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CRUDE SCALING LAWS FOR THE DYNAMIC APERTURE OF LHC FROM RANDOM NON-LINEAR ERRORS

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We discuss scaling laws for the dynamic aperture with respect to number of random multipoles in the machine, strength of the multipoles, and multipolarity-flavor n. Moreover we find rules of thumb for the combined effect of multipoles with different n on the dynamic aperture. These rules are tested for typically 10 seeds and are valid for large numbers of perturbing elements. These rules are then used to analyze tolerances for the random multipole errors in the LHC dipoles.

Keywords: Dynamic aperture; scaling law.

1 INTRODUCTION

Typically, dynamic aperture (DA) calculations are performed in order to assess the stability of a given optics, i.e. given the geometry and gradients of all linear and non-linear magnets single particles are mapped through the beam line for many turns and their survival is checked for given initial conditions. The larger the number of turns, the bigger the chance of hitting an unstable (usually small) portion of phase space, which causes the particle to escape to infinity. This procedure thus analyzes a given optics very carefully. On the other hand, the exact gradients of all magnets of an accelerator are not known very precisely due to finite construction tolerances, alignment errors and other effects. We therefore pursue a different path in this report. We investigate the short term DA over 1000 turns and perform ensemble averages over different seeds and deduce scaling laws for the DA with respect to strength, multipolarity, and discuss, how different multipoles jointly affect the DA.

We utilize the frequently used parameterization of the multipole errors of dipole magnets by the quantities a_n and b_n as given in the following equation

$$B_y + iB_x = B_0 \sum_n (b_n + ia_n) \left(\frac{x + iy}{R} \right)^{n-1} . \tag{1}$$

$B_{x/y}$ is the horizontal and vertical magnetic field, respectively. B_0 is the bending field of the dipole and R is a reference radius which is usually 10 mm.

2 NUMBER OF NON-LINEARITIES

In this report we limit ourselves to the short-term DA, because we are interested in coarse grained features of phase space and also found that the DA of a simple model varies by less than 3 % if the number of turns is varied between 10^3 and 10^6. We now turn to the effect of the number of random non-linear elements in the machine on 1000-turn DA. We do so by preparing a Henon map with tunes $Q_x = 68.28$ and $Q_y = 68.31$ and subdivide the beam line in 1024 pieces with equal phase advance and place a sextupole with random excitation between the linear pieces. The rms of the sextupole excitation is unity. Using the thus prepared lattice we calculate the diagonal DA, i.e. with starting conditions $x = y$ and $x' = y' = 0$. We then switch off every other sextupole and re-calculate the DA for that configuration and repeat this process of switching off every other remaining sextupole and DA calculation until only 4 sextupoles are left and plot the DA as a function of non-zero sextupoles in Figure 1. This is done for 10 different seeds which are labeled 0 through 9 in Figure 1.

Looking at each seed individually no clear trend is obvious, but fitting a straight line to the double-logarithmic representation of all data points reveals a slope of -0.4938 which is close to $-1/2$ and implies that averaged over many seeds the DA scales as $1/\sqrt{N}$ where N is the number of random sextupoles in the beam line. Since all sextupoles have the equal rms strength we may also say that the DA scales inversely to the integrated rms strength of the random sextupoles.[a]

[a] The author is grateful to J.-P. Koutchouk for pointing this out.

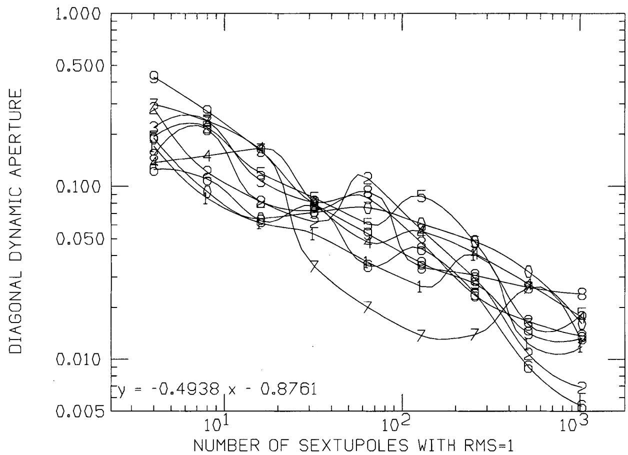

FIGURE 1 The effect of the number of sextupoles in the henon map on the dynamic aperture calculated for ten different seeds.

Re-doing the same exercise for decapoles we find that the DA scales as $1/N^{1/6}$ which is surprising at first sight. Since the rms of the decapoles is unity in these simulations we may write this as $1/(\sqrt{N}b_{5,\,\mathrm{rms}})^{1/(n-2)}$ with $n = 5$ and find again that it is the integrated rms strength that determines the DA. The value in the exponent $1/(n-2)$ will become obvious in the next section.

