Particle Accelerators, 1991, Vol. 35, pp. 1–14
Reprints available directly from the publisher
Photocopying permitted by license only

THE RESONANT THEORY OF LONGITUDINAL EMITTANCE BLOW-UP BY PHASE-MODULATED HIGH HARMONIC CAVITIES

V. V. BALANDIN, M. B. DYACHKOV and E. N. SHAPOSHNIKOVA

(Received 30 August 1989)

We consider some new aspects of controlled bunch dilution theory. The dependence of the blow-up rate on the high frequency cavity (HFC) parameters is theoretically accounted for. We found that the dilution rate has a resonant dependence on phase modulation frequency. In addition, it is strongly dependent on the phase modulation amplitude and the phase of the HFC voltage relative to the fundamental. All our theoretical conclusions have been confirmed by numerical simulation. We demonstrate how to affect different parts of the bunch and modify the particle distribution.

1. INTRODUCTION

The essential characteristic of kaon factory projects is high average intensity of the accelerated beam. This is attained by having a high circulating current and rapid cycling. Therefore, strong collective effects are expected. One of the most severe could be the longitudinal microwave instability. This can occur in a beam of a given intensity with a longitudinal emittance that is too small[1]. In the TRIUMF and Moscow Kaon Factory projects the upper bound on the longitudinal impedence of the main ring synchrotron Z/n is about 1–2 Ω[2,3], without an artificial increase of bunch area.

It is clear that this value is difficult to attain. Therefore, in the above projects, controlled emittance blow-up prior to acceleration in the main ring is proposed. The longitudinal emmitance has to be increased by a factor of 3 within 10–20 ms. This can be achieved by using additional high frequency cavities (HFCs) with phase modulation. Such cavities were successfully designed and constructed at the CERN Proton Synchrotron (PS)[4] in order to reduce particle loss during transition crossing. At the Brookhaven Alternating Gradient Synchrotron (AGS), a project to build a high frequency cavity is in progress[5].

From rf noise theory, it follows that emittance grows linearly with time due to particle diffusion[6,7]. Boussard applied rf noise theory to explain the longitudinal controlled blow-up in which HFCs were used with a sine-modulated phase[8]. This model correctly predicted some qualitative features observed during PS experiments and in numerical simulations[4,8]. According to this theory the dilution rate should be independent of phase modulation frequency. As has been shown later by numerical simulations[9], however, the dilution rate depends strongly on the phase modulation frequency. The author attemps to explain this dependence by taking into account the role of a parametric resonance area near the center of the bunch.

In this paper we consider some new aspects of controlled bunch dilution theory. The dependence of the blow-up rate on the HFC parameters is theoretically accounted for. We have found that the dilution rate has a resonant dependence on phase modulation frequency. It has also been shown that the dilution rate is strongly dependent on the phase modulation amplitude and the phase of the HFC voltage relative to the fundamental. It is possible to affect different parts of the bunch and modify the particle distribution. All our theoretical conclusions have been confirmed by numerical simulation.

2. BASIC EQUATIONS

2.1. General equations of motion with HFC

The longitudinal motion of the particles is governed by the following equations:

$$\dot{\varphi} = \frac{h\eta\omega_0}{\beta_s^2 E_s} W$$

$$\dot{W} = -\frac{e\omega_0}{2\pi} V. \tag{1}$$

Here φ denotes the deviation of a particle from the synchronous phase $\varphi_s = 0$;

$W = E - E_s$, where E_s is the synchronous particle energy;

h is the harmonic number ($f_{rf} = hf_0$);

$\beta_s c$ and $\omega_0 = 2\pi f_0$ are the velocity and revolution frequency of the synchronous particle; and

$\eta = 1/\gamma_t^2 - 1/\gamma^2$, where γ_t is the value of $\gamma = E/E_0$ at transition. The resulting voltage V affecting the particles can be represented as the sum of the main system voltage $V_0 \sin \varphi$ and a secondary voltage

$$V_1(\varphi, t) = V_1 \sin(N\varphi + \Phi(t)), \tag{2}$$

where V_1 and Nf_{rf} are the voltage and frequency applied to HFC, and $\Phi(t)$ is a phase modulation function. Here Nh should be an integer, though this need not be true for N.

Equations (1) are equivalent to one phase equation

$$\ddot{\varphi} + \omega_{s0}^2 \sin \varphi = \varepsilon\omega_{s0}^2 \sin(N\varphi + \Phi(t)), \tag{3}$$

where the parameter ε is the ratio of the secondary and primary voltage amplitudes and $\omega_{s0} = (\omega_0^2 |\eta| eV_0 h/2\pi\beta_s^2 E_s)^{1/2}$ is the synchrotron frequency of small phase oscillations.

Now let us introduce the new variables r and ψ (corresponding to the energy and phase of the synchrotron oscillations) in Eq. (3) with $\varepsilon = 0$.

$$\varphi = 2 \arcsin(r \operatorname{sn}(\omega_{s0}\psi/\omega(r), r))$$

$$\dot{\varphi} = 2\omega_{s0} r \operatorname{cn}(\omega_{s0}\psi/\omega(r), r). \tag{4}$$

Here $\omega(r) = \pi\omega_{s0}/2K(r)$ is the r-dependent synchrotron frequency, $K(r)$ is the elliptic integral of the first kind

$$K(r) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - r^2 \sin^2 \theta}},$$

and $\mathrm{sn}(u, r)$, $\mathrm{cn}(u, r)$ are Jacobian elliptic functions.

The variable r, given by

$$r = \frac{1}{\sqrt{2}} \left(1 - \cos\varphi + \frac{1}{2\omega_{s0}^2} \dot{\varphi}^2\right)^{1/2}$$

is the constant of motion of a nonperturbed system ($\varepsilon = 0$). Here r varies from zero in the center of the bucket to 1 on the separatrix.

