

## MICROWAVE INSTABILITY NEAR TRANSITION ENERGY\*

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**Abstract** Monte Carlo simulation for the microwave instability agrees with analytic calculation solving the Vlasov equation, provided that bunch shape distortion due to coupling is taken into account.

### INTRODUCTION

Electromagnetic self fields of wavelengths shorter than the bunch length, produced by the beam interacting with its environment may induce microwave instabilities when the frequency spread which provides Landau damping vanishes at the transition energy. Self fields of wavelengths comparable to the bunch length mainly distort bunch shape, causing mismatch and dilution. In order to understand more precisely<sup>1</sup> the various mechanisms, a computer algorithm is developed to simulate the motion of a beam by tracking a collection of macro-particles in longitudinal phase space. The program constructs self fields directly in the time domain according to the nature of the coupling and its frequency range. A quantitative agreement between the analytical expectation and computer simulation is achieved.

Longitudinal motion of a charged hadron can be described by an Hamiltonian

$$\mathcal{H} = \frac{h^2 \omega_0^2 \eta}{2E\beta^2} W^2 - \frac{qe\hat{V} \cos \phi_s}{4\pi h} \phi^2 - \int^\phi \dot{U}_{Z\parallel}(\phi') d\phi', \quad (1)$$

where  $W = \frac{\Delta E}{h\omega_0}$ ,  $\phi$  and  $\Delta E$  the RF phase and the energy deviation,  $\beta$  the synchronous velocity,  $h$  the harmonic number,  $\omega_0$  the revolution frequency, and  $U_{Z\parallel}$  the change in  $W$  during one revolution caused by beam-environment interaction. It is convenient to measure time relative to the instant that the synchronous particle crosses transition and to assume  $\gamma = \gamma_T + \dot{\gamma}t$ . With  $\eta \approx 2\dot{\gamma}t/\gamma_T^3$ , the time during which the motion is non-adiabatic<sup>2</sup> is characterised by  $T_c = \left( \frac{\pi E \beta^2 \gamma_T^3}{qe\hat{V} |\cos \phi_s| \dot{\gamma} h \omega_0^2} \right)^{1/3}$ .

Contribution from the self fields of a bunch of  $N_0$  particles may be written in terms of the longitudinal coupling impedance  $Z_{\parallel}(\omega)$ , as:<sup>3</sup>

$$\dot{U}_{Z\parallel} \approx \frac{q^2 e^2 \omega_0}{4\pi^2 h} \sum_{n=-\infty}^{+\infty} \int d\phi' \lambda(\phi', t) Z_{\parallel}(n\omega_0) e^{\frac{in}{h}(\phi - \phi')}, \quad (2)$$

where  $\lambda$  is related to the particle distribution  $\Psi$  as  $\lambda(\phi; t) = N_0 \int \Psi(\phi, W; t) dW$ . If collision and diffusion processes may be neglected,  $\Psi$  satisfies the Vlasov equation:

$$\frac{\partial \Psi}{\partial t} + \dot{\phi} \frac{\partial \Psi}{\partial \phi} + \dot{W} \frac{\partial \Psi}{\partial W} = 0. \quad (3)$$

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Since the Hamiltonian depends upon  $\Psi$  through  $U_{Z\parallel}$ , the Vlasov equation must be supplemented by the Maxwell equations with relevant boundary conditions to give a consistent solution. Suppose that under the coupling the action invariant is  $J^*$ .  $\Psi$  may be considered as a superposition of a distribution  $\Psi_0$  of contours along which particles have the same  $J^*$ , and a density fluctuation  $\Psi_1$ , as  $\Psi = \Psi_0(J^*) + \Psi_1$ . Correspondingly, the Hamiltonian  $\mathcal{H}$  may be written as  $\mathcal{H}(\phi, W; t) = \mathcal{H}_0(\phi, W; t; \Psi_0) + \mathcal{H}_1(\phi, W; t; \Psi_1)$ .  $\mathcal{H}_0$  causes deformation of the contours of motion and the distribution, while  $\mathcal{H}_1$  may induce coherent instabilities.