3 MULTIPOLARITY

In this section we deduce the (trivial and probably well known) scaling of the DA with respect to the strength of the non-linearities. For the sake of completeness we present the derivation of the scaling of the DA with multipole strength and begin by writing down the kick as experienced by the beam in an upright multipole

$$\Delta x' = \Delta\phi b_n \frac{x^{n-1}}{R^{n-1}} = \tilde{b}_n x^{n-1} \tag{2}$$

where we assume that the multipole is situated in a bending magnet with bending angle $\Delta\phi$ which is 4.868 mrad for a full length dipole of LHC, version 2 and the multipolar errors a_n and b_n are parameterized as shown in Equation 1. R is the reference radius referred to there.

Since the linear transport map between the non-linearities is scale invariant we only have to re-scale the variable x such that we obtain a unit kick and thus make the dynamics scale invariant with respect to the strength of the multipole. The new variable we call y and it is related to x by $y = \tilde{b}_n^p x$. Inserting this into Equation 2 we obtain

$$\frac{\Delta y'}{\tilde{b}_n^p} = \tilde{b}_n \frac{y^{n-1}}{(\tilde{b}_n^p)^{n-1}} \tag{3}$$

which can be rewritten as $\Delta y' = \tilde{b}_n^{p+1-np+p} y^{n-1}$. The requirement for a unit kick is thus transferred to the requirement that $p + 1 - np + p = 0$ which yields

$$p = \frac{1}{n-2} \tag{4}$$

and the DA in x-variables $d_{n,x}$ is related to that in y-variables $d_{n,y}$ by

$$d_{n,x} = \tilde{b}_n^{-1/(n-2)} d_{n,y} = R \left(\frac{R}{\Delta\phi b_n} \right)^{1/(n-2)} d_{n,y} \ . \tag{5}$$

Note that $d_{n,y}$ has the units of $\mathrm{m}^{-1/(n-2)}$. Equation 5 justifies the observed scaling at the end of the previous section.

Moreover we can check how the DA depends on the multipolarity n of the kick. We do so by tracking the simple 4-dimensional Henon map for LHC tunes over 1000 turns. In these runs the kick is always taken to be of the form $\Delta y' = y^{n-1}$, i.e. of unit strength. The DA thus found are shown in Table I in which the positive numbers correspond to upright multipoles and negative numbers to skew multipoles. Obviously the DA in rescaled variables $d_{n,y}$ depends only weakly on the multipolarity n. In the following we introduce parameters ζ and D that relate the $d_{n,y}$ by

$$d_{n,y} = D^{-1/(n-2)} \zeta \ . \tag{6}$$

TABLE I The 1000-turn diagonal dynamic aperture of the four-dimensional Henon map with $Q_x = 0.28$ and $Q_y = 0.31$. Note that the values do not differ considerably from 0.5. A positive number indicates an upright multipole and a negative number a skew multipole

n	$d_{n,y}$	n	$d_{n,y}$	n	$d_{n,y}$
3	0.3715	6	0.7815	9	0.7986
–3	0.4106	–6	0.7135	–9	0.7447
4	0.6870	7	0.6575	10	0.8494
–4	0.5468	–7	0.6341	–10	0.8172
5	0.6470	8	0.8750	11	0.7999
–5	0.6747	–8	0.7612	–11	0.7813

where ζ determines the overall magnitude and D serves a dual purpose. First, it introduces the weak variation with n and second, it has the units of meters in order to get the dimensions right. For example, ζ will be affected by the beta function at the point where the DA is calculated and the number of random non-linear elements as well as the tunes. For the Henon map data a simple fit to the data shown in Table I yields $D = 0.4$ m and $\zeta = 0.9$. These parameters will, however, be different for maps with other multipoles.

Using the relation given in Equation 6 we can deduce relations between DA and multipolar strength as a function of multipolarity n. Inserting Equation 6 into Equation 5 we obtain

$$d_{n,x} = \frac{\zeta R}{((D/R)\Delta\phi b_n)^{1/(n-2)}} = \frac{\zeta R}{(\kappa b_n)^{1/(n-2)}} \tag{7}$$

where we introduce the quantity $\kappa = (D/R)\Delta\phi$. After solving for b_n and taking logarithms this leads to

$$\ln b_n = \ln\left(\frac{d_{n,x}^2}{\zeta^2 R^2 \kappa}\right) + n \ln\left(\frac{\zeta R}{d_{n,x}}\right) . \tag{8}$$

