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Holomorphic superspace

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Abstract: We give a twisted holomorphic superspace description for the super-Yang-Mills theory, using holomorphic and antiholomorphic decompositions of twisted spinors. We consider the case of the $\mathcal{N}=1$ super-Yang-Mills theory in four dimensions. We solve the constraints in two different manners, without and with a prepotential. This might have further application for an holomorphic superspace description of $\mathcal{N}=1, d=10$ theory. We also explain how the $\mathcal{N}=1$ and $\mathcal{N}=2$ holomorphic superspaces are related.

Keywords: Supersymmetric gauge theory, Superspaces.

Contents

Introduction

The construction of a superspace path integral formulation for maximal supersymmetry is still an open question. To get a supersymmetry algebra that admits a functional representation on the fields is at the heart of the problem and it seems inevitable in dimensions $d \geqslant 7$ that this implies a breaking of the manifest Lorentz invariance.[1]

Such a functional representation was determined in [3] for the $\mathcal{N}=1, d=10$ theory, by a supersymmetry algebra made of 9 generators and a restriction of the ten-dimensional Lorentz group to $\mathrm{SO}(1,1)\times\mathrm{Spin}(7)\subset\mathrm{SO}(1,9)$. This led us to a reduced superspace with 9 fermionic coordinates. Covariant constraints were found, which do not imply equations of motion. They were solved in function of the fields of the component formalism and analogous results have been obtained for the $\mathcal{N}=2, d=4,8$ cases [4].

A path integral formulation was given for $\mathcal{N}=2, d=4$ in terms of the connection superfields themselves, which required an implementation of the constraints directly in the path integral. On the other hand, dimensional arguments show that the introduction of a prepotential is needed in the higher dimensional cases. Moreover, we expect such higher

[1] In lower dimensions, there is still the possibility to formulate the maximally supersymmetric YM theory in terms of a subalgebra of the whole super-Poincaré algebra, while maintaining manifest Lorentz invariance. In $d=4$ for instance, a harmonic superspace formulation was given preserving 3/4 of the supersymmetries [1], which then received a full quantum description [2].

dimensional cases to be formulated in terms of complex representations of SU(4) $\subset$ Spin(7). Using an SU(4) holomorphic formulation in 8 or 10 dimensions implies a framework that is formally "similar" to the holomorphic formulation in four dimensions that we study in this paper.

We thus display a holomorphic superspace formulation of the simple $\mathcal{N} = 1, d = 4$ super-Yang-Mills theory in its twisted form, by applying the general procedure of [4]. This superspace formulation involves 3 supercharges, a scalar and a $(1,0)$-vector. It completes the previous works for the $\mathcal{N} = 2,\ d = 4$ and $\mathcal{N} = 1,\ d = 10$ twisted superspace with 5 and 9 supercharges, respectively. We also provide a short discussion of the resolution of the constraints in terms of a prepotential. It must be noted that reality conditions are a delicate issue for the $\mathcal{N} = 1\ d = 4$ superspace in holomorphic coordinates. However, this question does not arise in the 10-dimensional formulation, so we will not discuss it here.

The first section defines the notations of the holomorphic $\mathcal{N} = 1, d = 4$ super-Yang-Mills theory and its formulation in components. The second gives its superspace formulation together with the coupling to matter. The third provides a discussion on the alternative formulation in superspace involving a prepotential. The fourth section is devoted to the $\mathcal{N} = 2$ case, both in components and superspace formulations.

1. Holomorphic $\mathcal{N} = 1, d = 4$ Yang-Mills supersymmetry

The twist procedure for the $\mathcal{N} = 1, d = 4$ super-Yang-Mills theory has been described in [5 − 7] in the context of topological field theory. For a hyperKähler manifold, one can use a pair of covariantly constant spinors $\zeta_\pm$, normalized by $\zeta_{-\dot\alpha}\zeta_+^{\dot\alpha} = 1$. They can be defined by $iJ^{m\bar n}\sigma_{m\bar n}\zeta_\pm = \pm\zeta_\pm$, where $J^{m\bar n}$ is the complex structure

$$J^{mn} = 0\,, \qquad J^{\bar m\bar n} = 0\,, \qquad J^{m\bar n} = ig^{m\bar n} \tag{1.1}$$

It permits one to decompose forms into holomorphic and antiholomorphic components. For the gauge connection 1-form A, one has

$$A = A_{(1,0)} + A_{(0,1)} \quad \text{with} \quad JA_{(1,0)} = iA_{(1,0)}\,, JA_{(0,1)} = -iA_{(0,1)} \tag{1.2}$$

and the decomposition of its curvature is $F = dA + AA = F_{(2,0)} + F_{(1,1)} + F_{(0,2)}$. A Dirac spinor decomposes as

$$\lambda_\alpha = \Psi_m\,\sigma^m_{\alpha\dot\alpha}\,\zeta_-^{\dot\alpha} \qquad \lambda^{\dot\alpha} = \eta\,\zeta_+^{\dot\alpha} + \chi_{\bar m\bar n}\,\sigma^{\bar m\bar n\dot\alpha}{}_{\dot\beta}\,\zeta_+^{\dot\beta} \tag{1.3}$$

In the case of a flat manifold, the twist is a mere rewritting of the Euclidean supersymmetric theory, obtained by mapping all spinors onto "holomorphic" and "antiholomorphic" forms after reduction of the Spin(4) covariance to SU(2). Notice that the Euclidean formulation of the $\mathcal{N} = 1$ theory is defined as the analytical continuation of the Minkowski theory. The Euclideanization procedure produces a doubling of the fermions [8], so that the complex fields $\eta, \chi_{\bar m\bar n}, \Psi_m$ are truly mapped onto a Dirac spinor λ. However, the twisted and untwisted actions do not depend on the complex conjugate fields and the path integral can be defined as counting only four real degrees of freedom.[2] The twist also maps the four $\mathcal{N}=1$ supersymmetry generators onto a (0,0)-scalar δ, a (0,1)-vector $\delta_{\bar{m}}$ and a (2,0)-tensor δ_{mn} generators. For formulating the "holomorphic superspace", we will only retain 3 of the four generators, the scalar one δ and the vector one $\delta_{\bar{m}}$. The invariance under δ and $\delta_{\bar{m}}$ has been shown to completely determine the supersymmetric action [7]. Moreover, the absence of anomaly for the tensor symmetry implies that this property can be conserved at the quantum level (at least at any given finite order in perturbation theory) [9].