## BUNCH-SHAPE DISTORTION

### Effects of a Reactive Coupling

A reactive coupling mainly changes the net focusing strength of the RF system. Consider an Hamiltonian with  $Z_{\parallel}/n$  independent of frequency  $\omega$ ,

$$\mathcal{H}_0 = \frac{\hbar^2 \omega_0^2 \eta}{2E\beta^2} W^2 - \frac{qe\hat{V} \cos \phi_s}{2\pi\hbar} \left\{ \frac{\phi^2}{2} - \frac{\epsilon_Z}{2} [\lambda_0(\phi) - \lambda_0(0)] \right\}, \quad (4)$$

where  $\epsilon_Z = -\frac{2qe\hbar^2 \omega_0 |Z_{\parallel}/n| \text{Sgn}(Z)}{\hat{V} \cos \phi_s}$ ,  $\lambda_0 = N_0 \int \Psi_0 dW$ , and  $\text{Sgn}(Z)$  equal to +1 if the impedance is capacitive and -1 if it is inductive. As an example, assume a parabolic distribution,

$$\Psi_0(\phi, W; t) = \begin{cases} \frac{3}{4\pi J_0} \sqrt{1 - J^*(\phi, W; t)/J_0} & \text{when } J^* \leq J_0 \\ 0 & \text{otherwise,} \end{cases} \quad (5)$$

with a bunch area  $2\pi J_0$ . Such a  $\Psi_0$  yields a density of quadratic form,  $\lambda_0(\phi) = \lambda_0(0) - \tilde{\lambda} \phi^2$ , where  $\tilde{\lambda}$  is to be determined. With \* labelling the coupling,  $dt = k_1^* d\tau^*$ , and  $k_1^* = -\frac{qe\hat{V} \cos \phi_s}{2\pi\hbar} (1 - \epsilon_Z \tilde{\lambda})$ ,  $\phi$  and  $W$  can be canonically transformed<sup>4-6</sup> to action-angle variables  $J^*$  and  $\varphi^*$  by means of a generating function  $F_3(W, \varphi^*; \tau^*) = -\frac{W^2}{2\beta_L^*} \left( \tan \varphi^* - \frac{\beta_L'^*}{2} \right)$ , where the longitudinal amplitude function  $\beta_L^*$  is defined as

$$\frac{1}{2} \beta_L^* \beta_L^{*''} - \frac{1}{4} \beta_L^{*2} + K^* \beta_L^{*2} = 1; \quad K^* = \frac{\hbar^2 \omega_0^2 \eta}{E\beta^2 k_1^*}. \quad (6) \quad (' \equiv \frac{d}{d\tau^*})$$

Since the new Hamiltonian ( $H_0 = J^*/\beta_L^*$ ) does not contain  $\varphi^*$ , the action

$$J^* = \frac{1}{2\beta_L^*} \left[ W^2 + \left( \beta_L^* \phi + \frac{\beta_L'^*}{2} W \right)^2 \right] \equiv J_0 \left( \alpha_{\phi\phi}^* \phi^2 + 2\alpha_{\phi W}^* \phi W + \alpha_{WW}^* W^2 \right), \quad (7)$$

is the constant of motion. From eq. 5, it is seen that  $\lambda_0(0) = \frac{3N_0}{4\sqrt{2}J_0} \sqrt{\frac{\beta_L^*}{1+\beta_L^{*2}/4}}$  and  $\tilde{\lambda} = \frac{3N_0}{4(2J_0)^{3/2}} \left( \frac{\beta_L^*}{1+\beta_L^{*2}/4} \right)^{3/2}$  both vary with time.  $\beta_L^*$ , as self-consistently solved from eq. 6 and  $\tilde{\lambda}$ , determines the motion and distribution of the particles.

At the instant of transition crossing, the angle  $\vartheta^*$  between the axis of the action ellipse and the phase axis deviates from its corresponding value in the absence of the coupling by  $\Delta\vartheta \approx \frac{4}{3} \epsilon_Z \tilde{\lambda} \frac{\alpha_{\phi W}^*(0) [\alpha_{\phi\phi}^*(0) + \alpha_{WW}^*(0)]}{[\alpha_{\phi\phi}^*(0) - \alpha_{WW}^*(0)]^2 + 4\alpha_{\phi W}^{*2}(0)}$ , where  $\alpha_{ij} \equiv \alpha_{ij}^*(\epsilon_Z \tilde{\lambda} = 0)$ , and  $\epsilon_Z \tilde{\lambda} \approx \frac{2qe\hbar^2 \omega_0 |Z_{\parallel}/n| \text{Sgn}(Z)}{\hat{V} \cos \phi_s} \left( \frac{3}{4} \right)^{5/2} N_0 \alpha_{\phi\phi}^{3/2}(0) \ll 1$ . With  $x = |t|/T_c$  and  $y = \frac{2}{3} x^{\frac{3}{2}}$ , it is found that  $\beta_L = |k_1^{-1}| T_c \hat{\beta}_L(x)$ , where,

