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FOKKER–PLANCK APPROACH TO STOCHASTIC MOMENTUM COOLING WITH A NOTCH FILTER

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A Fokker–Planck equation that describes stochastic momentum cooling with a notch filter is formulated. The Fokker–Planck coefficients are calculated for an idealized linear notch filter. The coherent energy correction is expressed explicitly with a parameter representing the difference of the particle's time-of-flight and the signal's transmission time from the pickup to the kicker. This formulation provides us with a guide to determine the system bandpass and the setting accuracy of the signal's transmission time for a given momentum spread of the beam. From the Fokker–Planck equation, a differential equation for the time variation of a standard deviation of energy error is derived to obtain the initial cooling time and the final momentum spread. This formulation is useful for experimental data analysis and for cooling system design, i.e., optimization of the system bandpass and gain, specification of the accuracy of the signal's transmission time, etc.

1. INTRODUCTION

In the stochastic momentum cooling process, a particle circulating in a storage ring has its energy error corrected by a signal produced by the particle itself. The particles are detected in a pickup, and the output signal is transformed by a notch filter so that the signal voltage is proportional to the energy error. This signal is applied to a kicker, where the particle is accelerated or decelerated, i.e., the energy error is corrected. For this correction, the signal and the particle must arrive at the kicker at the same time. The signal corrects the particle energy coherently during the cooling process.

If the particles were affected only by such coherent signals, the situation might be simple, and the momentum spread could be damped within one revolution time of a particle. In a real feedback system, however, the particle is also kicked by noises. We must consider two kinds of noise: One is the amplifier noise, and the other is the beam noise. The former is the thermal noise originating in the pickup and the preamplifier. The latter is the sum of signals from other particles. With a finite system bandwidth, the signal of a particle has a finite time width, which is much larger than the time interval between two particles. Therefore, the signal of one particle kicks other particles around it. Owing to these two incoherent noises, the motion of a particle is stochastic, like a Brownian motion. This situation requires us to examine the statistical behavior of the whole collection of particles. Our problem is simplified by the assumption that the noises are random. The amplifier noise is, in fact, inherently random, and the beam noise is approximately random because of the mixing effect, that is, the particles affecting a certain particle change with time owing to their momentum difference. With this assumption, we expect that the effect of the coherent signal will appear and that the energy spread of the beam will be damped after a cooling time much longer than the revolution time of the particle, because the average of the coherent signal over a long time exceeds that of the random noises. This is the basic principle of stochastic cooling invented by van der Meer.[1]

Van der Meer derived a differential equation that describes the damping of the beam emittance in the process of betatron cooling. On the assumption that the above noises are random, Sacherer pointed out that the kinetic equation is a Fokker–Planck equation and formulated the equation for momentum cooling with a notch filter.[2] On the other hand, Bisognano also derived a Fokker–Planck equation in another way: He developed his calculation accrding to BBGKY theory.[3] The Fokker–Planck equation turns out to be a powerful tool for investigating the stochastic cooling process and for optimizing the electronic characteristics of the feedback system.

The subject of this paper is to improve Sacherer's formulation by presenting the calculation of the Fokker–Planck coefficients more clearly. In our formulation, the coherent energy correction ΔE_c is first expressed with a parameter denoting the difference between the transmission time of a single particle's signal and the time-of-flight of the particle from the pickup to the kicker. This formulation enables us to investigate how crucially the setting error of the signal's transmission time affects the dependence of ΔE_c on the particle's energy error, and to determine the system bandwidth needed for a given momentum spread of the beam. Another subject of this paper is to express the initial cooling time and the final momentum spread as a function of the system gain. For this purpose, the Fokker–Planck equation is transformed to a differential equation describing the time variation of the standard deviation of energy error. Thus, the two important characteristics of the cooling system are calculable without solving the Fokker–Planck equation numerically.

In this paper, the beam is assumed to be an ion beam with a charge-to-mass ratio q/A.

In a forthcoming publication, we shall present the experimental results on momentum cooling at TARN and discuss the questions of signal suppression and parameter choices.

2. DERIVATION OF THE FOKKER–PLANCK EQUATION

Our aim is to derive a Fokker–Planck equation that describes the time evolution of the particle density function,

$$\psi(E, t) \equiv \frac{dN}{dE}, \tag{1}$$

where E is the energy deviation from the nominal kinetic energy per unit mass, E_0. It is convenient to consider the particle flux Φ, which is the number of particles crossing E per unit time. Then we have the continuity equation,

$$\frac{\partial \psi}{\partial t} + \frac{\partial \Phi}{\partial E} = 0. \tag{2}$$

When particle energy is corrected at the kicker by ΔE_k, the number of particles crossing E is given by the shaded area in Fig. 1; the flux is given by

$$\Phi = f_0\left[\psi(E)\Delta E_k - \frac{1}{2}\frac{\partial \psi}{\partial E}(\Delta E_k)^2\right]. \tag{3}$$