The preceding equation describes an exponential relation (with respect to n) between the multipolar excitation b_n and the DA $d_{n,x}$. Figure 2 shows the a_n and b_n that reduce the DA $d_{n,x}$ of LHC, version 2 at IP 1 with $\beta_x = \beta_y = 8$ m

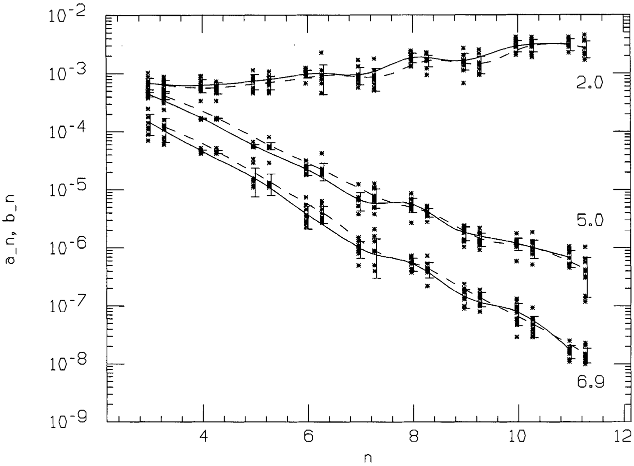

FIGURE 2 The a_n and b_n that reduces the dynamic aperture to 2, (top) 5, (middle) and 6.9 mm (bottom), respectively as function of the multipolarity n.

to 2, 5, and 6.9 mm. We clearly observe the linear dependence of a_n and b_n versus n in the logarithmic plot Figure 2 as predicted by Equation 8. Linear fits to the three curves yield the following estimates to the parameters D or κ and ζ. Note that we need two parameters in a fit to a straight line which justifies the parametrization choosen in Equation 6. We observe that the parameter ζ as deduced from the graphs is about 0.22, but the parameters D or κ vary quite drastically as determined from the three graphs. Bearing in mind that we are dealing with crude estimates we will use the values $D = 4000$ m or $\kappa = 2000$ and $\zeta = 0.22$ to characterize LHC in what follows and estimate the DA of LHC as resulting from random multipole errors of multipolarity n only by

$$d_{n,x} = \frac{0.22R}{(2000b_n)^{1/(n-2)}} \quad \text{in m for LHC, version 2.} \tag{9}$$

Note that the magnitude of κ only has a small effect on the DA from higher multipoles, because it enters under a root of order $n - 2$.

TABLE II The parameters ζ, D, and κ, deduced from linear fits to Figure 2

Dynamic Aperture *[mm]*	ζ	D *[m]*	κ
2.0	0.248	4800	2340
5.0	0.213	2660	1300
6.9	0.227	5290	2580

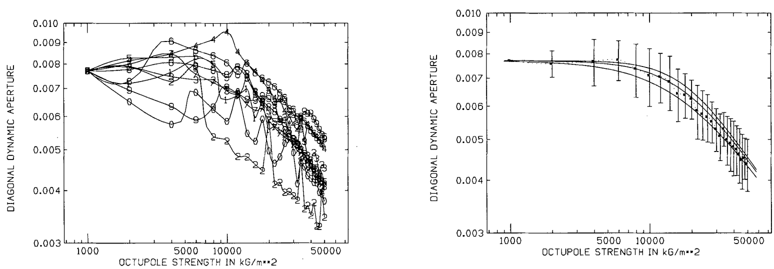

FIGURE 3 The 200-turn dynamic apertures of LHC, version 2 with a fixed sextupole seed and 10 different octupole seeds which are scaled up and the average over the ten seeds with fits to the data using exponents $\nu = 3,\ 4,\ 5$.

4 COMBINED EFFECT OF DIFFERENT MULTIPOLES

Whereas in the previous sections we looked at the effect of a single flavor of multipole only, we now turn to the effects generated by combining different multipoles in the same beam line. We first perform a numerical experiment in which we use LHC, version 2 and choose a random seed of sextupole components which are located at the dipoles, i.e. 1280 different sextupoles. We then choose 10 seeds, labeled 0 through 9, for octupoles situated at the same location and calculate the 200-turn diagonal DA for different excitations of the octupoles in which, for a given seed, all octupole excitations are increased by the same factor.

Figure 3 shows the resulting dynamic apertures as a function of the rms octupole excitation. The DA for each seed varies dramatically, but we can still see a trend. For small octupole excitations the DA is predominantly determined by the sextupoles and for large octupole excitation the DA decreases. In the transition region, however, strange things happen. In some cases (seed 4 and 6) the DA even improves as a result of the increased octupole strength. At present the mechanisms for this are unclear and need futher attention by calculating resonance driving terms and tune shifts with amplitude of these seeds and search for correlations between DA and those terms. This, however, is beyond the scope of this paper.

