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High energy factorization in nucleus-nucleus collisions. II. Multigluon correlations

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We extend previous results from the preceding paper on factorization in high energy nucleus-nucleus collisions by computing the inclusive multigluon spectrum to next-to-leading order. The factorization formula is strictly valid for multigluon emission in a slice of rapidity of width $\Delta Y \leq \alpha_s^{-1}$. Our results shows that often neglected disconnected graphs dominate the inclusive multigluon spectrum, and are crucial in order to achieve factorization for this quantity. These results provide a dynamical framework for the Glasma flux tube picture of the striking "ridge"-like correlation seen in heavy ion collisions.

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I. INTRODUCTION

In the previous article, henceforth referred to as Paper I [1], we investigated the formal basis for the application of the color glass condensate (CGC) framework [2–8] to the collision of two high energy nuclei. In Paper I, we focused on the formalism to compute the single gluon inclusive spectrum in the leading log x approximation. The main result of Paper I is a proof of the fact that the leading logarithms of $1/x_{1,2}$ that arise in all order loop corrections to this spectrum can be factorized in the distributions of color sources $W[\rho_{1,2}]$ in each of the two nuclei, evolved with the JIMWLK equation [9–16] from the beam rapidity to the rapidity of the measured gluon. One obtains for the single inclusive gluon distribution the result

$$\left\langle\frac{dN}{d^3\boldsymbol{p}}\right\rangle_{\rm LLog} = \int[D\rho_1][D\rho_2]W_{Y_{\rm beam}-Y}[\rho_1] \times W_{Y_{\rm beam}+Y}[\rho_2]\frac{dN}{d^3\boldsymbol{p}}\bigg|_{\rm LO}. \quad (1)$$

The W functionals are universal properties of the nuclear wave functions at high energies and (in analogy to the parton distribution functions of collinear factorization) can be extracted from deep inelastic scattering or proton-nucleus scattering experiments off nuclei. The inclusive single gluon spectrum $(dN/d^3\boldsymbol{p})_{\rm LO}$ that appears under the integral in the right-hand side (r.h.s.) is the leading order spectrum corresponding to one configuration of the sources $\rho_{1,2}$—it is obtained by solving the classical Yang-Mills equations for this fixed distribution of sources. This factorization theorem allows for considerable predictive power by relating measurements in a variety of scattering processes. It should be particularly useful at the CERN LHC, where the rapidity reach in proton-nucleus and nucleus-nucleus collisions will be considerable and the effects of energy evolution of the distribution of color sources clearly visible.

The derivation of the factorized expression in Eq. (1) relied on two essential steps:

(1) The 1-loop corrections to the gluon spectrum can, in the leading logarithm approximation, be expressed as the action of a certain linear operator on the leading order spectrum.[1]

(2) This operator acting on the initial color fields on the light cone is, again in the leading log approximation, the JIMWLK Hamiltonian.

In the present paper, we will show that a straightforward generalization of the first of these two steps is sufficient to extend our factorization result to inclusive *multigluon spectra* when all the measured gluons are located in a rapidity region of maximal width $\Delta Y \lesssim \alpha_s^{-1}$.

The paper is organized as follows. In Sec. II, we define a generating functional for multiparticle production in nucleus-nucleus collisions. This extends to the QCD case our previous results [17,18] for a similar object introduced for a ϕ^3 theory. We discuss key features of this generating functional and develop a diagrammatic interpretation of this object. We show how (at leading order) its first derivative can be expressed in terms of classical solutions of the Yang-Mills equations that obey both advanced and retarded boundary conditions. In Sec. III, we consider in detail the inclusive 2-gluon spectrum. We obtain an expression of this spectrum at next-to-leading order (NLO) using the previously defined generating functional. We end the section by showing that the leading logs of $1/x_{1,2}$ in this quantity can be factorized in the distributions of incoming color sources, provided the rapidity separation between the two gluons is small enough. We show that our formalism gives rise to the Glasma flux tube picture [19], which has been suggested as a mechanism to describe the ridgelike structure observed in heavy ion collisions at the Relativistic Heavy Ion Collider (RHIC) [20–24]. In Sec. IV, we generalize this factorization result to the case of the inclusive n-gluon spectrum. The knowledge of all the moments defines completely the distribution of proba-

[1]See the discussion after Eqs. (40) and (41) and at the end of Sec. 3.5 in [1].

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bilities. We demonstrate how the leading logarithmic corrections to the multiplicity distribution can be factorized into the JIMWLK evolution of the sources. We end with a brief summary. The three appendixes are devoted to the more technical aspects of our discussion.

II. GENERATING FUNCTIONAL

In Paper I, we developed the tools for studying at LO and NLO the single inclusive gluon spectrum in AA collisions in the CGC framework. Our goal is to generalize these techniques to obtain similar results for the n-gluon spectrum. Towards that purpose, we will define in this section a generating functional for n-gluon production, discuss its properties, and develop a diagrammatic interpretation. We then discuss the LO computation of the first derivative of this object in terms of solutions of classical Yang-Mills equations with both retarded and advanced boundary conditions.

A. Definition and properties

We define the generating functional as

$$\mathcal{F}[z(\boldsymbol{p})] \equiv \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\boldsymbol{p}_1} \cdots \int_{\boldsymbol{p}_n} z(\boldsymbol{p}_1) \cdots z(\boldsymbol{p}_n) \times |\langle \boldsymbol{p}_1 \cdots \boldsymbol{p}_{n\,\mathrm{out}} | 0_{\mathrm{in}} \rangle|^2, \tag{2}$$

where we use the following compact notation for phase-space integrals,[2]

$$\int_{\boldsymbol{p}} \cdots \equiv \int \frac{d^3\boldsymbol{p}}{(2\pi)^3 2E_{\boldsymbol{p}}} \cdots. \tag{3}$$

In this definition, $z(\boldsymbol{p})$ is an arbitrary function over the 1-gluon phase space. The matrix element squared that appears in the right-hand side is implicitly summed over the polarizations and colors of the produced gluons. Note that in this section, we consider the external current J^μ coupled to the gauge field to be fixed. This is the case in the CGC framework [8] where the fixed sources represent the large x light-cone color charge densities in the nuclear wave functions. We will address the issue of averaging over the external color sources later in this paper.

The generating functional generalizes the *generating function* $F(z)$ we introduced in Ref. [17]. This previously defined function is simply obtained as

$$\mathcal{F}[z(\boldsymbol{p}) \equiv z^*] = F(z^*), \tag{4}$$

where $z(\boldsymbol{p})$ is a constant z^*. Another obvious property of $\mathcal{F}[z(\boldsymbol{p})]$ is

$$\mathcal{F}[z(\boldsymbol{p}) \equiv 1] = 1 \tag{5}$$

which is a consequence of the fact that the theory is unitary.

The generating functional encapsulates the entire information content of the nuclear collision within the CGC framework. Indeed, if $\mathcal{F}[z(\boldsymbol{p})]$ were known, one could use it to build an event generator for the early Glasma [8,18,25] stage of nucleus-nucleus collisions. In particular, one can compute the inclusive multigluon spectra. For instance, the single inclusive[3] gluon spectrum is obtained as

$$\frac{dN}{d^3\boldsymbol{p}} = \frac{\delta \mathcal{F}[z]}{\delta z(\boldsymbol{p})}\bigg|_{z\equiv 1}. \tag{6}$$

Likewise, the inclusive 2-gluon spectrum is obtained by differentiating $\mathcal{F}[z]$ twice,

$$\frac{dN_2}{d^3\boldsymbol{p}d^3\boldsymbol{q}} = \frac{\delta^2 \mathcal{F}[z]}{\delta z(\boldsymbol{p})\delta z(\boldsymbol{q})}\bigg|_{z\equiv 1}, \tag{7}$$

where the integral over $\boldsymbol{p}$ and $\boldsymbol{q}$ on the left-hand side (l.h.s.) of this expression is the average value of $N(N-1)$. Physically, this quantity, in an event, corresponds to a histogram of all pairs of *distinct* gluons with momenta $(\boldsymbol{p}, \boldsymbol{q})$. We will discuss the average over all such events later. Equations (6) and (7) are the two simplest examples of the use of this generating functional, but, in principle, one can derive from it any observable that is related to the distribution of gluons produced in the collision. Equation (7) can be generalized to

$$\frac{d^nN_n}{d^3\boldsymbol{p}_1 \cdots d^3\boldsymbol{p}_n} = \frac{\delta^n \mathcal{F}[z]}{\delta z(\boldsymbol{p}_1) \cdots \delta z(\boldsymbol{p}_n)}\bigg|_{z\equiv 1}, \tag{8}$$

for the inclusive n-gluon spectrum. Note that the l.h.s., integrated over the n-particle phase space, is normalized to the average value of $N(N-1)\cdots(N-n+1)$.

From Eq. (8), it is possible to represent the generating functional $\mathcal{F}[z]$ as

$$\mathcal{F}[z(\boldsymbol{p})] = \sum_{n=0}^{\infty} \frac{1}{n!} \int \bigg[\prod_{i=1}^{n} d^3\boldsymbol{p}_i (z(\boldsymbol{p}_i) - 1)\bigg] \times \frac{d^nN_n}{d^3\boldsymbol{p}_1 \cdots d^3\boldsymbol{p}_n}. \tag{9}$$

This formula will later be the basis of our strategy to obtain an expression for $\mathcal{F}[z]$ at leading log. We will first obtain leading log expressions for the n-gluon spectra,[4] and will show that the infinite sum in Eq. (9) leads to a very simple expression.

Once we know $\mathcal{F}[z]$ (with a given accuracy), one can use the fact that its Taylor coefficients at $z(\boldsymbol{p}) = 0$ are the

[2]Whenever the integrand contains p_0 in such integrals, it should be replaced by the positive on-shell energy $p_0 = |\boldsymbol{p}|$.

[3]Note that setting $z(\boldsymbol{p})$ to zero instead, after taking the functional derivative, one obtains the differential probability for producing *exactly one* gluon in the collision,

$$\frac{dP_1}{d^3\boldsymbol{p}} = \frac{\delta \mathcal{F}[z]}{\delta z(\boldsymbol{p})}\bigg|_{z\equiv 0}.$$

[4]With the important limitation that the n gluons all sit in a rapidity slice of width $\Delta Y \lesssim \alpha_s^{-1}$.

differential probabilities for producing a fixed number of particles,[5]

$$\mathcal{F}[z(\boldsymbol{p})] = \sum_{n=0}^{\infty} \int \left[\prod_{i=1}^{n} d^3\boldsymbol{p}_i z(\boldsymbol{p}_i) \right] \frac{d^n P_n}{d^3\boldsymbol{p}_1 \cdots d^3\boldsymbol{p}_n}. \quad (10)$$

From this second representation of $\mathcal{F}[z]$, one can extract from $\mathcal{F}[z]$ detailed information about the distribution of produced gluons.

B. Diagrammatic interpretation of $\mathcal{F}[z]$

In order to see what diagrams contribute to $\mathcal{F}[z]$, let us first define

$$\mathcal{D} \equiv \int_{\boldsymbol{p}} \mathcal{D}_{\boldsymbol{p}}, \quad (11)$$

with

$$\mathcal{D}_{\boldsymbol{p}} \equiv \sum_\lambda \epsilon_\lambda^\mu(\boldsymbol{p})\epsilon_\lambda^\nu(\boldsymbol{p})^* \int d^4x d^4y e^{ip\cdot(x-y)} \square_x \square_y \times \frac{\delta}{\delta J_+^\mu(x)} \frac{\delta}{\delta J_-^\nu(y)}. \quad (12)$$

This operator has already been introduced in [17,18] to write P_n in terms of vacuum diagrams. The only difference here is that we extend its definition to the case of vector particles and QCD. The sum over the gluon polarizations λ spans the two physical polarization states. By mimicking the manipulations performed for scalar fields, one can prove that

$$\mathcal{F}[z(\boldsymbol{p})] = \exp\left[\int_{\boldsymbol{p}} z(\boldsymbol{p}) \mathcal{D}_{\boldsymbol{p}} \right] e^{iV[J_+^\mu]} e^{-iV^*[J_-^\mu]} \Big|_{J_+^\mu = J_-^\mu = J^\mu}, \quad (13)$$

where $iV[J^\mu]$ is the sum of the connected vacuum diagrams evaluated with the external current J^μ. It is easy to check that all the formulas we previously obtained in [17,18] for P_n or for the generating function $F(z)$ are all particular cases of this formula.