1.1 Pure $\mathcal{N}=1$ super-Yang-Mills theory

The bosonic fields content of the $\mathcal{N}=1$ pure super-Yang-Mills theory is made of the Yang-Mills field $A = A_m dz^m + A_{\bar{m}} dz^{\bar{m}}$, and an auxiliary scalar field T, while the fermionic fields are one scalar η, one $(1,0)$-form Ψ_m and one $(0,2)$-form $\chi_{\bar{m}\bar{n}}$. The transformation laws of the various fields in twisted representations are

$$\begin{aligned}
\delta\, A_m &= \Psi_m & \delta_{\bar{m}}\, A_n &= g_{\bar{m}n}\eta \\
\delta\, A_{\bar{m}} &= 0 & \delta_{\bar{m}}\, A_{\bar{n}} &= \chi_{\bar{m}\bar{n}} \\
\delta\, \Psi_m &= 0 & \delta_{\bar{m}}\, \Psi_n &= F_{\bar{m}n} - g_{\bar{m}n}T \\
\delta\, \eta &= T & \delta_{\bar{m}}\, \eta &= 0 \\
\delta\, T &= 0 & \delta_{\bar{m}}\, T &= D_{\bar{m}}\eta \\
\delta\, \chi_{\bar{m}\bar{n}} &= F_{\bar{m}\bar{n}} & \delta_{\bar{m}}\, \chi_{\bar{p}\bar{q}} &= 0
\end{aligned} \tag{1.4}$$

The three equivariant generators δ and $\delta_{\bar{m}}$ verify the following off-shell supersymmetry algebra

$$\delta^2 = 0\,, \quad \{\delta, \delta_{\bar{m}}\} = \partial_{\bar{m}} + \delta^{\,gauge}(A_{\bar{m}})\,, \quad \{\delta_{\bar{m}}, \delta_{\bar{n}}\} = 0 \tag{1.5}$$

The action for the pure $\mathcal{N}=1, d=4$ super-Yang-Mills is completely determined by the $\delta, \delta_{\bar{m}}$ invariance. It is given by [5, 7]

$$\mathcal{S}^{\mathcal{N}=1}_{YM} = \int \mathrm{d}^4x\sqrt{g}\,\mathrm{Tr}\,\left(\frac{1}{2}F^{mn}F_{mn} + T(T + iJ^{m\bar{n}}F_{m\bar{n}}) - \chi^{mn}D_m\Psi_n + \eta D^m\Psi_m\right) \tag{1.6}$$

The Wess and Zumino matter multiplet and its coupling to pure $\mathcal{N}=1$ super-Yang-Mills will only be discussed in the framework of superspace.

1.2 Elimination of gauge transformations in the closure relations

The algebra (1.5) closes on gauge transformations, due to the fact that in superspace, where supersymmetry is linearly realized, one breaks the super-gauge invariance to get the transformation laws of the component fields (1.4). To be consistent with supersymmetry, this in turn implies to modify the supersymmetry transformations by adding field dependent gauge transformations, resulting in non linear transformation laws. This super-gauge is analogous to the Wess and Zumino gauge in ordinary superspace and such an algebra is

[2] In the Euclideanization procedure, one also gives up hermiticity of the action, but a "formal complex conjugation" can be defined and extended in the twisted component formalism that restores hermiticity [7]

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usually referred as an algebra of the Wess and Zumino type. In this section, we show how the use of shadow fields makes it possible to remove these gauge transformations, by applying the general formalism of [9] to the $\mathcal{N}=1, d=4$ case. This in turn permits one to make contact with the general solution to the superspace constraints given in the next section.

To introduce the shadows, one replaces the knowledge of the $\delta, \delta_{\bar{m}}$ generators by that of graded differential operators Q and Q_κ, which represent supersymmetry in a nilpotent way. Let ω and $\kappa^{\bar{m}}$ be the commuting scalar and $(0,1)$-vector supersymmetry parameters, respectively. The actions of Q and Q_κ on the (classical) fields are basically supersymmetry transformations as in (1.4) minus a field dependent gauge transformation, that is

$$Q \equiv \omega\delta - \delta^{gauge}(\omega c)\,, \quad Q_\kappa \equiv \delta_\kappa - \delta^{gauge}(i_\kappa \gamma_1) \tag{1.7}$$

with $\delta_\kappa \equiv \kappa^{\bar{m}}\delta_{\bar{m}}$ and i_κ is the contraction operator along $\kappa^{\bar{m}}$. These operators obey $Q^2=0, Q_\kappa^2=0, \{Q, Q_\kappa\}=\omega\mathcal{L}_\kappa$. The scalar shadow field c and the $(0,1)$-form shadow field γ_1 are a generalization of the fields introduced in [10]. They carry a $\mathrm{U}(1)_R$ charge $+1$ and -1, respectively. The action of Q and Q_κ increases it by 1 and -1, respectively. Let moreover $\mathcal{Q} \equiv Q + Q_\kappa$. The property $\mathcal{Q}^2 = \omega\mathcal{L}_\kappa$ fixes the transformation laws of c and γ_1. In fact, the action of $\mathcal{Q}$ on all fields, classical and shadow ones, is given by the following horizontality equation

$$(d+\mathcal{Q}-\omega i_\kappa)(A+\omega c+i_\kappa\gamma_1)+(A+\omega c+i_\kappa\gamma_1)^2 = F+\omega\Psi_{(1,0)}+g(\kappa)\eta+i_\kappa\chi \tag{1.8}$$

together with its Bianchi identity

$$(d+\mathcal{Q}-\omega i_\kappa)(F+\omega\Psi_{(1,0)}+g(\kappa)\eta+i_\kappa\chi)+[A+\omega c+i_\kappa\gamma_1, F+\omega\Psi_{(1,0)}+g(\kappa)\eta+i_\kappa\chi]=0 \tag{1.9}$$

implied by $(d+\mathcal{Q}-\omega i_\kappa)^2=0$. Here and elsewhere $g(\kappa)\equiv g_{m\bar{m}}\kappa^{\bar{m}}dz^m$. The transformation laws (1.4) can indeed be recovered from these horizontality equations by expansion over form degree and $\mathrm{U}(1)_R$ number, modulo gauge transformations with parameters ωc or $i_\kappa\gamma_1$. The auxiliary T scalar field is introduced in order to solve the degenerate equation involving $Q\,g(\kappa)\eta + Q_\kappa\,\omega\Psi$, with $Q\eta=\omega T-[\omega c,\eta]$. Moreover, the fields in the r.h.s of (1.8) can be interpreted as curvature components.

Let us turn to the action of $\mathcal{Q}$ on the shadow fields. For the sake of notational simplicity, we will omit from now on the dependence on the scalar parameter ω. To recover its dependence, it is sufficient to remember that Q increases the $\mathrm{U}(1)_R$ number by one unit. The horizontality conditions imply three equations for the shadow fields

$$Qc=-c^2\,, \quad Q(i_\kappa\gamma_1)+Q_\kappa c+[c,i_\kappa\gamma_1]=i_\kappa A\,, \quad Q_\kappa(i_\kappa\gamma_1)=-(i_\kappa\gamma_1)^2 \tag{1.10}$$

Due to the nilpotency of i_κ, the third equation is defined modulo a contracted $(0,2)$ even form γ_2 of $\mathrm{U}(1)_R$ number -2, that is $Q_\kappa\gamma_1 = i_\kappa\gamma_2+\frac{1}{2}[\gamma_1,i_\kappa\gamma_1]$. To solve the second equation, we introduce an odd $(0,1)$-form c_1 of $\mathrm{U}(1)_R$ number zero. This gives $Q\gamma_1=c_1-[c,\gamma_1]$ and $Q_\kappa c = i_\kappa c_1 + i_\kappa A$. Since we must have $\mathcal{Q}^2=\mathcal{L}_\kappa$ on all fields, we find

$$\begin{aligned} Q\gamma_1 &= c_1-[c,\gamma_1] \\ Q\gamma_2 &= c_2-[c,\gamma_2]-\frac{1}{2}[c_1,\gamma_1] \end{aligned} \qquad \begin{aligned} Qc &= -c^2 \\ Qc_1 &= -[c,c_1] \\ Qc_2 &= -[c,c_2]-c_1^2 \end{aligned} \tag{1.11}$$

and

$$Q_\kappa \gamma_1 = i_\kappa \gamma_2 + \frac{1}{2}[\gamma_1, i_\kappa \gamma_1] \qquad\qquad Q_\kappa c = i_\kappa c_1 + i_\kappa A$$
$$Q_\kappa c_1 = i_\kappa c_2 + \mathscr{L}_\kappa \gamma_1 \tag{1.12}$$
$$Q_\kappa \gamma_2 = \frac{1}{2}[\gamma_1, i_\kappa \gamma_2] - \frac{1}{12}[\gamma_1, [\gamma_1, i_\kappa \gamma_1]] \qquad Q_\kappa c_2 = \mathscr{L}_\kappa \gamma_2 - \frac{1}{2}[\gamma_1, \mathscr{L}_\kappa \gamma_1]$$

with $\mathscr{L}_\kappa \equiv [i_\kappa, d_A]$.