Here we pursue a different path by constructing a heuristic model that explains the qualitative features of Figure 3.[b] In order to make the qualitative features of Figure 3 visible we plot the average over the ten seeds in Figure 3 as the dotted line which connects the asterisks. We then make the bold assumption that the DA of different multipoles of order n and m combine to yield a resulting DA d according to

$$\frac{1}{d^{\nu}} = \frac{1}{d_n^{\nu}} + \frac{1}{d_m^{\nu}} \tag{10}$$

where we omitted the index x for simplicity. In Figure 3 we show three curves which correspond the $\nu = 3, 4, 5$ in ascending order. Clearly the central curve with $\nu = 4$ fits best and we use $\nu = 4$ in all subsequent calculations. The fact that $\nu = 4$ has a surprising consequence. If we assume that the DA due to multipoles of order n d_n is inversely proportional to some power μ of the volume of holes in phase space V_n (near the particle's starting position when that is near the DA) V_n through which the particles can channel to infinity we may write

$$d_n \propto \frac{1}{V_n^{\mu}} . \tag{11}$$

Furthermore we may argue that in the limit of weak multipoles the different orders of multipoles do not interact, and the volumes V_n of different orders simply add, yielding $V = \sum_n V_n$. Inserting Equation 11 we get the result of Equation 10, provided that $\nu = 1/\mu$. Note that the proportionality constants all have to be the same, because otherwise the limit of a single multipole yields wrong results. With $\nu = 4$ we conclude that the DA d_n is inversely proportional to the linear dimension $V^{1/4}$ of the holes (resonances) in phase space, through which particles can escape. Note, however, that this reasoning is very handwaving and needs closer attention in the future.

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[b] Regarding this section the author profitted greatly from discussions with T. Trenkler.

TABLE III The estimated random errors for the LHC dipoles and the dynamic apertures calculated from Equation 9

n	3	4	5	6	7	8	9	10	11	
DA	10.5	10.0	7.6	7.6	6.5	6.3	5.6	5.4	5.2	mm

5 TOLERANCES FOR LHC

We now exploit the heuristic relation given in Equation 7 or Equation 9 to deduce the effect of the multipolar components in the dipoles of LHC on the DA and report the result in Table III where we use $\kappa = 2000$ and $\zeta R = 2.2\,\text{mm}$ for LHC. The values for random a_n and b_n are taken from Ref. 2. From this table we conclude that, averaged over many seeds, the higher multipoles contribute more than the lower multipoles and the tolerances given in Ref. 2 appear to be too loose.

In order to assess the combined effect of all multipoles we insert Equation 7 in Equation 10 and obtain under the assumption that all upright and skew multipoles add equally

$$d = \frac{\zeta R}{\left(\sum_{n=3} (\kappa(a_n + b_n))^{4/(n-2)}\right)^{1/4}} . \tag{12}$$

This equation can be used to estimate the effect of different random multipole components on the DA. Note, however, that it is not an exact equation that can predict the DA, but should merely serve as a rough guideline to the relative deteriorating effects of different random multipoles in the LHC dipoles. Inserting a_n and b_n from Table III we obtain

$$d = 3.7 \text{ mm} \tag{13}$$

which is considerably smaller than the DA resulting from e.g. sextupoles alone. The surprising fact, however, is, that the higher multipoles are the major culprits which reduce the DA as should be clear from Table III.

6 CONCLUSIONS

We developed scaling laws for the DA from random non-linear errors with respect to various parameters that can be expected from an ensemble average over many seeds. We found that for this crude analysis the 1000-turn DA is a suitable measure. The scaling laws with respect to the number of non-linearities of a given flavor were deduced and found to be determined by the integrated rms strength. This scaling is consistent with scaling laws with respect to the strength of the multipoles a_n, b_n. Moreover, scaling laws for the multipolarity flavor n were deduced. We then analyzed the combined effect of different multipoles in the same beam line and postulated a very bold heuristic model for that. We then applied the analysis to the random errors in LHC as given in Ref. 2 and found that the higher multipoles are the dominant restrictions of the short term DA.

The aforementioned parametrization by D or κ and ζ was determined from simple fits to a few tracking runs. It will certainly be interesting to determine these quantities from "first principles". In the same fashion the rule that the DA from different multipoles combine according to Equation 10 needs futher clarification beyond the truly handwaving discussion given in Section 4.

Acknowledgements

I am grateful to T. Trenkler and J.-P. Koutchouk for fruitful discussions.

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