From the interpretation of the operator $\mathcal{D}$ as an operator that makes cuts through vacuum diagrams, we see that the functional $\mathcal{F}[z(\boldsymbol{p})]$ is the sum of all the cut vacuum diagrams (connected or not) in which every cut propagator with momentum $\boldsymbol{p}$ is weighted by $z(\boldsymbol{p})$. Let us call $i\mathcal{W}[J_+^\mu, J_-^\mu; z]$ the sum of all such *connected* diagrams (before the currents J_+^μ and J_-^μ are set equal to the physical value J^μ):

$$e^{i\mathcal{W}[J_+^\mu, J_-^\mu; z]} \equiv \exp\left[\int_{\boldsymbol{p}} z(\boldsymbol{p}) \mathcal{D}_{\boldsymbol{p}} \right] e^{iV[J_+^\mu]} e^{-iV^*[J_-^\mu]}. \quad (14)$$

[5]Note that there is no $1/n!$ in this formula. A quick way to convince oneself that this is correct is to set $z(\boldsymbol{p}) = 1$; the integrals over the momenta $\boldsymbol{p}_i$ give the total probabilities P_n, which add up to unity.

It is useful to compute the first derivative of $\mathcal{F}[z(\boldsymbol{p})]$ with respect to $z(\boldsymbol{p})$,

$$\frac{\delta \mathcal{F}[z]}{\delta z(\boldsymbol{p})} = \frac{1}{(2\pi)^3 2E_{\boldsymbol{p}}} \mathcal{D}_{\boldsymbol{p}} e^{i\mathcal{W}[J_+^\mu, J_-^\mu; z]} \Big|_{J_+^\mu = J_-^\mu = J^\mu}. \quad (15)$$

Performing explicitly the derivatives contained in Eq. (12), this can be rewritten as

$$\frac{\delta \mathcal{F}[z]}{\delta z(\boldsymbol{p})} = \frac{1}{(2\pi)^3 2E_{\boldsymbol{p}}} \sum_\lambda \epsilon_\lambda^\mu(\boldsymbol{p})\epsilon_\lambda^\nu(\boldsymbol{p})^* \times \int d^4x d^4y e^{ip\cdot(x-y)} \square_x \square_y \left[\frac{\delta i\mathcal{W}}{\delta J_+^\mu(x)} \frac{\delta i\mathcal{W}}{\delta J_-^\nu(y)} + \frac{\delta^2 i\mathcal{W}}{\delta J_+^\mu(x) \delta J_-^\nu(y)} \right] e^{i\mathcal{W}[J_+^\mu, J_-^\mu; z]} \Big|_{J_+^\mu = J_-^\mu = J^\mu}. \quad (16)$$

The final exponential in this formula is nothing but $\mathcal{F}[z]$ itself. Therefore, we can write

$$\frac{\delta \ln \mathcal{F}[z]}{\delta z(\boldsymbol{p})} = \frac{1}{(2\pi)^3 2E_{\boldsymbol{p}}} \sum_\lambda \epsilon_\lambda^\mu(\boldsymbol{p})\epsilon_\lambda^\nu(\boldsymbol{p})^* \times \int d^4x d^4y e^{ip\cdot(x-y)} \square_x \square_y \left[\frac{\delta i\mathcal{W}}{\delta J_+^\mu(x)} \frac{\delta i\mathcal{W}}{\delta J_-^\nu(y)} + \frac{\delta^2 i\mathcal{W}}{\delta J_+^\mu(x) \delta J_-^\nu(y)} \right]_{J_+^\mu = J_-^\mu = J^\mu}. \quad (17)$$

This formula tells us that this quantity is made up of only connected diagrams since $i\mathcal{W}$ is a sum of connected diagrams. We also observe that this formula is very similar to the formula for the single inclusive particle spectrum with one very important difference: the function $z(\boldsymbol{p})$ is not set to 1 at the end, and therefore appears as a multiplicative factor attached to each cut propagator.

C. $\delta \ln \mathcal{F}[z]/\delta z(\boldsymbol{p})$ at leading order

Let us now show that, in the regime of strong external color sources, the expression in Eq. (17) can be expressed at LO in terms of classical solutions of the Yang-Mills equations.

First of all, note that the first derivatives $\delta\mathcal{W}/\delta J_\pm^\mu$ are of order[6] g^{-1}, while the second derivative $\delta^2\mathcal{W}/\delta J_+^\mu \delta J_-^\nu$ is order g^0. Thus the first term, composed of the product of two first derivatives, is the leading one. The second term begins to contribute only at NLO. At LO, we can thus write

$$\frac{\delta \ln \mathcal{F}[z]}{\delta z(\boldsymbol{p})} \Big|_{\rm LO} = \frac{1}{(2\pi)^3 2E_{\boldsymbol{p}}} \sum_\lambda \epsilon_{\lambda\mu}(\boldsymbol{p})\epsilon_{\lambda\nu}(\boldsymbol{p}) \times \int d^4x d^4y e^{ip\cdot(x-y)} \square_x \square_y \mathcal{A}_+^\mu(x) \mathcal{A}_-^\nu(y), \quad (18)$$

[6]Because $\mathcal{W}$ is the sum of connected vacuum graphs, in the presence of external sources $J_\pm^\mu \sim g^{-1}$, $\mathcal{W} \sim g^{-2}$.

where we denote[7]

$$\mathcal{A}^{\mu}_{\epsilon}(x) \underset{\text{tree}}{\equiv} \frac{\delta i \mathcal{W}}{\delta J^{\mu}_{\epsilon}(x)}\bigg|_{J^{\mu}_{+}=J^{\mu}_{-}=J^{\mu}}. \tag{19}$$

The "tree" here means we keep only tree diagrams in the expansion of $\delta\mathcal{W}/\delta J^{\mu}_{\pm}$ that defines $\mathcal{A}^{\mu}_{\epsilon}$.

All the arguments developed to compute the generating function $F(z)$ at leading order [18] can be extended trivially to the present situation, and one obtains the following results:

(i) $\mathcal{A}^{\mu}_{\epsilon}$ is a solution of the classical Yang-Mills equations,

$$[\mathcal{D}_{\mu}, \mathcal{F}^{\mu\nu}] = J^{\nu}. \tag{20}$$

(ii) If one decomposes $\mathcal{A}^{\mu}_{\epsilon}(x)$ in Fourier modes,

$$\mathcal{A}^{\mu}_{\epsilon}(x) \equiv \sum_{\lambda,a}\int_{p}\Big\{f^{(+)}_{\epsilon}(x_0; \boldsymbol{p}\lambda a)a^{0\mu}_{-\boldsymbol{p}\lambda a}(x) + f^{(-)}_{\epsilon}(x_0; \boldsymbol{p}\lambda a)a^{0\mu}_{+\boldsymbol{p}\lambda a}(x)\Big\}, \tag{21}$$

with $a^{0\mu}_{\pm\boldsymbol{p}\lambda a}(x) \equiv \epsilon^{\mu}_{\lambda}(\boldsymbol{p})T^{a}e^{\pm ip\cdot x}$, the boundary conditions obeyed by the classical field $\mathcal{A}^{\mu}_{\epsilon}(x)$ can be expressed as simple constraints on the Fourier coefficients,[8]

$$\begin{aligned} f^{(+)}_{+}(-\infty; \boldsymbol{p}\lambda a) &= f^{(-)}_{-}(-\infty; \boldsymbol{p}\lambda a) = 0,\\ f^{(+)}_{-}(+\infty; \boldsymbol{p}\lambda a) &= z(\boldsymbol{p})f^{(+)}_{+}(+\infty; \boldsymbol{p}\lambda a),\\ f^{(-)}_{+}(+\infty; \boldsymbol{p}\lambda a) &= z(\boldsymbol{p})f^{(-)}_{-}(+\infty; \boldsymbol{p}\lambda a). \end{aligned} \tag{22}$$

We see that the dependence of the classical fields $\mathcal{A}^{\mu}_{\pm}$ on the function $z(\boldsymbol{p})$ comes entirely from the boundary conditions,[9] since the Yang-Mills equations themselves do not explicitly contain $z(\boldsymbol{p})$. In terms of the Fourier coefficients $f^{(\pm)}_{\pm}$, Eq. (18) reads

$$\frac{\delta \ln\mathcal{F}[z]}{\delta z(\boldsymbol{p})}\bigg|_{\text{LO}} = \frac{1}{(2\pi)^3 2E_{\boldsymbol{p}}}\sum_{\lambda,a} f^{(+)}_{+}(+\infty; \boldsymbol{p}\lambda a)f^{(-)}_{-}(+\infty; \boldsymbol{p}\lambda a). \tag{23}$$

Note that it depends only on the Fourier coefficients of the fields at $x^0 = +\infty$.

Equations (20), (22), and (23) do not provide a practical way to obtain the LO generating functional $\mathcal{F}[z(\boldsymbol{p})]$ because the solutions depend on boundary conditions at both $\pm\infty$. It is not known at present how to solve Yang-Mills equations with simultaneous advanced and retarded boundary conditions. Nevertheless, the procedure outlined here provides a powerful theoretical tool to compute other quantities, which can be obtained as derivatives of the generating functional. A concrete illustration of this strategy is revealed in the case of the 2-gluon spectrum in the following section.

III. TWO-GLUON INCLUSIVE SPECTRUM

In this section, we will specialize our discussion of the generating functional in the previous section to the 2-gluon inclusive spectrum at LO and NLO. We will demonstrate that, just as in the case of the single gluon spectrum discussed in Paper I, the leading logarithm contributions that arise at NLO can be absorbed in the JIMWLK wave functionals of the two nuclei, provided the rapidity separation between the two gluons is small enough. As in Paper I, one obtains a factorized expression for the leading log 2-gluon inclusive spectrum. In the following section, this result will be extended to multigluon spectra.

A. Leading order

The inclusive 2-gluon spectrum is obtained by taking the second derivative of the generating functional $\mathcal{F}[z]$, and by setting the functions $z(\boldsymbol{p})$ and $z(\boldsymbol{q})$ to unity afterwards [see Eq. (7)]. Alternately, it is easy to obtain this derivative from the derivative of $\ln\mathcal{F}[z]$. We get

$$\frac{d^2N_2}{d^3\boldsymbol{p}d^3\boldsymbol{q}} = \frac{\delta \ln\mathcal{F}[z]}{\delta z(\boldsymbol{p})}\frac{\delta \ln\mathcal{F}[z]}{\delta z(\boldsymbol{q})} + \frac{\delta^2 \ln\mathcal{F}[z]}{\delta z(\boldsymbol{p})\delta z(\boldsymbol{q})}\bigg|_{z(\boldsymbol{p}),z(\boldsymbol{q})\equiv 1}. \tag{24}$$

The first term is simply the product of two single gluon spectra [see Eq. (6)], and therefore corresponds to the disconnected (independent) production of a gluon of momentum $\boldsymbol{p}$ and a gluon of momentum $\boldsymbol{q}$. In contrast, because $\ln\mathcal{F}[z]$ contains only connected diagrams, the second term corresponds to the two gluons being produced in the same graph. Note that these expressions correspond to the 2-gluon spectrum for a fixed configuration of the external sources $\rho_{1,2}$. When we average over these sources, some graphs that were disconnected prior to averaging become connected. Therefore, even the first term in Eq. (24) can lead to correlations in the measured 2-gluon spectrum.

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[7] $\mathcal{A}^{\mu}_{\pm}(x)$ depends on the function $z(\boldsymbol{p})$ as well, but we have omitted it from the notation to keep notations compact.

[8] The derivation of this result is analogous to the scalar case discussed in detail in Sec. 4.2 of Ref. [17].

