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MODULATED DIFFUSION FOR A SIMPLE LATTICE MODEL*

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A possible mechanism, which explains the diffusion in the phase space due to the ripples in the quadrupole currents, is studied on a simplified version of the SPS lattice used for experiments. We describe the diffusion driven by a single resonance in the space of the adiabatic invariant (action variable), by using the results of the Neishtadt's theory. Under suitable hypothesis, it is possible to introduce a random walk for the adiabatic invariant, which gives a quantitative description of the diffusion. The comparison with the numerical results turns out to be very effective.

Keywords: Lattices; diffusion.

1 INTRODUCTION

The study of the long term dynamics aperture in a particle accelerator is related to the stability of the orbits in a neighborhood of an elliptic fixed point of a symplectic map.[1,2] The multipolar components of the magnetic field correspond to nonlinear terms in the Taylor expansion of the map. The presence of Arnold diffusion[3] and overlapping of resonances[4] allows the possibility of finding unstable orbits, near the elliptic fixed point. However the numerical simulations show the following scenario: there is a threshold in the phase space after which a fast escape to infinity is observed (short term dynamics aperture); there is a neighborhood of the fixed point, where

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no diffusion can be detected; there is a region where some particular orbits escape to infinity after a big number of iterations, but usually the measure of the initial conditions of the unstable orbits is very small.

This situation can be explained by taking into account both the extreme slowness and sensitivity to the initial condition of Arnold diffusion and the limitation of the region of overlapping of resonances, which is a generic feature for polynomial symplectic maps. As a consequence to explain the slow diffusion of particles experimentally observed in accelerators, one has to introduce other effects in the model. Recent experiments on the SPS at CERN[5] and in other laboratories have shown that the ripples in the feeding currents of the quadrupoles due to harmonics of the 50 Hz frequency, coupled with the nonlinear components of the magnetic field, cause a slow decrease of the beam intensity.

The presence of ripples introduces a slow modulation in the coefficient of the one turn map, which describes the betatronic motion, so that we have to study the stability of the orbits of a non-autonomous symplectic map. However we can simplify the problem by using the results of the adiabatic theory for Hamiltonian systems,[6] since the ratio between the ripple frequency and the betatronic frequency is $10^{-3} \div 10^{-4}$ and can be used as a perturbation parameter. The numerical simulations show a qualitative agreement with the experimental data,[7] but a quantitative description of the *modulated diffusion* is still not available.

In this paper we present a description of the *modulated diffusion* due to a nonlinear resonance in the phase space: our approach is directly derived from the theory of the changing of the adiabatic invariant due to the crossing of a separatrix developed by A. Neishtadt[8,9] for Hamiltonian systems. We have applied the Neishtadt's theory to an area-preserving map, directly derived from a simplified model of the SPS lattice used in the experiments.

The plan of the paper is the following: in Section 2 we briefly summarize the results of the adiabatic theory and we introduce the hypothesis necessary to apply the results to our model; in Section 3 we describe the model and we discuss the comparison between the numerical results and the analytical approach; the concluding remarks are reported in Section 4.

2 ADIABATIC THEORY FOR HAMILTONIAN SYSTEMS

The adiabatic theory was mainly developed to describe the evolution of integrable or almost integrable Hamiltonian systems $H(q, p, \epsilon t)$ perturbed by a slow modulation: $\epsilon \ll 1$ and $H(q, p, \lambda)$ integrable and periodic in λ. The concept of slow modulation means that the period of modulation $T = 2\pi/\epsilon$ is much longer than the typical time scales of the unperturbed Hamiltonian $H(q, p, \lambda)$. The main idea of the adiabatic theory is to look for special dynamical variables $I(q, p, \lambda)$, called adiabatic invariant (a.i.), which have the following property:

$$|I(q, p, \epsilon t) - I_0| < C\epsilon \quad \forall\ t < \frac{1}{\epsilon} \tag{1}$$

where I_0 is the initial value of the a.i. and C is a constant. Far from a resonance region, the action variables for the unperturbed system are a.i. and one can prove that the perpetual adiabatic invariance (no limits on t) holds for a large set of initial conditions. As a consequence the instabilities due to a slow modulation are driven by the presence of resonances and in particular of separatrices in the phase space. The adiabatic description of the dynamics near to a resonance is based on the slow pulsation of the separatrix in the phase space due to the slow time dependence of the Hamiltonian (show Figure 1 left); starting from an initial condition out of the separatrix the actual trajectory will follow the unperturbed orbits of $H(q, p, \lambda)$ in order to preserve the value of the a.i. (i.e. the area of the unperturbed orbit for a 2D system); but since the separatrix is moving in the phase space, at a certain time it could happen that the orbit crosses the separatrix and starts to be trapped in the resonance region. In such a case the inequality (1) does not hold any more and it is necessary to compute the changing of the a.i. due to a crossing of a separatrix.[8,9] The main result of Neishtadt's theory is that if one considers an orbit in the region spanned by the separatrix, after one period of the slow modulation, the value of the a.i. will be changed by a quantity