2. $\mathcal{N} = 1$, $d = 4$ holomorphic superspace

2.1 Definition of holomorphic superspace

We now define a " twisted holomorphic " superspace for $\mathcal{N} = 1$ theories by extending the z_m, $z_{\bar m}$ bosonic space with three Grassmann coordinates, one scalar θ and two $(0,1)$ $\vartheta^{\bar p}$ $(m, \bar p = 1, 2)$. The supercharges are given by

$$\mathbb{Q} \equiv \frac{\partial}{\partial \theta} + \vartheta^{\bar m}\partial_{\bar m}, \qquad \mathbb{Q}_{\bar m} \equiv \frac{\partial}{\partial \vartheta^{\bar m}}$$
$$\mathbb{Q}^2 = 0, \qquad \{\mathbb{Q}, \mathbb{Q}_{\bar m}\} = \partial_{\bar m}, \qquad \{\mathbb{Q}_{\bar m}, \mathbb{Q}_{\bar n}\} = 0 \tag{2.1}$$

The covariant superspace derivatives and their anticommuting relations are

$$\nabla \equiv \frac{\partial}{\partial \theta} \qquad \nabla_{\bar m} \equiv \frac{\partial}{\partial \vartheta^{\bar m}} - \theta \partial_{\bar m}$$
$$\nabla^2 = 0 \qquad \{\nabla, \nabla_{\bar m}\} = -\partial_{\bar m} \qquad \{\nabla_{\bar m}, \nabla_{\bar n}\} = 0 \tag{2.2}$$

They anticommute with the supersymmetry generators. They can be gauge-covariantized by the introduction of connection superfields $\mathcal{A} \equiv (\mathbb{C}, \boldsymbol{\Gamma}_{\bar m}, \mathbb{A}_m, \mathbb{A}_{\bar m})$ valued in the adjoint of the gauge group of the theory

$$\hat\nabla \equiv \nabla + \mathbb{C}, \quad \hat\nabla_{\bar m} \equiv \nabla_{\bar m} + \boldsymbol{\Gamma}_{\bar m}, \quad \hat\partial_m \equiv \partial_m + \mathbb{A}_m, \quad \hat\partial_{\bar m} \equiv \partial_{\bar m} + \mathbb{A}_{\bar m} \tag{2.3}$$

The associated covariant superspace curvatures are defined as $(M = m, \bar m)$

$$\begin{aligned} \mathbb{F}_{MN} &\equiv [\hat\partial_M, \hat\partial_N] & \boldsymbol{\Sigma} &\equiv \hat\nabla^2 \\ \boldsymbol{\Psi}_M &\equiv [\hat\nabla, \hat\partial_M] & \mathbb{L}_{\bar m} &\equiv \{\hat\nabla, \hat\nabla_{\bar m}\} + \hat\partial_{\bar m} \\ \boldsymbol{\chi}_{\bar m N} &\equiv [\hat\nabla_{\bar m}, \hat\partial_N] & \bar{\boldsymbol{\Sigma}}_{\bar m \bar n} &\equiv \tfrac{1}{2}\{\hat\nabla_{\bar m}, \hat\nabla_{\bar n}\} \end{aligned} \tag{2.4}$$

so that

$$\begin{aligned} \mathbb{F}_{MN} &= \partial_M \mathbb{A}_N - \partial_N \mathbb{A}_M + [\mathbb{A}_M, \mathbb{A}_N] & \boldsymbol{\Sigma} &= \nabla \mathbb{C} + \mathbb{C}^2 \\ \boldsymbol{\Psi}_M &= \nabla \mathbb{A}_M - \partial_M \mathbb{C} - [\mathbb{A}_M, \mathbb{C}] & \mathbb{L}_{\bar m} &= \nabla \boldsymbol{\Gamma}_{\bar m} + \nabla_{\bar m}\mathbb{C} + \{\boldsymbol{\Gamma}_{\bar m}, \mathbb{C}\} + \mathbb{A}_{\bar m} \\ \boldsymbol{\chi}_{\bar m N} &= \nabla_{\bar m}\mathbb{A}_N - \partial_N \boldsymbol{\Gamma}_{\bar m} - [\mathbb{A}_N, \boldsymbol{\Gamma}_{\bar m}] & \bar{\boldsymbol{\Sigma}}_{\bar m \bar n} &= \nabla_{\{\bar m}\boldsymbol{\Gamma}_{\bar n\}} + \boldsymbol{\Gamma}_{\{\bar m}\boldsymbol{\Gamma}_{\bar n\}} \end{aligned} \tag{2.5}$$

Bianchi identities are given by $\Delta\mathcal{F} = -[\mathcal{A}, \mathcal{F}]$, where Δ and $\mathcal{F}$ denote collectively $(\nabla, \nabla_{\bar m}, \partial_m, \partial_{\bar m})$ and the superspace curvatures. The super-gauge transformations of the super-connection $\mathcal{A}$ and super-curvature $\mathcal{F}$ are

$$\mathcal{A} \to e^{-\alpha}(\Delta + \mathcal{A})e^{\alpha}, \quad \mathcal{F} \to e^{-\alpha}\mathcal{F}e^{\alpha} \tag{2.6}$$

where the gauge superparameter α can be any given general superfield valued in the Lie algebra of the gauge group. The "infinitesimal" gauge transformation is $\delta\mathcal{A} = \Delta\alpha + [\mathcal{A}, \alpha]$.

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2.2 Constraints and their resolution

The superfield interpretation of shadow fields is that they parametrize the general $\boldsymbol{\alpha}$-dependance of the solution of the superspace constraints, while in components they provide differential operators with no gauge transformations in their anticommutation relations. To eliminate superfluous degrees of freedom and make contact with the component formulation, we must impose the following gauge covariant superspace constraints

$$\boldsymbol{\Sigma} = \bar{\boldsymbol{\Sigma}}_{\bar m \bar n} = \mathbb{L}_{\bar m} = 0, \quad \boldsymbol{\chi}_{\bar m n} = \frac{1}{2} g_{\bar m n} \boldsymbol{\chi}^p{}_p \equiv g_{\bar m n} \boldsymbol{\eta} \tag{2.7}$$