As the kicker is excited by the coherent correction signal and the incoherent noise, ΔE_k is divided into corresponding two terms:

$$\Delta E_k = \Delta E_c + \Delta E_{ic}. \tag{4}$$

To evaluate Eq. (4), we consider the average values of ΔE_k and $(\Delta E_k)^2$, because we are not interested in the instantaneous motion of the particles, but rather in the motion over a time much longer than the revolution time of the particles. Then we have

$$\overline{\Delta E_k} \cong \Delta E_c, \tag{5}$$

$$\overline{(\Delta E_k)^2} \cong \overline{(\Delta E_{ic})^2}, \tag{6}$$

assuming

$$\overline{\Delta E_{ic}} = 0, \tag{7}$$

$$\overline{(\Delta E_{ic})^2} \gg \overline{(\Delta E_c)^2}. \tag{8}$$

With these averaged values, the flux is rewritten as

$$\Phi = f_0\left[\psi \Delta E_c - \frac{1}{2}\frac{\partial \psi}{\partial E}\overline{(\Delta E_{ic})^2}\right]. \tag{9}$$

We now introduce here coefficients F and D defined by

$$F \equiv f_0 \Delta E_c, \tag{10}$$

$$D \equiv \tfrac{1}{2} f_0 \overline{(\Delta E_{ic})^2}. \tag{11}$$

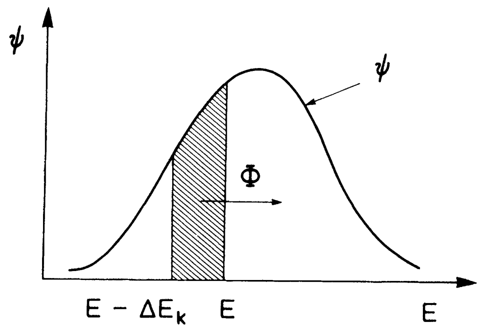


FIGURE 1 Energy distribution of particles in a bunch, showing those (hatched area) that cross a line corresponding to an energy E, given an energy correction ΔE_k at the kicker.

Then the continuity equation is expressed as

$$\frac{\partial \psi}{\partial t} + \frac{\partial}{\partial E}(F\psi) - \frac{\partial}{\partial E}\left(D\frac{\partial \psi}{\partial E}\right) = 0. \tag{12}$$

Inserting for F and D the electrical characteristics of the real feedback system, we obtain the required Fokker–Planck equation.

3. TIME EVOLUTION OF THE RMS ENERGY SPREAD

Prior to the calculation of F and D, we derive an equation describing the time evolution of the rms energy spread by representing F and D with simple equations.

We assume the coherent coefficient F to be given by

$$F = -\frac{E}{\tau_0}, \tag{13}$$

because ΔE_c should ideally be proportional to E; this is a postulate of stochastic cooling theory. The incoherent coefficient D is proportional to the noise power, which consists of two components. One is the amplifier noise, and the other the beam noise. The former remains constant during the cooling process; the latter is proportional to the square of the standard deviation of energy error,

$$\sigma^2(t) = \frac{1}{N}\int_{-\infty}^{\infty} E^2\psi(E, t)\, dE, \tag{14}$$

provided that the single-particle signal is proportional to E and that $\psi(E, t)$ is symmetrical with respect to $E = 0$. Therefore, we parametrize D as

$$D = D_a + d_b\sigma^2, \tag{15}$$

where D_a comes from the amplifier noise and $d_b\sigma^2$ from the beam noise.

With the above parametrized F and D, the Fokker–Planck equation is expressed as

$$\frac{\partial \psi}{\partial t} + \frac{\partial}{\partial E}\left(-\frac{E}{\tau_0}\psi\right) - (D_a + d_b\sigma^2)\frac{\partial^2 \psi}{\partial E^2} = 0. \tag{16}$$

Differentiating Eq. (14) and using Eq. (16), we have

$$\sigma\frac{d\sigma}{dt} = -\left(\frac{1}{\tau_0} - d_b\right)\sigma^2 + D_a. \tag{17}$$

The solution is

$$\sigma(t) = [(\sigma_0^2 - \tau D_a)e^{-2t/\tau} + \tau D_a]^{1/2}, \tag{18}$$

$$\frac{1}{\tau} \equiv \frac{1}{\tau_0} - d_b. \tag{19}$$

The final value of σ at $t = \infty$ is

$$\sigma(\infty) = (\tau D_a)^{1/2}, \tag{20}$$

and the initial cooling time τ_i is

$$\tau_i \equiv -\frac{\sigma_0}{\dot{\sigma}(0)} \tag{21}$$

$$= \tau\left[1 - \frac{\sigma^2(\infty)}{\sigma_0^2}\right]^{-1}, \tag{22}$$

where $\dot{\sigma} \equiv d\sigma/dt$. From Eq. (22), the final energy spread and the initial cooling time relate closely to each other. The relation between them is discussed later.

Setting $\partial\psi/\partial t = 0$ in Eq. (16), we have the final distribution function $\psi(E, \infty)$ of a Gaussian with a standard deviation given by Eq. (20).

In the absence of the noises, we have

$$\sigma(t) = \sigma_0 e^{-t/\tau_0}. \tag{23}$$

The energy spread decreases exponentially with a time constant τ_0, which we call the single-particle cooling time.

4. COOLING TERM

In this section, we calculate the coherent correction energy ΔE_c and the single-particle cooling time τ_0. The particles are assumed to be ions with a charge-to-mass ratio of q/A.