$$I_1 - I_0 = \epsilon f(I_0, \xi, \xi'; \log \epsilon) \tag{2}$$

where ξ and ξ' are two variables $\in [0, 1]$ related to the value of the phase variable at the crossing times (we have two crossings of the separatrix in one period); the leading term in Equation (2) is of order $\epsilon \log \epsilon$.

In order to understand a possible mechanism for the diffusion in the phase space, we consider the following scenario: the unperturbed system $H(q, p, \lambda)$ has a single resonance in the phase space for each value of λ and the region spanned by the separatrix when $\lambda = \epsilon t$ varies, is free from other resonances (this condition will be not true for quasi integrable systems, but in this case we suppose that the amplitude of other resonances is of order ϵ); the amplitude of the region spanned by the separatrix is much larger than ϵ. Let us consider a cluster of initial conditions in the region $\mathcal{A}$ spanned by the separatrix; the size of the cluster has to be larger than ϵ. The distribution function of the increments for the a.i. can be computed by using Equation (2) and considering ξ, ξ' as random variables uniformly distributed in [0, 1].[10] In spite of the complicated expression of the r.h.s. of Equation (2), we make the operative hypothesis that the increment $I_1 - I_0$ could be approximated by

$$I_1 - I_0 = \epsilon \hat{f}(I_0, \log \epsilon)\zeta \tag{3}$$

where ζ is an universal random variable whose distribution can be determined numerically, whereas the function $\hat{f}$ takes into account the dependence on the action of the diffusion coefficient and it is zero at the boundary of the region $\mathcal{A}$.

The main idea of our approach is to consider successive crossings of the separatrix as a random walk for the a.i. with independent increments; the probability space is defined by the set of the initial conditions. This approach is correct if we show that the values of phase variable after one period T of the modulation are uncorrelated. According to the results in Refs. 11 and 12, the relation between two successive phase values after a period T, can be expressed in term of the integral

$$\int_0^T \frac{\partial \Omega}{\partial I}(I_{n-1}, \epsilon t)(I_n - I_{n-1})dt \tag{4}$$

where $\Omega(I, \epsilon t)$ is the amplitude dependent rotation number (tune) and I_{n-1} and I_n the initial and final values of the a.i.. As a consequence if the integral (4) is ≈ 1, then the increments will be independent. Taking into account that $I_n - I_{n-1} \propto \epsilon \log \epsilon$ and that $T \propto 1/\epsilon$, one can estimate that the integral (4) is ≈ 1 if ϵ is sufficiently small and the derivative $\partial\Omega/\partial I$ is not too small, which is in general true near a separatrix. We have numerically checked the independence of the increments for the a.i. and our result seems to be generic for symplectic maps;[13] but for 2D symmetric Hamiltonian systems there are examples for which the condition (4) ≈ 1 fails and the phases are correlated for more than one period T.[14]

Therefore we can describe the diffusion of the a.i. due to the presence of an isolated single resonance by using the random walk

$$I_n - I_{n-1} = \epsilon \hat{f}(I_{n-1}, \log \epsilon)\zeta \tag{5}$$

where I_n is the value of the a.i. after n periods. The Equation (5) will be a crude approximation of the diffusion in the case that ϵ is not small or that the orbits are very close to boundary of the region $\mathcal{A}$. Moreover the diffusion described by the random walk (5) is a bounded diffusion, since the orbits cannot overcome the region $\mathcal{A}$ spanned by the separatrix, so that the final distribution function for the a.i. will be an uniform distribution in the region $\mathcal{A}$. In order to describe the escape to infinity of an orbit in the phase space we have to consider the overlapping of regions spanned by the separatrices of different resonances and to introduce the transition probabilities for the a.i. to go from one region to another. This mechanism is indeed observed in the numerical simulations[15] but up to now there is no analytical description.