[9] Note that, when $z(\boldsymbol{p}) \equiv 1$, the boundary conditions in Eqs. (22) become

$$\begin{aligned} f^{(+)}_{+}(-\infty; \boldsymbol{p}\lambda a) &= f^{(-)}_{-}(-\infty; \boldsymbol{p}\lambda a) = 0,\\ f^{(+)}_{-}(+\infty; \boldsymbol{p}\lambda a) &= f^{(+)}_{+}(+\infty; \boldsymbol{p}\lambda a),\\ f^{(-)}_{+}(+\infty; \boldsymbol{p}\lambda a) &= f^{(-)}_{-}(+\infty; \boldsymbol{p}\lambda a). \end{aligned}$$

The two conditions at $x^0 = +\infty$ imply that $\mathcal{A}_{+}(x) = \mathcal{A}_{-}(x)$ everywhere. The two conditions at $x^0 = -\infty$ then imply that $\lim_{x^0\to-\infty}\mathcal{A}_{\pm}(x) = 0$. Therefore, when $z(\boldsymbol{p}) \equiv 1$, the two classical fields $\mathcal{A}^{\mu}_{\pm}$ become identical to the retarded classical field with a vanishing initial condition in the remote past, and Eq. (18) gives the single inclusive gluon spectrum as expected.

The two terms in this expression do not begin at the same order in g^2. In our power counting,

$$\ln\mathcal{F}[z] = \frac{1}{g^2}[c_0 + c_1 g^2 + c_2 g^4 + \cdots]. \tag{25}$$

This implies that the first term in Eq. (24) is of order g^{-4}, while the second term is of order g^{-2} only. For the 2-gluon spectrum, "leading order" therefore means g^{-4}, and we simply have[10]

$$\left.\frac{d^2N_2}{d^3\boldsymbol{p}d^3\boldsymbol{q}}\right|_{\rm LO} = \left.\frac{dN}{d^3\boldsymbol{p}}\right|_{\rm LO}\left.\frac{dN}{d^3\boldsymbol{q}}\right|_{\rm LO}. \tag{26}$$

No new computations are necessary here because we know how to express the single gluon spectrum at LO in terms of classical solutions of the Yang-Mills equations with retarded boundary conditions. Note that at this order the $-N$ term contributing to N_2 is subleading relative to the N^2 contribution because it starts only at the order g^{-2} and therefore does not appear on the right-hand side of Eq. (27) which is of order g^{-4}.

B. Next-to-leading order—I

We shall now study the inclusive 2-gluon spectrum at NLO–the contribution at order g^{-2} in our power counting. At this order, the tree-level contribution to the second term in Eq. (24) must be included. We can therefore write

$$\left.\frac{d^2N_2}{d^3\boldsymbol{p}d^3\boldsymbol{q}}\right|_{\rm NLO} = \left.\frac{dN}{d^3\boldsymbol{p}}\right|_{\rm NLO}\left.\frac{dN}{d^3\boldsymbol{q}}\right|_{\rm LO} + \left.\frac{dN}{d^3\boldsymbol{p}}\right|_{\rm LO}\left.\frac{dN}{d^3\boldsymbol{q}}\right|_{\rm NLO} + \left.\frac{\delta^2\ln\mathcal{F}[z]}{\delta z(\boldsymbol{p})\delta z(\boldsymbol{q})}\right|_{\rm LO}. \tag{27}$$

The first two terms again do not require a new computation because we studied in great detail the single gluon spectrum at NLO in Paper I [1]. In particular, we recall here the previously derived formula

$$\left.\frac{dN}{d^3\boldsymbol{p}}\right|_{\rm NLO} = \Bigg[\underbrace{\int_\Sigma d^3\vec{\boldsymbol{u}}[\beta\cdot\mathbb{T}_{\boldsymbol{u}}]}_{\mathcal{L}_1} + \underbrace{\frac{1}{2}\sum_{\lambda,a}\int_{\boldsymbol{k}}\int_\Sigma d^3\vec{\boldsymbol{u}}d^3\vec{\boldsymbol{v}}[a_{-\boldsymbol{k}\lambda a}\cdot\mathbb{T}_{\boldsymbol{u}}][a_{+\boldsymbol{k}\lambda a}\cdot\mathbb{T}_{\boldsymbol{v}}]}_{\mathcal{L}_2}\Bigg] \times\left.\frac{dN}{d^3\boldsymbol{p}}\right|_{\rm LO} + \Delta N_{\rm NLO}(\boldsymbol{p}). \tag{28}$$

In this formula, $a_{\pm\boldsymbol{k}\lambda a}$ denotes small field fluctuations that propagate over the classical field $\mathcal{A}$. The subscripts indicate that these fluctuations begin in the remote past as plane waves of momentum $\pm k$, polarization λ, and color a. Similarly, β is also a small field fluctuation propagating on top of $\mathcal{A}$, but this fluctuation has a vanishing initial condition in the past and is driven by a nonzero source term. Σ is a surface on which the initial value of the classical fields are defined, and $d^3\vec{\boldsymbol{u}}$ is the measure on this surface. The operator $\mathbb{T}_{\boldsymbol{u}}$ is the generator of translations of the initial field at the point $\boldsymbol{u}\in\Sigma$. $\Delta N_{\rm NLO}(p)$ is a term contributing to the full expression. It will not be made more explicit here because it does not contain a leading logarithmic contribution—see the discussion of this term in [1]. Because we are interested here in these leading log contributions, this term will be dropped in all further equations in this paper.

At this point, we can rewrite the first two terms of the r.h.s. of Eq. (27) as

$$\left.\frac{dN}{d^3\boldsymbol{p}}\right|_{\rm NLO}\left.\frac{dN}{d^3\boldsymbol{q}}\right|_{\rm LO} + \left.\frac{dN}{d^3\boldsymbol{p}}\right|_{\rm LO}\left.\frac{dN}{d^3\boldsymbol{q}}\right|_{\rm NLO} = [\mathcal{L}_1+\mathcal{L}_2]_{\rm disc}\left.\frac{dN}{d^3\boldsymbol{p}}\right|_{\rm LO}\left.\frac{dN}{d^3\boldsymbol{q}}\right|_{\rm LO}, \tag{29}$$

where the subscript "disc" added to the operator between the square brackets indicates that when the combination $\mathbb{T}_{\boldsymbol{u}}\mathbb{T}_{\boldsymbol{v}}$ in $\mathcal{L}_2$ acts on the product $(dN/d^3\boldsymbol{p})(dN/d^3\boldsymbol{q})$, we keep only the terms where the two $\mathbb{T}$'s act on the same factor.[11] The subscript here reminds us that these terms are disconnected contributions that are the product of a function of $\boldsymbol{p}$ and a function of $\boldsymbol{q}$.

C. Next-to-leading order—II

The third term of Eq. (27), involving the second derivative of the log of the generating functional, is new and will be computed here. Fortunately, we need this term only at leading order—i.e. $\mathcal{O}(g^{-2})$. Therefore, our starting point in evaluating this term is Eq. (23). Differentiating this equation with respect to $z(\boldsymbol{q})$, we obtain

$$\left.\frac{\delta^2\ln\mathcal{F}[z]}{\delta z(\boldsymbol{p})\delta z(\boldsymbol{q})}\right|_{\rm LO} = \frac{1}{(2\pi)^3 2E_{\boldsymbol{p}}}\sum_{\lambda,a}\Bigg[\frac{\delta f_+^{(+)}(+\infty;\boldsymbol{p}\lambda a)}{\delta z(\boldsymbol{q})} \times f_-^{(-)}(+\infty;\boldsymbol{p}\lambda a) + f_+^{(+)}(+\infty;\boldsymbol{p}\lambda a)\frac{\delta f_-^{(-)}(+\infty;\boldsymbol{p}\lambda a)}{\delta z(\boldsymbol{q})}\Bigg]. \tag{30}$$

Further, differentiating Eq. (21) with respect to $z(\boldsymbol{q})$, one observes that the quantities $\delta f_\epsilon^{(\pm)}(+\infty;\boldsymbol{p}\lambda a)/\delta z(\boldsymbol{q})$ are the Fourier coefficients of the field

$$b^\mu_{\epsilon,\boldsymbol{q}}(x) \equiv \frac{\delta\mathcal{A}^\mu_\epsilon(x)}{\delta z(\boldsymbol{q})} \tag{31}$$

at $x^0 = +\infty$. The equation of motion obeyed by this object can be obtained by differentiating, with respect to $z(\boldsymbol{q})$, the

[10] One should keep in mind therefore that "LO" corresponds to different powers of g^2 for the single and double inclusive gluon spectra.

[11] $[\mathcal{L}_2]_{\rm disc}AB = [\mathcal{L}_2 A]B + A[\mathcal{L}_2 B]$.

equation of motion for $\mathcal{A}^\mu_\epsilon$. In order to do this, it is useful to start from the Yang-Mills equations written in a form that separates explicitly the kinetic and interaction terms[12] as

$$[\square_x g_\mu{}^\nu - \partial_{x\mu}\partial^\nu_x]\mathcal{A}^\mu_\epsilon(x) - \frac{\partial U(\mathcal{A}_\epsilon)}{\partial \mathcal{A}_{\epsilon,\nu}(x)} = J^\nu_\epsilon, \tag{32}$$

where $U(\mathcal{A})$ is the Yang-Mills potential in a gauge with a linear gauge condition. Differentiating this equation with respect to $z(\boldsymbol{q})$, we get

$$\left[\square_x g_\mu{}^\nu - \partial_{x\mu}\partial^\nu_x - \frac{\partial U(\mathcal{A}_\epsilon)}{\partial \mathcal{A}_{\epsilon\nu}(x)\partial \mathcal{A}^\mu_\epsilon(x)}\right] b^\mu_{\epsilon,\boldsymbol{q}}(x) = 0. \tag{33}$$

In other words, $b^\mu_{\epsilon,\boldsymbol{q}}(x)$ obeys the equation of motion of small fluctuations propagating on top of the classical field $\mathcal{A}_\epsilon$. The boundary conditions necessary in order to fully determine $b^\mu_{\epsilon,\boldsymbol{q}}(x)$ are easily obtained by differentiating Eqs. (22) with respect to $z(\boldsymbol{q})$:

$$\begin{aligned} b^{(+)}_{+,\boldsymbol{q}}(-\infty;\boldsymbol{p}\lambda a) &= b^{(-)}_{-,\boldsymbol{q}}(-\infty;\boldsymbol{p}\lambda a) = 0,\\ b^{(+)}_{-,\boldsymbol{q}}(+\infty;\boldsymbol{p}\lambda a) &= z(\boldsymbol{p}) b^{(+)}_{+,\boldsymbol{q}}(+\infty;\boldsymbol{p}\lambda a)\\ &\quad + \delta(\boldsymbol{p}-\boldsymbol{q}) f^{(+)}_{+}(+\infty;\boldsymbol{p}\lambda a),\\ b^{(-)}_{+,\boldsymbol{q}}(+\infty;\boldsymbol{p}\lambda a) &= z(\boldsymbol{p}) b^{(-)}_{-,\boldsymbol{q}}(+\infty;\boldsymbol{p}\lambda a)\\ &\quad + \delta(\boldsymbol{p}-\boldsymbol{q}) f^{(-)}_{-}(+\infty;\boldsymbol{p}\lambda a), \end{aligned} \tag{34}$$

where we have introduced the obvious notation

$$b^{(\eta)}_{\epsilon,\boldsymbol{q}}(x^0;\boldsymbol{p}\lambda a) \equiv \frac{\delta f^{(\eta)}_\epsilon(x^0;\boldsymbol{p}\lambda a)}{\delta z(\boldsymbol{q})} \tag{35}$$

for the Fourier coefficients of $b^\mu_{\epsilon,\boldsymbol{q}}$. We see that we have *nonhomogeneous boundary conditions*, which will lead to a nonzero $b^\mu_{\epsilon,\boldsymbol{q}}$ despite the fact that this fluctuation obeys a homogeneous equation of motion. Note also that at this point we can safely set $z(\boldsymbol{p}) = 1$ since we do not need to differentiate with respect to $z(\boldsymbol{p})$ again. This leads to the simplification that when $z(\boldsymbol{p}) = 1$, the classical fields $\mathcal{A}^\mu_+$ and $\mathcal{A}^\mu_-$ become identical—as can be checked from their boundary conditions (see footnote [9]). In fact, their common value is nothing but the classical field that vanishes when $x^0 \to -\infty$. We will simply denote by $\mathcal{A}^\mu$ the common value of these two fields and $f^{(\pm)}(x^0;\boldsymbol{p}\lambda a)$ its Fourier coefficients.