$$\hat{\beta}_L(x) = \frac{\pi}{3} \frac{1}{x^2} \left\{ \left[ 2J_{\frac{2}{3}}(y) - \frac{3}{2}yJ_{\frac{5}{3}}(y) \right]^2 + \left[ 2N_{\frac{2}{3}}(y) - \frac{3}{2}yN_{\frac{5}{3}}(y) \right]^2 \right\}, \tag{8}$$

$J_{\frac{2}{3}}$ , etc. are the Bessel functions. Since  $\epsilon_Z \tilde{\lambda}$  switches sign at  $t = 0$ , the nominal bunch shape before and after the crossing are mismatched by an angle  $2|\Delta\vartheta|$ . This discontinuity reflects the change of focusing property of the coupling upon the synchronous-phase switch-over. As a consequence, the bunch starts to tumble and dilute in longitudinal phase space after it crosses transition. The effective increase of the bunch area is estimated as  $\frac{\Delta J_0}{J_0} \Big|_{max} \approx \frac{|\alpha_{\phi\phi}^*(0) - \alpha_{\phi\phi}(0)|}{\alpha_{\phi\phi}(0)} \approx \frac{2}{3} |\epsilon_Z \tilde{\lambda}|$ .

Effects of a Resistive Coupling

If the coupling can be considered as a perturbation and  $\mathcal{R}$  as a constant,

$$\mathcal{H}_0 = \frac{h^2 \omega_0^2 \eta}{2E\beta^2} W^2 - \frac{qe\hat{V} \cos \phi_s}{2\pi h} \left[ \frac{1}{2} \phi^2 + \epsilon_R \lambda_0(0) \phi + \frac{\epsilon_R \tilde{\lambda}}{3} \phi^3 \right]; \quad \epsilon_R = -\frac{qe h \omega_0 \mathcal{R}}{\hat{V} \cos \phi_s}. \tag{9}$$

The linear term corresponds to an energy dissipation, while the cubic term corresponds to a bunch-shape distortion. Using a translation  $\tilde{\phi} = \phi - (\phi_F - \phi_s)$  to eliminate the linear term, the fixed point  $\phi_F$  is found to be  $\phi_F \approx \phi_s - \epsilon_R \lambda_0(0)$ , which implies that part of the energy gained through the RF acceleration compensates the dissipation. Consequently, the synchronous phase should be switched by an amount of  $\pi - 2\phi_F$  instead of  $\pi - 2\phi_s$  at transition.

The net effect after the compensation is a distortion of the particle distribution. The bunch is distorted differently below and above transition and thus mismatched at transition. Suppose that the coupling appears from time  $t = t_1 < 0$ . The amount of mismatch due to the sign-switching of  $\epsilon_R \tilde{\lambda}$  at transition is

$$\frac{\Delta J_0}{J_0} \Big|_{max} = \frac{\epsilon_R \tilde{\lambda}}{\sqrt{2}} \sqrt{\frac{J_0}{k_2}} I_{max}, \tag{10}$$

where,

$$I = \int_{x_1}^0 \hat{\beta}_L^{-3/2} \left\{ \left( 1 + \frac{\hat{\beta}_L^2}{4} \right) \cos \Theta dx' + \frac{3\hat{\beta}_L'}{2} \left( 1 + \frac{\hat{\beta}_L^2}{4} \right) \sin \Theta dx' + \left( 1 - \frac{3\hat{\beta}_L^2}{4} \right) \cos 3\Theta dx' + \frac{\hat{\beta}_L'}{2} \left( 3 - \frac{\hat{\beta}_L^2}{4} \right) \sin 3\Theta dx' \right\}; \quad \Theta = \varphi - \theta(0) + \theta(x'), \text{ and } \theta(x) = \int_{x_1}^x \frac{dx'}{\hat{\beta}_L(x')}.$$