4.1. *Transfer Functions of the Feedback System*

We consider a feedback system including a notch filter. Figure 2 shows the layout of the system from the pickup down to the kicker. The transfer functions of the elements are defined as follows.

Pickup. We define the coupling impedance of the pickup, Z_p, as the ratio of

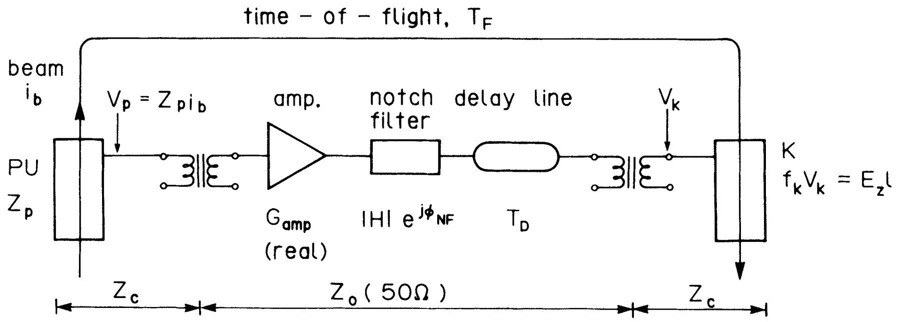


FIGURE 2 Layout of the feedback system from the pickup to the kicker.

the output voltage of the pickup, V_p, to the beam current i_b:

$$V_p = Z_p i_b. \tag{24}$$

The characteristic impedance is Z_c.

Transformers. For impedance matching between the 50-Ω electronics and the pickup or the kicker, transformers are inserted. Ideal transformers are assumed, that is, no electric power is lost in the transformers. We assume that the pickup and the kicker are identical in structure and that two identical transformers are used; therefore, we do not consider the transformers in the following calculation.

Amplifier. The gain of the amplifier, G_{amp}, is assumed to be real. The delay time in the amplifier is added to that of the following delay cable.

Notch filter. The transfer function of the notch filter is

$$H(f) = |H(f)| \exp[j\phi_{\mathrm{nf}}(f)]. \tag{25}$$

We assume an ideal notch filter, whose amplitude and phase responses are illustrated in Figs. 3a and 3b:

$$|H(f)| = g_{\mathrm{pole}} \left| \frac{f - nf_0}{f_0/2} \right| \quad [(n - \tfrac{1}{2})f_0 \leq f \leq (n + \tfrac{1}{2})f_0], \tag{26}$$

$$\phi_{\mathrm{nf}}(f) = \begin{cases} +\dfrac{\pi}{2} & [nf_0 \leq f \leq (n + \tfrac{1}{2})f_0], \\ -\dfrac{\pi}{2} & [(n - \tfrac{1}{2})f_0 \leq f \leq nf_0]. \end{cases} \tag{27}$$

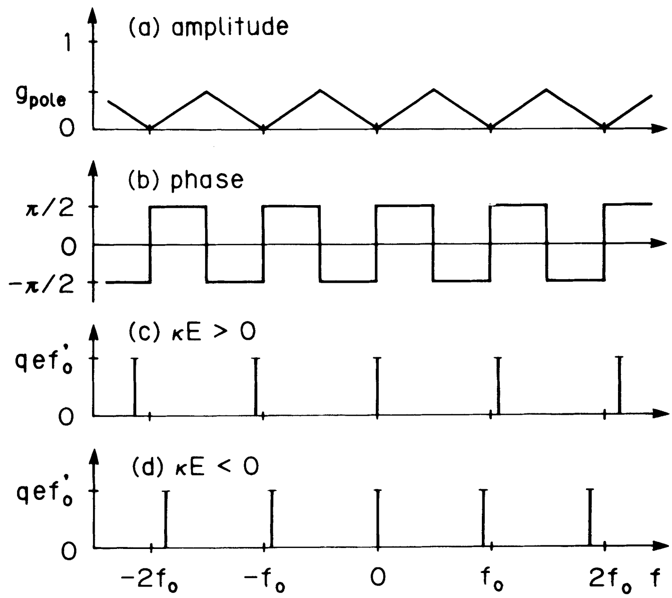


FIGURE 3 Amplitude and phase responses of the ideal notch filter [(a) and (b)]; and spectra for single particle current for $\kappa E > 0$ (c) and $\kappa E < 0$ (d). The frequency dispersion κ is defined by Eq. (31).

Delay line. The transmission time of the beam signal from the pickup to the kicker is expressed as the delay time in the delay cable, T_D. This time is set nearly equal to the time-of-flight T_F of the equilibrium particle with energy E_0:

$$T_D = T_F + \Delta T_D. \tag{28}$$

Assuming T_D is constant, though it varies generally with frequency, we have the phase shift of the signal in the delay cable, ϕ_D:

$$\phi_D(f) = -2\pi f T_D. \tag{29}$$

The time-of-flight of a particle with energy $E_0 + E$ differs from T_F by ΔT_F:

$$\frac{\Delta T_F}{T_F} = -\kappa \frac{E}{E_0}, \tag{30}$$

$$\kappa \equiv \frac{\eta_f \gamma}{1+\gamma}, \tag{31}$$

where γ is the relativistic factor and η_f is the dispersion of revolution frequency due to the momentum error of a particle:

$$\frac{\Delta f_0}{f_0} = \eta_f \frac{\Delta p}{p_0}. \tag{32}$$

Kicker. For energy correction of the particles, a longitudinal electric field E_z is induced in the kicker by an applied voltage to the kicker, V_k. We define the efficiency f_k of the kicker of length l as

$$\mathrm{Re}\,(f_k V_k) = E_z l. \tag{33}$$

With this correction, the energy per nucleon of an ion changes by

$$\Delta E_k = \frac{qe}{A} \mathrm{Re}\,(f_k V_k). \tag{34}$$

4.2. Calculation of ΔE_c

Our task here is to represent $V_k\ (t = T_F + \Delta T_F)$, induced by a particle passing the pickup at $t = 0$, with parameters defined above. Then we can obtain ΔE_c from Eq. (34).

The current of a single particle with energy $E_0 + E$ and revolution frequency $f_0' = f_0 + \Delta f_0$ is the sum of delta functions:

$$i_b = qe \sum_{n=-\infty}^{\infty} \delta\left(t - \frac{n}{f_0'}\right) \tag{35}$$

$$= qef_0' \sum_{n=-\infty}^{\infty} \exp\,(j2\pi n f_0' t). \tag{36}$$

Multiplying this current by the transfer functions of the pickup, the amplifier, the notch filter, and the delay cable, we have

$$V_k(t) = qef'_0 \sum_{n=-\infty}^{\infty} Z_p(nf'_0)G_{\text{amp}}(nf'_0)H(nf'_0)\exp\{j[2\pi nf'_0t + \phi_D(nf'_0)]\}. \quad (37)$$

It is straightforward to obtain ΔE_c by putting $t = T_F + \Delta T_F$ into this equation:

$$\Delta E_c(E) = \frac{(qe)^2}{A} f_0\left(1 + \kappa\frac{E}{E_0}\right) \sum_{n=-\infty}^{\infty} G_{\text{amp}}(nf_0)\,\text{Re}\,[Z_p(nf_0)f_k(nf_0)H(nf'_0)e^{-jn\xi}], \quad (38)$$

$$\xi \equiv 2\pi f_0 T_F\left(\kappa\frac{E}{E_0} + \frac{\Delta T_D}{T_F}\right). \quad (39)$$

In the derivation of Eq. (38), G_{amp}, Z_p, and f_k are assumed to vary smoothly with frequency.

A numerical calculation with the aid of a computer allows us to evaluate Eq. (38), which is a general expression of ΔE_c, because it is usually impossible to represent Z_p, f_k, H, and their product with simple functions. Therefore, we assume simplified transfer functions of the components in the feedback system. First, we employ the above ideal notch filter. In the case of no Schottky band overlap, the amplitude is (Figs. 3a–3d)

$$|H(nf'_0)| = 2g_{\text{pole}}\left|n\kappa\frac{E}{E_0}\right|, \quad (40)$$

and the phase is

$$\phi_{\text{nf}}(nf'_0) = \begin{cases} +\dfrac{\pi}{2} & (n\kappa E/E_0 > 0), \\ -\dfrac{\pi}{2} & (n\kappa E/E_0 < 0). \end{cases} \quad (41)$$

As a result, we have

$$H(nf'_0) = j2g_{\text{pole}}n\kappa\frac{E}{E_0}. \quad (42)$$

Second, we assume that the amplifier has a constant gain over the bandpass:

$$G_{\text{amp}}(f) = \begin{cases} G_{\text{amp}} & (n_1f_0 \le f \le n_2f_0), \\ 0 & (f < n_1f_0, f > n_2f_0), \end{cases} \quad (43)$$

and that Z_p and f_k are also constant. Consequently, dropping the E^2 term, we transform Eq. (38) to

$$\Delta E_c(E) = -4\frac{(qe)^2}{A} f_0 G\,|Z_p f_k|\,\kappa F_c(n_1, n_2, \alpha, \xi)\frac{E}{E_0}, \quad (44)$$

where

$$G \equiv G_{\text{amp}} g_{\text{pole}}, \tag{45}$$

$$\alpha \equiv \arg(Z_p f_k), \tag{46}$$

$$F_c(n_1, n_2, \alpha, \xi) \equiv -\cos\alpha \sum_{n=n_1}^{n_2} n \sin n\xi + \sin\alpha \sum_{n=n_1}^{n_2} n \cos n\xi. \tag{47}$$

Formulas for the summations of the circular functions are summarized in Appendix B.