Obviously, Eqs. (34) are not simple retarded boundary conditions. Our task is now to relate the fluctuations $b^\mu_{\epsilon,\boldsymbol{q}}$ and their Fourier coefficients to fluctuations that satisfy simple retarded boundary conditions. In order to achieve this, let us again use the small field fluctuations $a^\mu_{\pm\boldsymbol{k}\lambda a}$. They obey the equation of motion (33), and the boundary conditions

$$a^\mu_{\pm\boldsymbol{k}\lambda a}(x) \underset{x^0\to-\infty}{=} \epsilon^\mu_\lambda(\boldsymbol{k}) T^a e^{\pm ik\cdot x}. \tag{36}$$

Note that the fields $a^{0\mu}_{\pm\boldsymbol{k}\lambda a}$ introduced earlier are the analogue of the $a^\mu_{\pm\boldsymbol{k}\lambda a}$ in the absence of a background field. From this definition, $a_{+\boldsymbol{k}\lambda a}$ has only negative energy components at $x^0 \to -\infty$, while $a_{-\boldsymbol{k}\lambda a}$ has only positive energy components in this limit. Moreover, the fluctuations $a^\mu_{\pm\boldsymbol{k}\lambda a}$ provide a complete basis for the small field fluctuations that obey Eq. (33). From the boundary conditions of $a^\mu_{\epsilon,\boldsymbol{q}}$ at $x^0 = -\infty$, we see that we must have

$$b^\mu_{\pm,\boldsymbol{q}}(x) = \sum_{\lambda,a}\int_{\boldsymbol{k}} \gamma^{\boldsymbol{k}\lambda a}_{\pm,\boldsymbol{q}} a^\mu_{\pm\boldsymbol{k}\lambda a}(x). \tag{37}$$

The coefficients $\gamma^{\boldsymbol{k}\lambda a}_{\pm,\boldsymbol{q}}$ in these linear decompositions do not depend on space or time. The boundary conditions at $x^0 = -\infty$ do not constrain further the coefficients $\gamma^{\boldsymbol{k}\lambda a}_{\pm,\boldsymbol{q}}$, but they can be determined from the boundary conditions at $x^0 = +\infty$. To achieve this end, we introduce the Fourier decomposition of the functions $a^\mu_{\pm\boldsymbol{k}\lambda a}(x)$,

$$\begin{aligned} a^\mu_{\pm\boldsymbol{k}\lambda a}(x) \equiv \sum_{\zeta,b}\int_{\boldsymbol{p}} \{& h^{(+)}_{\pm\boldsymbol{p}\zeta b}(x^0;\boldsymbol{k}\lambda a) a^{0\mu}_{-\boldsymbol{p}\zeta b}(x)\\ &+ h^{(-)}_{\pm\boldsymbol{p}\zeta b}(x^0;\boldsymbol{k}\lambda a) a^{0\mu}_{+\boldsymbol{p}\zeta b}(x)\}. \end{aligned} \tag{38}$$

It is then a simple exercise to rewrite the boundary conditions at $x^0 = +\infty$ as

$$\begin{aligned} &\sum_{\lambda,a}\int_{\boldsymbol{k}} [\gamma^{\boldsymbol{k}\lambda a}_{-,\boldsymbol{q}} h^{(+)}_{-\boldsymbol{k}\lambda a}(\boldsymbol{p}\zeta b) - \gamma^{\boldsymbol{k}\lambda a}_{+,\boldsymbol{q}} h^{(+)}_{+\boldsymbol{k}\lambda a}(\boldsymbol{p}\zeta b)]\\ &\quad = \delta(\boldsymbol{p}-\boldsymbol{q}) f^{(+)}(\boldsymbol{p}\zeta b),\\ &\sum_{\lambda,a}\int_{\boldsymbol{k}} [\gamma^{\boldsymbol{k}\lambda a}_{+,\boldsymbol{q}} h^{(-)}_{+\boldsymbol{k}\lambda a}(\boldsymbol{p}\zeta b) - \gamma^{\boldsymbol{k}\lambda a}_{-,\boldsymbol{q}} h^{(-)}_{-\boldsymbol{k}\lambda a}(\boldsymbol{p}\zeta b)]\\ &\quad = \delta(\boldsymbol{p}-\boldsymbol{q}) f^{(-)}(\boldsymbol{p}\zeta b), \end{aligned} \tag{39}$$

where, to keep the expressions compact, we have omitted the argument $x^0 = +\infty$ in all the Fourier coefficients. This can be seen as a system of linear equations for the coefficients $\gamma^{\boldsymbol{k}\lambda a}_{\pm,\boldsymbol{q}}$. The solution of this system of linear equations is obtained in Appendix A.

Inserting the results of Eqs. (A3) for $\gamma^{\boldsymbol{k}\lambda a}_{\pm,q}$ into Eq. (37), one can easily determine the Fourier coefficients of $b^\mu_{\pm,\boldsymbol{q}}(x)$ at $x^0 = +\infty$ [Eq. (35)]. Inserting these into Eq. (30), we obtain[13,14]

[12]Note that the differentiation with respect to $z(\boldsymbol{q})$ does not modify the gauge fixing condition, provided it is linear. Thus, $b^\mu_{\epsilon,\boldsymbol{q}}$ obeys the same gauge condition as $\mathcal{A}^\mu_\epsilon$.

[13]We additionally use Eqs. (A2) to symmetrize the formula with respect to $(\boldsymbol{p},\boldsymbol{q})$.

[14]The part of this formula which is bilinear in the objects $h^{(\pm)}{}_{\pm\boldsymbol{k}\lambda a}$ generalizes to the dense regime (i.e. to the presence of a strong background field) 2-gluon production vertex obtained in [26,27] in the course of the derivation of the Balitsky-Fadin-Kuraev-Lipatov equation at NLO.

$$\frac{\delta^2 \ln\mathcal{F}[z]}{\delta z(\boldsymbol{p})\delta z(\boldsymbol{q})}\Big|_{\mathrm{LO};z(\boldsymbol{p}),z(\boldsymbol{q})=1} = \frac{1}{2}\frac{1}{(2\pi)^6 4E_{\boldsymbol{p}}E_{\boldsymbol{q}}}\sum_{\lambda,a}\sum_{\xi,b}\sum_{\zeta,c}\int_{\boldsymbol{k}}\{(h^{(-)}_{+\boldsymbol{k}\lambda a}(\boldsymbol{p}\xi b)h^{(-)}_{-\boldsymbol{k}\lambda a}(\boldsymbol{q}\zeta c)$$
$$+ h^{(-)}_{-\boldsymbol{k}\lambda a}(\boldsymbol{p}\xi b)h^{(-)}_{+\boldsymbol{k}\lambda a}(\boldsymbol{q}\zeta c))f^{(+)}(\boldsymbol{p}\xi b)f^{(+)}(\boldsymbol{q}\zeta c) + (h^{(+)}_{+\boldsymbol{k}\lambda a}(\boldsymbol{p}\xi b)h^{(+)}_{-\boldsymbol{k}\lambda a}(\boldsymbol{q}\zeta c)$$
$$+ h^{(+)}_{-\boldsymbol{k}\lambda a}(\boldsymbol{p}\xi b)h^{(+)}_{+\boldsymbol{k}\lambda a}(\boldsymbol{q}\zeta c))f^{(-)}(\boldsymbol{p}\xi b)f^{(-)}(\boldsymbol{q}\zeta c) + (h^{(-)}_{+\boldsymbol{k}\lambda a}(\boldsymbol{p}\xi b)h^{(+)}_{-\boldsymbol{k}\lambda a}(\boldsymbol{q}\zeta c)$$
$$+ h^{(-)}_{-\boldsymbol{k}\lambda a}(\boldsymbol{p}\xi b)h^{(+)}_{+\boldsymbol{k}\lambda a}(\boldsymbol{q}\zeta c))f^{(+)}(\boldsymbol{p}\xi b)f^{(-)}(\boldsymbol{q}\zeta c) + (h^{(+)}_{+\boldsymbol{k}\lambda a}(\boldsymbol{p}\xi b)h^{(-)}_{-\boldsymbol{k}\lambda a}(\boldsymbol{q}\zeta c)$$
$$+ h^{(+)}_{-\boldsymbol{k}\lambda a}(\boldsymbol{p}\xi b)h^{(-)}_{+\boldsymbol{k}\lambda a}(\boldsymbol{q}\zeta c))f^{(-)}(\boldsymbol{p}\xi b)f^{(+)}(\boldsymbol{q}\zeta c)\} - \frac{1}{(2\pi)^3 2E_{\boldsymbol{p}}}\delta(\boldsymbol{p}-\boldsymbol{q})\sum_{\zeta,c}f^{(+)}(\boldsymbol{p}\zeta c)f^{(-)}(\boldsymbol{p}\zeta c). \tag{40}$$

We have therefore obtained an expression for the connected piece of the 2-gluon spectrum entirely in terms of Fourier modes of the classical field ($f^{(\pm)}$) and the small fluctuation field ($h^{(\pm)}_{\pm\boldsymbol{k}\lambda a}$). The former can be determined by solving the Yang-Mills equations with retarded boundary conditions, while the latter can be determined by solving the equations for small fluctuations about the classical field, also with retarded boundary conditions.

The last term in Eq. (40), proportional to $\delta(\boldsymbol{p}-\boldsymbol{q})$ times the single particle spectrum, arises because the quantity $dN_2/d^3\boldsymbol{p}d^3\boldsymbol{q}$ is defined in such a way that its integral over $\boldsymbol{p}$ and $\boldsymbol{q}$ gives the average value[15] of $N(N-1)$. This term provides the $-N$ contribution to this quantity. Because the logs in the multiplicity N arise only at the order $\mathcal{O}(g^0)$, this term cannot provide any leading log in the 2-gluon spectrum and can thus be dropped.

D. Leading log resummation of the 2-gluon spectrum

Combining the results in Eqs. (28) and (40) in Eq. (27), we now have a formula for the 2-gluon spectrum, including both LO and NLO contributions. As mentioned previously, it can, in principle, be evaluated, in full generality, by numerical solutions of small fluctuation partial differential equations with retarded boundary conditions. However, if one is interested primarily in the leading logarithmic piece of the NLO contributions, we can go significantly further analytically. Indeed, as we will now show by using the information obtained thus far, we can compute the leading logarithmic contributions to the 2-gluon spectrum in perturbation theory.

The first step in this derivation is to obtain an even more compact form for Eq. (40) by using the linear operator $\mathbb{T}_{\boldsymbol{u}}$ that we used previously in the expression for the 1-loop corrections to the single particle spectrum—see Eq. (28). In Paper I, we demonstrated explicitly that this operator allows one to express the value of a retarded fluctuation at a point x in terms of the value of the classical field at the same point as

$$a^\mu(x) = \int_\Sigma d^3\vec{\boldsymbol{u}}[a\cdot\mathbb{T}_{\boldsymbol{u}}]\mathcal{A}^\mu(x), \tag{41}$$

where Σ is the initial surface on which we know the value of the fluctuation. (The point x is located above this surface.) Performing the Fourier decomposition of both sides of this relation, we obtain simply the relation between the Fourier coefficients (at $x^0 = +\infty$) of the small fluctuation and the classical field to be

$$h^{(\epsilon)}(+\infty;\boldsymbol{p}\lambda a) = \int_\Sigma d^3\vec{\boldsymbol{u}}[a\cdot\mathbb{T}_{\boldsymbol{u}}]f^{(\epsilon)}(+\infty;\boldsymbol{p}\lambda a). \tag{42}$$

Applying Eq. (42) to the various fluctuations that appear in Eq. (40), and using the $z(p)=1$ simplification of Eq. (23),

$$\frac{dN}{d^3\boldsymbol{p}}\Big|_{\mathrm{LO}} = \frac{\delta\ln\mathcal{F}[z]}{\delta z(\boldsymbol{p})}\Big|_{z=1,\mathrm{LO}} = \frac{1}{(2\pi)^3 2E_{\boldsymbol{p}}}\sum_{\zeta,c}f^{(+)}(\boldsymbol{p}\zeta c)f^{(-)}(\boldsymbol{p}\zeta c), \tag{43}$$

it is a matter of simple algebra to check that

$$\frac{\delta^2\ln\mathcal{F}[z]}{\delta z(\boldsymbol{p})\delta z(\boldsymbol{q})}\Big|_{\mathrm{LO};z(\boldsymbol{p}),z(\boldsymbol{q})=1} = -\delta(\boldsymbol{p}-\boldsymbol{q})\frac{dN}{d^3\boldsymbol{p}}\Big|_{\mathrm{LO}} + [\mathcal{L}_2]_{\mathrm{connected}}\frac{dN}{d^3\boldsymbol{p}}\Big|_{\mathrm{LO}}\frac{dN}{d^3\boldsymbol{q}}\Big|_{\mathrm{LO}}. \tag{44}$$

The subscript "connected" indicates that one of the $\mathbb{T}$ operators in the expression $\mathcal{L}_2$ appearing in Eq. (28) must act on the $\boldsymbol{p}$-dependent factor and the other on the $\boldsymbol{q}$-dependent factor. (Terms where they both act on the same factor should be excluded.)