Using the coordinate transformation (4) we can rewrite the phase Eq. (3) in the form

$$\begin{aligned} \dot{r} &= \frac{\varepsilon\omega_{s0}}{2} \sin(2N \arcsin(r\, \mathrm{sn}(\omega_{s0}\psi/\omega(r), r)) + \Phi(t))\, \mathrm{cn}(\omega_{s0}\psi/\omega(r), r), \\ \dot{\psi} &= \omega(r) - \varepsilon \frac{1}{2r} \omega(r) \sin(2N \arcsin(r\, \mathrm{sn}(\omega_{s0}\psi/\omega(r), r)) + \Phi(t)) \\ &\quad \times \frac{\partial}{\partial r} (\arcsin(r\, \mathrm{sn}(\omega_{s0}\psi/\omega(r), r)) \end{aligned} \tag{5}$$

To simplify Eqs. (5) let us neglect the terms of order ε^z in the right-hand sides of these expressions. For this purpose we use the Fourier series for elliptic functions. Analysis of the Fourier coefficients shows that, for ε in a range (0.1–0.3) and for $r < 2/3$, it is sufficient to take into account only the first Fourier harmonic. This is a good approximation when the length of the bunch does not exceed 120°. Then Eqs. (5) can be written as

$$\begin{aligned} \dot{r} &= \frac{\varepsilon\omega(r)}{4Nr} \sum_k k J_k(2Nr)[\sin(k\psi + \Phi(t)) + (-1)^k \sin(k\psi - \Phi(t))] \\ \dot{\psi} &= \omega(r) - \frac{\varepsilon\omega(r)}{2r} \left\{ J_1(2Nr) \cos\Phi(t) + \tfrac{1}{2} \sum_k [J_{k+1}(2Nr) - J_{k-1}(2Nr)] \right. \\ &\quad \left. \times [\cos(k\psi + \Phi(t)) + (-1)^k \cos(k\psi - \Phi(t))] \right\}. \end{aligned}$$

Here $J_k(x)$ is the Bessel function of order k.

2.2. *HFC With Harmonic Phase Modulation*

For HFCs with harmonic phase modulation, we can write

$$\Phi(t) = \alpha \sin(\Omega t + \theta_1) + \theta_2, \tag{7}$$

where α and Ω are the phase modulation amplitude and frequency, respectively, and θ_1 and θ_z are constants.

Expanding again the right-hand sides of Eqs. (6) in a Fourier series with respect to phase $(k\psi - l\Omega t)$ and neglecting high order oscillation terms, we get

$$\dot{r} = \frac{\varepsilon\omega(r)}{4Nr} \sum_k \sum_l (-1)^l k A_{k,l}(a_{kl} \sin\theta_2 \cos\Psi_{kl} + b_{kl} \cos\theta_2 \sin\Psi_{kl})$$

$$\dot{\psi} = \omega(r) - \frac{\varepsilon\omega(r)}{2r} A_{1,0} \cos\theta_2 - \frac{\varepsilon\omega(r)}{4r} \sum_k \sum_l (-1)^l (A_{k+1,l} - A_{k-1,l}) \tag{8}$$

$$\times (a_{kl} \sin\theta_2 \sin\Psi_{kl} + b_{kl} \cos\theta_2 \cos\Psi_{kl}),$$

where

$$A_{kl} = J_k(2Nr)J_l(\alpha),$$

$$a_{kl} = [1 - (-1)^{k+l}], \qquad b_{kl} = [1 + (-1)^{k+l}],$$

$$\Psi_{kl} = k\psi - l\Omega t - l\theta_1.$$

If one or more resonant conditions for Ψ_{kl},

$$k\omega(r_0) = l\Omega, \tag{9}$$

are satisfied, then we obtain nonoscillating terms in the right-hand sides of Eqs. (8), which lead to systematic changes of r and ψ with time. It can easily be shown that at a given point r_0 only integer multiple resonances can exist

$$\frac{k}{l} = \frac{k_0}{l_0}, \quad \text{or } k = mk_0,\ l = ml_0, \quad \text{where } m = 1, 2, \ldots$$

Since the frequency $\omega(r)$ changes slightly in the relevant range we can use a one-resonance model:

$$\dot{r} = \frac{\varepsilon\omega(r)}{4Nr} \sum_m mk_0(-1)^{ml_0} A_{mk_0,ml_0}[b_{mk_0,ml_0} \cos\theta_2 \sin(m\Psi_r)$$

$$+ a_{mk_0,ml_0} \sin\theta_2 \cos(m\Psi_r)]. \tag{10}$$

$$\dot{\Psi} = k_0\omega(r) - l_0\Omega + O(\varepsilon), \tag{11}$$

where $\Psi_r = k_0\psi - l_0\Omega t - l_0\theta_1$ is the resonant phase. The terms of order $O(\varepsilon)$ in Eq. (11) can be obtained from Eqs. (8).

The number of terms in the series (10) that contribute significantly inside the bunch $0 < r < r_{\max}$ depends on the value of N. For a Bessel function of high order $J_k(x_1)$, the position of the first maximum can be roughly expressed in the form

$$x_1 \cong k + 0.8k^{1/3}.$$

Therefore, the mth harmonic is significant if the condition

$$mk_0 + 0.8(mk_0)^{1/3} \lesssim 2Nr_{\max} \tag{13}$$

is satisfied.