MICROWAVE INSTABILITY

Consider a disturbance of a single phase component,  $\Psi_1 = f_1(W)e^{i(\frac{\pi}{k}\phi - \Omega t)}$ . The corresponding density amplitude is  $\lambda_1 = N_0 \int f_1(W)dW$ . With a capacitive coupling, the azimuthal electric field propagating with the beam below the cutoff frequency is  $\mathcal{E} \approx \frac{qeh\omega_0 |Z_{||}|}{2\pi R} \lambda_1 e^{i(\frac{\pi}{k}\phi - \Omega t + \frac{\pi}{2})}$ . It lags the density wave by a phase  $\frac{\pi}{2}$ . Particles tend to move away from the density wave crest when below transition, and towards the crest when above. Instabilities may happen only above transition. The situation is reversed if the coupling is inductive.

With a resistive coupling, the field  $\mathcal{E} \approx -qeh\omega_0 \mathcal{R} \lambda_1 e^{i(\frac{\pi}{k}\phi - \Omega t)}$  lags the density wave by a phase  $\pi$ . Particles near the density wave crest are shifted in the direction of positive phase, while particles near the wave trough are shifted in the direction

of negative phase below transition. Particles tend to move toward the middle of the front slope, and thus the position of the wave crest is shifted. The same thing happens above transition, when particles tend to move towards the middle of the back slope. In both cases, the amplitude of the density wave is enhanced and, at the same time, the location is continuously shifted; the motion is unstable.

Generally, with a disturbance  $\Psi_1 = e^{-i\Omega t} f_1(\phi, W)$ , the Vlasov equation can be expressed as

$$\Psi_1 = -\frac{i\beta_L^*}{2 \sin \pi \Omega_\tau \beta_L^*} \frac{\partial \Psi_0}{\partial J^*} \int_{\varphi^*}^{\varphi^*+2\pi} \frac{\partial H_1(\varphi^{*'})}{\partial \varphi^{*'}} e^{-i\Omega_\tau \beta_L^* (\varphi^{*'} - \varphi^* - \pi)} d\varphi^{*'}; \quad \Omega_\tau = \Omega/k_1^*, \quad (11)$$

which is a linear equation in  $\Psi_1$  according to eq. 2. It may be written in a matrix form by defining  $\rho_m = \int \int d\varphi^* dJ^* f_1(\varphi^*, J^*) e^{-i\frac{m}{h}\phi(\varphi^*, J^*)}$ , as:

$$\rho_n = \sum_{m=-\infty}^{+\infty} T_{nm} \rho_m, \quad n = \text{all integers}. \quad (12)$$

Solving eq. 11 is equivalent to obtaining the eigenvalues of a matrix  $\mathbf{T}$  of infinite dimensions, i.e.  $\det(\delta_{nm} - T_{nm}) = 0$ . The so called fast blow-up regime<sup>7</sup> is defined as the one in which the rise time of the instability is short compared with the period of synchrotron oscillation and long compared with the period of the disturbing fields. In this regime,  $T_{nm}$  can be simplified<sup>1</sup> as

$$T_{nm} = \frac{3N_0 q^2 e^2 \omega_0 i Z_{||}(m\omega_0) k_1^* \beta_L^*}{8\pi^2 h (2J_0)^{3/2}} \left[ \frac{1 + \beta_L^{*2}/4}{\beta_L^*} \right]^{3/2} \int d\bar{W} \frac{J_0 \left[ \frac{m-n}{hB(\bar{W})} \right] \bar{W}}{\beta_L^* \Omega_\tau - \frac{m}{h} \frac{1}{\beta_L^*} (1 + \beta_L^{*2}/4) \bar{W}},$$

where  $\bar{\phi} = \phi$ ,  $\bar{W} = W - \frac{\beta_L^* \beta_L^{*'}}{2(1 + \beta_L^{*2}/4)} \phi$ ,  $J_0$  is the Bessel function of the 0th order, and  $B^2(\bar{W}) = \frac{1}{2J_0} \frac{\beta_L^*}{1 + \beta_L^{*2}/4} (1 - \frac{1 + \beta_L^{*2}/4}{2J_0 \beta_L^*} \bar{W}^2)^{-1}$ . Instability that most likely happens has the largest eigenfrequency, which,<sup>8</sup> since  $\sum_{m=-\infty}^{\infty} J_0 \left[ \frac{m-n}{hB(\bar{W})} \right] \approx 2hB(\bar{W})$ , satisfies

