such a term can exist in rigid $\mathcal{N}=1$ supersymmetry without quantum anomalies.

(ii) If the chiral fermion spectrum is anomalous under the gauge group, the anomalous triangle diagrams lead to a non-gauge invariance of the quantum effective action Γ for the gauge symmetry: $\delta(\Lambda)\Gamma = \int \mathrm{d}^4x \Lambda^\Lambda \mathcal{A}_\Lambda$ of the form

$$\mathcal{A}_\Lambda = -\frac{1}{4}\varepsilon^{\mu\nu\rho\sigma}\left[2d_{\Omega\Sigma\Lambda}\partial_\mu A_\nu{}^\Sigma + \left(d_{\Omega\Sigma\Gamma}X_{\Lambda\Xi}{}^\Sigma + \frac{3}{2}d_{\Omega\Sigma\Lambda}X_{\Gamma\Xi}{}^\Sigma\right)A_\mu{}^\Gamma A_\nu{}^\Xi\right]\partial_\rho A_\sigma{}^\Omega\,, \tag{2.8}$$

with a symmetric[8] tensor $d_{\Omega\Lambda\Sigma}$. If

$$X^{(\mathrm{s})}_{\Omega\Lambda\Sigma} = d_{\Omega\Lambda\Sigma}\,, \tag{2.9}$$

this quantum anomaly cancels the symmetric part of (2.5). This is the Green-Schwarz mechanism.

In [26], it was studied to what extent a general gauge theory of the above type (i.e., with gauged axionic shift symmetries, GCS terms and quantum gauge anomalies) can be compatible with $\mathcal{N}=1$ supersymmetry. The results can be summarized as follows: if one takes as one's starting point the matter-coupled supergravity Lagrangian in eq. (5.15) of reference [44], an axionic shift symmetry with $X_{\Lambda\Sigma\Omega}\neq 0$ satisfying the closure condition (2.4) can be gauged in a way consistent with $\mathcal{N}=1$ supersymmetry if

(i) a GCS term (2.7) with $X^{(\mathrm{CS})}_{\Omega\Lambda\Sigma} = X^{(\mathrm{m})}_{\Omega\Lambda\Sigma}$ is added,

(ii) an additional term bilinear in the gaugini, $\lambda^\Sigma(x)$, and linear in the vector fields is added:[9]

$$\mathcal{L}_{\mathrm{extra}} = -\frac{1}{4}\mathrm{i}A_\mu{}^\Omega X_{\Omega\Lambda\Sigma}\bar{\lambda}^\Lambda\gamma_5\gamma^\mu\lambda^\Sigma, \tag{2.10}$$

(iii) the fermions in the chiral multiplets give rise to quantum anomalies with $d_{\Omega\Lambda\Sigma} = X^{(\mathrm{s})}_{\Omega\Lambda\Sigma}$. The consistent gauge anomaly, $\mathcal{A}_\Lambda$ is of the form (2.8). The exact result for the supersymmetry anomaly can be found in [28] or eq. (5.8) of [26]. These quantum anomalies precisely cancel the classical gauge and supersymmetry variation of the new Lagrangian $\mathcal{L}_{\mathrm{old}} + \mathcal{L}_{\mathrm{GCS}} + \mathcal{L}_{\mathrm{extra}}$, where $\mathcal{L}_{\mathrm{old}}$ denotes the original Lagrangian of [44].

[8] More precisely, the anomalies have a scheme dependence. As reviewed in [18] one can choose a scheme in which the anomaly is proportional to a symmetric $d_{\Omega\Lambda\Sigma}$. Choosing a different scheme is equivalent to the choice of another GCS term (see item (i)). We will always work with a renormalization scheme in which the quantum anomaly is indeed proportional to the symmetric tensor $d_{\Omega\Lambda\Sigma}$ according to (2.8).

[9] A superspace expression for the sum $\mathcal{L}_{\mathrm{GCS}} + \mathcal{L}_{\mathrm{extra}}$ is known only for the case $X^{(\mathrm{s})}_{\Lambda\Sigma\Omega} = 0$, i.e., for the case without quantum anomalies [41].

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3. The embedding tensor and the symplectically covariant formalism

In this section, we recapitulate the results of [42], which describe a symplectically covariant formulation of (classically) gauge invariant field theories. Correspondingly, we will assume the absence of quantum anomalies in this section.

3.1 Electric/magnetic duality and the conventional gauging

In the absence of charged fields, a gauge invariant four-dimensional Lagrangian of n Abelian vector fields $A_\mu{}^\Lambda (\Lambda = 1, \dots, n)$ only depends on their curls $F_{\mu\nu}{}^\Lambda \equiv 2\partial_{[\mu} A_{\nu]}{}^\Lambda$. Defining the dual magnetic field strengths

$$G_{\mu\nu\,\Lambda} \equiv \varepsilon_{\mu\nu\rho\sigma} \frac{\partial \mathcal{L}}{\partial F_{\rho\sigma}{}^\Lambda}\,, \tag{3.1}$$

the Bianchi identities and field equations read

$$\partial_{[\mu} F_{\nu\rho]}{}^\Lambda = 0\,, \tag{3.2}$$

$$\partial_{[\mu} G_{\nu\rho]\,\Lambda} = 0\,. \tag{3.3}$$

The equations of motion (3.3) imply the existence of magnetic gauge potentials, $A_{\mu\,\Lambda}$, via $G_{\mu\nu\,\Lambda} = 2\partial_{[\mu} A_{\nu]\Lambda}$. These magnetic gauge potentials are related to the electric vector potentials, $A_\mu{}^\Lambda$, by nonlocal field redefinitions. The electric Abelian field strengths, $F_{\mu\nu}{}^\Lambda$, and their magnetic duals, $G_{\mu\nu\,\Lambda}$, can be combined into a $2n$-plet, $F_{\mu\nu}{}^M$, such that $F^M = (F^\Lambda, G_\Lambda)$. This allows us to write (3.2) and (3.3) in the following compact way:

$$\partial_{[\mu} F_{\nu\rho]}{}^M = 0\,. \tag{3.4}$$

Apparently, equation (3.4) is invariant under general linear transformations

$$F^M \to F'^M = \mathcal{S}^M{}_N F^N\,, \quad \text{where } \mathcal{S}^M{}_N = \begin{pmatrix} U^\Lambda{}_\Sigma & Z^{\Lambda\Sigma} \\ W_{\Lambda\Sigma} & V_\Lambda{}^\Sigma \end{pmatrix}\,, \tag{3.5}$$

but only for symplectic matrices $\mathcal{S}^M{}_N \in \mathrm{Sp}(2n, \mathbb{R})$ a relation of the type (3.1) is possible. The admissible rotations $\mathcal{S}^M{}_N$ thus form the group $\mathrm{Sp}(2n, \mathbb{R})$:

$$\mathcal{S}^T \Omega \mathcal{S} = \Omega, \tag{3.6}$$

with the symplectic metric, Ω_{MN}, given by

$$\Omega_{MN} = \begin{pmatrix} 0 & \Omega_\Lambda{}^\Sigma \\ \Omega^\Lambda{}_\Sigma & 0 \end{pmatrix} = \begin{pmatrix} 0 & \delta_\Lambda^\Sigma \\ -\delta_\Sigma^\Lambda & 0 \end{pmatrix}\,. \tag{3.7}$$

We define Ω^{MN} via $\Omega^{MN}\Omega_{NP} = -\delta^M{}_P$. Note that the components of Ω^{MN} should not be written as $\Omega^\Lambda{}_\Sigma$ etc., as these are different from (3.7).

Starting with a kinetic Lagrangian of the form (2.1), an electric/magnetic duality transformation leads to a new Lagrangian, $\mathcal{L}'(F')$, which is of a similar form, but with a new gauge kinetic function

$$\mathcal{N}_{\Lambda\Sigma} \to \mathcal{N}'_{\Lambda\Sigma} = (V\mathcal{N} + W)_{\Lambda\Omega} \left[(U + Z\mathcal{N})^{-1}\right]^\Omega{}_\Sigma\,. \tag{3.8}$$

The subset of $\mathrm{Sp}(2n,\mathbb{R})$ symmetries (of field equations and Bianchi identities) for which the Lagrangian remains unchanged in the sense that $\mathcal{L}'(F'(F)) = \mathcal{L}(F)$ and (3.8) is implemented by transformations of the fields on which $\mathcal{N}$ depends, are *invariances* of the action. In a different duality frame, the Lagrangian might have a different set of invariances.

From the spacetime point of view, these are all rigid ("global") symmetries. Sometimes these global symmetries can be turned into local ("gauge") symmetries. For the conventional gaugings one has to restrict to the transformations that leave the Lagrangian invariant, which implies that $Z^{\Lambda\Sigma}$ in the matrices $\mathcal{S}^M{}_N$ of (3.5) has to vanish. In the context of symplectically covariant gaugings [42], however, this restriction can be lifted, and we will come back to these in section 3.2. The standard way to perform a gauging of a symmetry of interest is therefore to first switch to a symplectic duality frame in which the symmetries of interest act on ${F_{\mu\nu}}^M = ({F_{\mu\nu}}^\Lambda, G_{\mu\nu\,\Lambda})$ by lower block triangular matrices (i.e. those with $Z = 0$) such that they become (as rigid symmetries) invariances of the action.

The gauging requires the introduction of gauge covariant derivatives and field strengths and can be implemented solely with the electric vector fields ${A_\mu}^\Omega$ and the corresponding electric gauge parameters Λ^Ω. The gaugeable symplectic transformation, $\mathcal{S}$, must be of the infinitesimal form

$$\mathcal{S}^M{}_N = \delta^M{}_N - \Lambda^\Omega \mathcal{S}_\Omega{}^M{}_N\,. \tag{3.9}$$

According to our definition (3.5), these infinitesimal symplectic transformations act on the field strengths by multiplication with the matrices $\mathcal{S}_\Lambda{}^M{}_N$ from the left. Following the conventions of [42], however, we will use matrices $X_{\Omega M}{}^N$ to describe the infinitesimal symplectic action via multiplication from the right:

$$\delta {F_{\mu\nu}}^M = {F'_{\mu\nu}}^M - {F_{\mu\nu}}^M = -\Lambda^\Omega {F_{\mu\nu}}^N X_{\Omega N}{}^M\,, \qquad \text{i.e.} \qquad X_{\Omega N}{}^M = S_\Omega{}^M{}_N\,. \tag{3.10}$$

For standard electric gaugings, we then have

$$\delta \begin{pmatrix} F^\Lambda_{\mu\nu} \\ G_{\mu\nu\,\Lambda} \end{pmatrix} = -\Lambda^\Omega \begin{pmatrix} X_{\Omega\Xi}{}^\Lambda & 0 \\ X_{\Omega\Lambda\Xi} & X_\Omega{}^\Xi{}_\Lambda \end{pmatrix} \begin{pmatrix} F^\Xi_{\mu\nu} \\ G_{\mu\nu\,\Xi} \end{pmatrix}\,, \tag{3.11}$$

where $X_{\Omega\Sigma}{}^\Lambda = -X_\Omega{}^\Lambda{}_\Sigma = f_{\Omega\Sigma}{}^\Lambda$ are the structure constants of the gauge algebra, and $X_{\Sigma\Xi\Gamma} = X_{\Sigma(\Xi\Gamma)}$ give rise to the axionic shifts mentioned in section 2 (compare (3.8) with (2.3) for the particular choice of $\mathcal{S}$ given in (3.9)).