They can be solved in terms of component fields as follows. The super-gauge symmetry (2.6) allows one to choose a super-gauge so that every antisymmetric as well as the first component of $\boldsymbol{\Gamma}_{\bar m}$ is set to zero. We also fix the first component $\mathbb{C}|_0 = 0$, so that we are left with the ordinary gauge degree of freedom corresponding to $\boldsymbol{\alpha}|_0$. The constraint $\bar{\boldsymbol{\Sigma}}_{\bar m \bar n} = 0$ then implies that the whole $\boldsymbol{\Gamma}_{\bar m}$ super-connection is zero. The constraint $\boldsymbol{\Sigma} = 0$ implies that one must have

$$\mathbb{C} = \tilde A - \theta \tilde A^2, \quad \tilde A|_0 = 0 \tag{2.8}$$

where $\tilde A$ is a function of the $\vartheta_{\bar m}$. One defines $(\frac{\partial}{\partial \vartheta^{\bar m}} \tilde A)|_0 \equiv -A_{\bar m}$. The constraint $\mathbb{L}_{\bar m} = 0$ implies

$$\mathbb{A}_{\bar m} = -\nabla_{\bar m} \mathbb{C} = -\frac{\partial}{\partial \vartheta^{\bar m}} \tilde A + \theta \Big(\partial_{\bar m} \tilde A - \frac{\partial}{\partial \vartheta^{\bar m}} \tilde A^2 \Big) \tag{2.9}$$

Then, with $(\frac{\partial}{\partial \vartheta^{\bar m}} \frac{\partial}{\partial \vartheta^{\bar n}} \tilde A)|_0 \equiv \chi_{\bar m \bar n}$, we have

$$\boldsymbol{\chi}_{\bar m \bar n} = \nabla_{\bar m} \mathbb{A}_{\bar n} = -\chi_{\bar m \bar n} - \theta F_{\bar m \bar n} \tag{2.10}$$

It follows that

$$\mathbb{C} = -\vartheta^{\bar m} A_{\bar m} - \frac{1}{2} \vartheta^{\bar m} \vartheta^{\bar n} \chi_{\bar m \bar n} - \theta \Big(\frac{1}{2} \vartheta^{\bar m} \vartheta^{\bar n} [A_{\bar m}, A_{\bar n}] \Big) \tag{2.11}$$

It only remains to determine the field component content of $\mathbb{A}_m$. We define $\mathbb{A}_m|_0 \equiv A_m$, $(\frac{\partial}{\partial \theta} \mathbb{A}_m)|_0 \equiv \Psi_m$ and $\boldsymbol{\eta}|_0 \equiv \eta$. The trace constraint on $\boldsymbol{\chi}_{\bar m n} = \nabla_{\bar m} \mathbb{A}_n$ implies

$$\mathbb{A}_m = A_m + \vartheta^{\bar p} g_{\bar p m} \eta + \theta \Big(\Psi_m - \vartheta^{\bar p} (\partial_{\bar p} A_m + g_{\bar p m} T) + \vartheta^{\bar p} \vartheta^{\bar q} g_{m[\bar p} \partial_{\bar q]} \eta \Big) \tag{2.12}$$

We see that the whole physical content in the component fields stand in the θ independant part of the curvature superfield $\boldsymbol{\Psi}_m$,

$$\boldsymbol{\Psi}_m|_{\theta=0} = \Psi_m + \vartheta^{\bar p} (F_{\bar p m} - g_{\bar p m} T) + \frac{1}{2} \vartheta^{\bar p} \vartheta^{\bar q} (2 g_{m[\bar p} D_{\bar q]} \eta - D_m \chi_{\bar p \bar q}) \tag{2.13}$$

The general solution to the constraints can be obtained by a super-gauge transformation, whose superfield parameter has vanishing first component. It can be parametrized in various manners. The following one allows one to recover the transformation laws that we computed in components in the section (1.2) for the full set of fields, including the scalar and vectorial shadows

$$e^{\boldsymbol{\alpha}} = e^{\theta \vartheta^{\bar m} \partial_{\bar m}} e^{\tilde\gamma} e^{\theta \tilde c} = e^{\tilde \gamma} \big(1 + \theta (\tilde c + e^{-\tilde\gamma} \vartheta^{\bar m} \partial_{\bar m} e^{\tilde \gamma}) \big) \tag{2.14}$$

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where $\tilde{\gamma}$ and $\tilde{c}$ are respectively commuting and anticommuting functions of $\vartheta^{\bar{m}}$ and the coordinates $z^m, z^{\bar{m}}$, with the condition $\tilde{\gamma}|_0 = 0$. These fields appear here as the longitudinal degrees of freedom in superspace. The transformation laws given in eqs. (1.4) are recovered for $\tilde{\gamma} = \tilde{c} = 0$, modulo field-dependent gauge-restoring transformations.

2.3 Pure $\mathcal{N} = 1, d = 4$ super-Yang-Mills action

To express the pure super-Yang-Mills action in the twisted superspace, we observe that the Bianchi identity $\nabla\Psi_m + [\mathbb{C}, \Psi_m]$ implies that the gauge invariant function Tr $\Psi_m\Psi_n$ is θ independent. Its component in $\vartheta^{\bar{m}}\vartheta^{\bar{n}}$ can thus be used to write an equivariant action as an integral over the full superspace

$$\mathcal{S}_{\mathrm{EQ}} = \int \mathrm{d}\vartheta^m \mathrm{d}\vartheta^n \ \mathrm{Tr}\ \Big(\mathbf{\Psi}_m \mathbf{\Psi}_n\Big) = \int \mathrm{d}\vartheta^m \mathrm{d}\vartheta^n \ \mathrm{d}\theta\, \mathrm{Tr}\ \Big(\mathbb{A}_m\, \mathbf{\Psi}_n - \mathbb{C}\partial_m \mathbb{A}_n\Big)$$
$$= \int \mathrm{d}\vartheta^m \mathrm{d}\vartheta^n \ \mathrm{d}\theta\, \mathrm{Tr}\ \Big(\mathbb{A}_m\, \nabla\mathbb{A}_n - \mathbb{C}\mathbb{F}_{mn}\Big) \qquad (2.15)$$

Berezin integration is defined as $\int \mathrm{d}\vartheta^m \mathrm{d}\vartheta^n\, \mathbb{X}_{mn} \equiv -\frac{1}{2}\frac{\partial}{\partial\vartheta_m}\frac{\partial}{\partial\vartheta_n}\mathbb{X}_{mn}$, where $\mathbb{X}_{mn}$ is a $(2,0)$-form superfield. By use of the identity Tr $(-\frac{1}{2}F^m{}_n F^n{}_m + \frac{1}{2}F^m{}_m F^n{}_n) =$ Tr $(\frac{1}{2}F_{mn}F^{mn}) +$ "surface term", one recovers after implementation of the constraints the twisted form of the $\mathcal{N} = 1$ supersymmetric Yang-Mills action (1.6), up to a total derivative [11].