We see now that Eqs. (29) and (44) can be combined very easily, because the sum of "disconnected" and connected terms is equivalent to the unrestricted action of $\mathbb{T}_{\boldsymbol{u}}\mathbb{T}_{\boldsymbol{v}}$ on the product $(dN/d^3\boldsymbol{p})(dN/d^3\boldsymbol{q})$. We thus obtain

[15]This can easily be checked on a Poisson distribution, for which the second derivative $\delta\ln\mathcal{F}[z]/\delta z(\boldsymbol{p})\delta z(\boldsymbol{q})$ is exactly zero. When we insert this in Eq. (24) and integrate over $\boldsymbol{p}$ and $\boldsymbol{q}$, we obtain $\langle N(N-1)\rangle = \langle N\rangle^2$—as expected for a Poisson distribution.

$$\frac{d^2N_2}{d^3\boldsymbol{p}d^3\boldsymbol{q}}\bigg|_{\rm NLO} = -\delta(\boldsymbol{p}-\boldsymbol{q})\frac{dN}{d^3\boldsymbol{p}}\bigg|_{\rm LO} + [\mathcal{L}_1+\mathcal{L}_2]\frac{dN}{d^3\boldsymbol{p}}\bigg|_{\rm LO}\frac{dN}{d^3\boldsymbol{q}}\bigg|_{\rm LO}, \tag{45}$$

where $\mathcal{L}_1$ and $\mathcal{L}_2$ were both introduced previously in Eq. (28).

We shall now discuss the logarithmic singularities in this expression. First, $\delta(\boldsymbol{p}-\boldsymbol{q})(dN/d^3\boldsymbol{p})_{\rm LO}$ does not contain large logarithms in x because these logs start appearing at NLO in the single gluon spectrum. Because we are restricting our discussion to leading logs, we can therefore discard this term henceforth. The logarithmic divergences in the second and third terms of the r.h.s. of Eq. (45) can be extracted straightforwardly by using the main result of Paper I,

$$\mathcal{L}_1+\mathcal{L}_2 \underset{\rm LLog}{=} \ln\left(\frac{\Lambda^+}{M^+}\right)\mathcal{H}_1 + \ln\left(\frac{\Lambda^-}{M^-}\right)\mathcal{H}_2. \tag{46}$$

Here $\mathcal{H}_{1,2}$ are the JIMWLK Hamiltonians of the nuclei moving in the $+z$ and $-z$ directions, respectively [1,5–7]; $\Lambda^\pm$ represent the longitudinal momenta that separate the static color sources $\rho_{1,2}$ in each of the nuclei, respectively, from the gauge fields that produce gluons at the rapidity of interest; and $M^\pm$ correspond to the typical longitudinal momentum scales of the object (the 2-gluon spectrum in this case) to which the operator is applied. From Eq. (46) we obtain

$$\frac{d^2N_2}{d^3\boldsymbol{p}d^3\boldsymbol{q}}\bigg|_{\rm LO+NLO} \underset{\rm LLog}{=} \left[1+\ln\left(\frac{\Lambda^+}{M^+}\right)\mathcal{H}_1+\ln\left(\frac{\Lambda^-}{M^-}\right)\mathcal{H}_2\right] \times\frac{dN}{d^3\boldsymbol{p}}\bigg|_{\rm LO}\frac{dN}{d^3\boldsymbol{q}}\bigg|_{\rm LO}. \tag{47}$$

All of our discussion thus far has been for a fixed distribution of sources $\rho_{1,2}$ in the two nuclei. The CGC effective theory [2–4,9–16] prescribes to average physical quantities over all the possible configurations $\rho_{1,2}$ of the fast color sources representing the projectiles, with gauge invariant weight functionals $W[\rho_{1,2}]$ that describe the probability of each configuration. When we integrate Eq. (47) over $\rho_{1,2}$, we can exploit the Hermiticity of the JIMWLK Hamiltonians $\mathcal{H}_{1,2}$ in order to integrate by parts, so that the Hamiltonians are now acting on the distributions $W[\rho_{1,2}]$. By reproducing the arguments developed in Paper I for the single gluon spectrum, we finally obtain the factorization formula for inclusive 2-gluon production,

$$\left\langle\frac{d^2N_2}{d^3\boldsymbol{p}d^3\boldsymbol{q}}\right\rangle_{\rm LLog} = \int[D\rho_1][D\rho_2]W_{Y_1}[\rho_1]W_{Y_2}[\rho_2]\frac{dN}{d^3\boldsymbol{p}}\bigg|_{\rm LO} \times\frac{dN}{d^3\boldsymbol{q}}\bigg|_{\rm LO}, \tag{48}$$

at leading log accuracy. Here the distributions $W[\rho_{1,2}]$ obey the JIMWLK equation

$$\frac{\partial W_Y[\rho]}{\partial Y} = \mathcal{H}W_Y[\rho], \tag{49}$$

and are evolved thus from nonperturbative initial conditions at the beam rapidities to the rapidities $Y_1 = \ln(\sqrt{s}/M^+)$ and $Y_2 = \ln(\sqrt{s}/M^-)$, respectively. In the regime where gluon radiation between the two tagged gluons is small, this formula resums all leading logarithms of $1/x_{1,2}$ as well as all the rescattering corrections in $(g\rho_{1,2})^n$ to all orders.

We now address the primary limitation of the present calculation. As the previous discussion hints, it is valid when the momenta $\boldsymbol{p}$ and $\boldsymbol{q}$ of the two observed gluons are close enough in rapidity so that they have similar longitudinal components. More precisely, we need to have

$$\alpha_s\ln\left(\frac{p^+}{q^+}\right)\ll 1, \qquad \alpha_s\ln\left(\frac{p^-}{q^-}\right)\ll 1. \tag{50}$$

If this is the case, we can simply take $M^\pm$ to be the common value[16] of $p^\pm$, $q^\pm$. Physically, the condition of Eq. (50) means that the probability of radiating a gluon between the two measured gluons is small. When the rapidity separation between the two gluons is large such that Eq. (50) is violated, we need to resum gluon emissions between the tagged gluons; this would require a generalization of the present formalism, which is not discussed here.

E. Factorization and the ridge in AA collisions

A striking ridge structure has been revealed in studies of the near-side spectrum of correlated pairs of hadrons by the STAR Collaboration [20–22]. The spectrum of correlated pairs on the near side of the detector (defined by an accompanying unquenched jet spectrum) extends across the entire detector acceptance in pseudorapidity of order $\Delta\eta\sim 2$ units but is strongly collimated for azimuthal angles $\Delta\phi$. Preliminary analyses of measurements by the PHENIX [23] and PHOBOS [24] collaborations appear to corroborate the STAR results. In the latter case, with a high momentum trigger, the ridge is observed to span the wider PHOBOS acceptance in pseudorapidity of $\Delta\eta\sim 4$ units.

In Ref. [19], it was argued that the ridge is formed as a consequence of both long range rapidity correlations that are generic in hadronic and nuclear collisions at high energies, plus the radial flow of the hot partonic matter that is specific to high energy nuclear collisions. Let us first focus on the long range correlations that are essential to this picture—how are they generated?

In the leading order formalism of the CGC, classical solutions of Yang-Mills equations are boost invariant [28–31]. Real-time numerical simulations [32–40] also demon-

[16]It is of course not necessary that p^+ and q^+ be equal, just that they are close enough so that it does not matter which value we choose between p^+ and q^+.

strate that the Yang-Mills fields form flux tubes of a typical transverse size $1/Q_s$ (where Q_s is the saturation scale) with parallel chromo-electric and chromo-magnetic field strengths. (An important consequence is that these Glasma fields [25] have nontrivial topological charge [41].) In Sec. III A, we showed that the leading order 2-gluon spectrum, for a fixed configuration of sources, was given by Eq. (26). Because each of the single particle distributions is boost invariant, the two-particle spectrum is also, at this order, independent of the rapidity separation of the gluons. While the two gluons are uncorrelated for a fixed configuration of sources, correlations are provided by the averaging over the source distributions. In Ref. [19], the source distribution was assumed to be Gaussian as in the McLerran-Venugopalan (MV) model [2–4]. The ridge spectrum was shown to have the simple form

$$\frac{\Delta\rho}{\sqrt{\rho_{\rm ref}}} \equiv C(\boldsymbol{p},\boldsymbol{q})\frac{\langle\frac{dN}{dy}\rangle}{\langle\frac{dN}{dy_p p dp d\phi_p}\rangle\langle\frac{dN}{dy_q q dq d\phi_q}\rangle} = \frac{K_N}{\alpha_s(Q_s)}, \quad (51)$$

where

$$C(\boldsymbol{p},\boldsymbol{q}) \equiv \left\langle\frac{dN_2}{dy_p d^2\boldsymbol{p}_\perp dy_q d^2\boldsymbol{q}_\perp}\right\rangle - \left\langle\frac{dN}{dy_p d^2\boldsymbol{p}_\perp}\right\rangle \times\left\langle\frac{dN}{dy_q d^2\boldsymbol{q}_\perp}\right\rangle, \quad (52)$$

and K_N is a number of order unity. For further details, we refer the reader to Ref. [19].

There are several conceptual issues in this context. First, how does one justify this averaging procedure for the 2-gluon spectrum from first principles? Second, how does one take into account the energy evolution of the sources? And finally, do NLO contributions spoil this picture? The results in this paper solve most of these conceptual issues. Our result in Eq. (48) shows that the trivial LO result of Eq. (26) can be promoted to a full leading log result simply by averaging it over the sources $\rho_{1,2}$—with distributions of sources that evolve according to the JIMWLK equation. Most importantly, this shows that the higher order corrections, to leading logs in $x_{1,2}$, do not spoil the form in Eq. (51) of the Glasma flux tube picture. Moreover, this factorization provides compelling evidence that it is a robust result beyond LO. As discussed previously, this picture will have to be modified when the rapidity separation between the gluons is greater than $\alpha_s^{-1}(Q_s)$.

These initial state considerations are not affected by the final state transverse flow of the Glasma flux tubes, which is the other important feature determining the near-side ridge seen in heavy ion collisions. It has been shown very recently that a proper treatment of flow and hadronization effects of the Glasma flux tubes provides excellent quantitative agreement with the RHIC data on the dependence of the ridge amplitude on centrality and as a function of energy, as well as the angular width of the ridge as a function of centrality [42]. Further sophisticated treatments of both the initial state effects discussed here and the final state effects discussed in Ref. [42] therefore open the door to quantitative 3-D imaging of heavy ion collisions. A deeper relation between initial and final state effects, as outlined in Paper I, can be obtained by studying quantum fluctuations at NLO that are not accompanied by logs in $x_{1,2}$, but grow rapidly in time [43,44] in a manner analogous to plasma instabilities [45].

We should also mention that the initial state effects described here are also present in proton/deuteron-nucleus collisions [46–48], without the final state effects characteristic of the ridge in nucleus-nucleus collisions. These collisions are therefore useful in order to isolate the initial state effects and to corroborate the framework of multiparticle production in high energy QCD developed here.

IV. MULTIGLUON INCLUSIVE SPECTRUM

In this section, we will show how the results of the previous section modify multigluon probability distributions, with the caveat, as previously, that these gluons are emitted in a narrow rapidity window. We will also derive a simple expression for the differential probability of producing n gluons.