The gauging then proceeds in the usual way by introducing covariant derivatives $(\partial_\mu - {A_\mu}^\Lambda \delta_\Lambda)$, where the δ_Λ are the gauge generators in a suitable representation of the matter fields. One also introduces covariant field strengths and possibly GCS terms as described in section 2. As we assume the absence of quantum anomalies in this section, we have to require $X_{(\Lambda\Sigma\Gamma)} = 0$.

3.2 The symplectically covariant gauging

We will now turn to the more general gauging of symmetries. The group that will be gauged is a subgroup of the rigid symmetry group. What we mean by the rigid symmetry group is a bit more subtle in $\mathcal{N} = 1$ supergravity (or theories without supergravity) than in

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extended supergravities. This is due to the fact that in extended supergravities the vectors are supersymmetrically related to scalar fields, and therefore their rigid symmetries are connected to the symmetries of scalar manifolds.

In $\mathcal{N}=1$ supersymmetry, the rigid symmetry group, $G_{\rm rigid}$, is a subset of the product of the symplectic duality transformations that act on the vector fields and the isometry group of the scalar manifold of the chiral multiplets: $G_{\rm rigid} \subseteq \mathrm{Sp}(2n,\mathbb{R}) \times \mathrm{Iso}(\mathcal{M}_{\rm scalar})$. The relevant isometries are those that respect the Kähler structure (i.e. generated by holomorphic Killing vectors) and that also leave the superpotential invariant (in supergravity, the superpotential should transform according to the Kähler transformations). Elements $(g_1,\, g_2)$ of $\mathrm{Sp}(2n,\mathbb{R}) \times \mathrm{Iso}(\mathcal{M}_{\rm scalar})$ that are compatible with (3.8) in the sense that the symplectic action (3.8) of g_1 on the matrix $\mathcal{N}$ is induced by the isometry g_2 on the scalar manifold, are rigid ("global") symmetries provided they also leave the rest of the theory (deriving from scalar potentials, etc.) invariant [45]. The rigid symmetry group, $G_{\rm rigid}$, is thus a subgroup of $\mathrm{Sp}(2n,\mathbb{R}) \times \mathrm{Iso}(\mathcal{M}_{\rm scalar})$.[10]

The generators of $G_{\rm rigid}$ will be denoted by δ_α, $\alpha = 1, \ldots, \dim(G_{\rm rigid})$. These generators act on the different fields of the theory either via Killing vectors $\delta_\alpha = K_\alpha = K_\alpha^i \frac{\partial}{\partial \phi^i}$ defining infinitesimal isometries on the scalar manifold, or with certain matrix representations,[11] e.g. $\delta_\alpha \phi^i = -\phi^j (t_\alpha)_j{}^i$.

On the field strengths $F_{\mu\nu}{}^M = (F_{\mu\nu}{}^\Lambda,\, G_{\mu\nu\,\Lambda})$, these rigid symmetries must act by multiplication with infinitesimal symplectic matrices[12] $(t_\alpha)_M{}^P$, i.e., we have

$$(t_\alpha)_{[M}{}^P \Omega_{N]P} = 0\,. \tag{3.12}$$

In order to gauge a subgroup, $G_{\rm local} \subset G_{\rm rigid}$, the $2n$-dimensional vector space spanned by the vector fields $A_\mu{}^M$ has to be projected onto the Lie algebra of $G_{\rm local}$, which is formally done with the so-called embedding tensor $\Theta_M{}^\alpha = (\Theta_\Lambda{}^\alpha, \Theta^{\Lambda\,\alpha})$. Equivalently, $\Theta_M{}^\alpha$ completely determines the gauge group $G_{\rm local}$ via the decomposition of the gauge group generators, which we will denote by $\tilde{X}_M$, into the generators of the rigid invariance group $G_{\rm rigid}$:

$$\tilde{X}_M \equiv \Theta_M{}^\alpha \delta_\alpha. \tag{3.13}$$

The gauge generators $\tilde{X}_M$ enter the gauge covariant derivatives of matter fields,

$$\mathcal{D}_\mu = \partial_\mu - A_\mu{}^M \tilde{X}_M = \partial_\mu - A_\mu{}^\Lambda \Theta_\Lambda{}^\alpha \delta_\alpha - A_{\mu\Lambda} \Theta^{\Lambda\alpha} \delta_\alpha\,, \tag{3.14}$$

where the generators δ_α are meant to either act as representation matrices on the fermions or as Killing vectors on the scalar fields, as mentioned above. On the field strengths of the vector potentials, the generators δ_α act by multiplication with the matrices $(t_\alpha)_N{}^P$,

[10]Note that this may include cases where either the symplectic transformation g_1 or the isometry g_2 is trivial. Another special case is when the isometry g_2 is non-trivial, but $\mathcal{N}$ does not transform under it, as happens, e.g, when $\mathcal{N} = \mathrm{i}\mathbb{1}$ is constant. $G_{\rm rigid}$ is in general a genuine subgroup of $\mathrm{Sp}(2n,\mathbb{R}) \times \mathrm{Iso}(\mathcal{M}_{\rm scalar})$, even in the latter case of constant $\mathcal{N}$.

[11]The structure constants defined by $[\delta_\alpha, \delta_\beta] = f_{\alpha\beta}{}^\gamma \delta_\gamma$ lead for the matrices to $[t_\alpha, t_\beta] = -f_{\alpha\beta}{}^\gamma t_\gamma$.

[12]These matrices might be trivial, e.g., for Abelian symmetry groups that only act on the scalars (and/or the fermions) and that do not give rise to axionic shifts of the kinetic matrix $\mathcal{N}_{\Lambda\Sigma}$.

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so that (3.13) is represented by matrices $(X_M)_N{}^P$ whose elements we denote as $X_{MN}{}^P$, see (1.2), and whose antisymmetric part in the lower indices appears in the field strengths

$$\mathcal{F}_{\mu\nu}{}^M = 2\partial_{[\mu}A_{\nu]}{}^M + X_{[NP]}{}^M A_\mu{}^N A_\nu{}^P\,, \qquad X_{NP}{}^M = \Theta_N{}^\alpha (t_\alpha)_P{}^M\,. \tag{3.15}$$

The symplectic property (3.12) implies

$$X_{M[N}{}^Q\Omega_{P]Q} = 0\,, \qquad X_{MQ}{}^{[N}\Omega^{P]Q} = 0\,. \tag{3.16}$$

In the remainder of this paper, the symmetrized contraction $X_{(MN}{}^Q\Omega_{P)Q}$ will play an important rôle. We therefore give this tensor a special name and denote it by D_{MNP}:

$$D_{MNP} \equiv X_{(MN}{}^Q\Omega_{P)Q}\,. \tag{3.17}$$

Note that this is really just a definition and no new constraint. Using the definition (3.17), one can check that

$$\begin{aligned} 2X_{(MN)}{}^Q\Omega_{RQ} + X_{RM}{}^Q\Omega_{NQ} &= 3D_{MNR}\,, \\ \text{i.e.} \qquad X_{(MN)}{}^P &= \frac{1}{2}\Omega^{PR}X_{RM}{}^Q\Omega_{NQ} + \frac{3}{2}D_{MNR}\Omega^{RP}\,. \end{aligned} \tag{3.18}$$

3.2.1 Constraints on the embedding tensor

The embedding tensor $\Theta_M{}^\alpha$ has to satisfy a number of consistency conditions. Closure of the gauge algebra and locality require, respectively, the quadratic constraints

$$\text{closure:} \qquad f_{\alpha\beta}{}^\gamma\Theta_M{}^\alpha\Theta_N{}^\beta = (t_\alpha)_N{}^P\Theta_M{}^\alpha\Theta_P{}^\gamma\,, \tag{3.19}$$

$$\text{locality:} \qquad \Omega^{MN}\Theta_M{}^\alpha\Theta_N{}^\beta = 0 \quad \Leftrightarrow \quad \Theta^{\Lambda[\alpha}\Theta_\Lambda{}^{\beta]} = 0\,, \tag{3.20}$$

where $f_{\alpha\beta}{}^\gamma$ are the structure constants of the rigid invariance group G_{rigid}, see footnote 11. Another constraint, besides (3.19) and (3.20), was inferred in [42] from supersymmetry constraints in $\mathcal{N} = 8$ supergravity

$$D_{MNR} \equiv X_{(MN}{}^Q\Omega_{R)Q} = 0\,. \tag{3.21}$$

This constraint eliminates some of the representations of the rigid symmetry group and is therefore sometimes called the "representation constraint". As we pointed out in the introduction, one can show that the locality constraint is not independent of (3.19) and (3.21), apart from specific cases where $(t_\alpha)_M{}^N$ has a trivial action on the vector fields.

However, we will neither use the locality constraint (3.20) nor the representation constraint (3.21). We will, instead, need another constraint in section 3.2.4, whose meaning we will discuss in section 4. Before coming to that new constraint, we thus only use the closure constraint (3.19). This constraint reflects the invariance of the embedding tensor under G_{local} and it implies for the matrices X_M the relation

$$[X_M, X_N] = -X_{MN}{}^P X_P\,. \tag{3.22}$$

This clearly shows that the gauge group generators commute into each other with 'structure constants' given by $X_{[MN]}{}^P$. However, note that $X_{MN}{}^P$ in general also contains a non-trivial symmetric part, $X_{(MN)}{}^P$. The antisymmetry of the left hand side of (3.22) only requires that the contraction $X_{(MN)}{}^P \Theta_P{}^\alpha$ vanishes, as is also directly visible from (3.19). Therefore one has

$$X_{(MN)}{}^P \Theta_P{}^\alpha = 0 \qquad \rightarrow \qquad X_{(MN)}{}^P X_{PQ}{}^R = 0\,. \tag{3.23}$$

Writing (3.22) explicitly gives

$$X_{MQ}{}^P X_{NP}{}^R - X_{NQ}{}^P X_{MP}{}^R + X_{MN}{}^P X_{PQ}{}^R = 0\,. \tag{3.24}$$

Antisymmetrizing in $[MNQ]$, we can split the second factor of each term into the antisymmetric and symmetric part, $X_{MN}{}^P = X_{[MN]}{}^P + X_{(MN)}{}^P$, and this gives a violation of the Jacobi identity for $X_{[MN]}{}^P$ as

$$\begin{aligned} & X_{[MN]}{}^P X_{[QP]}{}^R + X_{[QM]}{}^P X_{[NP]}{}^R + X_{[NQ]}{}^P X_{[MP]}{}^R \\ & \qquad = -\frac{1}{3}\left(X_{[MN]}{}^P X_{(QP)}{}^R + X_{[QM]}{}^P X_{(NP)}{}^R + X_{[NQ]}{}^P X_{(MP)}{}^R\right)\,. \end{aligned} \tag{3.25}$$