3 DESCRIPTION OF THE MODEL AND NUMERICAL RESULTS

In the 1991 experiments at CERN to study the diffusion of the beam due to the ripples in the quadrupole magnets, the SPS had a special configuration with eight sextupoles strongly exited to produce intense nonlinear fields.[5] The betatronic motion for a flat beam is well described by the composition of eight quadratic area-preserving maps, which take into account the effect of the strong sextupoles. This is the model we have used for the numerical simulations; the same model was considered to study the diffusion of the nonlinear invariant, when a stochastic noise is introduced in the linear tune.[16] We cannot claim that our model is realistic, but we expect that it is sufficiently generic to perform all the features of a realistic model in the flat beam approximation.

We have fixed the unperturbed linear tune $\omega_0 = 26.61$, so that there is a 5-order resonance in the phase space (see Figure 1 (right)). We have checked that the conditions required by Neishtadt's theory are satisfied and we have introduced a slow periodic modulation in the linear tune. In the Courant-Snyder coordinates, the map can be written in the form

$$(X_n, P_n) = \mathcal{R}(-a\cos(2\pi\epsilon n)) \circ \mathcal{R}(\omega_0) \circ \mathcal{M}_{NL}(X_{n-1}.P_{n-1}) \tag{6}$$

where $\mathcal{R}$ is a rotation matrix and $\mathcal{M}_{NL}$ is a nonlinear map, which starts with the identity. We have used the following parameters in the numerical

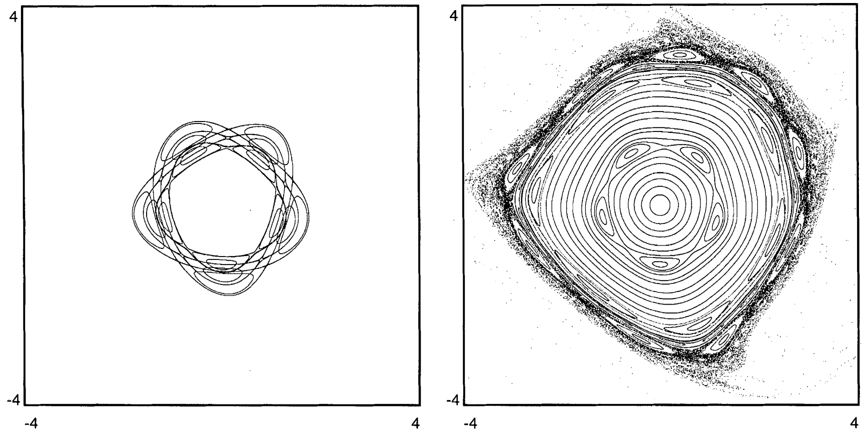

FIGURE 1 Plot of the separatrices of the resonance 5 in the phase space (left) for different values of $\lambda = \epsilon n$ in the map (6); phase space of the map (6) (right) with $\lambda = 0$.

simulations: modulation frequency $\epsilon = 0.0005$, amplitude of the modulation $a = 0.003$. In Figure 1 (left) we show the region $\mathcal{A}$ spanned by the separatrix of the 5-order resonance; by comparing the region $\mathcal{A}$ and the unperturbed phase space, we can see that no other macroscopic resonance intersects the region $\mathcal{A}$ for our choice of the parameter value a. The initial conditions are distributed in a ring of unperturbed orbits, whose amplitude in action is ≈ 0.07; this is a compromise between the amplitude in action of the region $\mathcal{A}$ (≈ 0.7) and the request that the spread in action of the initial conditions cannot be too small in order to have a good statistics. We have considered a population of 40,000 particles centered at different values of action in the region $\mathcal{A}$, to compute numerically the function $\epsilon \hat{f}(I, \log \epsilon)$ and the distribution of the random variable ζ (see Equation (5)). The value of the a.i. (unperturbed action) is given by the integral

$$I(X, P, \lambda) = \frac{1}{2\pi} \oint P dX \tag{7}$$

which is evaluated by using the unperturbed orbits. We have numerically computed the function $I(X, P, 0)$ by constructing a lattice of points in the phase space and by using an interpolation technique (see Ref. 17). In our computations, the relative error in the action is $\approx 10^{-4} \div 10^{-5} < \epsilon$.