Here, the constraints (2.7) have been solved in terms of component fields without using a prepotential. They must be implemented directly in the path integral when one quantizes the theory, which run over the unconstrained potentials. This is performed by the following superspace integral depending on Lagrange multipliers superfields

$$\mathcal{S}_C = \int \mathrm{d}\vartheta^m \mathrm{d}\vartheta^n\, \mathrm{d}\theta\, \Omega_{mn} \mathrm{Tr}\ \Big(\bar{\mathbb{B}}\, \mathbf{\Sigma} + \bar{\mathbb{B}}^{\bar{m}\bar{n}}\, \Sigma_{\bar{m}\bar{n}} + \bar{\mathbb{K}}^{\bar{m}}\, \mathbb{L}_{\bar{m}} + \bar{\mathbf{\Psi}}^{m\bar{n}}\, \chi_{m\bar{n}}\Big) \qquad (2.16)$$

where $\bar{\mathbb{B}}^{\bar{m}\bar{n}}$ is symmetric and $\bar{\mathbf{\Psi}}^{m\bar{n}}$ is traceless. The resolution of the constraints is such that the formal integration over the above auxiliary superfields gives rise to the non-manifestly supersymmetric formulation of the theory in components, without introducing any determinant contribution in the path-integral. However, due to the Bianchi identities, $\bar{\mathbb{B}}$, $\bar{\mathbb{B}}^{\bar{m}\bar{n}}$ and $\bar{\mathbf{\Psi}}^{m\bar{n}}$ admit a large class of zero modes that must be considered in the manifestly supersymmetric superspace Feynman rules. They can be summarized by the following invariance of the action

$$\delta^{\mathrm{zero}}\, \bar{\mathbb{B}} = \hat{\nabla}\, \lambda\,, \quad \delta^{\mathrm{zero}}\, \bar{\mathbb{B}}^{\bar{m}\bar{n}} = \hat{\nabla}_{\bar{p}}\, \lambda^{(\bar{m}\bar{n}\bar{p})} - \partial_p\, \lambda^{p\bar{m}\bar{n}}\,, \quad \delta^{\mathrm{zero}}\, \bar{\mathbf{\Psi}}^{m\bar{n}} = \hat{\nabla}_{\bar{p}}\, \lambda^{m\bar{n}\bar{p}} \qquad (2.17)$$

where $\lambda^{(\bar{m}\bar{n}\bar{p})}$ is completely symmetric and $\lambda^{m\bar{n}\bar{p}}$ is traceless in its $m\bar{n}$ indices and symmetric in $\bar{n}\bar{p}$. This feature is peculiar to twisted superspace and the appearance of this infinitely degenerated gauge symmetry was already underlined in [4] and is detailed in [9]. We will not go in further details in this paper, and let the reader see in [9] how it may be possible to deal with this technical subtelty by use of suitable projectors in superspace.

One needs a gauge fixing-action $\mathcal{S}_{\mathrm{GF}}$. It is detailed for the analogous $\mathcal{N} = 2$ twisted superspace in [4, 9] as a superspace generalization of the Landau gauge fixing action in components. One also needs a gauge-fixing part $\mathcal{S}_{\rm CGF}$ for the action of constraints (2.16), and the total action for $\mathcal{N}=1, d=4$ super-Yang-Mills in holomorphic superspace reads

$$\mathcal{S}^{\mathcal{N}=1}_{SYM} = \mathcal{S}_{\rm EQ} + \mathcal{S}_C + \mathcal{S}_{\rm GF} + \mathcal{S}_{\rm CGF} \tag{2.18}$$

2.4 Wess and Zumino model

We then turn to the matter content of the theory and consider as a first step the Wess and Zumino superfield formulation. We introduce two scalar superfields $\mathbf{\Phi}$ and $\bar{\mathbf{\Phi}}$, and one $(2,0)$-superfield $\bar{\boldsymbol{\chi}}_{mn}$. These superfields correspond to the scalar chiral and anti-chiral superfields of ordinary superspace. They take their values in arbitrary representations of the gauge group. The chirality constraints of the super-Poincaré superspace are replaced by the following constraints

$$\nabla \mathbf{\Phi} = 0, \quad \nabla_{\bar m} \bar{\mathbf{\Phi}} = 0, \quad \nabla_{\bar p} \bar{\boldsymbol{\chi}}_{mn} = 2\, g_{\bar p [m} \partial_{n]} \bar{\mathbf{\Phi}} \tag{2.19}$$

We define the following component fields corresponding to the unconstrained components of the superfields as $\bar{\boldsymbol{\chi}}_{mn}|_0 \equiv \bar\chi_{mn}, (\frac{\partial}{\partial\theta}\bar{\boldsymbol{\chi}}_{mn})|_0 \equiv T_{mn}, \bar{\mathbf{\Phi}}|_0 \equiv \bar\Phi, (\frac{\partial}{\partial\theta}\bar{\mathbf{\Phi}})|_0 \equiv \bar\eta, \mathbf{\Phi}|_0 \equiv \Phi, (\frac{\partial}{\partial\vartheta^m}\mathbf{\Phi})|_0 \equiv -\bar\Psi_{\bar m}, (\frac{\partial}{\partial\vartheta^m}\frac{\partial}{\partial\vartheta^n}\mathbf{\Phi})|_0 \equiv \bar T_{\bar m\bar n}$. We then deduce

$$\begin{aligned}
\mathbf{\Phi} &= \Phi - \vartheta^{\bar m}\bar\Psi_{\bar m} - \frac{1}{2}\vartheta^{\bar m}\vartheta^{\bar n}\bar T_{\bar m\bar n} \\
\bar{\mathbf{\Phi}} &= \bar\Phi + \theta\Big(\bar\eta - \vartheta^{\bar m}\partial_{\bar m}\bar\Phi\Big) \\
\bar{\boldsymbol{\chi}}_{mn} &= \bar\chi_{mn} + 2\vartheta_{[m}\partial_{n]}\bar\Phi + \theta\Big(T_{mn} + \vartheta^{\bar m}(-\partial_{\bar m}\bar\chi_{mn} + 2g_{\bar m[m}\partial_{n]}\bar\eta) + \vartheta_m\vartheta_n\partial_{\bar p}\partial^{\bar p}\bar\Phi\Big)
\end{aligned} \tag{2.20}$$

The free Wess and Zumino action can be written as

$$\begin{aligned}
\mathcal{S}_{\rm WZ} &= \int {\rm d}\vartheta^m {\rm d}\vartheta^n\, d\theta\,\Big(-\mathbf{\Phi}\,\bar{\boldsymbol{\chi}}_{mn}\Big) \\
&= \int {\rm d}^4x\sqrt{g}\,{\rm Tr}\,\Big(\frac{1}{2}T^{\bar m\bar n}\bar T_{\bar m\bar n} - \chi^{\bar m\bar n}\partial_{\bar m}\bar\Psi_{\bar n} + \bar\eta\partial_m\bar\Psi^m - \bar\Phi\partial_m\partial^m\Phi\Big)
\end{aligned} \tag{2.21}$$

2.5 Gauge coupling to matter

In order to get the matter coupling to the pure super-Yang-Mills action, we covariantize the constraints. This can be shown to be consistent with those of (2.7). We thus have

$$\hat\nabla \mathbf{\Phi} = 0, \quad \hat\nabla_{\bar m} \bar{\mathbf{\Phi}} = 0, \quad \hat\nabla_{\bar p} \bar{\boldsymbol{\chi}}_{mn} = 2\, g_{\bar p [m} \hat\partial_{n]} \bar{\mathbf{\Phi}} \tag{2.22}$$

In order to fulfil these new constraints, we modify the matter superfields as follows

$$\begin{aligned}
\mathbf{\Phi} &= \Phi - \vartheta^{\bar m}\bar\Psi_{\bar m} - \frac{1}{2}\vartheta^{\bar m}\vartheta^{\bar n}T_{\bar m\bar n} + \theta\Big(\vartheta^{\bar m}A_{\bar m}\Phi + \vartheta^{\bar m}\vartheta^{\bar n}\Big(\frac{1}{2}\chi_{\bar m\bar n}\Phi - A_{\bar m}\Psi_{\bar n}\Big)\Big) \\
\bar{\boldsymbol{\chi}}_{mn} &= \bar\chi_{mn} + 2\vartheta_{[m}D_{n]}\bar\Phi + \vartheta_m\vartheta_n\eta\bar\Phi + \theta\Big(T_{mn} + \vartheta^{\bar m}(-\partial_{\bar m}\bar\chi_{mn} + 2g_{\bar m[m}D_{n]}\bar\eta \\
&\qquad -2g_{\bar m[m}\Psi_{n]}\bar\Phi) + \vartheta_m\vartheta_n(\partial_{\bar p}D^{\bar p}\bar\Phi + \eta\bar\eta - h\bar\phi)\Big)
\end{aligned} \tag{2.23}$$