Other relevant consequences of (3.24) can be obtained by (anti)symmetrizing in MQ. This gives, using also (3.23), the two equations

$$\begin{aligned} X_{(MQ)}{}^P X_{NP}{}^R - X_{NQ}{}^P X_{(MP)}{}^R - X_{NM}{}^P X_{(QP)}{}^R &= 0\,, \\ X_{[MQ]}{}^P X_{NP}{}^R - X_{NQ}{}^P X_{[MP]}{}^R + X_{NM}{}^P X_{[QP]}{}^R &= 0\,. \end{aligned} \tag{3.26}$$

3.2.2 Gauge transformations

The violation of the Jacobi identity (3.25) is the prize one has to pay for the symplectically covariant treatment in which both electric and magnetic vector potentials appear at the same time. In order to compensate for this violation and in order to make sure that the number of propagating degrees of freedom is the same as before, one imposes an additional gauge invariance in addition to the usual non-Abelian transformation $\partial_\mu \Lambda^M + X_{[PQ]}{}^M A_\mu{}^P \Lambda^Q$ and extends the gauge transformation of the vector potentials to

$$\delta A_\mu{}^M = \mathcal{D}_\mu \Lambda^M - X_{(NP)}{}^M \Xi_\mu{}^{NP}\,, \qquad \mathcal{D}_\mu \Lambda^M = \partial_\mu \Lambda^M + X_{PQ}{}^M A_\mu{}^P \Lambda^Q\,, \tag{3.27}$$

where we introduced the covariant derivative $\mathcal{D}_\mu \Lambda^M$, and new vector-like gauge parameters $\Xi_\mu{}^{NP}$, symmetric in the upper indices. The extra terms $X_{(PQ)}{}^M A_\mu{}^P \Lambda^Q$ and the Ξ-transformations contained in (3.27) allow one to gauge away the vector fields that correspond to the directions in which the Jacobi identity is violated, i.e., directions in the kernel of the embedding tensor (see (3.23)).

It is important to notice that the modified gauge transformations (3.27) still close on the gauge fields and thus form a Lie algebra. Indeed, if we split (3.27) into two parts,

$$\delta A_\mu{}^M = \delta(\Lambda) A_\mu{}^M + \delta(\Xi) A_\mu{}^M\,, \tag{3.28}$$

the commutation relations are

$$\begin{aligned} \left[\delta(\Lambda_1), \delta(\Lambda_2)\right] A_\mu{}^M &= \delta(\Lambda_3) A_\mu{}^M + \delta(\Xi_3) A_\mu{}^M, \\ \left[\delta(\Lambda), \delta(\Xi)\right] A_\mu{}^M &= \left[\delta(\Xi_1), \delta(\Xi_2)\right] A_\mu{}^M = 0\,, \end{aligned} \tag{3.29}$$

with

$$\begin{aligned}\Lambda_3^M &= X_{[NP]}{}^M \Lambda_1^N \Lambda_2^P\,,\\ \Xi_{3\mu}{}^{PN} &= \Lambda_1^{(P}\mathcal{D}_\mu \Lambda_2^{N)} - \Lambda_2^{(P}\mathcal{D}_\mu \Lambda_1^{N)}\,.\end{aligned} \tag{3.30}$$

To prove that the terms that are quadratic in the matrices X_M in the left-hand side of (3.29) follow this rule, one uses (3.26).

Due to (3.23) and (3.27), however, the usual properties of the field strength

$$\mathcal{F}_{\mu\nu}{}^M = 2\partial_{[\mu}A_{\nu]}{}^M + X_{[PQ]}{}^M A_\mu{}^P A_\nu{}^Q \tag{3.31}$$

are changed. In particular, it will no longer fulfill the Bianchi identity, which now must be replaced by

$$\mathcal{D}_{[\mu}\mathcal{F}_{\nu\rho]}{}^M = X_{(NP)}{}^M A_{[\mu}{}^N \mathcal{F}_{\nu\rho]}{}^P - \frac{1}{3}X_{(PN)}{}^M X_{[QR]}{}^P\, A_{[\mu}{}^N A_\nu{}^Q A_{\rho]}{}^R\,. \tag{3.32}$$

Furthermore, $\mathcal{F}_{\mu\nu}{}^M$ does not transform covariantly under a gauge transformation (3.27). Instead, we have

$$\begin{aligned}\delta\mathcal{F}_{\mu\nu}{}^M &= 2\mathcal{D}_{[\mu}\delta A_{\nu]}{}^M - 2X_{(PQ)}{}^M A_{[\mu}{}^P \delta A_{\nu]}{}^Q\\ &= X_{NQ}{}^M\, \mathcal{F}_{\mu\nu}{}^N \Lambda^Q - 2X_{(NP)}{}^M \mathcal{D}_{[\mu}\Xi_{\nu]}{}^{NP} - 2X_{(PQ)}{}^M A_{[\mu}{}^P \delta A_{\nu]}{}^Q\,,\end{aligned} \tag{3.33}$$

where the covariant derivative is (both expressions are useful and related by (3.26))

$$\begin{aligned}X_{(NP)}{}^M \mathcal{D}_\mu \Xi_\nu{}^{NP} &= \partial_\mu\left(X_{(NP)}{}^M \Xi_\nu{}^{NP}\right) + A_\mu{}^R X_{RQ}{}^M X_{(NP)}{}^Q \Xi_\nu{}^{NP}\,,\\ \mathcal{D}_\mu \Xi_\nu{}^{NP} &= \partial_\mu \Xi_\nu{}^{NP} + X_{QR}{}^P A_\mu{}^Q \Xi_\nu{}^{NR} + X_{QR}{}^N A_\mu{}^Q \Xi_\nu{}^{PR}\,.\end{aligned} \tag{3.34}$$

Therefore, if we want to deform the original Lagrangian (2.1) and accommodate electric and magnetic gauge fields, $\mathcal{F}_{\mu\nu}{}^M$ cannot be used to construct gauge-covariant kinetic terms.

For this reason, the authors of [42] introduced tensor fields $B_{\mu\nu\,\alpha}$, later in [46] to be described by $B_{\mu\nu}{}^{MN}$, symmetric in (MN), and with them modified field strengths

$$\mathcal{H}_{\mu\nu}{}^M = \mathcal{F}_{\mu\nu}{}^M + X_{(NP)}{}^M B_{\mu\nu}{}^{NP}\,. \tag{3.35}$$

We will consider gauge transformations of the antisymmetric tensors of the form

$$\delta B_{\mu\nu}{}^{NP} = 2\mathcal{D}_{[\mu}\Xi_{\nu]}{}^{NP} + 2A_{[\mu}{}^{(N}\delta A_{\nu]}{}^{P)} + \Delta B_{\mu\nu}{}^{NP}\,, \tag{3.36}$$

where $\Delta B_{\mu\nu}{}^{NP}$ depends on the gauge parameter Λ^Q, but we do not fix it further at this point. Together with (3.33), this then implies[13]

$$\delta\mathcal{H}_{\mu\nu}{}^M = X_{NQ}{}^M \Lambda^Q \mathcal{H}_{\mu\nu}{}^N + X_{(NP)}{}^M \Delta B_{\mu\nu}{}^{NP}\,. \tag{3.37}$$

[13]Note that $\mathcal{F}_{\mu\nu}{}^N$ in the second line of (3.33) can be replaced by $\mathcal{H}_{\mu\nu}{}^N$ due to (3.23).

3.2.3 The kinetic lagrangian

The first step towards a gauge invariant action is to replace $\mathcal{F}_{\mu\nu}{}^{\Lambda}$ in $\mathcal{L}_{\text{g.k.}}$, (2.1), by $\mathcal{H}_{\mu\nu}{}^{\Lambda}$, which then yields the new kinetic Lagrangian

$$\mathcal{L}_{\text{g.k.}} = \frac{1}{4} e \mathcal{I}_{\Lambda\Sigma} \mathcal{H}_{\mu\nu}{}^{\Lambda} \mathcal{H}^{\mu\nu\Sigma} - \frac{1}{8} \mathcal{R}_{\Lambda\Sigma} \varepsilon^{\mu\nu\rho\sigma} \mathcal{H}_{\mu\nu}{}^{\Lambda} \mathcal{H}_{\rho\sigma}{}^{\Sigma} \,, \tag{3.38}$$

where again $\mathcal{I}_{\Lambda\Sigma}$ and $\mathcal{R}_{\Lambda\Sigma}$ denote, respectively, $\text{Im}\,\mathcal{N}_{\Lambda\Sigma}$ and $\text{Re}\,\mathcal{N}_{\Lambda\Sigma}$. Using

$$\mathcal{G}_{\mu\nu\,\Lambda} \equiv \varepsilon_{\mu\nu\rho\sigma} \frac{\partial \mathcal{L}}{\partial \mathcal{H}_{\rho\sigma}{}^{\Lambda}} = \mathcal{R}_{\Lambda\Gamma} \mathcal{H}_{\mu\nu}{}^{\Gamma} + \frac{1}{2} e \varepsilon_{\mu\nu\rho\sigma} \mathcal{I}_{\Lambda\Gamma} \mathcal{H}^{\rho\sigma\,\Gamma} \,, \tag{3.39}$$

the Lagrangian and its transformations can be written as

$$\begin{aligned}
\mathcal{L}_{\text{g.k.}} &= -\frac{1}{8} \varepsilon^{\mu\nu\rho\sigma} \mathcal{H}^{\Lambda}_{\mu\nu} \mathcal{G}_{\rho\sigma\,\Lambda} \,, \\
\delta\mathcal{L}_{\text{g.k.}} &= -\frac{1}{4} \varepsilon^{\mu\nu\rho\sigma} \mathcal{G}_{\mu\nu\,\Lambda} \delta \mathcal{H}^{\Lambda}_{\rho\sigma} \\
&\quad + \frac{1}{8} \varepsilon^{\mu\nu\rho\sigma} \Lambda^{Q} \left(\mathcal{H}^{\Lambda}_{\mu\nu} X_{Q\Lambda\Sigma} \mathcal{H}^{\Sigma}_{\rho\sigma} - 2 \mathcal{H}^{\Lambda}_{\mu\nu} X_{Q\Lambda}{}^{\Sigma} \mathcal{G}_{\rho\sigma\,\Sigma} - \mathcal{G}_{\mu\nu\,\Lambda} X_{Q}{}^{\Lambda\Sigma} \mathcal{G}_{\rho\sigma\,\Sigma} \right) \,,
\end{aligned} \tag{3.40}$$

where, in the third line, we used the infinitesimal form of (3.8):

$$\delta(\Lambda)\mathcal{N}_{\Lambda\Sigma} = \Lambda^{M} \Big[- X_{M\Lambda\Sigma} + 2\, X_{M(\Lambda}{}^{\Gamma} \mathcal{N}_{\Sigma)\Gamma} + \mathcal{N}_{\Lambda\Gamma}\, X_{M}{}^{\Gamma\Xi}\, \mathcal{N}_{\Xi\Sigma} \Big] . \tag{3.41}$$