A. *n*-gluon spectrum at LO and NLO

Our starting point in evaluating the inclusive n-gluon spectrum is Eq. (8). Because we have thus far obtained expressions up to NLO for the first and second derivatives of $\ln\mathcal{F}[z]$, it is convenient to rewrite this expression as[17]

$$\underbrace{\frac{d^nN_n}{d^3\boldsymbol{p}_1\cdots d^3\boldsymbol{p}_n}}_{\mathcal{O}(\frac{1}{g^{2n}}+\cdots)} = \underbrace{\prod_{i=1}^n\frac{\delta\ln\mathcal{F}[z]}{\delta z(\boldsymbol{p}_i)}}_{\mathcal{O}(\frac{1}{g^{2n}}+\cdots)} + \underbrace{\sum_{i<j}\frac{\delta^2\ln\mathcal{F}[z]}{\delta z(\boldsymbol{p}_i)\delta z(\boldsymbol{p}_j)}\prod_{k\neq i,j}\frac{\delta\ln\mathcal{F}[z]}{\delta z(\boldsymbol{p})k}}_{\mathcal{O}(\frac{1}{g^{2(n-1)}}+\cdots)} + \cdots \quad (53)$$

Because $\ln\mathcal{F}[z] = \mathcal{O}(g^{-2})$ in our power counting, the LO term in the r.h.s. is of order g^{-2n}, the NLO term is of order $g^{-2(n-1)}$, and next-to-next-to-leading order and higher terms represented by the ellipses are omitted at the level of the present discussion. The n-gluon spectra on the l.h.s. of Eq. (53) are quantities that, for $n > 1$, are given by the first term on the r.h.s. By computing them to NLO we gain access to the first correction to the Poisson distribution, the deviation of the variance of the multiplicity distribution from the Poissonian result $\langle N(N-1)\rangle = \langle N\rangle^2$, and the corresponding modifications for the higher moments of

[17]This formula is obtained by replacing $\mathcal{F}[z]$ by $\exp(\ln\mathcal{F}[z])$ in Eq. (8).

the distribution. We refer to Appendix C for a more detailed discussion of the interpretation of our result for the probability distribution of the gluon multiplicity.

At leading order, only the first term contributes, and we obtain (for a fixed distribution of sources)

$$\frac{d^n N_n}{d^3\boldsymbol{p}_1\cdots d^3\boldsymbol{p}_n}\Big|_{\rm LO} = \prod_{i=1}^{n}\frac{dN}{d^3\boldsymbol{p}_i}\Big|_{\rm LO}. \tag{54}$$

At next-to-leading order, we have

$$\frac{d^n N_n}{d^3\boldsymbol{p}_1\cdots d^3\boldsymbol{p}_n}\Big|_{\rm NLO} = \sum_{i=1}^{n}\frac{dN}{d^3\boldsymbol{p}_i}\Big|_{\rm NLO}\prod_{j\neq i}\frac{dN}{d^3\boldsymbol{p}_j}\Big|_{\rm LO} + \sum_{i<j}\frac{\delta^2\ln\mathcal{F}[z]}{\delta z(\boldsymbol{p}_i)\delta z(\boldsymbol{p}_j)}\Big|_{\rm LO}\prod_{k\neq i,j}\frac{dN}{d^3\boldsymbol{p}_k}\Big|_{\rm LO}. \tag{55}$$

All the objects that appear in this equation are known already from the discussion in Paper I and the previous section. In [1], we showed that

$$\frac{dN}{d^3\boldsymbol{p}}\Big|_{\rm NLO} = [\mathcal{L}_1 + \mathcal{L}_2]\frac{dN}{d^3\boldsymbol{p}}\Big|_{\rm LO} + \Delta N_{\rm NLO}(\boldsymbol{p}), \tag{56}$$

where $\mathcal{L}_1$ and $\mathcal{L}_2$ are defined in Eq. (28). In the previous section, we showed that[18]

$$\frac{\delta^2\ln\mathcal{F}[z]}{\delta z(\boldsymbol{p})\delta z(\boldsymbol{q})}\Big|_{\rm LO} = [\mathcal{L}_2]_{\rm connected}\frac{dN}{d^3\boldsymbol{p}}\Big|_{\rm LO}\frac{dN}{d^3\boldsymbol{q}}\Big|_{\rm LO}, \tag{57}$$

where we remind the reader that the subscript "connected" attached to the operator $\mathcal{L}_2$ indicates that the two operators $\mathbb{T}$ it contains do not simultaneously act on the same object.

B. Leading log resummation

If we combine the terms in Eqs. (56) and (57), we get simply

$$\frac{d^n N_n}{d^3\boldsymbol{p}_1\cdots d^3\boldsymbol{p}_n}\Big|_{\rm NLO} \underset{\rm LLog}{=} [\mathcal{L}_1 + \mathcal{L}_2]\prod_{i=1}^{n}\frac{dN}{d^3\boldsymbol{p}_i}\Big|_{\rm LO}. \tag{58}$$

Using again Eq. (46) and following the steps that lead from Eq. (47) to Eq. (48), we arrive at the all-order leading log n-gluon spectrum

$$\Big\langle\frac{d^n N_n}{d^3\boldsymbol{p}_1\cdots d^3\boldsymbol{p}_n}\Big\rangle_{\rm LLog} = \int[D\rho_1][D\rho_2]W_{Y_1}[\rho_1]W_{Y_2}[\rho_2] \times\frac{dN}{d^3\boldsymbol{p}_1}\Big|_{\rm LO}\cdots\frac{dN}{d^3\boldsymbol{p}_n}\Big|_{\rm LO}. \tag{59}$$

Once again, one needs all the rapidity differences between the n-measured gluons to be much smaller than α_s^{-1}, to ensure all leading logarithmic contributions are resummed by this formula.

[18] We are ignoring the term $-\delta(\boldsymbol{p}-\boldsymbol{q})\frac{dN}{d^3\boldsymbol{p}}|_{\rm LO}$ because it does not contribute in the leading logarithmic approximation in x as discussed previously.

C. Generating functional in a small rapidity slice

Equation (60) provides a complete description of gluon production in the leading log x approximation when one considers a slice in rapidity of width $\Delta Y \ll \alpha_s^{-1}$. One can summarize these results into a generating functional $\mathcal{F}_{Y,\Delta Y}[z(\boldsymbol{p})]$ defined from the "master" $\mathcal{F}[z(\boldsymbol{p})]$ as

$$\mathcal{F}_{Y,\Delta Y}[z(\boldsymbol{p})] = \mathcal{F}[z^*(\boldsymbol{p})] \quad \text{with} \quad \begin{cases} z^*(\boldsymbol{p}) &= z(\boldsymbol{p}) \quad \text{if } y_p \in [Y-\frac{\Delta Y}{2}, Y+\frac{\Delta Y}{2}] \\ z^*(\boldsymbol{p}) &= 1 \quad \text{otherwise.}\end{cases} \tag{60}$$

Setting the argument of the generating functional to unity outside of the phase-space region of interest means that we define observables that are completely inclusive with respect to this unobserved part of the phase space.

We see from Eq. (9) that $\mathcal{F}[z^*(\boldsymbol{p})]$ can be obtained by multiplying Eq. (59) by $(z^*(\boldsymbol{p}_1)-1)\cdots(z^*(\boldsymbol{p}_n)-1)/n!$, integrating over the n-gluon phase space and summing over n. Because $z^*(\boldsymbol{p})$ is unity outside of the strip of width ΔY in rapidity, the n-gluon spectrum outside of the regime of validity of Eq. (59) is not needed. This procedure leads to a simple exponentiation of the leading log factorized formula for the generating functional $\mathcal{F}_{Y,\Delta Y}$ as

$$\langle\mathcal{F}_{Y,\Delta Y}[z(\boldsymbol{p})]\rangle_{\rm LLog} = \int[D\rho_1][D\rho_2]W_{Y_{\rm beam}-Y}[\rho_1] \times W_{Y_{\rm beam}+Y}[\rho_2]\times\exp\Big[\int_{Y-(\Delta Y/2)}^{Y+(\Delta Y/2)} \times d^3\boldsymbol{p}(z(\boldsymbol{p})-1)\frac{dN}{d^3\boldsymbol{p}}\Big|_{\rm LO}\Big]. \tag{61}$$

This leading log result for the generating functional, in turn, allows us to extract the corresponding formula for the differential probability of producing exactly n gluons in the rapidity slice of interest. This gives

$$\Big\langle\frac{d^n P_n}{d^3\boldsymbol{p}_1\cdots d^3\boldsymbol{p}_n}\Big\rangle_{\rm LLog} = \int[D\rho_1][D\rho_2]W_{Y_{\rm beam}-Y}[\rho_1] \times W_{Y_{\rm beam}+Y}[\rho_2]\frac{1}{n!}\frac{dN}{d^3\boldsymbol{p}_1}\Big|_{\rm LO} \times\cdots\frac{dN}{d^3\boldsymbol{p}_n}\Big|_{\rm LO} \times\exp\Big[-\int_{Y-(\Delta Y/2)}^{Y+(\Delta Y/2)}d^3\boldsymbol{p}\frac{dN}{d^3\boldsymbol{p}}\Big|_{\rm LO}\Big]. \tag{62}$$

This simple result, valid, we emphasize, in the leading log approximation, suggests that the particle distribution in a small rapidity slice can be written as the average over $\rho_{1,2}$ of a Poisson distribution with the leading log corrections completely factorized into the JIMWLK evolution of the sources. Note that, despite the fact that the integrand contains a Poisson distribution, the l.h.s. of Eq. (62) is not a Poisson distribution after the integration over the sources, because particles produced uncorrelated in each configuration of ρ_1 and ρ_2 are correlated in the averaged distribution because of the correlations among the color sources.[19]

In general, even for a fixed distribution of sources, the probability distribution is not Poissonian [17]. To some extent, the fact that we get a Poissonian functional form in the integrand of Eq. (62) is a consequence of the way we have organized our calculation. In Eq. (53) we are performing a weak coupling expansion of the moments $\langle N(N-1)\cdot\ldots\cdot(N-n+1)\rangle$, that includes the orders g^{-2n} and the leading log part of the order $g^{-2(n-1)}$. Terms starting at the order $g^{-2(n-2)}$ are beyond the accuracy of our calculation, and therefore their values in our formulas are arbitrary. The arbitrariness of these subleading terms influences the precise form of the resulting generating functional. For example, if we had performed the weak coupling expansion of $\langle N^n\rangle$ instead of $\langle N(N-1)\ldots(N-n+1)\rangle$, we would have obtained a different generating functional. Of course, the two generating functionals so obtained would lead to the same moments of the distribution to the order of our calculation. The nontrivial aspect of our result in Eq. (62) is that all the deviations from a Poisson distribution that result from the large logarithms of x at NLO can be factorized into the JIMWLK evolution of the sources. Equation (62) shows how these corrections modify the n-gluon production probabilities. The Poissonian nature of the multiplicity distribution and deviations from it are discussed in more detail in Appendix C.

V. CONCLUSION AND OUTLOOK

We demonstrated in this paper that our result of Paper I on initial state JIMWLK factorization for the single inclusive gluon spectrum in nucleus-nucleus collisions can be extended to inclusive multigluon spectra. Our result is valid provided all the gluons are produced in a rapidity window of width $\Delta Y \lesssim \alpha_s^{-1}$. Our final result for the generating functional for multigluon production, in the leading logarithmic approximation in $x_{1,2}$, is very simple; the distribution of gluons produced in the stated rapidity window can be written as the average over the JIMWLK-evolved distributions of sources of a Poisson distribution. It is important to keep in mind that the result of this source average is not a Poisson distribution, due to the correlations between the evolved color sources.

As we discussed in Sec. III E, our results are of great interest in detailed imaging of the space-time evolution of nucleus-nucleus collisions. An important ingredient in future studies will be to extend the present result to the case of correlations between gluons produced at rapidity differences $\alpha_s^{-1} \lesssim \Delta Y$. A full leading log computation of these initial long range rapidity correlations requires that one identifies and resums the additional large logarithmic corrections that may arise when the rapidities in the 2-gluon spectrum are widely separated. In the dilute regime, it was noted in [49] that $k_\perp$-factorization is broken when two gluons are produced with a large rapidity separation between them; therefore, it will be interesting to see whether the same conclusion holds in the JIMWLK framework discussed here.