Equations (10), (11) have an integral of motion, which is accurate to $O(\varepsilon^2)$:

$$H(r, \Psi_r) = \int_0^r (k_0\omega(r) - l\Omega_0)dr + \varepsilon \sum_m [C_m(r)\cos(m\Psi_r) + D_m(r)\sin(m\Psi_r)], \tag{13}$$

where

$$C_m(r) = \frac{\omega(r)k_0}{4Nr} A_{mk_0, ml_0}(-1)^{ml_0} a_{mk_0, ml_0} \cos\theta_2,$$

$$D_m(r) = \frac{\omega(r)k_0}{4Nr} A_{mk_0, ml_0}(-1)^{ml_0} b_{mk_0, ml_0} \sin\theta_2.$$

In what follows we shall suppose that Eqs. (10) and (11) are canonical with Hamiltonian $H(r, \Psi_r)$. This assumption is correct up to terms $\sim\varepsilon$ in Eq. (11) for resonant phase.

2.3. *Evolution of Particle Distribution*

Let us consider a distribution function $f(r, \Psi_r, t)$. The equation that governs the time evolution of the local density distribution is

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial r}\left(\frac{\partial H}{\partial \Psi_r} f\right) - \frac{\partial}{\partial \Psi_r}\left(\frac{\partial H}{\partial r} f\right). \tag{14}$$

Now we assume that the initial distribution depends solely on r: $f(r, \Psi_r, 0) = F_0(r)$. Then we can represent the solution of Eq. (14) in the form

$$f(r, \Psi_r, t) \cong f_0(r, t) + \varepsilon \sum_m [P_m(r, t)\sin(m\Psi_r) + Q_m(r, t)\cos(m\Psi_r)]. \tag{15}$$

Using Eqs. (14) and (15) we get

$$\frac{\partial P_m}{\partial t} = -mC_m \frac{\partial f_0}{\partial r} + m\delta Q_m, \tag{16}$$

$$\frac{\partial Q_m}{\partial t} = mD_m \frac{\partial f_0}{\partial r} + m\delta P_m, \tag{17}$$

$$P_m(r, 0) = Q_m(r, 0) = 0, \qquad m = 1, 2, \ldots, \tag{18}$$

$$\frac{\partial f_0}{\partial t} = \frac{\varepsilon^2}{2} \sum_m m \frac{\partial}{\partial r}(Q_m D_m - P_m C_m), \tag{19}$$

where $\delta = k_0\omega(r) - l_0\Omega$. For simplicity we choose the center of the resonance at the point $r_0 = 0$. If we take into account that

$$Q_m D_m - C_m P_m \approx \frac{\sin(m\delta t)}{\delta} \frac{\partial F_0}{\partial r}(C_m^2 + D_m^2),$$

then Eq. (19) can be reduced to

$$f_0(r, t) \cong F_0(r) + \frac{\varepsilon^2}{2} \sum_m m^2 \frac{\partial}{\partial r} \left\{ (C_m^2 + D_m^2) \frac{\partial F_0}{\partial r} \frac{[1 - \cos(m\delta t)]}{(m\delta)^2} \right\}. \tag{20}$$

Using Eq. (20) we get from Eq. (15) the description of the time evolution of the initial distribution. It has been obtained in chaotic phase approximation. Since the coefficients P_m and Q_m change linearly with time, we can see from Eq. (15) that phase inhomogeneity also increases linearly with time proportional to ε. However, the phase-averaged distribution function $f_0(r, t)$ defined by Eq. (20) varies as ε^2.

3. THE DEPENDENCE OF THE DILUTION RATE ON THE HFC PARAMETERS: COMPUTER SIMULATION

From the analysis of Eqs. (10) and (11) one can determine the dependence of the longitudinal emittance blow-up rate on the individual HFC parametrs: ε, Ω, α, N, θ_1 and θ_2 (see expression (7)). Let us consider the consequences of Eqs. (10) and (11) in detail.

1. The blow-up rate does not depend on the phase θ_1.

2. The dilution rate does not depend on the sign of the phase modulation amplitude α (one can consider $\theta_1 + \pi$). However, it follows from Eq. (10) that a strong dependence on the value of α exists. The value $\alpha = \pi$, which was used in papers [8, 9 and 10] is close to the positions of the maxima of the orders ($m = 1, 2\ldots$) of the functions $J_{ml_0}(\alpha)$ with $l_0 = 1$. So this choice of α is very convenient for resonances of the form $\Omega = kw(r_0)$. On the other hand, the resonances with $l_0 > 1$ are less effective because it is impossible to obtain the significant contribution from $J_{ml_0}(\alpha)$ simultaneously. In practice for resonances with $l_0 > 1$ it is necessary to use cavities with low quality factor Q, because of the large frequency modulation.

3. If the phase $\theta_2 = 0$, then for resonances with an odd value of $(k+l)$ the first term ($m = 1$) of the series in Eq. (10) equals zero. However, for even value of $(k + l)$, the harmonics with $m = 1$ give significant contributions. So for $\theta_2 = 0$ the resonances with $l_0 = 1$ and even k ($k = 2, 4, \ldots$) are suppressed. In the case $\theta_2 = \pi/2$, $l_0 = 1$ we have the opposite situation. There the resonances with odd $k = 1, 3, \ldots$ are suppressed.