As described above, the ideal $\Delta E_c(E)$ for cooling is proportional to E, that is, $G\kappa F_c(n_1, n_2, \alpha, \xi)$ in Eq. (44) should be a positive constant. For this purpose, we must set n_1, n_2, and ΔT_D at adequate values so that $F_c(n_1, n_2, \alpha, \xi)$ remains almost constant over a range of ξ given by

$$\xi_c - \pi f_0 T_{FK} \kappa \frac{\Delta E_0}{E_0} \leq \xi \leq \xi_c + \pi f_0 T_{FK} \kappa \frac{\Delta E_0}{E_0}, \tag{48}$$

where ξ_0 is the central value of ξ:

$$\xi_c \equiv 2\pi f_0 \Delta T_D, \tag{49}$$

and ΔE_0 is the full width of the beam's energy spread. To satisfy the condition of no Schottky band overlap, ΔE_0 must be smaller than a critical energy error E_c:

$$E_c \equiv \frac{E_0}{n_2 \kappa}. \tag{50}$$

This is one of the conditions that determines the upper limit of the system bandwidth for a given energy spread of the beam:

$$n_{2,\max} = \frac{1}{\kappa}\left(\frac{E_0}{\Delta E_0}\right). \tag{51}$$

Now we investigate the behavior of $F_c(n_1, n_2, \alpha, \xi)$ for the case of TARN. The parameters necessary here are summarized in Table I. With these values, we have $E_c/E_0 = 0.0336$. Figure 4 illustrates profiles of $F_c(n_1, n_2, \alpha, \xi)$ for $\alpha = 0$, $\pi/4$, $\pi/2$, and $3\pi/4$. When $\alpha = \pi/2$, for example (the solid line in the figure), $F_c(n_1, n_2, \alpha, \xi)$ takes its maximum value at $\xi = 0$:

$$\begin{aligned} F_c(n_1, n_2, \pi/2, 0) &\cong \tfrac{1}{2}(n_2^2 - n_1^2) \\ &= 3366. \end{aligned} \tag{52}$$

We can obtain a nearly constant F_c over the required range of ξ by setting $\xi_c = 0$, or $\Delta T_D = 0$. With this delay time and at $E = \pm E_c/2$ ($\xi = \pm 0.0142$), $F_c(n_1, n_2, \pi/2, \xi)$ decreases to 2200, about two-thirds of the maximum value. As a result, ΔE_c is approximately proportional to E, as shown by the solid line in Fig. 5, where $F_c(n_1, n_2, \pi/2, \xi) \cdot E/E_c$ for $\Delta T_D = 0$, ± 1, and 2 ns are illustrated. As is apparent in the figure, the setting error of ΔT_D within ± 1 ns is acceptable, but the error of 2 ns is not. The full width of the acceptable setting error of ΔT_D is about a few tenths of $1/(f_{\min} + f_{\max})$, as will be discussed in Section 7.

TABLE I
Main Parameters of the Cooling System at TARN

Ions	Protons, α particles
Kinetic energy, E_0	7 MeV/u
Beam velocity, $\beta = v/c$	0.1215
Number of particles, N	$\sim 10^7$–10^8
Revolution frequency, f_0	1.13 MHz
Time-of-flight from pickup to kicker, T_F	335 ns
Frequency dispersion, η_f	0.705
κ	0.354
Momentum spread (full width), $\Delta p_0/p_0$	0.01
Bandpass ($f_{\min}, f_{\max}$)	20–95 MHz
Minimum harmonic number, n_1	18
Maximum harmonic number, n_2	84
System bandwidth, W	75 MHz

At $\xi = \pm 0.05$, $F_c(n_1, n_2, \pi/2, \xi)$ also takes an extreme value of -2200, as shown in Fig. 4. This enables us to achieve cooling by setting ξ_c to ± 0.05, or ΔT_D to ± 7 ns, and reversing the polarity of the amplifier gain, though the cooling efficiency decreases to two-thirds of that with $\Delta T_D = 0$ ns.

Also, for other values of α, we can keep $F_c(n_1, n_2, \alpha, \xi)$ almost constant by

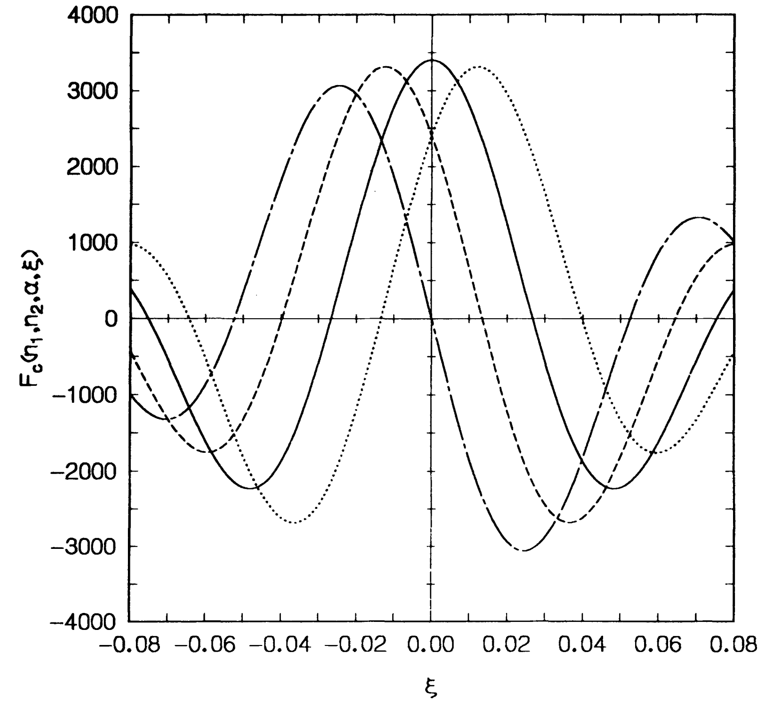


FIGURE 4 Profiles of $F_c(18, 84, \alpha, \xi)$ for $\alpha = 0$ (– · –), $\pi/4$ (– – –), $\pi/2$ (——), and $3\pi/4$ (· · ·). The maximum value of $|F_c|$ is approximately $\frac{1}{2}(84^2 - 18^2) = 3366$, independent of α.