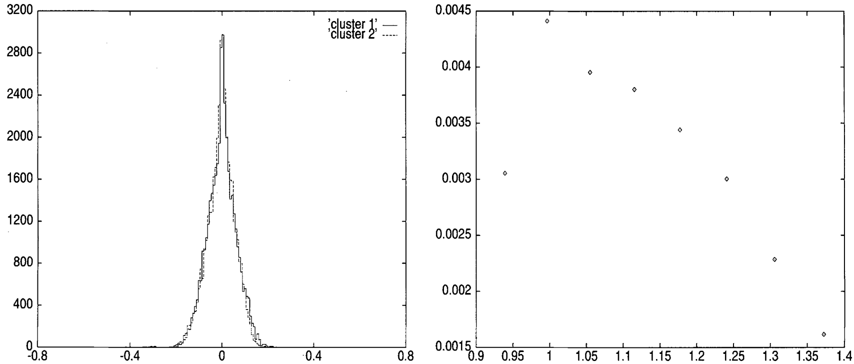

FIGURE 2 Normalized distribution of the action increments after one period T for clusters of 40,000 particles centered at different values of action (left); second moment of the distribution for the action increments as a function of action: 8 different clusters of 40,000 of particles are considered (right).

In Figure 2 (left), we compare the distribution functions of the variable ζ, obtained by normalizing the distribution of the action increments after one period T, for two initial values of the action; we observe that the two distributions are in a very good agreement, so that the hypothesis on the existence of an universal random variable ζ seems to be numerically correct. Then we have computed the function $\epsilon \hat{f}(I, \log \epsilon)$ by using the square root of the second moments of the distribution for the action increments (see Figure 2 (right)). We observe that the $\epsilon \hat{f}(I, \log \epsilon)$ goes towards 0 as we approach the boundary of the region $\mathcal{A}$. We recall that the adiabatic theory cannot be applied at the boundary of the region $\mathcal{A}$, so that we expect some discrepancy between the numerical simulations and the analytical results as we consider actions very near to the boundaries.

In order to check if the random walk (5) can give a quantitative description of the diffusion of the a.i., we have iterated a cluster of 40,000 particles centered in the middle of the region $\mathcal{A}$ for 8 periods of the modulation; in Figure 3 we compare the distributions function of the increments $I_n - I_0$ obtained from the numerical simulations (continuous curve) and the random walk (5) (dashed curve). We have numerically computed the correlations between the increments at different periods, obtaining a correlation of order $\approx 10^{-2}$, which is consistent with the hypothesis of independence of the

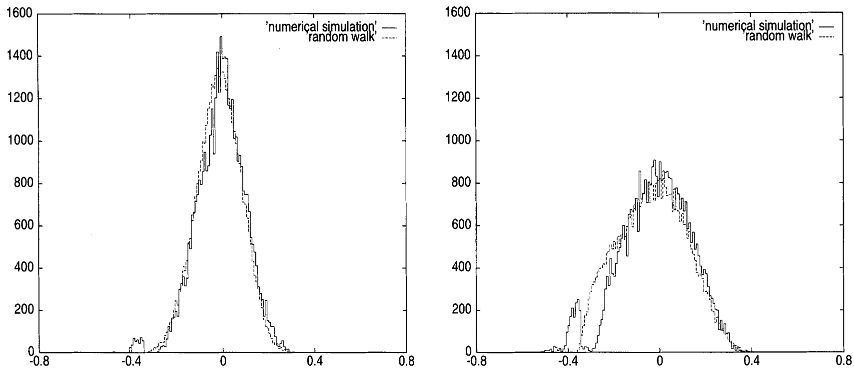

FIGURE 3 Distribution functions of the action increments computed by using a numerical simulation (continuous curve) and the random walk (dotted curve); a cluster of 40,000 particles centered in the middle of the region $\mathcal{A}$ is considered; we have plotted the distribution after 4 periods T (left) and after 10 periods T (right).

increments according to statistical errors. After 4 periods (Figure 3 (left)), we have a very good agreement between the two distributions, whereas after 8 periods (Figure 3 (right)) we see that a small peak appears in the distribution computed by using the numerical simulations, so that the left side of the distributions does not agree any more. The reason of this discrepancy is that in the numerical simulations the orbits have reached the internal boundary of the region $\mathcal{A}$, and the adiabatic description of the dynamics with a trapping and detrapping mechanisms from the resonance region is no more valid: in particular the small peak on the left of the distribution means that there are particles which reach an internal region and remain there for more than one period T, so that their dynamics cannot be described by the random walk (5). This phenomenon could be very relevant in the diffusion of particles from one resonance to another, but its understanding goes beyond the purpose of this paper.

4 CONCLUSIONS

We have shown the possibility to describe the diffusion of the orbits of a symplectic maps in the region spanned by a separatrix when a parameter is slowly modulated, by using a random walk in the space of the adiabatic invariant. In spite of the local character of our analysis, this is the first step to understand the diffusion due to the ripples of quadrupole currents in a particle accelerator. The next step will be to study what happens at the boundary of the region spanned by the separatrix in order to understand how the particles can communicate from one resonance to another.

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