To numerically see the influence of different parameters on the dilution speed, a computer program based on Eqs. (1) was developed. To characterize the beam dilution we choose[9] a parameter

$$R = \frac{1}{n} \sum_{i=1}^{n} r_i^2,$$

where n is the number of particles. The value $R^{1/2}$ is the rms size of the bunch in phase space. Figure 1 shows the dependence of the relative change in R after the first 2.5 ms on the normalized phase modulation frequency Ω/ω_{s0} for $\theta_2 = 0$ (Fig. 1a) and $\theta_2 = \pi/2$ (Fig. 1b). The other HFC parameters are set to $\varepsilon = 0.2$, $N = 10.5$, $\alpha = \pi$. For

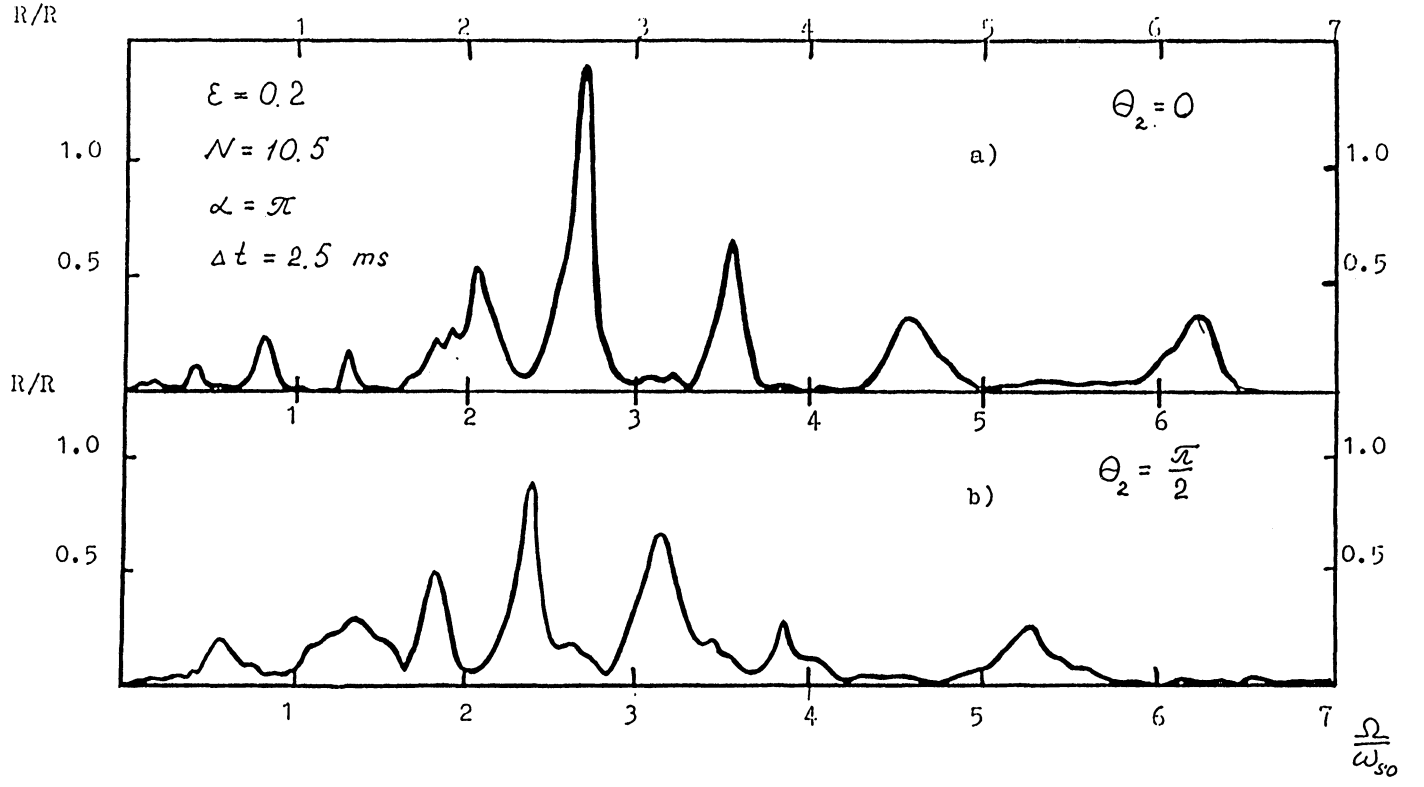


FIGURE 1. The dependence of dilution on phase modulation frequency Ω/ω_{s0} for $\theta_2 = 0$ (a) and $\theta_2 = \pi/2$ (b).

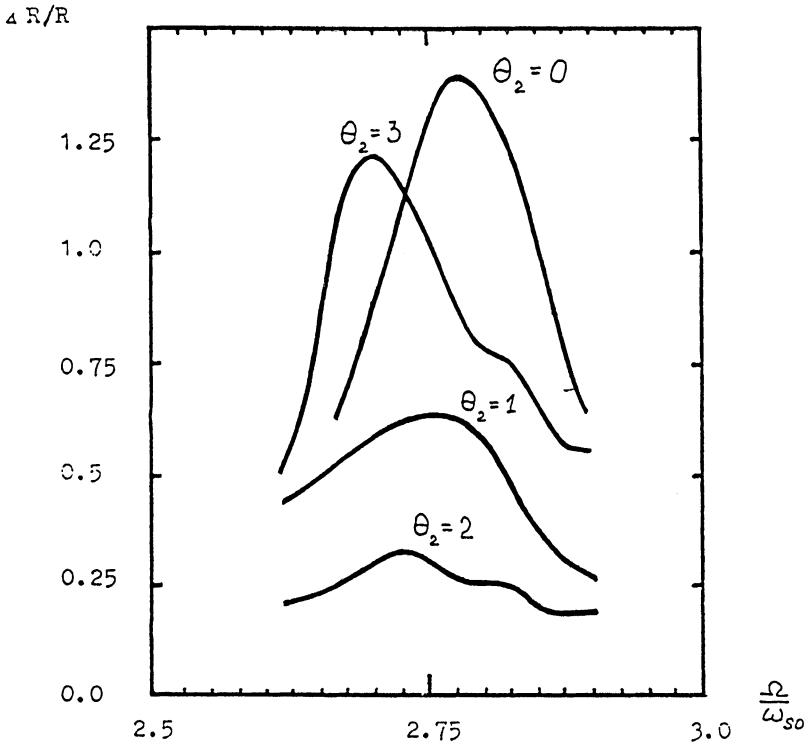


FIGURE 2 The dependence of dilution on phase modulation frequency Ω/ω_{s0} for various θ_2.