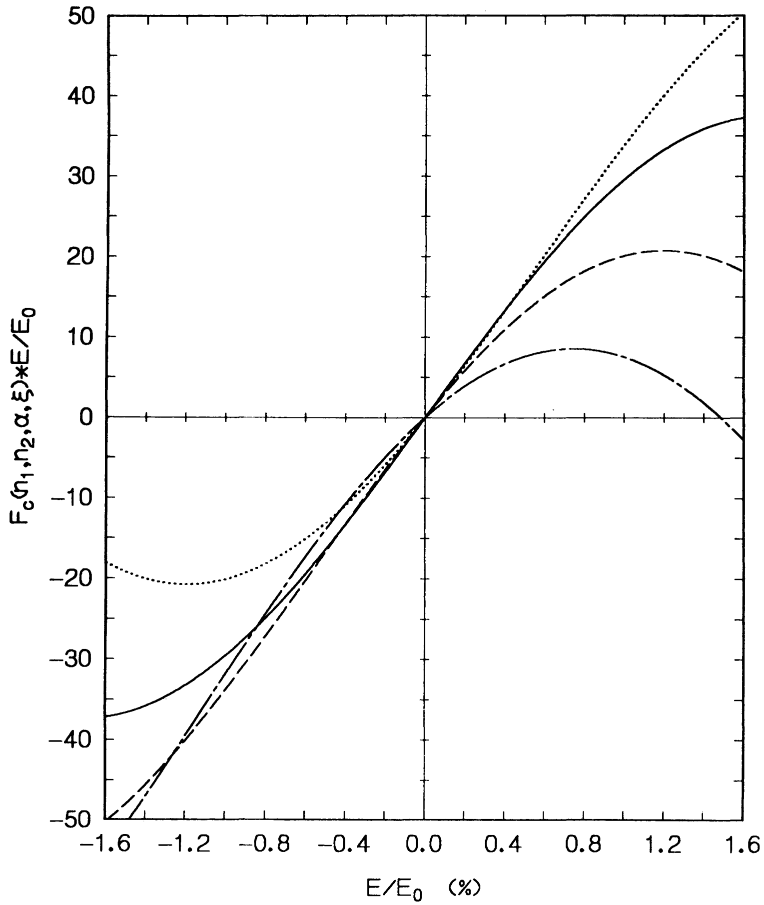


FIGURE 5 Profiles of $F_c(18, 84, \pi/2, \xi) \cdot E/E_0$ for $\Delta T_D = 0$ ns (——), 1 ns (– – –), 2 ns (– · –), and −1 ns (· · ·). ΔE_c is nearly proportional to E with $\Delta T_D = 0$ ns. A setting error of ΔT_D equal to ±1 ns is acceptable, but one of 2 ns is not.

setting ΔT_D to

$$\Delta T_{D\text{opt}} = \frac{\xi_{\text{opt}}}{2\pi f_0}, \tag{53}$$

where ξ_{opt} is the value at which $|F_c(n_1, n_2, \alpha, \xi)|$ takes is maximum value. The maximum value, $|F_c(n_1, n_2, \alpha, \xi_{\text{opt}})|$, hardly varies with α, as shown in Fig. 4:

$$F_c(n_1, n_2, \alpha, \xi_{\text{opt}}) \cong \frac{1}{2}(n_2^2 - n_1^2). \tag{54}$$

Therefore, independent of α, we have an approximate expression for $\Delta E_c(E)$:

$$\Delta E_c(E) \cong -2\frac{(qe)^2}{A}(n_2^2 - n_1^2) f_0 \, |\kappa G Z_p f_k| \frac{E}{E_0}. \tag{55}$$

Defining the system bandwidth W and the central frequency of the bandpass, W_c, as

$$W \equiv (n_2 - n_1)f_0, \tag{56}$$

$$W_c \equiv \tfrac{1}{2}(n_2 + n_1)f_0, \tag{57}$$

we have the single-particle cooling time defined by Eq. (13):

$$\tau_0 = \left[4\frac{(qe)^2}{A} WW_c\, |\kappa G Z_p f_k| \frac{1}{E_0}\right]^{-1}. \tag{58}$$

In Appendix A, we discuss the coherent energy correction with another type of a notch filter, whose phase response is not a square wave [as defined by Eq. (41)] but rather a sawtooth shape. With this notch filter, the single-particle cooling time of Eq.(58) is also derived, though the region of E where ΔE_c is proportional to E is appreciably reduced.

5. DIFFUSION TERM

The kicker is excited by noises—both the beam noise and the amplifier noise. The calculation of the electric power of the noises leads to $\overline{(\Delta E_{ic})^2}$:

$$\overline{(\Delta E_{ic})^2} = \frac{(qe)^2}{A^2}|f_k|^2 Z_c(P_a + P_b), \tag{59}$$

where P_a is the power of the amplifier noise, and P_b the beam noise. The powers are obtained by integrating the noise spectrums over the bandpass.