The total action of super-Yang-Mills coupled to matter then reads

$$\mathcal{S}_{SYM+Matter} = \int {\rm d}\vartheta^m {\rm d}\vartheta^n\, d\theta\,\Big({\rm Tr}\,(\mathbb{A}_m\,\nabla\mathbb{A}_n - \mathbb{C}\mathbb{F}_{mn}) - \mathbf{\Phi}\,\bar{\boldsymbol{\chi}}_{mn}\Big) \tag{2.24}$$

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which matches that of [7]. A WZ superpotential can be added in the twisted superspace formalism as the sum of two terms, one which is written as an integral over $\mathrm{d}\theta$ and the other as an integral over $\mathrm{d}\vartheta^m \mathrm{d}\vartheta^n$.

3. Prepotential

We now turn to the study of a twisted superspace formulation for the pure $\mathcal{N}=1$ super-Yang-Mills theory that involves a prepotential. It is sufficient to consider here the abelian case. The super-connections $(\mathbb{C}, \boldsymbol{\Gamma}_{\bar{m}}, \mathbb{A}_{\bar{m}}, \mathbb{A}_m)$ count altogether for $(1+2+2+2)\cdot 2^3 = 56$ degrees of freedom, 8 of which are longitudinal degrees of freedom associated to the gauge invariance in superspace (2.6). The constraints (2.7) for $\boldsymbol{\Sigma}$ and $\boldsymbol{\Sigma}_{\bar{m}\bar{n}}$ can be solved by the introduction of unconstrained prepotentials as

$$\mathbb{C} = \nabla \mathbb{D}\,, \quad \boldsymbol{\Gamma}_{\bar{m}} = \nabla_{\bar{m}} \Delta \tag{3.1}$$

which reduces the 16 degrees of freedom in $\boldsymbol{\Gamma}_{\bar{m}}$ to the 8 degrees of freedom in Δ. Gauge invariance for the prepotentials now reads

$$\mathbb{D} \to \mathbb{D} + \boldsymbol{\alpha}\,, \quad \Delta \to \Delta + \boldsymbol{\alpha} \tag{3.2}$$

Owning to their definition, the prepotentials are not uniquely defined. Indeed, they can be shifted by the additional transformations $\mathbb{D} \to \mathbb{D} + \mathbf{S}$ and $\Delta \to \Delta - \mathbf{T}$, where $\mathbf{S}$, $\mathbf{T}$ obey $\nabla \mathbf{S} = 0$ respectively $\nabla_{\bar{m}} \mathbf{T} = 0$. The constraint $\mathbb{L}_{\bar{m}} = 0$ implies that $\mathbb{A}_{\bar{m}}$ can be expressed in terms of the prepotentials, $\mathbb{A}_{\bar{m}} = -\nabla\nabla_{\bar{m}}\Delta - \nabla_{\bar{m}}\nabla\mathbb{D}$. Its gauge invariance is given by

$$\mathbb{A}_{\bar{m}} \to -\nabla\nabla_{\bar{m}}(\Delta + \boldsymbol{\alpha} - \mathbf{T}) - \nabla_{\bar{m}}\nabla(\mathbb{D} + \boldsymbol{\alpha} + \mathbf{S}) = \mathbb{A}_{\bar{m}} + \partial_{\bar{m}}\boldsymbol{\alpha} \tag{3.3}$$

We now perform a gauge choice and choose $\boldsymbol{\alpha} = -\Delta$ so that we fix $\boldsymbol{\Gamma}_{\bar{m}} = 0$. The remaining gauge invariance is then given by $\mathbb{D} \to \mathbb{D} + \mathbf{S} + \mathbf{T}$. The last constraint $\boldsymbol{\chi}_{\bar{m}n} = g_{\bar{m}n}\boldsymbol{\eta}$ is then solved by introducing

$$\mathbb{A}_m = -\nabla^n \mathbb{P}_{nm} \tag{3.4}$$

so that $\boldsymbol{\chi}_{\bar{m}n} = \nabla_{\bar{m}}\mathbb{A}_n = \frac{1}{2} g_{\bar{m}n} \nabla_{\bar{p}}\nabla_{\bar{q}}\mathbb{P}^{\bar{p}\bar{q}}$. Although the residual gauge invariance is well established in the case when we consider the theory with the full set of generators, it is still unclear how exactly it happens in reduced superspace. But as a matter of fact, we are left with the two unconstrained prepotentials $\mathbb{D}$ and $\mathbb{P}_{mn}$, counting for 16 degrees of freedom, which permits one to write the classical action. We consider the curvature $\boldsymbol{\Psi}_m$, which reads in terms of the prepotentials as

$$\boldsymbol{\Psi}_m = \nabla\nabla^n \mathbb{P}_{mn} - \partial_m \nabla \mathbb{D} \tag{3.5}$$

It obeys trivialy the Bianchi identity $\nabla\boldsymbol{\Psi}_m = 0$, and its first component is a $(1,0)$-vector of canonical dimension $3/2$, that we identify with $\Psi_m - \partial_m c$ of the previous section, when the super-gauge invariance is restored. Since the first component of the superfield (3.5) is the same as that of $\boldsymbol{\Psi}_m$ in the previous section, both superfields are equal. It follows that the classical $\mathcal{N}=1, d=4$ super-Yang-Mills action can also be written as an integral over the full superspace, in function of an unconstrained prepotential

$$\mathcal{S}_{\rm EQ}=\int \mathrm{d}\vartheta^m \mathrm{d}\vartheta^n \;\mathrm{Tr}\,\Big(\boldsymbol{\Psi}_m\boldsymbol{\Psi}_n\Big)=\int \mathrm{d}\vartheta^m \mathrm{d}\vartheta^n\, \mathrm{d}\theta\,\mathrm{Tr}\,\Big(\mathbb{P}_{mp}\nabla^p\nabla\nabla^q\mathbb{P}_{nq}+\mathbb{P}_{mp}\partial_n\nabla^p\nabla\mathbb{D}\Big) \quad (3.6)$$

In fact, it corresponds to the twisted version of the super-Poincaré superspace action, once the tensorial coordinate has been eliminated. We are currently studying how this formulation could be extended to a $\mathcal{N}=2, d=8$ superspace in a SU(4) formulation [9].

4. $\mathcal{N}=2$, $d=4$ holomorphic Yang-Mills supersymmetry

We now define the holomorphic formulation of the $\mathcal{N}=2$, $d=4$ Yang-Mills supersymmetry and see how it can be decomposed into that of the $\mathcal{N}=1$ supersymmetry. We first focus on the component formulation and afterwards we give its superspace version. The latter will involve 5 fermionic coordinates, as compared to the 3 fermionic coordinates of the $\mathcal{N}=1$ twisted superspace.