When we introduce

$$\mathcal{G}_{\mu\nu}{}^{M} = \left(\mathcal{G}_{\mu\nu}{}^{\Lambda} \,,\, \mathcal{G}_{\mu\nu\Lambda} \right) \quad \text{with} \quad \mathcal{G}_{\mu\nu}{}^{\Lambda} \equiv \mathcal{H}_{\mu\nu}{}^{\Lambda} \,, \tag{3.42}$$

we can rewrite the second line of (3.40) in a covariant expression, and when we also use (3.37) we get

$$\begin{aligned}
\delta\mathcal{L}_{\text{g.k.}} = \varepsilon^{\mu\nu\rho\sigma} \Bigg[&-\frac{1}{4} \mathcal{G}_{\mu\nu\,\Lambda} \left(\Lambda^{Q} X_{PQ}{}^{\Lambda} \mathcal{H}_{\rho\sigma}{}^{P} + X_{(NP)}{}^{\Lambda} \Delta B_{\rho\sigma}{}^{NP} \right) \\
&+ \frac{1}{8} \mathcal{G}_{\mu\nu}{}^{M} \mathcal{G}_{\rho\sigma}{}^{N} \Lambda^{Q} X_{QM}{}^{R} \Omega_{NR} \Bigg] .
\end{aligned} \tag{3.43}$$

Clearly, the newly proposed form for $\mathcal{L}_{\text{g.k.}}$ in (3.38) is still not gauge invariant. This should not come as a surprise because (3.41) contains a constant shift (i.e., the term proportional to $X_{M\Lambda\Sigma}$), which requires the addition of extra terms to the Lagrangian as was reviewed in section 2 for purely electric gaugings. Also the last term on the right hand side of (3.41) gives extra contributions that are quadratic in the kinetic function. In the next steps we will see that besides GCS terms, also terms linear and quadratic in the tensor field are required to restore gauge invariance. We start with the discussion of the latter terms.

3.2.4 Topological terms for the B-field and a new constraint

The second step towards gauge invariance is made by adding topological terms linear and quadratic in the tensor field $B_{\mu\nu}{}^{NP}$ to the gauge kinetic term (3.38), namely

$$\mathcal{L}_{\text{top},B} = \frac{1}{4} \varepsilon^{\mu\nu\rho\sigma}\, X_{(NP)}{}^{\Lambda}\, B_{\mu\nu}{}^{NP} \left(\mathcal{F}_{\rho\sigma\,\Lambda} + \frac{1}{2}\, X_{(RS)\Lambda}\, B_{\rho\sigma}{}^{RS} \right) . \tag{3.44}$$

Note that for pure electric gaugings $X_{(NP)}{}^{\Lambda}=0$, as we saw in (3.11). Therefore, in this case this term vanishes, implying that the tensor fields decouple.

We recall that, up to now, only the closure constraint (3.19) has been used. We are now going to impose one *new constraint*:

$$X_{(NP)}{}^{M}\Omega_{MQ}X_{(RS)}{}^{Q}=0\,. \tag{3.45}$$

We will later show that this constraint is implied by the locality constraint (3.20) and the original representation constraint of [42], i.e. (1.3), but also by the locality constraint and the modified constraint (1.4) that we discussed in the introduction. The constraint thus says that

$$X_{(NP)}{}^{\Lambda}X_{(RS)\Lambda}=X_{(NP)\Lambda}X_{(RS)}{}^{\Lambda}\,. \tag{3.46}$$

A consequence of this constraint that we will use below follows from the first of (3.18) and (3.23):

$$X_{(PQ)}{}^{R}D_{MNR}=0\,. \tag{3.47}$$

The variation of $\mathcal{L}_{\mathrm{top},B}$ is

$$\begin{aligned}\delta\mathcal{L}_{\mathrm{top},B}&=\frac{1}{4}\,\varepsilon^{\mu\nu\rho\sigma}X_{(NP)}{}^{\Lambda}\left[\mathcal{H}_{\mu\nu\Lambda}\,\delta B_{\rho\sigma}{}^{NP}+B_{\rho\sigma}{}^{NP}\delta\mathcal{F}_{\mu\nu\Lambda}\right]\\&=\frac{1}{4}\,\varepsilon^{\mu\nu\rho\sigma}X_{(NP)}{}^{\Lambda}\left[\mathcal{H}_{\mu\nu\Lambda}\,\delta B_{\rho\sigma}{}^{NP}+2B_{\rho\sigma}{}^{NP}\left(\mathcal{D}_{\mu}\delta A_{\nu\Lambda}-X_{(RS)\Lambda}A_{\mu}^{R}\delta A_{\nu}^{S}\right)\right]\,.\end{aligned} \tag{3.48}$$

3.2.5 Generalized Chern-Simons terms

As in [42], we introduce a generalized Chern-Simons term of the form (these are the last two lines in what they called $\mathcal{L}_{top}$ in their equation (4.3))

$$\mathcal{L}_{\mathrm{GCS}}=\varepsilon^{\mu\nu\rho\sigma}A_{\mu}{}^{M}A_{\nu}{}^{N}\left(\frac{1}{3}X_{MN\,\Lambda}\,\partial_{\rho}A_{\sigma}{}^{\Lambda}+\frac{1}{6}X_{MN}{}^{\Lambda}\partial_{\rho}A_{\sigma\Lambda}+\frac{1}{8}X_{MN\,\Lambda}X_{PQ}{}^{\Lambda}A_{\rho}{}^{P}A_{\sigma}{}^{Q}\right)\,. \tag{3.49}$$

Modulo total derivatives one can write its variation as (using (3.24) antisymmetrized in $[MNQ]$ and the definition of D_{MNP} in (3.17))

$$\begin{aligned}\delta\mathcal{L}_{\mathrm{GCS}}=\varepsilon^{\mu\nu\rho\sigma}\ &\Big[\frac{1}{2}\mathcal{F}_{\mu\nu}{}^{\Lambda}\mathcal{D}_{\rho}\delta A_{\sigma\Lambda}-\frac{1}{2}\mathcal{F}_{\mu\nu\Lambda}X_{(NP)}{}^{\Lambda}A_{\rho}{}^{N}\delta A_{\sigma}{}^{P}\\&-D_{MNP}A_{\mu}{}^{M}\delta A_{\nu}{}^{N}\left(\partial_{\rho}A_{\sigma}{}^{P}+\frac{3}{8}X_{RS}{}^{P}A_{\rho}{}^{R}A_{\sigma}{}^{S}\right)\Big]\,.\end{aligned} \tag{3.50}$$

These variations can be combined with (3.48) to

$$\begin{aligned}\delta\left(\mathcal{L}_{\mathrm{top},B}+\mathcal{L}_{\mathrm{GCS}}\right)=\varepsilon^{\mu\nu\rho\sigma}&\Big[\frac{1}{2}\mathcal{H}_{\mu\nu}{}^{\Lambda}\mathcal{D}_{\rho}\delta A_{\sigma\Lambda}+\frac{1}{4}\mathcal{H}_{\mu\nu\Lambda}X_{(NP)}{}^{\Lambda}\left(\delta B_{\rho\sigma}{}^{NP}-2A_{\rho}{}^{N}\delta A_{\sigma}{}^{P}\right)\\&-D_{MNP}A_{\mu}{}^{M}\delta A_{\nu}{}^{N}\left(\partial_{\rho}A_{\sigma}{}^{P}+\frac{3}{8}X_{RS}{}^{P}A_{\rho}{}^{R}A_{\sigma}{}^{S}\right)\Big]\,,\end{aligned} \tag{3.51}$$

3.2.6 Variation of the total action

We are now ready to discuss the symmetry variation of the total Lagrangian

$$\mathcal{L}_{VT} = \mathcal{L}_{\rm g.k.} + \mathcal{L}_{{\rm top},B} + \mathcal{L}_{\rm GCS}\,, \tag{3.52}$$

built from (3.38), (3.44) and (3.49). We first check the invariance of (3.52) with respect to the Ξ-transformations. We see directly from (3.43) that the gauge-kinetic terms are invariant. The second line of (3.51) also clearly vanishes inserting (3.27) and using (3.47). This leaves us with the first line of (3.51), which, using (3.36) and (3.27), can be written in a symplectically covariant form:

$$\delta_\Xi \mathcal{L}_{VT} = -\frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\mathcal{H}_{\mu\nu}{}^M X_{(NP)}{}^Q \Omega_{MQ}\mathcal{D}_\rho \Xi_\sigma{}^{NP}\,. \tag{3.53}$$

The B-terms in $\mathcal{H}$, see (3.35), are proportional to $X_{(RS)}{}^M$ and thus give a vanishing contribution due to our new constraint (3.45). For the $\mathcal{F}$ terms we can perform an integration by parts[14] and then (3.32) gives again only terms proportional to $X_{(RS)}{}^M$ leading to the same conclusion. We therefore find that the Ξ-variation of the total action vanishes.

We can thus further restrict to the Λ^M gauge transformations. According to (3.33), the $\mathcal{D}_\rho \delta A_{\sigma\Lambda}$-term in (3.51) can then be replaced by $\frac{1}{2}\Lambda^Q X_{NQ\Lambda}\mathcal{H}_{\rho\sigma}{}^N$ (see again footnote 13), which can then be combined with the first term of (3.43) to form a symplectically covariant expression (the first term on the right hand side of (3.54) below). Adding also the remaining terms of (3.51) and (3.43), one obtains, using (3.36),

$$\begin{aligned}\delta\mathcal{L}_{VT} = \varepsilon^{\mu\nu\rho\sigma}\ \Big[&\frac{1}{4}\mathcal{G}_{\mu\nu}{}^M\Lambda^Q X_{NQ}{}^R\Omega_{MR}\mathcal{H}_{\rho\sigma}{}^N + \frac{1}{8}\mathcal{G}_{\mu\nu}{}^M\mathcal{G}_{\rho\sigma}{}^N\Lambda^Q X_{QM}{}^R\Omega_{NR}\\ &+\frac{1}{4}(\mathcal{H}-\mathcal{G})_{\mu\nu\,\Lambda}X_{(NP)}{}^\Lambda \Delta B_{\rho\sigma}{}^{NP}\\ &-D_{MNP}A_\mu{}^M\mathcal{D}_\nu\Lambda^N\left(\partial_\rho A_\sigma{}^P + \frac{3}{8}X_{RS}{}^P A_\rho{}^R A_\sigma{}^S\right)\Big]\,.\end{aligned} \tag{3.54}$$