An important caveat (also applicable to our previous study of the single gluon spectrum in nucleus-nucleus collisions) is that final state effects, related to the growth of unstable fluctuations, need to be resummed. While the details are still unknown, the structure of the result is known. The result of the resummation of unstable fluctuations, as shown in Paper I, can be expressed as

$$\begin{aligned}\langle \mathcal{O}\rangle_{\rm LLog+LInst} = &\int [D\tilde{\mathcal{A}}_1^+][D\tilde{\mathcal{A}}_2^-]W_{Y_1}[\tilde{\mathcal{A}}_1^+]W_{Y_2}[\tilde{\mathcal{A}}_2^-]\\ &\times\int [Da(\vec{\boldsymbol{u}})]\tilde{Z}[a(\vec{\boldsymbol{u}})]\\ &\times \mathcal{O}_{\rm LO}[\tilde{\mathcal{A}}_1^+ + a, \tilde{\mathcal{A}}_2^- + a]. \end{aligned} \tag{63}$$

Here, we have traded the sources $\tilde{\rho}_{1,2}$ in covariant gauge for the corresponding gauge fields $\tilde{\mathcal{A}}^{\pm}_{1,2} \equiv \frac{1}{\nabla_\perp^2}\tilde{\rho}_{1,2}$. The functional $\tilde{Z}[a(\vec{\boldsymbol{u}})]$ is the spectrum of small fluctuations of the classical field on the forward light cone. In Paper I, $\mathcal{O}$ corresponded to the single inclusive spectrum, but this formula also applies to the multigluon spectrum because the proof does not depend on the nature of the observable being measured. However, the complete functional form of $\tilde{Z}[a(\vec{\boldsymbol{u}})]$ is still unknown—for a first attempt, see Ref. [50].

These considerations are eased somewhat if we take the "dilute-dense" limit of proton/deuteron-nucleus collisions because we do not expect instabilities to play a major role in that case. Several studies have been performed in this limit [46–48,51–53]. A particular focus is on the applicability of the so-called Abramovsky-Gribov-Kancheli cutting rules [49,54–56]. We plan to address these issues in a future work.

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APPENDIX A: FOURIER COEFFICIENTS OF SMALL FLUCTUATION FIELDS

We will outline here the solution to the system of equations

[19]For instance, two color sources may be correlated because they result from the splitting of a common "ancestor" in the course of JIMWLK evolution.

$$\sum_{\lambda,a}\int_{\boldsymbol{k}}[\gamma^{\boldsymbol{k}\lambda a}_{-,\boldsymbol{q}}h^{(+)}_{-\boldsymbol{k}\lambda a}(\boldsymbol{p}\zeta b)-\gamma^{\boldsymbol{k}\lambda a}_{+,\boldsymbol{q}}h^{(+)}_{+\boldsymbol{k}\lambda a}(\boldsymbol{p}\zeta b)] = \delta(\boldsymbol{p}-\boldsymbol{q})f^{(+)}(\boldsymbol{p}\zeta b),$$
$$\sum_{\lambda,a}\int_{\boldsymbol{k}}[\gamma^{\boldsymbol{k}\lambda a}_{+,\boldsymbol{q}}h^{(-)}_{+\boldsymbol{k}\lambda a}(\boldsymbol{p}\zeta b)-\gamma^{\boldsymbol{k}\lambda a}_{-,\boldsymbol{q}}h^{(-)}_{-\boldsymbol{k}\lambda a}(\boldsymbol{p}\zeta b)] = \delta(\boldsymbol{p}-\boldsymbol{q})f^{(-)}(\boldsymbol{p}\zeta b), \tag{A1}$$

that was obtained in Eq. (39). We had previously derived analogous equations in the case of a simpler scalar theory in [17]. However, in [17], we did not manage to solve these equations, and suggested that one may have to solve them numerically. It turns out that one can in fact obtain an analytical solution of Eqs. (39), thanks to the relations

$$\begin{aligned}
&\sum_{\lambda,a}\int_{\boldsymbol{k}}[h^{(+)}_{-\boldsymbol{k}\lambda a}(\boldsymbol{p}\xi b)h^{(-)}_{+\boldsymbol{k}\lambda a}(\boldsymbol{q}\zeta c)-h^{(+)}_{+\boldsymbol{k}\lambda a}(\boldsymbol{p}\xi b)h^{(-)}_{-\boldsymbol{k}\lambda a}(\boldsymbol{q}\zeta c)] = (2\pi)^3\delta_{\xi\zeta}\delta_{bc}2E_{\boldsymbol{p}}\delta(\boldsymbol{p}-\boldsymbol{q}),\\
&\sum_{\lambda,a}\int_{\boldsymbol{k}}[h^{(-)}_{+\boldsymbol{k}\lambda a}(\boldsymbol{p}\xi b)h^{(+)}_{-\boldsymbol{k}\lambda a}(\boldsymbol{q}\zeta c)-h^{(-)}_{-\boldsymbol{k}\lambda a}(\boldsymbol{p}\xi b)h^{(+)}_{+\boldsymbol{k}\lambda a}(\boldsymbol{q}\zeta c)] = (2\pi)^3\delta_{\xi\zeta}\delta_{bc}2E_{\boldsymbol{p}}\delta(\boldsymbol{p}-\boldsymbol{q}),\\
&\sum_{\lambda,a}\int_{\boldsymbol{k}}[h^{(+)}_{+\boldsymbol{k}\lambda a}(\boldsymbol{p}\xi b)h^{(+)}_{-\boldsymbol{k}\lambda a}(\boldsymbol{q}\zeta c)-h^{(+)}_{-\boldsymbol{k}\lambda a}(\boldsymbol{p}\xi b)h^{(+)}_{+\boldsymbol{k}\lambda a}(\boldsymbol{q}\zeta c)] = 0,\\
&\sum_{\lambda,a}\int_{\boldsymbol{k}}[h^{(-)}_{+\boldsymbol{k}\lambda a}(\boldsymbol{p}\xi b)h^{(-)}_{-\boldsymbol{k}\lambda a}(\boldsymbol{q}\zeta c)-h^{(-)}_{-\boldsymbol{k}\lambda a}(\boldsymbol{p}\xi b)h^{(-)}_{+\boldsymbol{k}\lambda a}(\boldsymbol{q}\zeta c)] = 0.
\end{aligned}\tag{A2}$$

These relations are the mathematical consequence of the unitary temporal evolution of small fluctuations on top of the classical field $\mathcal{A}(x)$. In particular, an orthonormal basis of solutions of Eq. (33) remains orthonormal at any later time. A proof of these formulas is presented in Appendix B. Thanks to these relations, it is easy to invert the system of equations (39), and one gets

$$\gamma^{\boldsymbol{k}\lambda a}_{+,\boldsymbol{q}} = \frac{1}{(2\pi)^3 2E_{\boldsymbol{q}}}\sum_{\zeta,b}[h^{(-)}_{-\boldsymbol{k}\lambda a}(\boldsymbol{q}\zeta b)f^{(+)}(\boldsymbol{q}\zeta b) + h^{(+)}_{-\boldsymbol{k}\lambda a}(\boldsymbol{q}\zeta b)f^{(-)}(\boldsymbol{q}\zeta b)],$$
$$\gamma^{\boldsymbol{k}\lambda a}_{-,\boldsymbol{q}} = \frac{1}{(2\pi)^3 2E_{\boldsymbol{q}}}\sum_{\zeta,b}[h^{(-)}_{+\boldsymbol{k}\lambda a}(\boldsymbol{q}\zeta b)f^{(+)}(\boldsymbol{q}\zeta b) + h^{(+)}_{+\boldsymbol{k}\lambda a}(\boldsymbol{q}\zeta b)f^{(-)}(\boldsymbol{q}\zeta b)]. \tag{A3}$$

APPENDIX B: UNITARY EVOLUTION OF SMALL FLUCTUATIONS

Consider the partial differential equation

$$\left[(\Box_x g_\mu{}^\nu - \partial_{x\mu}\partial_x^\nu)\delta^{ab} - \frac{\partial U(\mathcal{A}_\epsilon)}{\partial\mathcal{A}_{\epsilon a\nu}(x)\partial\mathcal{A}^\mu_{\epsilon b}(x)}\right]a^{\mu b}(x) = 0, \tag{B1}$$

where we have written explicitly all the color indices. We assume that the background color field in which the wave propagates is real. For a generic solution $a(x)$ of this equation, define the following vectors:

$$|\boldsymbol{a}\rangle \equiv \begin{pmatrix} a^{\mu a}(x)\\ \dot a^{\mu a}(x)\end{pmatrix}, \qquad \langle\boldsymbol{a}| \equiv \begin{pmatrix} a^{*\mu a}(x) & \dot a^{*\mu a}(x)\end{pmatrix}, \tag{B2}$$

where the dot means a derivative with respect to time. Then, it is trivial to check that the following "scalar product,"

$$\langle\boldsymbol{a}_1|\boldsymbol{\sigma}_1|\boldsymbol{a}_2\rangle \equiv ig_{\mu\nu}\delta_{ab}\int d^3\boldsymbol{x}[\dot a_1^{*\mu a}(x)a_2^{\nu b}(x) - a_1^{*\mu a}(x)\dot a_2^{\nu b}(x)], \tag{B3}$$

where $\boldsymbol{\sigma}_2$ is the second Pauli matrix, is independent of time when a_1^μ and a_2^μ are two solutions of Eq. (B1).

Then, if the $a_{\pm\boldsymbol{k}\lambda a}(x)$ are the retarded solutions of Eq. (B1) whose initial conditions at $x^0\to-\infty$ are $\epsilon^\mu_\lambda(\boldsymbol{k})T^a e^{\pm ik\cdot x}$, one can check explicitly that

$$\begin{aligned}
&\langle\boldsymbol{a}_{+\boldsymbol{k}\lambda a}|\boldsymbol{\sigma}_2|\boldsymbol{a}_{+\boldsymbol{k}'\lambda'a'}\rangle = (2\pi)^3 2E_{\boldsymbol{k}}\delta_{\lambda\lambda'}\delta_{aa'}\delta(\boldsymbol{k}-\boldsymbol{k}'),\\
&\langle\boldsymbol{a}_{-\boldsymbol{k}\lambda a}|\boldsymbol{\sigma}_2|\boldsymbol{a}_{-\boldsymbol{k}'\lambda'a'}\rangle = -(2\pi)^3 2E_{\boldsymbol{k}}\delta_{\lambda\lambda'}\delta_{aa'}\delta(\boldsymbol{k}-\boldsymbol{k}'),\\
&\langle\boldsymbol{a}_{+\boldsymbol{k}\lambda a}|\boldsymbol{\sigma}_2|\boldsymbol{a}_{-\boldsymbol{k}'\lambda'a'}\rangle = \langle\boldsymbol{a}_{-\boldsymbol{k}\lambda a}|\boldsymbol{\sigma}_2|\boldsymbol{a}_{+\boldsymbol{k}'\lambda'a'}\rangle = 0.
\end{aligned}\tag{B4}$$

(Since all these scalar products are time independent, it is sufficient to check these relations by calculating the integral in the r.h.s. of Eq. (B3) for the corresponding initial conditions.)