$$1 = (U - iV) \int_{-1}^1 dx \frac{1}{x - x_1} \frac{dg(x)}{dx}; \quad x_1 = \frac{h\beta_L^* \Omega_\tau}{n\sqrt{2J_0}} \sqrt{\frac{\beta_L^*}{1 + \beta_L^{*2}/4}}, \quad (13)$$

where  $U$  and  $V$  are both real,  $U - iV = \frac{3N_0 q^2 e^2 h \omega_0 i Z_{||} k_1^*}{8\pi (2J_0)^{3/2}} \frac{\beta_L^{*3/2}}{\sqrt{1 + \beta_L^{*2}/4}}$  and, for the parabolic distribution,  $g(x) = \frac{2}{\pi} \sqrt{1 - x^2}$ ,  $|x| \leq 1$ . In the case of a reactive coupling,  $V = 0$ , and the condition for instabilities to occur is  $1 + 2U \leq 0$ .

Away from transition,  $\beta_L^{*'} \approx 0$ , and  $\beta_L^* \approx 1/\sqrt{K^*}$ . The instability condition becomes

$$\frac{qe\beta^2 \hat{I}_0 |Z_{||}/n| \text{Sgn}(Z)}{2\pi E\eta (\hat{\epsilon}_0/\sqrt{2})^2} (1 - \epsilon_Z \tilde{\lambda})^{-1/4} \geq 1, \quad (14)$$

with  $\hat{\phi}_0$  and  $\hat{\epsilon}_0$  the phase and fractional energy spread,  $\hat{I}_0 = \frac{3\pi h I}{2\phi_0}$  and  $\bar{I} = \frac{N_0 q e \omega_0}{2\pi}$  the peak and average current, respectively. The quantity  $(1 - \epsilon_Z \tilde{\lambda})^{-1/4}$  represents the contribution from the bunch-shape deformation caused by the coupling. It can be shown<sup>1</sup> that such a contribution always tends to enhance Landau damping.

Near transition,  $\eta \sim 0$ , and instabilities are likely to occur for lack of Landau damping. Neglecting the bunch-shape deformation, the instability condition is

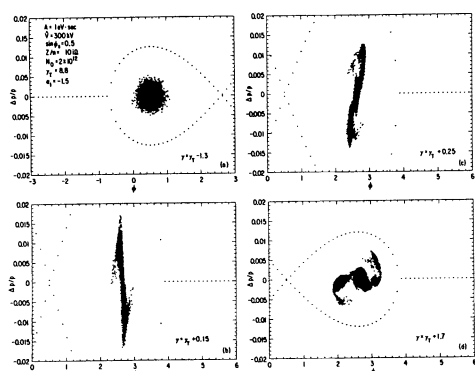


Figure 1.

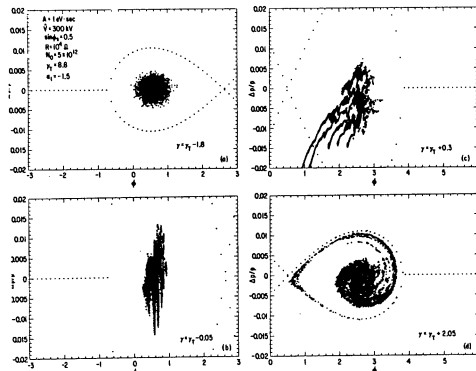


Figure 2.

$$|D(0)| \equiv \left| -\frac{3\pi h \bar{I} |Z_{||}/n| \text{Sgn}(Z)}{\hat{V} \cos \phi_s} \left(\frac{T_c}{2J_0 k_1}\right)^{3/2} \frac{\hat{\beta}_L^{3/2}(x)}{\sqrt{1 + \hat{\beta}_L^2(x)/4}} \right|_{x=0} \geq 1, \quad (15)$$

where, from eq. 8,  $\frac{\hat{\beta}_L^{3/2}(0)}{\sqrt{1 + \hat{\beta}_L^2(0)/4}} \approx 1.73$ . For a capacitive impedance, instabilities occur after the beam crosses transition if eq. 15 is true. The instabilities last for a period of  $T_{mw} = x_{mw} T_c$  until  $D(x_{mw}) = 1$ . With eq. 8,  $T_{mw}$  is found to be

$$T_{mw} \approx \begin{cases} \frac{2\hat{\beta}_L(0)}{3|\hat{\beta}_L'(0)|} [D(0) - 1] T_c \approx 0.92 [D(0) - 1] T_c & (T_{mw} \ll T_c) \\ \frac{q e \beta^2 \hat{I}_0 |Z_{||}/n| \gamma_T^3}{2\pi E \hat{\epsilon}_0^2 \hat{\gamma}} & (T_{mw} \gg T_c) \end{cases} \quad (16)$$