We observe that if the $\mathcal{H}$ in the first line was a $\mathcal{G}$, eqs. (3.16) and (3.18) would allow one to write the first line as an expression proportional to D_{MNP}. This leads to the first line in (3.55) below. The second observation is that the identity $(\mathcal{H}-\mathcal{G})^\Lambda = 0$ allows one to rewrite the second line of (3.54) in a symplectically covariant way, so that, altogether, we have

$$\begin{aligned}\delta\mathcal{L}_{VT} = \varepsilon^{\mu\nu\rho\sigma}\ \Big[&\frac{1}{4}\mathcal{G}_{\mu\nu}{}^M\Lambda^Q X_{NQ}{}^R\Omega_{MR}(\mathcal{H}-\mathcal{G})_{\rho\sigma}{}^N + \frac{3}{8}\mathcal{G}_{\mu\nu}{}^M\mathcal{G}_{\rho\sigma}{}^N\Lambda^Q D_{QMN}\\ &-\frac{1}{4}(\mathcal{H}-\mathcal{G})_{\mu\nu}{}^M\Omega_{MR}X_{(NP)}{}^R\Delta B_{\rho\sigma}{}^{NP}\\ &-D_{MNP}A_\mu{}^M\mathcal{D}_\nu\Lambda^N\left(\partial_\rho A_\sigma{}^P + \frac{3}{8}X_{RS}{}^P A_\rho{}^R A_\sigma{}^S\right)\Big]\,.\end{aligned} \tag{3.55}$$

By choosing

$$\Delta B_{\rho\sigma}{}^{NP} = -\Lambda^N\mathcal{G}_{\rho\sigma}{}^P - \Lambda^P\mathcal{G}_{\rho\sigma}{}^N\,, \tag{3.56}$$

[14] Integration by parts with the covariant derivatives is allowed as (3.24) can be read as the invariance of the tensor X and (3.16) as the invariance of Ω.

the result (3.55) becomes

$$\delta\mathcal{L}_{VT} = \varepsilon^{\mu\nu\rho\sigma}\ \Bigg[\frac{3}{8}\Lambda^Q D_{MNQ}\left(2\mathcal{G}_{\mu\nu}{}^M(\mathcal{H}-\mathcal{G})_{\rho\sigma}{}^N + \mathcal{G}_{\mu\nu}{}^M\mathcal{G}_{\rho\sigma}{}^N\right) \\ - D_{MNP}A_\mu{}^M\mathcal{D}_\nu\Lambda^N\left(\partial_\rho A_\sigma{}^P + \frac{3}{8}X_{RS}{}^P A_\rho{}^R A_\sigma{}^S\right)\Bigg]\,, \qquad (3.57)$$

which is then proportional to D_{MNP}, and hence zero when the original representation constraint (3.21) of [42] is imposed.

Our goal is to generalize this for theories with quantum anomalies. These anomalies depend only on the gauge vectors. The field strengths $\mathcal{G}$, (3.39), however, also depend on the matrix $\mathcal{N}$ which itself generically depends on scalar fields. Therefore, we want to consider modified transformations of the antisymmetric tensors such that $\mathcal{G}$ does not appear in the final result.

To achieve this, we would like to replace (3.56) by a transformation such that

$$X_{(NP)}{}^R\Delta B_{\rho\sigma}{}^{NP} = -2X_{(NP)}{}^R\Lambda^N\mathcal{G}_{\rho\sigma}{}^P + \frac{3}{2}\Omega^{RM}D_{MNQ}\Lambda^Q(\mathcal{H}-\mathcal{G})_{\rho\sigma}{}^N\,. \qquad (3.58)$$

Indeed, inserting this in (3.55) would lead to

$$\delta\mathcal{L}_{VT} = \varepsilon^{\mu\nu\rho\sigma}\ \Bigg[\frac{3}{8}\Lambda^Q D_{MNQ}\mathcal{F}_{\mu\nu}{}^M\mathcal{F}_{\rho\sigma}{}^N \\ - D_{MNP}A_\mu{}^M\mathcal{D}_\nu\Lambda^N\left(\partial_\rho A_\sigma{}^P + \frac{3}{8}X_{RS}{}^P A_\rho{}^R A_\sigma{}^S\right)\Bigg]\,, \qquad (3.59)$$

where we have used (3.47) to delete contributions coming from the $B_{\mu\nu}{}^{NP}$ term in $\mathcal{H}_{\mu\nu}{}^M$ (cf. (3.35)).

The first term on the right hand side of (3.58) would follow from (3.56), but the second term cannot in general be obtained from assigning transformations to $B_{\rho\sigma}{}^{NP}$ (compare with (3.18)). Indeed, self-consistency of (3.58) requires that the second term on the right hand side be proportional to $X_{(NP)}{}^R$, which imposes a further constraint on D_{MNP}. We will see in section 4.3 how we can nevertheless justify the transformation law (3.58) by introducing other antisymmetric tensors. For the moment, we just accept (3.58) and explore its consequences.

Expanding (3.59) using (3.15) and (3.27) and using a partial integration, (3.59) can be rewritten as

$$\delta\mathcal{L}_{VT} = -\mathcal{A}[\Lambda]\,, \qquad (3.60)$$

where

$$\mathcal{A}[\Lambda] = -\frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\Lambda^P D_{MNP}\partial_\mu A_\nu{}^M\partial_\rho A_\sigma{}^N \\ -\frac{1}{4}\varepsilon^{\mu\nu\rho\sigma}\Lambda^P\left(D_{MNR}X_{[PS]}{}^N + \frac{3}{2}D_{MNP}X_{[RS]}{}^N\right)\partial_\mu A_\nu{}^M A_\rho{}^R A_\sigma{}^S\,. \qquad (3.61)$$

This expression formally looks like a symplectically covariant generalization of the electric consistent anomaly (2.8). Notice, however, that at this point this is really only a formal analogy, as the tensor D_{MNP} has, a priori, no connection with quantum anomalies. We will study the meaning of this analogy in more detail in the next section. To prove (3.60), one uses (3.47) and the preservation of D_{MNP} under gauge transformations, which follows from preservation of X, see (3.24), and of Ω, see (3.16), and reads

$$X_{M(N}{}^{P}\,D_{QR)P}=0\,. \tag{3.62}$$

For the terms quartic in the gauge fields, one needs the following consequence of (3.62):

$$\begin{aligned}(X_{RS}{}^{M}\,X_{PQ}{}^{N}\,D_{LMN})_{[RSPL]} &= -(X_{RS}{}^{M}\,X_{PM}{}^{N}\,D_{LQN}+X_{RS}{}^{M}\,X_{PL}{}^{N}\,D_{QMN})_{[RSPL]}\\ &= -(X_{RS}{}^{M}\,X_{PL}{}^{N}\,D_{QMN})_{[RSPL]}\,,\end{aligned} \tag{3.63}$$

where the final line uses (3.25) and again (3.47).

Let us summarize the result of our calculation up to the present point. We have used the action (3.52) and considered its transformations under (3.27) and (3.36), where $\Delta B_{\mu\nu}{}^{NP}$ was undetermined. We used the closure constraint (3.19) and one new constraint (3.45). We showed that the choice (3.56) leads to invariance if D_{MNP} vanishes, which is the representation constraint (3.21) used in the anomaly-free case studied in [42]. However, when we use instead the more general transformation (3.58) in the case $D_{MNP}\neq 0$, we obtain the non-vanishing classical variation (3.60). The corresponding expression (3.61) formally looks very similar to a symplectically covariant generalization of the electric consistent quantum anomaly.

In order to fully justify and understand this result, we are then left with the following three open issues, which we will discuss in the following section:

(i) The expression (3.61) for the non-vanishing classical variation of the action has to be related to quantum anomalies so that gauge invariance can be restored at the level of the quantum effective action, in analogy to the electric case described in section 2. This will be done in section 4.1.

(ii) The meaning of the new constraint (3.45) that was used to obtain (3.60) has to be clarified. This is subject of section 4.2.

(iii) We have to show how the transformation (3.58), which also underlies the result (3.60), can be realized. This will be done in section 4.3.

4. Gauge invariance of the effective action with anomalies

4.1 Symplectically covariant anomalies

In section 3, we discussed the algebraic constraints that were imposed on the embedding tensor in ref. [42] and that allowed the construction of a gauge invariant Lagrangian with electric and magnetic gauge potentials as well as tensor fields. Two of these constraints, (3.19) and (3.20), had a very clear physical motivation and ensured the closure of the gauge algebra and the mutual locality of all interacting fields. The physical origin of the third constraint, the representation constraint, (3.21), on the other hand, remained a bit obscure. In order to understand its meaning, we specialize it to its purely electric components,

$$X_{(\Lambda\Sigma\Omega)} = 0\,. \tag{4.1}$$

Given that the components $X_{\Lambda\Sigma\Omega}$ generate axionic shift symmetries (remember the first term on the right hand side of (3.41)), we can identify them with the corresponding symbols $X_{\Lambda\Sigma\Omega}$ in section 2, and recognize (4.1) as the condition for the absence of quantum anomalies for the electric gauge bosons (see (2.9)). It is therefore suggestive to interpret (3.21) as the condition for the absence of quantum anomalies for all gauge fields (i.e. for the electric and the magnetic gauge fields), and one expects that in the presence of quantum anomalies, this constraint can be relaxed. We will show that the relaxation consists in assuming that the symmetric tensor D_{MNP} defined by (3.17) is of the form[15]

$$D_{MNP} = d_{MNP}\,, \tag{4.2}$$

for a symmetric tensor d_{MNP} which describes the quantum gauge anomalies due to anomalous chiral fermions. In fact, one expects quantum anomalies from the loops of these fermions, ψ, which interact with the gauge fields via minimal couplings

$$\bar{\psi}\gamma^\mu(\partial_\mu - A_\mu{}^\Lambda\Theta_\Lambda{}^\alpha\delta_\alpha - A_{\mu\Lambda}\Theta^{\Lambda\alpha}\delta_\alpha)\psi\,. \tag{4.3}$$

Therefore, the anomalies contain — for each external gauge field (or gauge parameter) — an embedding tensor, i.e. d_{MNP} has the following particular form:

$$d_{MNP} = \Theta_M{}^\alpha\Theta_N{}^\beta\Theta_P{}^\gamma d_{\alpha\beta\gamma}\,, \tag{4.4}$$

with $d_{\alpha\beta\gamma}$ being a constant symmetric tensor. In the familiar context of a theory with a flat scalar manifold, constant fermionic transformation matrices, t_α, and the corresponding minimal couplings, the tensor d_{MNP} is simply proportional to

$$d_{MNP} \propto \Theta_M{}^\alpha\Theta_N{}^\beta\Theta_P{}^\gamma \mathrm{Tr}(\{t_\alpha, t_\beta\}t_\gamma\}, \tag{4.5}$$

where the trace is over the representation matrices of the fermions.[16]

We showed that the generalization of the consistent anomaly (2.8) in a symplectically covariant way leads to an expression of the form (3.61) with the D_{MNP}-tensor replaced by d_{MNP}. Indeed, the constraint (4.2) implies the cancellation of this quantum gauge anomaly by the classical gauge variation (3.60). Note that it is necessary for this cancellation that the anomaly tensor d_{MNP} is really constant (i.e., independent of the scalar fields). We expect this constancy to be generally true for the same topological reasons that imply the constancy of $d_{\Lambda\Gamma\Omega}$ in the conventional electric gaugings [27, 28]. In this way we have already addressed the first issue of the end of the previous section. We are now going to show how the constraint (4.2) suffices also to address the other two issues, (ii) and (iii).

---

[15]The possibility to impose a relation such as (4.2) is by no means guaranteed for all types of gauge groups (see e.g. [47] for a short discussion in the purely electric case studied in [26]).