5.1. Beam Noise

The spectrum of the beam current is illustrated in Figs. 3c and 3d. The rms current in a Schottky band is

$$i_{\rm rms} = \sqrt{2N}\, qef_0, \tag{60}$$

because the current amplitude of one particle is $2ef_0$, and the phase is random. The density of the noise power induced by this current is

$$2(qe)^2 f_0^2 \frac{|Z_p|^2}{Z_c}\frac{dN}{df} \tag{61}$$

at the output of the pickup. This noise is amplified in the feedback system, and at the kicker,

$$P_b = \int_{n_1 f_0}^{n_2 f_0} 2(qe)^2 f_0^2 \frac{|Z_p|^2}{Z_c}\frac{dN}{df} G_{\rm amp}^2 |H(f)|^2\, df. \tag{62}$$

For simplicity, we assume a flat distribution of E; then we have

$$\frac{dN}{df} = \frac{N}{n\Delta f_0} \tag{63}$$

at the nth harmonic. Then, with the ideal linear notch filter, we have

$$P_b = \frac{2}{3}(qe)^2 \frac{|Z_p|^2}{Z_c} NG^2(\Delta f_0)^2 \sum_{n=n_1}^{n_2} n^2. \tag{64}$$

With an approximation,

$$\sum_{n=n_1}^{n_2} n^2 \cong \frac{W}{f_0^3}\left(W_c^2 + \frac{1}{12}W^2\right), \tag{65}$$

we have

$$P_b = \frac{2}{3}(qe)^2 N \frac{|Z_p|^2}{Z_c} G^2 \frac{W}{f_0}\left(W_c^2 + \frac{1}{12}W^2\right)\kappa^2\left(\frac{\Delta E_0}{E_0}\right)^2. \tag{66}$$

The resulting diffusion coefficient is

$$d_b = \frac{1}{4}\frac{N}{W}\frac{1}{\tau_0^2}\left(1 + \frac{1}{12}\frac{W^2}{W_c^2}\right), \tag{67}$$

where the following relation is used:

$$\Delta E_0 = \sqrt{12}\,\sigma_E(0) \text{ (for the flat distribution).} \tag{68}$$

5.2. *Amplifier Noise*

For an amplifier with a noise temperature T_n, the effective power density of the thermal noise at the input of the amplifier is

$$k(T + T_n), \tag{69}$$

where T is the ambient temperature, and k is Boltzmann's constant, 8.617×10^{-5} eV/K $= 1.381 \times 10^{-23}$ J/K. At the normal temperature $T_0 = 290$ K, the thermal noise power is expressed with a noise figure NF:

$$10^{NF/10} kT_0. \tag{70}$$

Integrating the power density in the same way as above, we have

$$P_a = \frac{1}{3}k(T + T_n)G^2 W. \tag{71}$$

We define noise-to-signal ratio at the pickup, U_p, then the ratio of the amplifier noise power to the beam noise power:

$$U_p \equiv \frac{k(T + T_n)f_0}{2(qe)^2 f_0^2 N \frac{|Z_p|^2}{Z_c}}. \tag{72}$$

The numerator and the denominator are powers per Schottky band at the input of the amplifier. Then we have the diffusion coefficient

$$D_a = \frac{1}{48}\frac{N}{W}\frac{1}{\tau_0^2}\frac{U_p f_0^2 E_0^2}{W_c^2 \kappa^2}. \tag{73}$$

6. INITIAL COOLING TIME, FINAL ENERGY SPREAD, AND SYSTEM GAIN

The coefficients necessary to calculate the final energy spread and the initial cooling time have been obtained. Their dependences on the amplifier gain are presented here.

It is convenient to define a noise-to-signal ratio at the kicker,

$$U_k \equiv \frac{P_a}{P_b}, \tag{74}$$

in order to measure which noise is dominant. This value grows large with the reduction of the energy spread of the beam; its initial value is given by

$$U_k(0) = \frac{D_a}{d_b \sigma_E^2(0)} \tag{75}$$

$$= \left(1 + \frac{1}{12}\frac{W^2}{W_c^2}\right)^{-1} \left(\frac{f_0}{\kappa W_c}\frac{E_0}{\Delta E_0}\right)^2 U_p. \tag{76}$$

The initial cooling time defined by Eq. (71) is a function of the system gain G and takes its minimum value

$$\tau_{i,\min} = \frac{N}{W}\left(1 + \frac{1}{12}\frac{W^2}{W_c^2}\right)(1 + U_k(0)), \tag{77}$$

with

$$|G_{\max}| = \frac{1}{2}\frac{A}{(qe)^2}\frac{1}{NW_c}\left(1 + \frac{1}{12}\frac{W^2}{W_c^2}\right)^{-1}\frac{E_0}{|\kappa Z_p f_k|}\frac{1}{1 + U_k(0)}. \tag{78}$$

At this maximum system gain, the electric power applied to the kicker at $t = 0$ is

$$P_{\max} = \frac{1}{3}k(T + T_n)WG_{\max}^2\left(1 + \frac{1}{U_k(0)}\right), \tag{79}$$

and the final energy spread is

$$\frac{\sigma_E(\infty)}{\sigma_E(0)} = \left(\frac{U_k(0)}{2U_k(0) + 1}\right)^{1/2} (|G| = |G_{\max}|). \tag{80}$$