$0 < \theta_2 < \pi/2$ we have intermediate cases (see Fig. 2), which can occur for different bunches on the orbit if N is not an integer: $N = \text{Int}[N] + \Delta$. Indeed if $\theta_2 = 0$ for a certain bunch then for the next one $\theta_2 = 2\pi\Delta$ and so on. Hence, for the choice $N = 22 + 1/3$ of Ref. [9], the phase θ_n for a sequence of bunches looks like 0, $\pi/3$, $\pi/3$, 0, ... Therefore, for two bunches among three, the resonance is reduced, so the dilution rates for different bunches are not identical. Note that the numerical simulation in Ref. [9] deals with the case $\theta_2 = 0$ only. The choice $\Delta = 1/4$ is the worst for that case because half of the bunches are in the suppressed resonance region. In order to raise the microwave instability threshold it is necessary to increase the emittance of every bunch on the orbit. This requirement is satisfied for $\Delta = 0, 1/2$, whereNh-integer. To provide the stability for the coupled bunch modes (at frequency Nhf_0) during dilution we should also set $\Delta = 0, 1/2$.

4. Resonances of the form

$$l_0\Omega = \omega(r),\ k_0 = 1,$$

are not parametric and give only linear growth of r with time in the vicinity of $r = 0$. Hence they should reveal themselves as weaker than the resonances of lower orders of the form

$$l_0\Omega = k_0\omega(r),\ k_0 > 1,$$

The inefficiency of the use of resonances with $l_0 > 1$ can be seen in Fig. 1.

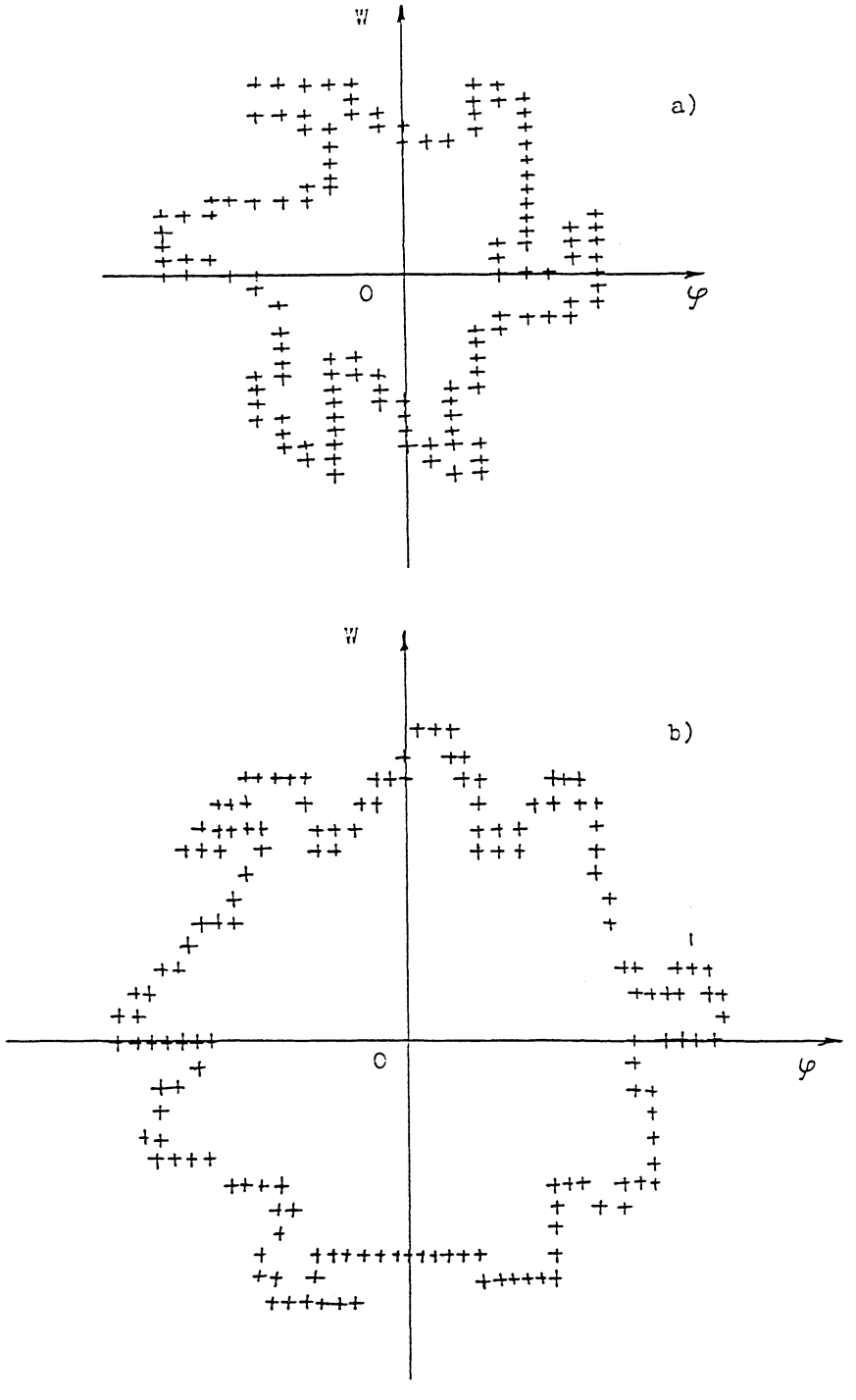
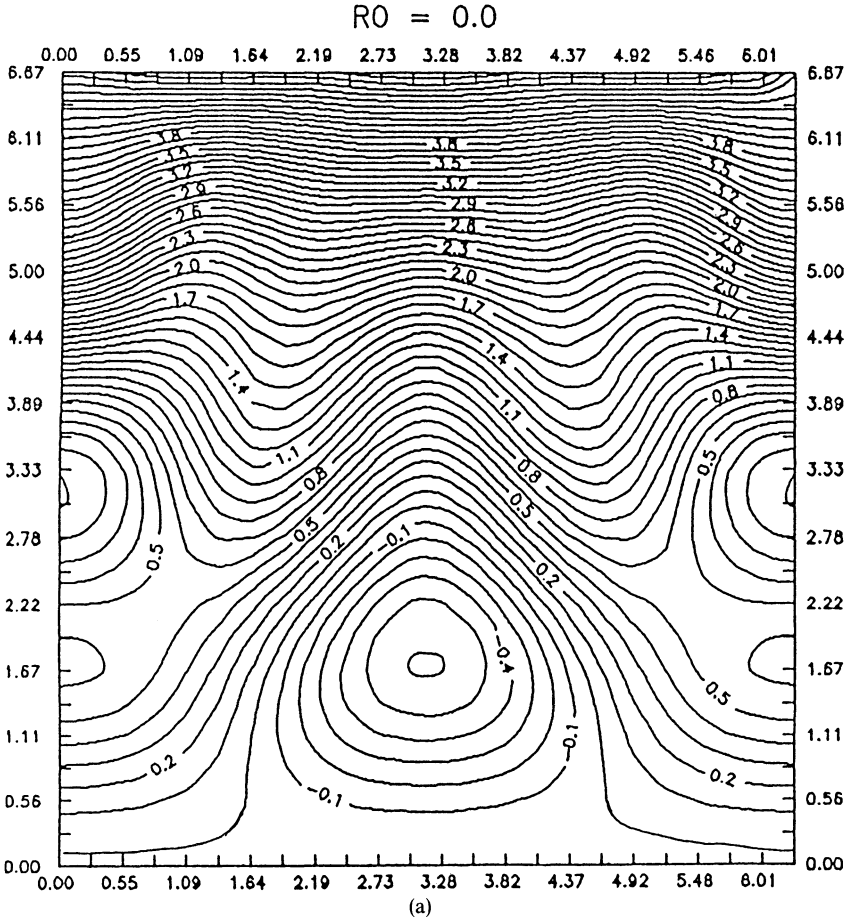