[16]One might wonder how the magnetic vector fields $A_{\mu\Lambda}$ can give rise to anomalous triangle diagrams, as they have no propagator due to the lack of a kinetic term. However, it is the *amputated* diagram with internal fermion lines that one has to consider.

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4.2 The new constraint

We now comment on the constraint (3.45):

$$X_{(NP)}{}^{M}\Omega_{MQ}X_{(RS)}{}^{Q}=0\,. \tag{4.6}$$

We will show that this equation holds if the locality constraint is satisfied, and (4.2) is imposed on D_{MNP} with d_{MNP} of the particular form given in (4.4). To clarify this, we introduce as in [42] the 'zero mode tensor'[17]

$$Z^{M\alpha}=\frac{1}{2}\Omega^{MN}\Theta_{N}{}^{\alpha}\,,\qquad \text{i.e.}\qquad \begin{cases} Z^{\Lambda\alpha}=\frac{1}{2}\Theta^{\Lambda\alpha}\,,\\ Z_{\Lambda}{}^{\alpha}=-\frac{1}{2}\Theta_{\Lambda}{}^{\alpha}\,.\end{cases} \tag{4.7}$$

One then obtains, using (3.18), the definition of X in (3.15) and (4.4) that

$$X_{(NP)}{}^{M}=Z^{M\alpha}\Delta_{\alpha NP}\,, \tag{4.8}$$

for some tensor $\Delta_{\alpha NP}=\Delta_{\alpha PN}$. Due to the fact that we allow the symmetric tensor D_{MNP} in (3.17) to be non-zero and impose the constraint (4.2), this tensor $\Delta_{\alpha NP}$ is not the analogous quantity called $d_{\alpha MN}$ in [42],[18] but can be written as

$$\Delta_{\alpha NP}=(t_{\alpha})_{N}{}^{Q}\Omega_{PQ}-3d_{\alpha\beta\gamma}\Theta_{N}{}^{\beta}\Theta_{P}{}^{\gamma}\,. \tag{4.9}$$

However, the explicit form of this expression will not be relevant. We will only need that $X_{(NP)}{}^{M}$ is proportional to $Z^{M\alpha}$.

Now we will finally use the locality constraint (3.20), which implies

$$Z^{\Lambda[\alpha}Z_{\Lambda}{}^{\beta]}=0\,,\qquad \text{i.e.}\qquad Z^{M\alpha}Z^{N\beta}\Omega_{MN}=0\,. \tag{4.10}$$

This then leads to the desired result (4.6).

The tensor $Z^{M\alpha}$ can be called zero-mode tensor as e.g. the violation of the usual Jacobi identity (second line of (3.25)) is proportional to it. We now show that it also defines zero modes of D_{MNR}. Indeed, another consequence of the locality constraint is

$$X_{MN}{}^{P}\Omega^{MQ}\Theta_{Q}^{\alpha}=0\qquad\rightarrow\qquad X_{MN}{}^{P}Z^{M\alpha}=0\,,\qquad X_{QM}{}^{P}\Omega^{QS}X_{SN}{}^{R}=0\,. \tag{4.11}$$

With (3.18) and (3.23) this implies

$$D_{MNR}Z^{R\alpha}=0\,. \tag{4.12}$$

Note that we did not need (4.2) to achieve this last result, but that the equation is consistent with it.

[17] Note that the components of Ω^{MN} have signs opposite to those of Ω_{MN} as given in (3.7).

[18] We use $\Delta_{\alpha MN}$ in this paper to denote the analogue (or better: generalization) of what was called $d_{\alpha MN}$ in [42], because $d_{\alpha MN}$ is reserved in the present paper to denote the quantity $\Theta_{M}{}^{\beta}\Theta_{N}{}^{\gamma}d_{\alpha\beta\gamma}$ (cf eq. (4.20)) related to the quantum anomalies.

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4.3 New antisymmetric tensors

Finally, in this section we will justify the transformation (3.58), without requiring further constraints on the D-tensor. That transformation gives an expression for $X_{(NP)}{}^R \Delta B_{\rho\sigma}{}^{NP}$ that is not obviously a contraction with the tensor $X_{(NP)}{}^R$ (due to the second term on the right hand side of (3.58)). We can therefore in general not assign a transformation of $B_{\rho\sigma}{}^{NP}$ such that its contraction with $X_{(NP)}{}^R$ gives (3.58). To overcome this problem, we will have to change the set of independent antisymmetric tensors. The $B_{\mu\nu}{}^{MN}$ cannot be considered as independent fields in order to realize (3.58). We will, as in [42], introduce a new set of independent antisymmetric tensors, denoted by $B_{\mu\nu\,\alpha}$ for any α denoting a rigid symmetry.

The fields $B_{\mu\nu}{}^{NP}$ and their associated gauge parameters Ξ^{NP} appeared in the relevant formulae in the form $X_{(NP)}{}^M B_{\mu\nu}{}^{NP}$ or $X_{(NP)}{}^M \Xi^{NP}$, see e.g. in (3.27), (3.33), (3.35) and (3.44). With the form (4.8) that we now have, this can be written as

$$X_{(NP)}{}^M B_{\mu\nu}{}^{NP} = Z^{M\alpha} \Delta_{\alpha NP} B_{\mu\nu}{}^{NP} . \tag{4.13}$$

We will therefore replace the tensors $B_{\mu\nu}{}^{MN}$ by new tensors $B_{\mu\nu\,\alpha}$ using

$$\Delta_{\alpha MN} B_{\mu\nu}{}^{MN} \quad \rightarrow \quad B_{\mu\nu\,\alpha} . \tag{4.14}$$

and consider the $B_{\mu\nu\,\alpha}$ as the independent antisymmetric tensors. There is thus one tensor for every generator of the rigid symmetry group. The replacement thus implies that

$$X_{(NP)}{}^M B_{\mu\nu}{}^{NP} \quad \rightarrow \quad Z^{M\alpha} B_{\mu\nu\,\alpha} . \tag{4.15}$$

We also introduce a corresponding set of independent gauge parameters $\Xi_{\mu\,\alpha}$ through the substitution:

$$\Delta_{\alpha MN} \Xi_{\mu}{}^{MN} \quad \rightarrow \quad \Xi_{\mu\,\alpha} . \tag{4.16}$$

This allows us to reformulate all the equations in the previous sections in terms of $B_{\mu\nu\,\alpha}$ and $\Xi_{\mu\,\alpha}$. For instance we will write:

$$\delta A_\mu{}^M = \mathcal{D}_\mu \Lambda^M - Z^{M\alpha} \Xi_{\mu\,\alpha} , \tag{4.17}$$

$$\mathcal{H}_{\mu\nu}{}^M = \mathcal{F}_{\mu\nu}{}^M + Z^{M\alpha} B_{\mu\nu\,\alpha} , \tag{4.18}$$

$$\mathcal{L}_{\text{top},B} = \frac{1}{4} \varepsilon^{\mu\nu\rho\sigma} Z^{\Lambda\alpha} B_{\mu\nu\,\alpha} \left(\mathcal{F}_{\rho\sigma\,\Lambda} + \frac{1}{2} Z_\Lambda{}^\beta B_{\rho\sigma\,\beta} \right) . \tag{4.19}$$

We will show that considering $B_{\mu\nu\,\alpha}$ as the independent variables, we are ready to solve the remaining third issue mentioned at the end of section 3. To this end, we first note that all the calculations in section 3 remain valid when we use (4.15) and (4.17)–(4.19) to express everything in terms of the new variables $B_{\mu\nu\,\alpha}$ and $\Xi_{\mu\,\alpha}$, because the equations (3.45) and (3.47) we used in section 3 are now simply replaced by (4.10) and (4.12), respectively.

If we now set, following (4.4),

$$d_{MNP} = \Theta_M{}^\alpha d_{\alpha NP} , \qquad d_{\alpha NP} = d_{\alpha\beta\gamma} \Theta_N{}^\beta \Theta_P{}^\gamma , \tag{4.20}$$

then we can define (bearing in mind (4.8))

$$\begin{aligned} \delta B_{\mu\nu\,\alpha} &= 2\,\mathcal{D}_{[\mu}\Xi_{\nu]\alpha} + 2\Delta_{\alpha\,NP} A_{[\mu}{}^{N}\delta A_{\nu]}{}^{P} + \Delta B_{\mu\nu\,\alpha}\,, \\ \Delta B_{\mu\nu\,\alpha} &= -2\Delta_{\alpha NP}\Lambda^{N}\mathcal{G}_{\mu\nu}{}^{P} + 3d_{\alpha NP}\Lambda^{N}(\mathcal{H}-\mathcal{G})_{\mu\nu}{}^{P}\,, \end{aligned} \tag{4.21}$$

to reproduce (3.58), where the left-hand side of (3.58) is replaced according to (4.15). Here the covariant derivative is defined as

$$\mathcal{D}_{[\mu}\Xi_{\nu]\alpha} = \partial_{[\mu}\Xi_{\nu]\,\alpha} + f_{\alpha\beta}{}^{\gamma}\Theta_{P}{}^{\beta}A_{[\mu}{}^{P}\Xi_{\nu]\,\gamma}\,. \tag{4.22}$$

Of course, (4.21) is only fixed modulo terms that vanish upon contraction with the embedding tensor.

4.4 Result

In this section we have seen, so far, that it is possible to relax the representation constraint (3.21) used in ref. [42] to the more general condition (4.2) if one allows for quantum anomalies. The physical interpretation of the original representation constraint (3.21) of [42] is thus the absence of quantum anomalies.

Due to these constraints we obtained the equation (4.8), which allowed us to introduce the $B_{\mu\nu\,\alpha}$ as independent variables. All the calculations of section 3.2 are then valid with the substitutions given in (4.15) and (4.16). We did not impose (4.8) in section 3.2, and therefore we could at that stage only work with $B_{\mu\nu}{}^{NP}$. However, now we conclude that we need the $B_{\mu\nu\,\alpha}$ as independent fields and will further only consider these antisymmetric tensors.

The results of this section can alternatively be viewed as a covariantization of the results of [18, 26] with respect to electric/magnetic duality transformations.[19] To further check the consistency of our results, we will in the next section reduce our treatment to a purely electric gauging and show that the results of [26] can be reproduced.

4.5 Purely electric gaugings

Let us first explicitly write down D_{MNP} in its electric and magnetic components:

$$\begin{aligned} D_{\Lambda\Sigma\Gamma} &= X_{(\Lambda\Sigma\Gamma)}\,, \\ 3D^{\Lambda}{}_{\Sigma\Gamma} &= X^{\Lambda}{}_{\Sigma\Gamma} - 2X_{(\Sigma\Gamma)}{}^{\Lambda}\,, \\ 3D^{\Lambda\Sigma}{}_{\Gamma} &= -X_{\Gamma}{}^{\Lambda\Sigma} + 2X^{(\Lambda\Sigma)}{}_{\Gamma}\,, \\ D^{\Lambda\Sigma\Gamma} &= -X^{(\Lambda\Sigma\Gamma)}\,. \end{aligned} \tag{4.23}$$

In the case of a purely electric gauging, the only non-vanishing components of the embedding tensor are electric:

$$\Theta_{M}{}^{\alpha} = (\Theta_{\Lambda}{}^{\alpha},\ 0)\ . \tag{4.24}$$

[19] We have not discussed the complete embedding into $\mathcal{N}=1$ supersymmetry here, which would include all fermionic terms as well as the supersymmetry transformations of all the fields. This is beyond the scope of the present paper.