Consider now a generic solution $a^\mu(x)$ of Eq. (B1). Since the solutions $a^\mu_{\pm\boldsymbol{k}\lambda a}(x)$ span the entire space of solutions, we can write

$$|\boldsymbol{a}\rangle \equiv \sum_{\lambda,a}\int_{\boldsymbol{k}}[\gamma_-^{\boldsymbol{k}\lambda a}|\boldsymbol{a}_{-\boldsymbol{k}\lambda a}\rangle + \gamma_+^{\boldsymbol{k}\lambda a}|\boldsymbol{a}_{+\boldsymbol{k}\lambda a}\rangle], \tag{B5}$$

where the coefficients $\gamma_\pm^{\boldsymbol{k}\lambda a}$ do not depend on time. By using the orthogonality relations obeyed by the vectors $|\boldsymbol{a}_{\pm\boldsymbol{k}\lambda a}\rangle$, one obtains

$$\gamma_-^{\boldsymbol{k}\lambda a} = -\langle\boldsymbol{a}_{-\boldsymbol{k}\lambda a}|\boldsymbol{\sigma}_2|\boldsymbol{a}\rangle, \qquad \gamma_+^{\boldsymbol{k}\lambda a} = \langle\boldsymbol{a}_{+\boldsymbol{k}\lambda a}|\boldsymbol{\sigma}_2|\boldsymbol{a}\rangle. \tag{B6}$$

Inserting these relations back into Eq. (B5), one gets the following identity,

$$\sum_{\lambda,a}\int_{\boldsymbol{k}}[|\boldsymbol{a}_{+\boldsymbol{k}\lambda a}\rangle\langle\boldsymbol{a}_{+\boldsymbol{k}\lambda a}| - |\boldsymbol{a}_{-\boldsymbol{k}\lambda a}\rangle\langle\boldsymbol{a}_{-\boldsymbol{k}\lambda a}|] = \boldsymbol{\sigma}_2 g^{\mu\nu}\delta^{bc}, \tag{B7}$$

which is valid over the space of solutions of Eq. (B1). [The Lorentz indices μ, ν and color indices b, c do not appear explicitly in the l.h.s., but are part of the definition of the vectors $|\boldsymbol{a}\rangle$ and $\langle\boldsymbol{a}|$—see Eq. (B2).] This relation is valid at all times, and is the expression of the fact that the unitary evolution of small fluctuations preserves the completeness of the set of states $|\boldsymbol{a}_{\pm\boldsymbol{k}\lambda a}\rangle$.

Let us now introduce states $|\boldsymbol{a}^0_{\pm\boldsymbol{k}\lambda a}\rangle$, that are the analogue of the states $|\boldsymbol{a}_{\pm\boldsymbol{k}\lambda a}\rangle$ in the vacuum (i.e. when the background field is zero). Naturally, they are just plane waves $a^{0\mu}_{\pm\boldsymbol{k}\lambda a} = \epsilon^\mu_\lambda(\boldsymbol{k})T^a e^{\pm ik\cdot x}$ that we have introduced in order to perform the Fourier decomposition of classical fields and small fluctuations. The Fourier coefficients $h^{(\pm)}_{\pm\boldsymbol{k}\lambda a}(\boldsymbol{p}\zeta c)$ of the fluctuations $a^\mu_{\pm\lambda a}$ can be obtained as

$$\begin{aligned} h^{(+)}_{\pm\boldsymbol{k}\lambda a}(\boldsymbol{p}\zeta c) &= -\langle\boldsymbol{a}^0_{-\boldsymbol{p}\zeta c}|\boldsymbol{\sigma}_2|\boldsymbol{a}_{\pm\boldsymbol{k}\lambda a}\rangle,\\ h^{(-)}_{\pm\boldsymbol{k}\lambda a}(\boldsymbol{p}\zeta c) &= \langle\boldsymbol{a}^0_{+\boldsymbol{p}\zeta c}|\boldsymbol{\sigma}_2|\boldsymbol{a}_{\pm\boldsymbol{k}\lambda a}\rangle. \end{aligned} \tag{B8}$$

(These relations are valid only in the regions where the interactions are switched off, i.e. when $x^0 \to \pm\infty$. In the rest of the discussion, we are only interested in these Fourier coefficients in the limit $x^0 \to +\infty$.) By multiplying Eq. (B7) by $\langle\boldsymbol{a}^0_{\epsilon\boldsymbol{p}\xi b}|\boldsymbol{\sigma}_2$ on the left and by $\boldsymbol{\sigma}_2|\boldsymbol{a}^0_{\epsilon'\boldsymbol{q}\zeta c}\rangle$ on the right and using $(h^{\epsilon}_{\epsilon'\boldsymbol{k}\lambda a}(\boldsymbol{p}\zeta c))^* = h^{-\epsilon}_{-\epsilon'\boldsymbol{k}\lambda a}(\boldsymbol{p}\zeta c)$, we obtain the following relation among these Fourier coefficients:

$$\begin{aligned} \sum_{\lambda,a}\int_{\boldsymbol{k}}&[h^{(-\epsilon)}_{+\boldsymbol{k}\lambda a}(\boldsymbol{p}\xi b)h^{(+\epsilon')}_{-\boldsymbol{k}\lambda a}(\boldsymbol{q}\zeta c)\\ &- h^{(-\epsilon)}_{-\boldsymbol{k}\lambda a}(\boldsymbol{p}\xi b)h^{(+\epsilon')}_{+\boldsymbol{k}\lambda a}(\boldsymbol{q}\zeta c)]\\ &= \delta_{\epsilon\epsilon'}\epsilon(2\pi)^3\delta_{\xi\zeta}\delta_{bc}2E_{\boldsymbol{p}}\delta(\boldsymbol{p}-\boldsymbol{q}), \end{aligned} \tag{B9}$$

which is nothing but a compact way of writing the four equations (A2).

APPENDIX C: POISSON DISTRIBUTION

At first sight, Eq. (62) appears to be the average over the distributions of sources of a Poisson distribution. This seems to contradict a result we stressed in [17], that the distribution of multiplicities calculated in a fixed configuration of sources $\rho_{1,2}$ is not a Poisson distribution. For the sake of the discussion in this appendix, let us introduce the generating function $F(z)$ for the multiplicity distribution in the region of rapidity $[Y - \Delta Y/2, Y + \Delta Y/2]$. In the language of the present paper, it is obtained by using in Eq. (61) a constant function $z(\boldsymbol{p})$ whose value is equal to the number z.

Consider first this generating function for a given configuration $\rho_{1,2}$ of the external color sources. In [17], $F(z)$ was parametrized as[20]

$$\ln F(z) \equiv \sum_{r=1}^{\infty} b_r(z^r - 1), \tag{C1}$$

and we had obtained the formula for the probability P_n of producing n particles in the portion of phase space under consideration to be

$$P_n = e^{-\sum_r b_r}\sum_{p=1}^{n}\frac{1}{p!}\sum_{r_1+\cdots+r_p=n} b_{r_1}\cdots b_{r_p}. \tag{C2}$$

In Ref. [17], we also showed that b_r is the sum of all the cut connected vacuum graphs, where exactly r internal lines are cut. Because b_r is a sum of *connected* graphs, it has a perturbative expansion that starts at the order $1/g^2$,

$$b_r = \frac{1}{g^2}\oplus 1 \oplus g^2 \oplus \cdots \tag{C3}$$

In particular, all the b_r have *a priori* the same order of magnitude. However, it is easy to see that Eq. (C2) is a Poisson distribution only in the exceptional case where[21]

$$b_1 \neq 0, \qquad b_r = 0 \quad \text{for } r \geq 2. \tag{C4}$$

Since for a generic field theory the b_r for $r \geq 2$ have no reason to vanish or to be smaller than b_1, the distribution of the multiplicities in a fixed configuration of sources is, in general, not a Poisson distribution. Moreover, since $b_{2,3,\cdots}$ are of the same order in g^2 as b_1, the deviations from a Poisson distribution are an effect of order unity, not a subleading correction.

In order to make the connection with the present paper easier, it is preferable to parametrize $F(z)$ as

$$\ln F(z) \equiv \sum_{k=1}^{\infty} c_k(z-1)^k. \tag{C5}$$

[This series starts at the index $k = 1$, because $F(1) = 0$.] The numbers c_k are related to the numbers b_k by

$$b_r = \sum_{k=r}^{\infty}\binom{k}{r}(-1)^{k-r}c_k, \qquad c_k = \sum_{r=k}^{\infty}\binom{r}{k}b_r, \tag{C6}$$

where the $\binom{k}{r}$ are the binomial coefficients. The derivatives of $\ln F(z)$ evaluated at $z = 1$ are best expressed in terms of the coefficients c_k as

$$\left.\frac{\partial^k \ln F(z)}{\partial z^k}\right|_{z=1} = k!c_k. \tag{C7}$$

[20]Compared to the notations used in [17], we absorb the factors of $1/g^2$ into the definition of the numbers b_r.

[21]From Eq. (C2) and the definition of the b_r, we have $F(z) \equiv \sum_n z^n P_n$. Then, it is immediate to check that $\ln F(z)$ should be a polynomial of degree 1 in the case of a Poisson distribution.

Let us now rephrase our results in this language. The inclusive n-particle spectrum is the nth derivative of $F(z)$ at $z = 1$. These derivatives read

$$F^{(1)}(1) = c_1, \qquad F^{(2)}(1) = c_1^2 + 2c_2, \\ F^{(3)}(1) = c_1^3 + 6c_1c_2 + 6c_3, \cdots \tag{C8}$$

All the coefficients c_k are sums of connected vacuum graphs, and therefore start at the order $1/g^2$, up to logarithms. At leading order, we thus keep only

$$F^{(n)}(1)|_{\rm LO} = [c_1]^n_{\rm LO}. \tag{C9}$$

At this order of truncation, one can obviously get a Poisson distribution, since this approximation is compatible with $c_2 = c_3 = \cdots = 0$, i.e. $b_2 = b_3 = \cdots = 0$. However, the coefficients $b_{2,3,\ldots}$ could have any value of order g^{-2} without affecting our leading order truncation. The arbitrary choice one is allowed to make for these subleading terms, in general, alters the Poissonian nature of the distribution.

The actual paradox arises only at the next-to-leading order. There, one keeps the terms

$$F^{(n)}(1)|_{\rm LO+NLO} = \underbrace{[c_1]^n_{\rm LO}}_{g^{-2n}} + \underbrace{n[c_1]^{n-1}_{\rm LO}[c_1]_{\rm NLO} + n![c_1]^{n-2}_{\rm LO}[c_2]_{\rm LO}}_{g^{-2(n-1)}\times\log}. \tag{C10}$$

This does not correspond to a Poisson distribution anymore, since one needs a nonzero b_2 in order to obtain these formulas. In fact, at this order of truncation, one has $b_2 = c_2$ while the higher b_r's are still zero. Even worse, our calculation of the second derivative of $\ln F$ shows that c_2 is enhanced by a large logarithm, and is actually of order $g^{-2}\ln(1/x_{1,2})$ rather than the naive expectation g^{-2}. Therefore, not only is the distribution not Poissonian, but the deviations from a Poisson distribution are logarithmically large.

However, the main result of the present paper is that one can obtain the NLO corrections to the inclusive n-particle spectra by the action of a certain operator on the product of n 1-particle spectra at LO. In the present language, this reads

$$F^{(n)}(1)|_{\rm LO+NLO} = [1 + \mathcal{L}_1 + \mathcal{L}_2][c_1]^n_{\rm LO}. \tag{C11}$$

Remember that, so far, all the discussion is for a fixed configuration of the sources $\rho_{1,2}$. Then, by averaging over these sources and by using the Hermiticity of the operator $\mathcal{L}_1 + \mathcal{L}_2$, one can transfer the action of this operator from the quantity $[c_1]^n_{\rm LO}$ to the distribution of sources. As we have seen, this amounts to letting the distribution of sources evolve according to the JIMWLK equation. In other words, Eq. (C10) deviates strongly from a Poisson distribution, but does so in such a way that all correlations can be interpreted as coming from correlations among the sources that are generated by the JIMWLK evolution.

Let us end this appendix with a word of caution in the interpretation of Eq. (61). Strictly speaking, our leading log approximation gives us control only over the $g^{-2}\ln(1/x_{1,2})$ part of the coefficient b_2, but not over its g^{-2} part (without a log). The latter would only show up in a next-to-leading-log calculation. This means that, in principle, one could modify the argument of the exponential in the integrand of Eq. (61) by a term of second degree in $z(\boldsymbol{p}) - 1$ and with a coefficient of order g^{-2}, without affecting any of our results for the inclusive gluon spectra at the order at which we calculate them. Obviously, such a modification of the integrand in Eq. (61) would be a deviation from a Poisson distribution. Thus, the statement according to which the deviations from a Poisson distribution come from the JIMWLK evolution of the distributions of the sources $\rho_{1,2}$ is true only for the largest of these deviations—i.e. those that are enhanced by large logarithms of the momentum fractions $x_{1,2}$. Other deviations from Poisson exist, that are not enhanced by such logarithms—these are beyond the scope of the present calculation.

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