4.1 Component formulation

The component formulation of $\mathcal{N}=2, d=4$ super-Yang-Mills in terms of complex representations has been discussed in [6, 12, 13]. We consider a reduction of the euclidean rotation group $\mathrm{SU}(2)_L\times \mathrm{SU}(2)_R$ to $\mathrm{SU}(2)_L\times \mathrm{U}(1)_R$, with $\mathrm{U}(1)_R\subset \mathrm{SU}(2)_R$. The two dimensional representation of $\mathrm{SU}(2)_R$ decomposes under $\mathrm{U}(1)_R$ as a sum of one dimensional representations with opposite charges. In particular, the scalar and vector supersymmetry generators decompose as $\boldsymbol{\delta}=\delta+\bar\delta$ and $\boldsymbol{\delta}_K=\delta_\kappa+\bar\delta_{\bar\kappa}$, where $\bar\kappa$ is the complex conjugate of κ, so that $|\kappa|^2=i_{\bar\kappa}g(\kappa)$. The subsets (δ,δ_κ) and $(\bar\delta,\bar\delta_\kappa)$ form two $\mathcal{N}=1$ subalgebras of the $\mathcal{N}=2$ supersymmetry, (δ,δ_κ) being related to those of the previous sections

$$\begin{aligned} &\delta^2=0\,, && \{\delta,\bar\delta\}=\delta^{\,gauge}(\Phi)\,, && \bar\delta^2=0\\ &\{\delta_m,\delta_n\}=0\,, && \{\delta_m,\delta_{\bar m}\}=g_{m\bar m}\delta^{\,gauge}(\bar\Phi)\,, && \{\delta_{\bar m},\delta_{\bar n}\}=0 \end{aligned} \quad (4.1)$$

Quite concretely, the transformation laws for pure $\mathcal{N}=2$ super-Yang-Mills can be obtained from the holomorphic and antiholomorphic decomposition of the horizontality equation in $\mathrm{SU}(2)\times\mathrm{SU}(2)'$ twisted formulation [7]. This equation involves the graded differential operator

$$\mathcal{Q}\equiv Q+Q_K \quad (4.2)$$

which verify $(d+Q+Q_K-\varpi i_K)^2=0$. One defines

$$\mathcal{A}=A_{(1,0)}+A_{(0,1)}+\varpi c+i_K\gamma_1 \quad (4.3)$$

where c is a scalar shadow field and $\gamma_1=\gamma_{1\,(1,0)}+\gamma_{1\,(0,1)}$ involves "holomorphic" and "antiholomorphic" vector shadow fields. Q and Q_K are constructed out of the five $\boldsymbol{\delta}$, δ_κ and $\bar\delta_{\bar\kappa}$ supersymmetry generators with shadow dependent gauge transformations

$$Q\equiv\varpi\boldsymbol{\delta}-\delta^{\,gauge}(\varpi c)\,,\quad Q_K\equiv\boldsymbol{\delta}_K-\delta^{\,gauge}(i_K\gamma_1) \quad (4.4)$$

An antiselfdual 2-form splits in holomorphic coordinates as $\chi \to (\chi_{(2,0)}, \chi_{(0,2)}, \chi_{(1,1)})$ where $\chi_{(1,1)}$ is subjected to the condition $\chi_{m\bar{n}} = \frac{1}{2} J_{m\bar{n}} J^{p\bar{q}} \chi_{p\bar{q}}$. We thus define a scalar χ as $\chi_{m\bar{n}} = g_{m\bar{n}}\chi$ and the holomorphic horizontality equation can be written as

$$\begin{aligned}\mathcal{F} &\equiv \mathcal{Q}\mathcal{A} + \mathcal{A}\mathcal{A} \\ &= F_{(2,0)} + F_{(1,1)} + F_{(0,2)} + \varpi\Psi + g(\kappa)(\eta+\chi) + g(\bar{\kappa})(\eta-\chi) \\ &\quad + i_{\bar{\kappa}}\chi_{(2,0)} + i_{\kappa}\chi_{(0,2)} + \varpi^2\Phi + |\kappa|^2\bar{\Phi}\end{aligned} \tag{4.5}$$

with Bianchi identity

$$\mathcal{Q}\mathcal{F} = -[\mathcal{A}, \mathcal{F}] \tag{4.6}$$

By expansion over form degrees and $\mathrm{U}(1)_R$ number, one gets transformation laws for $\mathcal{N}=2$ super-Yang-Mills in holomorphic and antiholomorphic components. In order to recover the transformation laws for $\mathcal{N}=1$ supersymmetry (1.4) together with those of the matter multiplet in the adjoint representation $\varphi = (T_{\bar{m}\bar{n}}, T_{mn}, \Psi_{\bar{m}}, \chi_{mn}, \bar{\eta}, \Phi, \bar{\Phi})$, one can proceed as follows. One first derive from (4.4), (4.5), (4.6) the transformation laws for the physical fields under the equivariant operator δ_K. One then obtains the action of the $\mathcal{N}=1$ vector generator by restricting the constant vector K to its antiholomorphic component $\kappa^{\bar{m}}$, so that $\delta_K \to \kappa^{\bar{m}}\delta_{\bar{m}}$. Finally, the action of the holomorphic component δ of the scalar operator $\boldsymbol{\delta}$ on the various fields is completely determined by the requirement that it satisfies the $\mathcal{N}=1$ subalgebra $\delta^2=0$ and $\{\delta, \delta_{\bar{m}}\} = \partial_{\bar{m}} + \delta^{\,gauge}(A_{\bar{m}})$.

4.2 $\mathcal{N}=2$ holomorphic superspace

We extend the superspace of the $\mathcal{N}=1$ case into one with complex bosonic coordinates z_m, $z_{\bar{m}}$ and five Grassmann coordinates, one scalar θ, two "holomorphic" ϑ^m and two "antiholomorphic" $\vartheta^{\bar{m}}$ $(m, \bar{m} = 1,2)$. The supercharges are now given by

$$\mathbb{Q} \equiv \frac{\partial}{\partial\theta} + \vartheta^m\partial_m + \vartheta^{\bar{m}}\partial_{\bar{m}}, \qquad \mathbb{Q}_m \equiv \frac{\partial}{\partial\vartheta^m}, \qquad \mathbb{Q}_{\bar{m}} \equiv \frac{\partial}{\partial\vartheta^{\bar{m}}} \tag{4.7}$$

They verify

$$\mathbb{Q}^2 = 0, \qquad \{\mathbb{Q}, \mathbb{Q}_M\} = \partial_M, \qquad \{\mathbb{Q}_M, \mathbb{Q}_N\} = 0 \tag{4.8}$$

with $M = m, \bar{m}$. The covariant derivatives and their anticommuting relations are

$$\begin{aligned}&\nabla \equiv \frac{\partial}{\partial\theta} && \nabla_M \equiv \frac{\partial}{\partial\vartheta^M} - \theta\partial_M && \\ &\nabla^2 = 0 && \{\nabla, \nabla_M\} = -\partial_M && \{\nabla_M, \nabla_N\} = 0\end{aligned} \tag{4.9}$$

They anticommute with the supersymmetry generators. The gauge covariant superderivatives are

$$\hat{\nabla} \equiv \nabla + \mathbb{C}, \quad \hat{\nabla}_M \equiv \nabla_M + \Gamma_M, \quad \hat{\partial}_M \equiv \partial_M + \mathbb{A}_M \tag{4.10}$$

from which we define the covariant superspace curvatures

$$\begin{aligned}\mathbb{F}_{MN} &\equiv [\hat{\partial}_M, \hat{\partial}_N] & \boldsymbol{\Sigma} &\equiv \hat{\nabla}^2 \\ \boldsymbol{\Psi}_M &\equiv [\hat{\nabla}, \hat{\partial}_M] & \mathbb{L}_M &\equiv \{\hat{\nabla}, \hat{\nabla}_M\} + \hat{\partial}_M \\ \boldsymbol{\chi}_{MN} &\equiv [\hat{\nabla}_M, \hat{\partial}_N] & \bar{\boldsymbol{\Sigma}}_{MN} &\equiv \tfrac{1}{2}\{\hat{\nabla}_M, \hat{\nabla}_N\}\end{aligned} \tag{4.11}$$

They are subjected to Bianchi identities and the super-gauge transformations of the various connections and curvatures follow analogously to the $\mathcal{N}=1$ case.