Therefore also $X^{\Lambda}{}_N{}^P = 0$ and (4.4) implies that the only non-zero components of $D_{MNP} = d_{MNP}$ are $D_{\Lambda\Sigma\Omega}$. Therefore, (4.23) reduce to

$$D_{\Lambda\Sigma\Omega} = X_{(\Lambda\Sigma\Omega)}\,, \qquad X_{(\Sigma\Omega)}{}^{\Lambda} = 0\,, \qquad X_{\Omega}{}^{\Lambda\Sigma} = 0\,. \tag{4.25}$$

The non-vanishing entries of the gauge generators are $X_{\Lambda\Sigma\Gamma}$ and $X_{\Sigma\Omega}{}^{\Lambda} = -X_{\Sigma}{}^{\Lambda}{}_{\Omega} = X_{[\Sigma\Omega]}{}^{\Lambda}$, the latter satisfying the Jacobi identities since the right hand side of (3.25) for $MNQR$ all electric indices vanishes. The $X_{[\Sigma\Omega]}{}^{\Lambda}$ can be identified with the structure constants of the gauge group that were introduced e.g. in (2.2). The $X_{\Lambda\Sigma\Omega}$ correspond to the shifts in (2.2). The first relation in (4.25) then corresponds to (2.9).

The locality constraint is trivially satisfied and the closure relation reduces to (2.4) as expected.

At the level of the action $\mathcal{L}_{\rm VT}$, all tensor fields drop out since, when we express everything in terms of the new tensors $B_{\mu\nu\,\alpha}$, these tensors always appear contracted with a factor $\Theta^{\Lambda\,\alpha} = 0$. In particular, the topological terms $\mathcal{L}_{{\rm top},B}$ vanish and the modified field strengths for the electric vector fields $\mathcal{H}_{\mu\nu}{}^{\Lambda}$ reduce to ordinary field strengths:

$$\mathcal{H}_{\mu\nu}{}^{\Lambda} = 2\partial_{[\mu}A_{\nu]}{}^{\Lambda} + X_{[\Omega\Sigma]}{}^{\Lambda}A_{\mu}{}^{\Omega}A_{\nu}{}^{\Sigma}\,. \tag{4.26}$$

Also the GCS terms (3.49) reduce to their purely electric form (2.7) with $X^{\rm (CS)}_{\Omega\Lambda\Sigma} = X^{\rm (m)}_{\Omega\Lambda\Sigma}$. Finally, the gauge variation of $\mathcal{L}_{\rm VT}$ reduces to minus the ordinary consistent gauge anomaly, as we presented it in (2.8).

This concludes our reinvestigation of the electric gauging with axionic shift symmetries, GCS terms and quantum anomalies as it follows from our more general symplectically covariant treatment. We showed that the more general theory reduces consistently to the known case of a purely electric gauging.

4.6 On-shell covariance of $\mathcal{G}_{\mu\nu}{}^M$

For completeness, we will show in this section that $\mathcal{G}_{\mu\nu}{}^M$ (as defined in (3.39) and (3.42)) is the object that transforms covariantly on-shell, rather than $\mathcal{H}_{\mu\nu}{}^M$. We consider the total action (3.52), where now $\mathcal{L}_{{\rm top},B}$ is given by (4.19), and in $\mathcal{L}_{\rm g.k.}$, the expression (4.18) is used. We write the general variation of this action under generic variations $\delta A_{\mu}{}^M$, $\delta B_{\mu\nu\,\alpha}$ of $A_{\mu}{}^M$, $B_{\mu\nu\,\alpha}$. The variation of $\mathcal{L}_{\rm g.k.}$ has a contribution only from $\mathcal{H}^{\Lambda}$, since the matrix $\mathcal{N}$ is inert under variations of $A_{\mu}{}^M$ and $B_{\mu\nu\,\alpha}$, and thus will be given by the first term in the expression of $\delta\mathcal{L}_{\rm g.k.}$ in (3.40). Summing this variation with the variation of the topological terms (3.51) we find:

$$\begin{aligned}\delta\mathcal{L}_{VT} = \varepsilon^{\mu\nu\rho\sigma}\Bigg[&\frac{1}{2}\mathcal{G}_{\mu\nu}{}^{M}\mathcal{D}_{\rho}\delta A_{\sigma}{}^{N}\,\Omega_{MN}\\ &+\frac{1}{4}\left(\mathcal{H}_{\mu\nu\Lambda}-\mathcal{G}_{\mu\nu\Lambda}\right)\left(Z^{\Lambda\alpha}\delta B_{\rho\sigma\,\alpha} - 2X_{(NP)}{}^{\Lambda}A_{\rho}{}^{N}\delta A_{\sigma}{}^{P}\right)\\ &-D_{MNP}A_{\mu}{}^{M}\delta A_{\nu}{}^{N}\left(\partial_{\rho}A_{\sigma}{}^{P}+\frac{3}{8}X_{RS}{}^{P}A_{\rho}{}^{R}A_{\sigma}{}^{S}\right)\Bigg]\,. \end{aligned}\tag{4.27}$$

This allows us to determine the equations of motion for the independent tensor fields $B_{\mu\nu\,\alpha}$:

$$\frac{\delta\mathcal{L}_{VT}}{\delta B_{\mu\nu\,\alpha}} \approx 0 \qquad \Leftrightarrow \qquad (\mathcal{H}-\mathcal{G})_{\mu\nu\Lambda}\,Z^{\Lambda\alpha} = \frac{1}{2}(\mathcal{H}-\mathcal{G})_{\mu\nu\Lambda}\,\Theta^{\Lambda\alpha} \approx 0\,, \tag{4.28}$$

which tells us that the equations of motion imply[20] that just some $\mathcal{H}_{\mu\nu\,\Lambda}$ are identified on-shell with the corresponding $\mathcal{G}_{\mu\nu\Lambda}$. More precisely, these are the tensors $\mathcal{H}_{\mu\nu\,\Lambda}$ that are singled out by the contraction with $\Theta^{\Lambda\alpha}$; they thus correspond to those magnetic vectors $A_{\mu\Lambda}$ that enter the action. From (4.28), together with the constraint (4.2) and the particular form (4.4) for d_{MNP}, we also see that

$$(\mathcal{H}_{\mu\nu}{}^{P} - \mathcal{G}_{\mu\nu}{}^{P})\, D_{PMN} \approx 0\,. \tag{4.29}$$

The properties (4.28) and (4.29) will be used next to prove that the tensor which is actually on-shell covariant under gauge-induced duality transformations is $\mathcal{G}_{\mu\nu}{}^{M}$ and not $\mathcal{H}_{\mu\nu}{}^{M}$.

Given the complete gauge variation for the antisymmetric tensor fields (4.21), we can write down the explicit gauge transformation properties of $\mathcal{H}_{\mu\nu}{}^{M}$ and $\mathcal{G}_{\mu\nu}{}^{M}$, which generalize those found in [42, 36] for $D_{MNP} = 0$:

$$\begin{aligned}
\delta\mathcal{H}_{\mu\nu}{}^{M} &= -\Lambda^{Q}\, X_{QP}{}^{M}\, \mathcal{H}_{\mu\nu}{}^{P} + \Lambda^{Q}\left(2\, X_{(QP)}{}^{M} + \frac{3}{2}\,\Omega^{MN}\, D_{NPQ}\right)(\mathcal{H}_{\mu\nu}{}^{P} - \mathcal{G}_{\mu\nu}{}^{P})\,,\\
\delta\mathcal{G}_{\mu\nu}{}^{\Lambda} &= -\Lambda^{Q}\, X_{QP}{}^{\Lambda}\, \mathcal{G}_{\mu\nu}{}^{P} + \Lambda^{Q}\, \hat{X}_{PQ}{}^{\Lambda}\,(\mathcal{H}_{\mu\nu}{}^{P} - \mathcal{G}_{\mu\nu}{}^{P})\,,\\
\delta\mathcal{G}_{\mu\nu\Lambda} &= -\Lambda^{Q}\, X_{QP\Lambda}\mathcal{G}_{\mu\nu}{}^{P} + \frac{1}{2}\,\varepsilon_{\mu\nu\rho\sigma}\,\mathcal{I}_{\Lambda\Sigma}\,\Lambda^{Q}\,\hat{X}_{PQ}{}^{\Sigma}\,(\mathcal{H}^{\rho\sigma P} - \mathcal{G}^{\rho\sigma P})\\
&\quad + \mathcal{R}_{\Lambda\Sigma}\,\Lambda^{Q}\,\hat{X}_{PQ}{}^{\Sigma}\,(\mathcal{H}_{\mu\nu}{}^{P} - \mathcal{G}_{\mu\nu}{}^{P})\,,
\end{aligned} \tag{4.30}$$

where we have used the following short-hand notation:

$$\hat{X}_{PQ}{}^{M} \equiv X_{PQ}{}^{M} + \frac{3}{2}\,\Omega^{MN}\, D_{NPQ}\,. \tag{4.31}$$

The first line of (4.30) follows from (3.37) and (3.58). The second transformation is a component of the first one since $\mathcal{G}_{\mu\nu}{}^{\Lambda} = \mathcal{H}_{\mu\nu}{}^{\Lambda}$, and for the transformation of $\mathcal{G}_{\mu\nu\Lambda}$ we use (3.41).