FIGURE 3. Distributions of particles after dilution with $\Omega = 3\omega(r_0 = 0)$, $N = 10.5$, $\varepsilon = 0.2$. Particles were placed initially at circle with $r = 0.3$ (a) and $r = 0.5$ (b).

5. Let us consider the dependence of the dilution rate on N. From Eq. (9) it follows that $\dot{r}$ is proportional to $1/N$. On the other hand, for bunches with a given size, the number of harmonics affecting the particles rises with increasing N; see Eq. (12). Therefore, the dilution rate is maximal for some magnitude of N. For example, for bunch dilution using the phase modulation frequency $\Omega = 3\omega$ (r) the quantity N is of the order of 10. With an increase in the resonance number k_0 this number N grows too.

As follows from Eq.(10), we can influence different parts of the bunch by varying the parameters α and N. Figure 3a and 3b show the distributions of particles placed initially on a circle with $r = 0.3$ and $r = 0.5$. They correspond to the resonance $\Omega = 3\omega(r)$; $N = 10.5$ and are described approximately by the following equation:

$$\dot{r} = \frac{\varepsilon\omega_{s0}}{2Nr}[3J_3(2Nr)J_1(\alpha)\sin(3\psi_r) + 6J_6(2Nr)J_2(\alpha)\sin(6\psi_r) + 9J_9(2Nr)J_3(\alpha)\sin(9\psi_r) + \cdots].$$



(a)

FIGURE 4 Curves H = constant for $r_0 = 0$ (a), $r_0 = 0.25$ (b) and $r_0 = 0.5$ (c).