The growth rate of the unstable density amplitudes near frequency  $n\omega_0$  may be evaluated<sup>1</sup> from eq. 13, as  $Im\Omega_\tau \approx -\frac{n\sqrt{2J_0}}{h\beta_L\Omega_\tau} \sqrt{\frac{1+\beta_L^2/4}{\beta_L}} (1 + 2U)$ . The total growth is  $G_{mw} = e^{\int Im\Omega_\tau(\tau) d\tau}$ ,  $0 \leq \tau \leq |k_1^{-1}| T_{mw}$  only. Generally,  $G_{mw}$  can be calculated by solving eq. 13 numerically<sup>2</sup> with a certain  $g(x)$  for consecutive instants, and by summing the growth over the whole period that the instabilities last.

### COMPARISON WITH COMPUTER SIMULATION

The linear theory no longer holds when the unstable density amplitudes become comparable to the original density. An alternative way of treating this problem is to simulate the particle motion using a computer. In obtaining a quantitative comparison, statistical accuracy of the simulation has been estimated. In the case of a reactive coupling, the standard deviation was found<sup>9</sup> to be inversely proportional to the square root of the product of the total number of macro-particles and the cube of the bin length. This fact makes it practically impossible to simulate couplings of very high frequencies, which requires a very short bin length and thus a great number of macro-particles. Figs.1a-d show the effects of space charge ( $Z_{||}/n \sim 10i\Omega$ ) associated with a bunch of  $10^{12}$  protons. Fig.1b shows how the momentum-spread in the bunch is increased by the space-charge fields.

The different features of the coherent instabilities due to a reactive and a resistive coupling can be easily illustrated. It is seen from figs.1a-d that due to

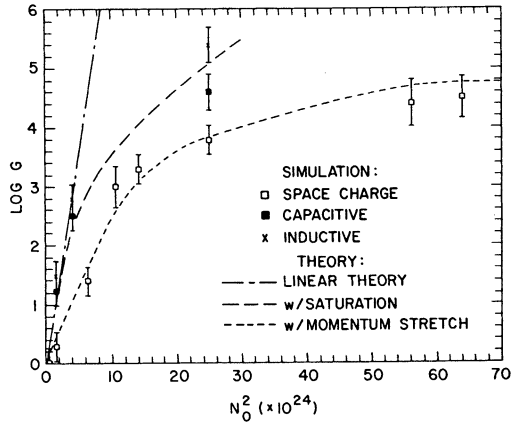


Figure 3: A comparison between the linear theory and the simulation.

the space charge, microwave modes break the bunch into many short pieces after crossing. The dots in fig.3 represent the growth as a function of the bunch intensity, as calculated from simulation. The errors come mostly from the statistical fluctuation. The dotted line represents the results calculated by solving eq. 13 numerically, without considering the bunch-shape distortion caused by the space charge, while the dashed line represents the results when this distortion is incorporated. The linear theory agrees with the simulation within statistical accuracy in a linear regime. Beyond that, the instability growth as calculated from the simulation is significantly smaller. This saturation phenomenon is mostly due to the fact that the Landau damping caused by bunch-area blow-up counteracts the instabilities, and the fact that the original bunch breaks into smaller pieces with few particles in between. The effect of a broad-band reactive coupling differs from that of the space charge, since the bunch-shape distortion becomes less significant.

Figs. 2a-d show the diagrams for a proton bunch under a broad-band resistive impedance near the cutoff frequency. As predicted in the previous section, pure micro-bunching does not occur, but the bunch suffers smearing due to the enhancement and shifting of the density disturbances. Instabilities start when the beam energy is close to but below the transition energy. They continue to grow after transition until there is enough Landau damping.

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