From (4.28) and (4.29) we see that, on-shell, the terms containing $(\mathcal{H}_{\mu\nu}{}^{P} - \mathcal{G}_{\mu\nu}{}^{P})\,\hat{X}_{PQ}{}^{M}$ vanish. Therefore we conclude that, as opposed to $\mathcal{H}_{\mu\nu}{}^{M}$, the tensor $\mathcal{G}_{\mu\nu}{}^{M}$ is on-shell gauge covariant and the gauge algebra closes on it modulo field equations. Consistency of course requires that field equations transform into field equations, and indeed it can be shown that:

$$\begin{aligned}
\delta(\mathcal{H}_{\mu\nu\Lambda} - \mathcal{G}_{\mu\nu\Lambda}) &= \Lambda^{Q}\left(\hat{X}_{PQ\Lambda} + \mathcal{R}_{\Lambda\Sigma}\,\hat{X}_{PQ}{}^{\Sigma}\right)(\mathcal{H}_{\mu\nu}{}^{P} - \mathcal{G}_{\mu\nu}{}^{P})\\
&\quad + \frac{1}{2}\,\varepsilon_{\mu\nu\rho\sigma}\,\mathcal{I}_{\Lambda\Sigma}\,\Lambda^{Q}\,\hat{X}_{PQ}{}^{\Sigma}\,(\mathcal{H}^{\rho\sigma P} - \mathcal{G}^{\rho\sigma P})\,.
\end{aligned} \tag{4.32}$$

5. A simple nontrivial example

Let us now briefly illustrate the above results by means of a simple example. We consider a theory with a rigid symmetry group embedded in the electric/magnetic duality group $\mathrm{Sp}(2,\mathbb{R})$. The embedding in the symplectic transformations is given by

$$t_{1M}{}^{N} = \begin{pmatrix} 1 & 0\\ 0 & -1\end{pmatrix}\,,\qquad t_{2M}{}^{N} = \begin{pmatrix} 0 & 0\\ 1 & 0\end{pmatrix}\,,\qquad t_{3M}{}^{N} = \begin{pmatrix} 0 & 1\\ 0 & 0\end{pmatrix}\,, \tag{5.1}$$

[20] Identifications on shell are indicated by $\approx$.

i.e. $t_2{}^{11} = 1$. Let us consider the following subset of duality transformations:

$$\mathcal{S}^M{}_N = \delta^M{}_N - \Lambda^P X_{PN}{}^M\,, \quad \text{with generators} \quad X_{PM}{}^N = \begin{pmatrix} 0 & 0 \\ X_P{}^{11} & 0 \end{pmatrix}, \tag{5.2}$$

where Λ^P is the rigid transformation parameter. The tensor X is related to the embedding of the symmetries in the symplectic algebra using the embedding tensor,

$$X_{PM}{}^N = \sum_{\alpha=1}^{3} \Theta_P{}^\alpha t_{\alpha M}{}^N\,. \tag{5.3}$$

We have thus chosen the embedding tensor

$$\Theta_P{}^1 = 0\,, \qquad \Theta_P{}^2 = X_P{}^{11}\,, \qquad \Theta_P{}^3 = 0\,. \tag{5.4}$$

We now want to promote $\mathcal{S}^M{}_N$ to be a gauge transformation, i.e., we take the $\Lambda^N = \Lambda^N(x)$ spacetime dependent and the $X_{PM}{}^N$ are the gauge generators. This obviously corresponds to a magnetic gauging, as (4.25) is violated, and therefore requires the formalism that was developed in [42] and reviewed in section 3.2. The locality constraint (3.20) is automatically satisfied, as only the index value $\alpha = 2$ appears, and closure of the gauge algebra spanned by the $X_{PM}{}^N$ requires that we impose (3.19), where only the right-hand side is non-trivial. It requires $\Theta_1{}^2 = 0$, and thus the only gauge generators that are consistent with this constraint are

$$X_{PM}{}^N = (X_{1M}{}^N,\, X^1{}_M{}^N)\,, \quad \text{with} \quad X_{1M}{}^N = 0\,, \quad X^1{}_M{}^N = \begin{pmatrix} 0 & 0 \\ X^{111} & 0 \end{pmatrix}. \tag{5.5}$$

Note that this choice still violates the original linear representation constraint (3.21), as (4.23) gives $D^{111} = -X^{111} \neq 0$. However, as we saw in section 3, this does not prevent us from performing the gauging with generators $X_{PM}{}^N$ given in (5.5). We introduce a vector $A_\mu{}^M$ which contains an electric and a magnetic part, $A_\mu{}^1$ and $A_{\mu 1}$. Note that only the magnetic vector couples to matter via covariant derivatives since the embedding tensor projects out the electric part. In what follows, we also assume the presence of anomalous couplings between the magnetic vector and chiral fermions. As we will now review, this justifies the nonzero $X^{111} \neq 0$, since it will give rise to anomaly cancellation terms in the classical gauge variation of the action. More precisely, we will have to require that

$$\Theta^{12} = X^{111}\,, \qquad -X^{111} = d^{111} = (X^{111})^3 \tilde{d}_{222}\,, \tag{5.6}$$

where we introduced $\tilde{d}_{222}$ as the component of $d_{\alpha\beta\gamma}$.

To show this, we first introduce a kinetic term for the electric vector fields:

$$\mathcal{L}_{\text{g.k.}} = \frac{1}{4}\, e\, \mathcal{I}\, \mathcal{H}_{\mu\nu}{}^1 \mathcal{H}^{\mu\nu\,1} - \frac{1}{8}\, \mathcal{R}\, \varepsilon^{\mu\nu\rho\sigma} \mathcal{H}_{\mu\nu}{}^1 \mathcal{H}_{\rho\sigma}{}^1, \tag{5.7}$$

where we introduced the modified field strength (4.18)

$$\mathcal{H}_{\mu\nu}{}^1 = 2\partial_{[\mu} A_{\nu]}{}^1 + \frac{1}{2} X^{111} B_{\mu\nu 2}\,, \tag{5.8}$$

which depends on a tensor field $B_{\mu\nu 2}$ and therefore transforms covariantly under

$$\begin{aligned}
\delta A_\mu{}^1 &= \partial_\mu \Lambda^1 + X^{111} A_{\mu\,1}\Lambda_1 - \frac{1}{2} X^{111}\Xi_{\mu 2}\,,\\
\delta B_{\mu\nu 2} &= 2\partial_{[\mu}\Xi_{\nu]2} + 4A_{[\mu\,1}\partial_{\nu]}\Lambda_1 - 6\Lambda_1\partial_{[\mu}A_{\nu]\,1} - \Lambda_1\mathcal{G}_{\mu\nu\,1}\,,\\
\delta A_{\mu 1} &= \partial_\mu \Lambda_1\,.
\end{aligned} \tag{5.9}$$

This follows from (4.21) since the only nonzero component of Δ_{2MN} is $\Delta_2{}^{11} = 2$ and for d_{2MN} we have only $d_2{}^{11} = -1$. One can check that

$$\begin{aligned}
\delta\mathcal{H}_{\mu\nu}{}^1 &= -\frac{1}{2} X^{111}\Lambda_1(\mathcal{H}+\mathcal{G})_{\mu\nu\,1}\,, \quad \text{with}\\
\mathcal{H}_{\mu\nu\,1} &= \mathcal{F}_{\mu\nu\,1} = 2\partial_{[\mu}A_{\nu]1}\,,\\
\mathcal{G}_{\mu\nu\,1} &\equiv \mathcal{R}\mathcal{H}_{\mu\nu}{}^1 + \frac{1}{2} e\mathcal{I}\varepsilon_{\mu\nu\rho\sigma}\mathcal{H}^{\rho\sigma\,1}\,.
\end{aligned} \tag{5.10}$$

Under gauge variations, the real and imaginary part of the kinetic function transform as follows (cf. (3.41)):

$$\delta\mathcal{I} = 2\Lambda_1 X^{111}\mathcal{R}\mathcal{I}\,, \qquad \delta\mathcal{R} = \Lambda_1 X^{111}\left(\mathcal{R}^2 - \mathcal{I}^2\right)\,. \tag{5.11}$$

Then it's a short calculation to show that

$$\delta\mathcal{L}_{\text{g.k.}} = \frac{1}{4}\varepsilon^{\mu\nu\rho\sigma}\Lambda_1 X^{111}\mathcal{G}_{\mu\nu\,1}\partial_\rho A_{\sigma 1}\,. \tag{5.12}$$

This is consistent with (3.43).

In a second step, we add the topological term (4.19)

$$\mathcal{L}_{\text{top},B} = \frac{1}{4}\varepsilon^{\mu\nu\rho\sigma} X^{111} B_{\mu\nu 2}\partial_{[\rho}A_{\sigma]\,1}\,. \tag{5.13}$$

The gauge variation of this term is equal to (up to a total derivative)

$$\delta\mathcal{L}_{\text{top},B} = -\frac{1}{4}\Lambda_1 X^{111}\varepsilon^{\mu\nu\rho\sigma}\left(\partial_\mu A_{\nu\,1}\right)\left(2\partial_\rho A_{\sigma\,1} + \mathcal{G}_{\rho\sigma\,1}\right)\,. \tag{5.14}$$

The generalized Chern-Simons term (3.49) vanishes in this case. Combining (5.12) and (5.14), one derives

$$\delta\left(\mathcal{L}_{\text{g.k.}} + \mathcal{L}_{\text{top},B}\right) = -\frac{1}{2}\Lambda_1 X^{111}\left(\partial_\mu A_{\nu\,1}\right)\left(\partial_\rho A_{\sigma\,1}\right)\varepsilon^{\mu\nu\rho\sigma}\,. \tag{5.15}$$

This cancels the magnetic gauge anomaly whose form can be derived from (3.61),

$$\mathcal{A}[\Lambda] = -\frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\Lambda_1 d^{111}\left(\partial_\mu A_{\nu\,1}\right)\left(\partial_\rho A_{\sigma\,1}\right)\,, \tag{5.16}$$

if we remember that $X^{111} = -D^{111} = -d^{111}$. Note that the electric gauge fields do not appear which corresponds to the fact that the electric gauge fields do not couple to the chiral fermions.

A simple fermionic spectrum that could yield such an anomaly (5.16) is given by, e.g., three chiral fermions with canonical kinetic terms and quantum numbers $Q = (-1), (-1), (+2)$ under the U(1) gauged by $A_{\mu\,1}$. Indeed, with this spectrum, we would have $\text{Tr}(Q) = 0$, i.e., vanishing gravitational anomaly, but a cubic Abelian gauge anomaly $d^{111} \propto \text{Tr}(Q^3) = +6$.

6. Conclusions

In this paper we have shown how general gauge theories with axionic shift symmetries, generalized Chern-Simons terms and quantum anomalies [26] can be formulated in a way that is covariant with respect to electric/magnetic duality transformations. This generalizes previous work of [42], in which only *classically* gauge invariant theories with anomaly-free fermionic spectra were considered. Whereas the work [42] was modelling extended (and hence automatically anomaly-free) gauged supergravity theories, our results here can be applied to general $\mathcal{N} = 1$ gauged supergravity theories with possibly anomalous fermionic spectra. Such anomalous fermionic spectra are a natural feature of many string compactifications, notably of intersecting brane models in type II orientifold compactifications, where also GCS terms frequently occur [18]. Especially in combination with background fluxes, such compactifications may naturally lead to 4D actions with tensor fields and gaugings in unusual duality frames. Our formulation accommodates all these non-standard formulations, just as ref. [42] does in the anomaly-free case.

At a technical level, our results were obtained by relaxing the so-called representation constraint to allow for a symmetric three-tensor d_{MNP} that parameterizes the quantum anomaly. In contrast to the other constraints for the embedding tensor, this modified representation constraint is not homogeneous in the embedding tensor, which is a novel feature in this formalism. Also our treatment gave an interpretation for the physical meaning of the "representation" constraint: In its original form used in [42], it simply states the absence of quantum anomalies. It is interesting, but in retrospect not surprising, that the extended supergravity theories from which the original constraint has been derived in [42], need this constraint for their internal classical consistency.

It would be interesting to embed our results in a manifestly supersymmetric framework. Likewise, it would be interesting to study explicit $\mathcal{N} = 1$ string compactifications within the framework used in this paper, making use of manifest duality invariances. Another topic we have not touched upon are Kähler anomalies [48 – 58] in $\mathcal{N} = 1$ supergravity or gravitational anomalies. We hope to return to some of these questions in the future.

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