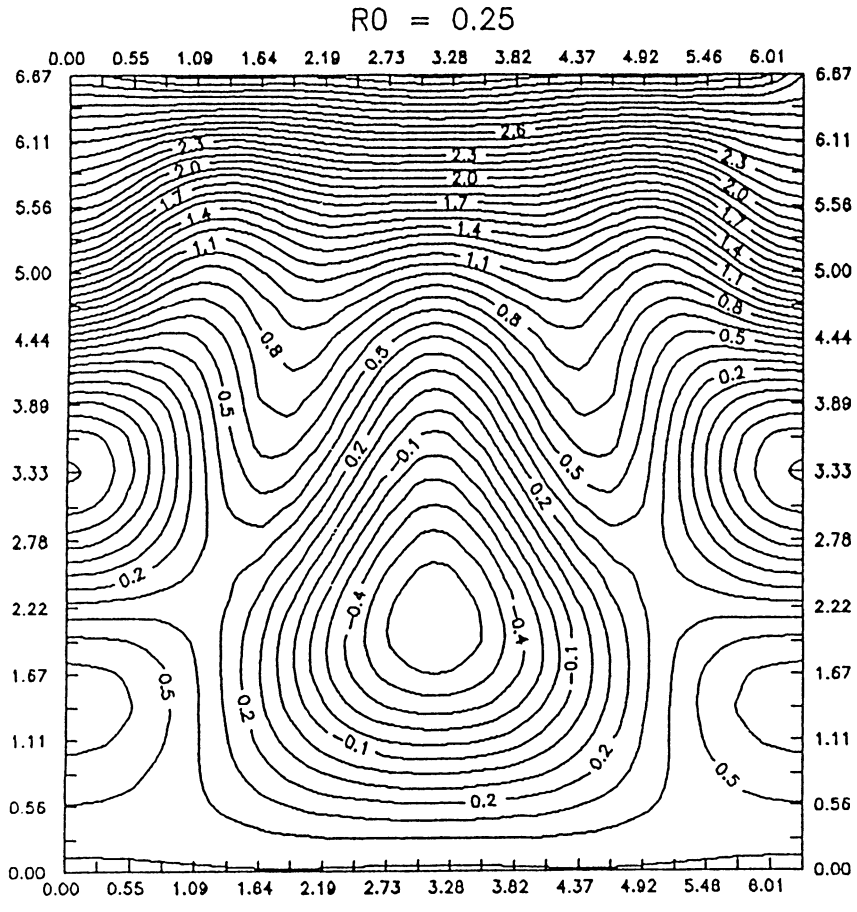


FIGURE 4b

In the first case (Fig. 3a) the second term in this expression is dominant, while in the second one (Fig. 3b) the third term is the largest. As follows from (12) it is desirable to use a large N to obtain a smoother distribution of particles in longitudinal phase space.

6. To investigate the influence of the location of the resonance center r_0 ($k_0\Omega = l_0\omega(r_0)$), let us consider the Hamiltonian H (see Eq. (13)). Using an expansion of the resonance frequency in the vicinity of point r_0 we obtain

$$H \cong -\frac{\omega_{s0}k_0}{4}[r_0(r-r_0)^2 + \tfrac{1}{3}(r-r_0)^3] + \varepsilon\sum_m [C_m\cos(m\psi_r) + D_m\sin(m\psi_r)]. \tag{21}$$

At small ε the resonance capture region has a width $\Delta r \sim (\varepsilon/r_0)^{1/2}$ and its center at r_0. The size of the capture region decreases with an increase in r_0. The curves H = constant are illustrated in Fig. 4a,b and c for $r_0 = 0$, 0.25 and 0.5 respectively.

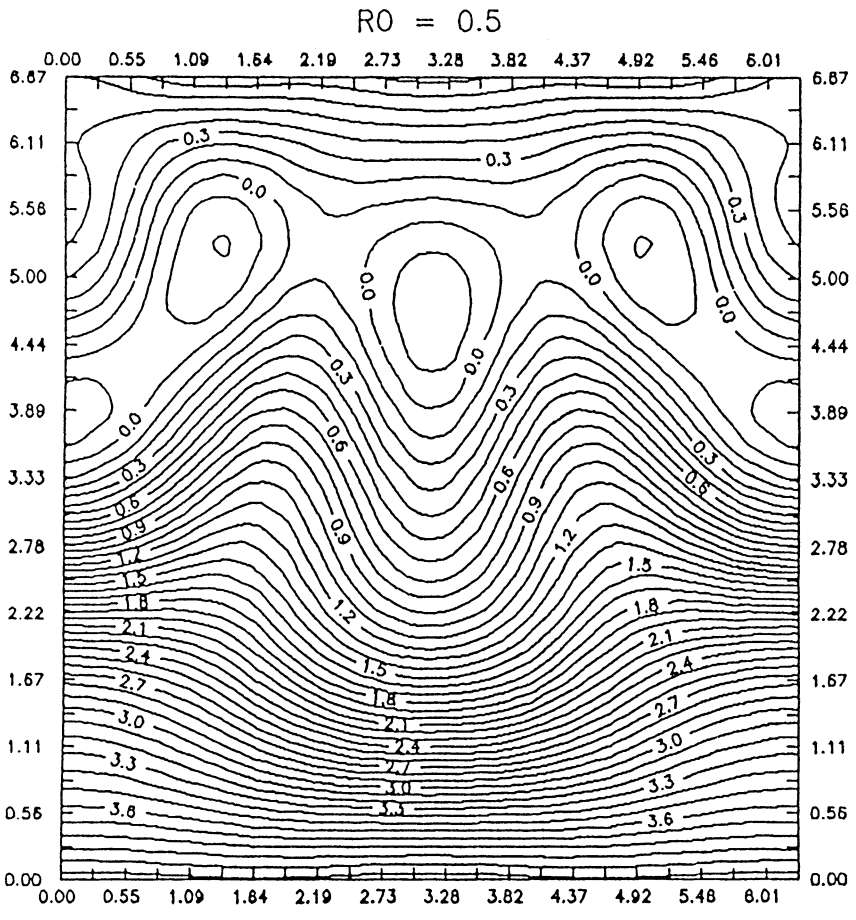


FIGURE 4c

It follows from Figure 4 that the dilution regime where r_0 varies from 0 to the boundary of the bunch r_{max} is the most effective.

In Figure 5 we show the distribution functions obtained from initially uniform distributions after dilution over 10 ms for two cases: a) $\Omega = 3\omega_{s0}$ ($r_0 = 0$) and b) $\Omega = 2.8\omega_{s0}$. So it is possible to transform the particle distribution in a desirable way.