The constraints for the $\mathcal{N}=2$ case are

$$\mathbb{L}_M = 0, \quad \bar{\boldsymbol{\Sigma}}_{mn} = \bar{\boldsymbol{\Sigma}}_{\bar{m}\bar{n}} = 0, \quad \bar{\boldsymbol{\Sigma}}_{m\bar{n}} = \frac{1}{2} g_{m\bar{n}} \bar{\boldsymbol{\Sigma}}_p{}^p, \quad \boldsymbol{\chi}_{\bar{m}n} = \frac{1}{2} g_{\bar{m}n} \boldsymbol{\chi}_p^p \tag{4.12}$$

Their solution can be directly deduced from [4, 9], by decomposition into holomorphic and antiholomorphic coordinates. The full physical vector supermultiplet now stands in the scalar odd connexion, which in the Wess-Zumino-like gauge is

$$\mathbb{C} = \tilde{A} + \theta(\tilde{\Phi} - \tilde{A}^2) \tag{4.13}$$

where

$$\begin{aligned}\tilde{A} &= -\vartheta^m A_m - \vartheta^{\bar{m}} A_{\bar{m}} - \frac{1}{2}\vartheta^m \vartheta^n \bar{\chi}_{mn} - \frac{1}{2}\vartheta^{\bar{m}}\vartheta^{\bar{n}} \chi_{\bar{m}\bar{n}} - \vartheta^m \vartheta_m \chi \\ &+ \frac{1}{2}\vartheta^n \vartheta_n (\vartheta^m D_m \bar{\Phi} + \vartheta^{\bar{m}} D_{\bar{m}} \bar{\Phi}) - \vartheta^m \vartheta_m \vartheta^{\bar{m}} \vartheta_{\bar{m}} [\bar{\Phi}, \eta]\end{aligned} \tag{4.14}$$

There is an analogous decomposition for $\tilde{\Phi}$ in [4]. The general solution to the constraints is recovered by the following super-gauge-transformation

$$e^{\boldsymbol{\alpha}} = e^{\theta(\vartheta^{\bar{m}}\partial_{\bar{m}} + \vartheta^m \partial_m)} e^{\tilde{\gamma}} e^{\theta\tilde{c}} = e^{\tilde{\gamma}}\big(1 + \theta(\tilde{c} + e^{-\tilde{\gamma}}(\vartheta^{\bar{m}}\partial_{\bar{m}} + \vartheta^m \partial_m)e^{\tilde{\gamma}})\big) \tag{4.15}$$

where $\tilde{\gamma}$ and $\tilde{c}$ are respectively commuting and anticommuting functions of $\vartheta^{\bar{m}}, \vartheta^m$ and the coordinates $z^m, z^{\bar{m}}$, with the condition $\tilde{\gamma}|_0 = 0$. Transformation laws in components can then be recovered, which match those in (4.5) and (4.6).

The action is then given by

$$\mathcal{S}^{\mathcal{N}=2}_{\mathrm{SYM}} = \int \mathrm{d}^4\vartheta \, \mathrm{d}\theta \, \mathrm{Tr} \left(\mathbb{C}\nabla\mathbb{C} + \frac{2}{3}\mathbb{C}^3 \right) \tag{4.16}$$

To recover the previous results of the $\mathcal{N}=1$ super-Yang-Mills theory with matter in the adjoint representation, one first integrates (4.16) over the θ variable, which gives

$$\mathcal{S}^{\mathcal{N}=2}_{\mathrm{SYM}} = \int \mathrm{d}^4\vartheta \, \mathrm{Tr} \, \boldsymbol{\Sigma}\boldsymbol{\Sigma} \tag{4.17}$$

Further integration over the ϑ^m variables, or equivalently derivation with ∇_m, yields two terms which are both invariant under the $\mathcal{N}=1$ scalar supersymmetry generator. In turn, they can be expressed as an integral over the full twisted $\mathcal{N}=1$ superspace, which yields the superspace action given in (2.24) with matter in the adjoint representation.

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References

[1] A. Galperin, E. Ivanov, S. Kalitsyn, V. Ogievetsky and E. Sokatchev, *Unconstrained off-shell* $\mathcal{N}=3$ *supersymmetric Yang-Mills theory*, *Class. and Quant. Grav.* **2** (1985) 155.

[2] F. Delduc and J. McCabe, *The quantization of* $\mathcal{N}=3$ *super Yang-Mills off-shell in harmonic superspace*, *Class. and Quant. Grav.* **6** (1989) 233.

[3] L. Baulieu, N.J. Berkovits, G. Bossard and A. Martin, *Ten-dimensional super-Yang-Mills with nine off-shell supersymmetries*, *Phys. Lett.* **B 658** (2008) 249 [arXiv:0705.2002].

[4] L. Baulieu, G. Bossard and A. Martin, *Twisted superspace*, *Phys. Lett.* **B 663** (2008) 275 [arXiv:0802.1980].

[5] A. Johansen, *Twisting of* $\mathcal{N}=1$ *SUSY gauge theories and heterotic topological theories*, *Int. J. Mod. Phys.* **A 10** (1995) 4325 [hep-th/9403017].

[6] E. Witten, *Supersymmetric Yang-Mills theory on a four manifold*, *J. Math. Phys.* **35** (1994) 5101 [hep-th/9403195].

[7] L. Baulieu and G. Bossard, *Reconstruction of* $\mathcal{N}=1$ *supersymmetry from topological symmetry*, *Phys. Lett.* **B 632** (2006) 138 [hep-th/0507004].

[8] H. Nicolai, *A possible constructive approach to (Super* ϕ^3*) in four-dimensions.* 1. *Euclidean formulation of the model*, *Nucl. Phys.* **B 140** (1978) 294.

[9] L. Baulieu, G. Bossard and A. Martin, in preparation.

[10] L. Baulieu, G. Bossard and S.P. Sorella, *Shadow fields and local supersymmetric gauges*, *Nucl. Phys.* **B 753** (2006) 273 [hep-th/0603248].

[11] A. Galperin and O. Ogievetsky, *Holonomy groups, complex structures and* $D=4$ *topological Yang-Mills theory*, *Commun. Math. Phys.* **139** (1991) 377.

[12] M. Mariño, *The geometry of supersymmetric gauge theories in four dimensions*, hep-th/9701128.

[13] J.-S. Park, $\mathcal{N}=2$ *topological Yang-Mills theory on compact Kähler surfaces*, *Commun. Math. Phys.* **163** (1994) 113 [hep-th/9304060].

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