4. CONCLUSIONS

We have shown that for the effective bunch dilution it is desirable to choose the modulation frequency

$$\Omega = k_0 \omega(r_0)$$

where k_0 is integer; k_0 should be odd for $\theta_2 = 0$, and may be either odd or even for $\theta_2 = \pi/2$. Nevertheless with increasing k_0 the value of r (for which influence is effective)

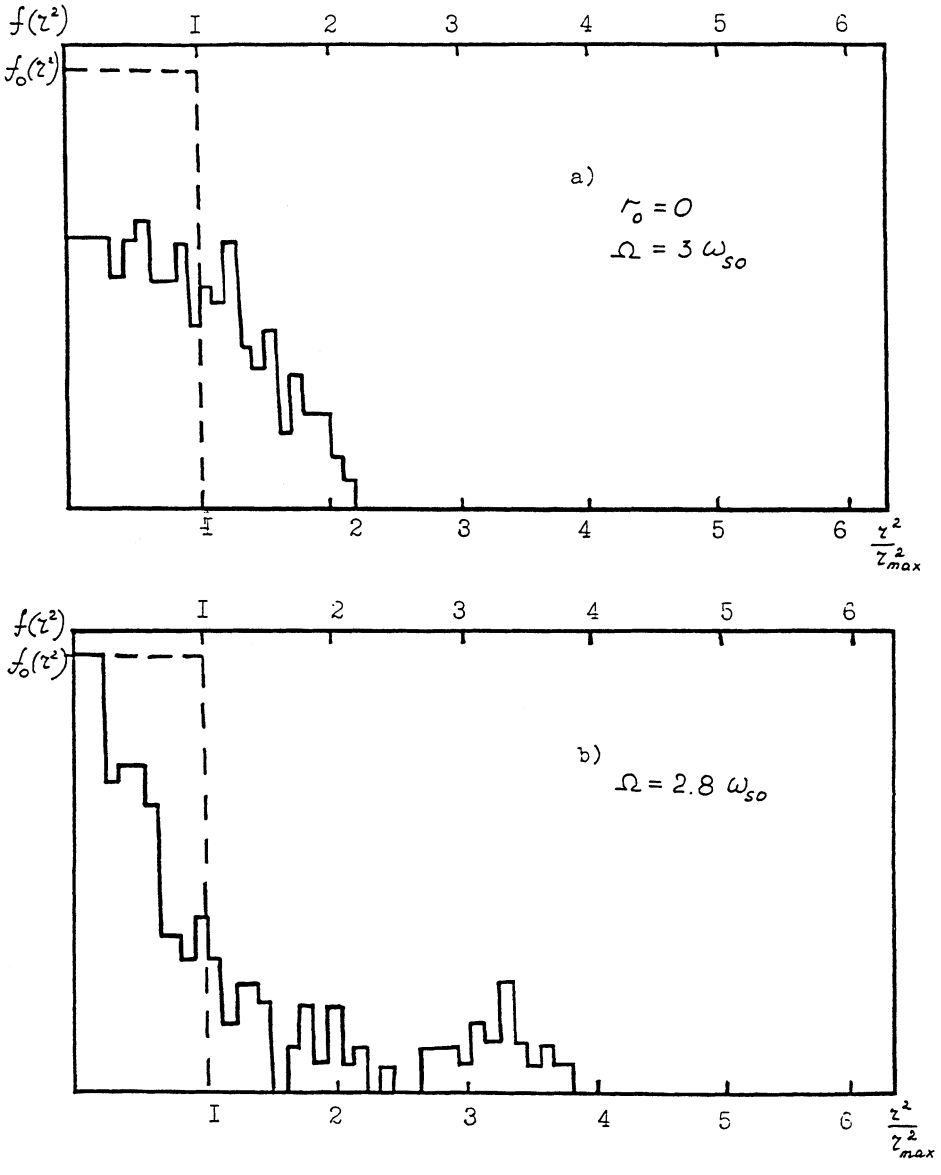


FIGURE 5. Distribution functions obtained from initially uniform distributions after dilution over 10 ms for $\Omega = 3\omega_{s0}$ (a) and $\Omega = 2.8\omega_{s0}$ (b).

also increases. So the use of high k_0 (as well as $r_0 \neq 0$) can produce tails in a particle distributions. By varying r_0 inside the bunch more particles are captured at resonance.

REFERENCES

1. D. Boussard, *Observation of Microwave Longitudinal Instabilities in the SPS*, CERN-LABII/RF/Int./75-2.

2. R. Baartman, H. Schonauer, and T. Suzuki, *Instabilities in the Proposed TRIUMF KAON Factory Synchrotrons*, IEEE Trans. Nucl. Sci. **NS-32**, N5, October 1985, p. 2395.
3. A. G. Chursin *et al.*, *The Accelerator Complex of Kaon Factory of Institute for Nuclear Research of USSR AS*, *Proc.* XI All Union Conf. on Charge Particle Accel., Dubna, 1988.
4. E. Brouzet, R. Cappi, R. Garoby, and H. Schonauer, Compte Rendu de MD sur le PS, Mesures sur le Grossissement Longitudinal Obtenu avec le Systeme 200 MHz (rev. 12–7–85)
5. J. M. Brennan *et al.*, *A High Harmonic Cavity for Controlled Longitudinal Dilution in the AGS*, EPAC, Rome, Italy, June 7–11, 1988.
6. S. Hansen *et al.*, *Longitudinal Bunch Dilution due to RF Noise.* IEEE Tr. on Nucl. Science, Vol. NS24, N3, June 1977.
7. D. Boussard, G. Dome, and C. Graziani. *The Influence of RF Noise on the Lifetime of Bunched Proton Beams*, Int. Conf. on High Energy Accelerators, CERN, Geneva, July 1980, p. 620.
8. D. Boussard, *The RF Noise Theory Applied to the PS Longitudinal Controlled Blow-up.* SPS/arf/note 85–7, 12th July, 1985.
9. J. M. Kats, *Particle Longitudinal Diffusion Produced by a High Frequency Cavity*, 1987 Particle Accelerator Conf., Washington, D.C., 3/16–19/87.