It is usual to set the system gain lower than $|G_{\max}|$ for the following reasons. First, in the case of a high-energy beam and a broad bandwidth, $P_{\max}$ exceeds a practically available value. Second, in the case of a large noise-to-signal ratio, $U_k(0) \gg 1$, the final energy spread is as large as $\sigma_E(0)/\sqrt{2}$. Defining the normalized system gain as

$$G_N \equiv \frac{G}{G_{\max}}, \tag{81}$$

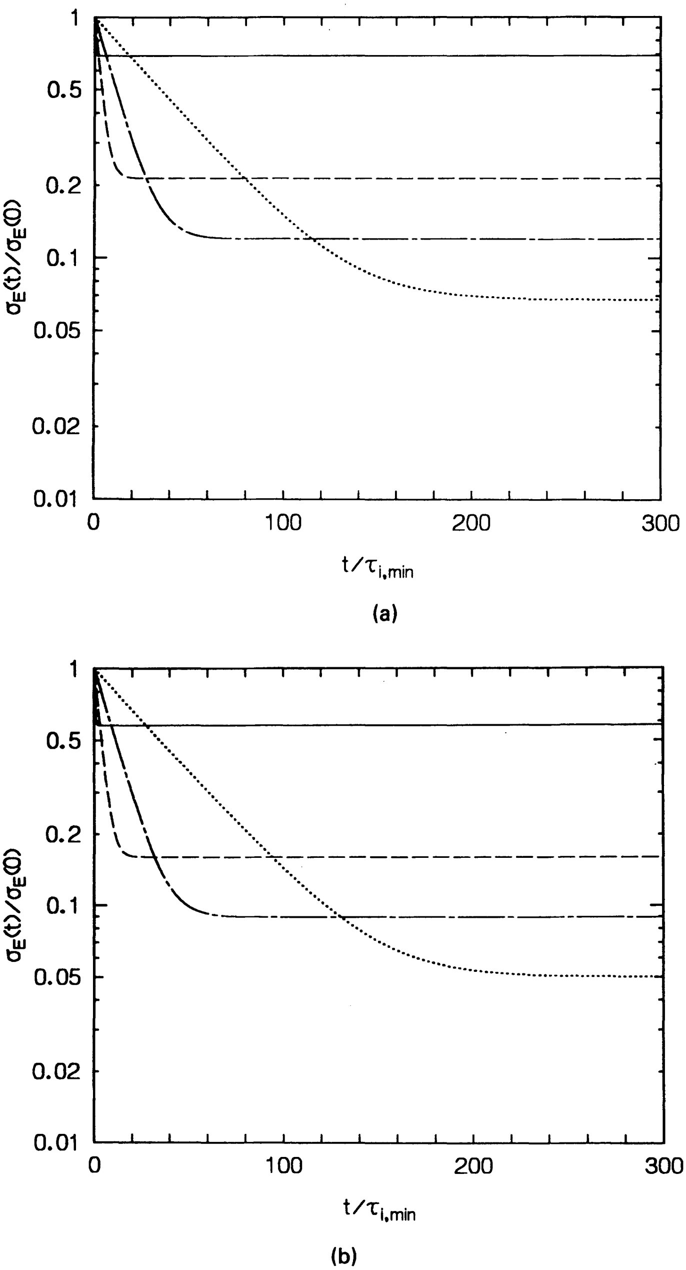
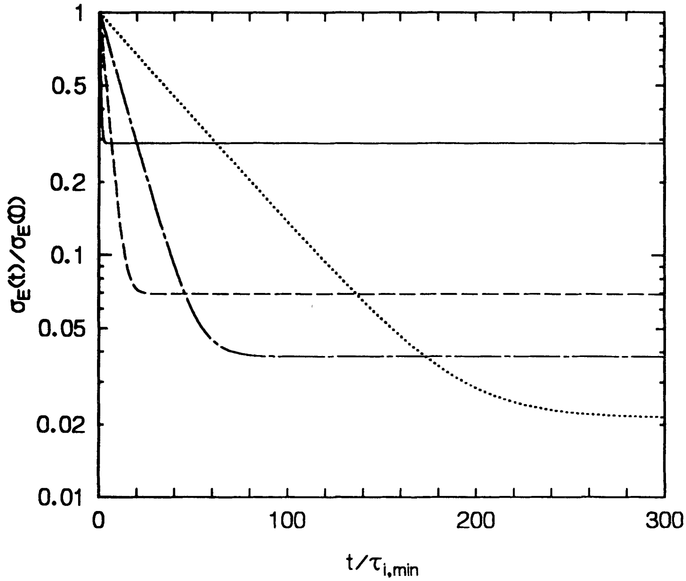


FIGURE 6 (a) Profiles of $\sigma_E(t)/\sigma_E(0)$ for $U_k(0)=10$, $G_N=1$ (——), 0.1 (– – –), 0.0316 (– · –), and 0.01 (· · ·); (b) profiles of $\sigma_E(t)/\sigma_E(0)$ for $U_k(0)=1$, $G_N=1$ (——), 0.1 (– – –), 0.0316 (– · –), and 0.01 (· · ·); and (c) profiles of $\sigma_E(t)/\sigma_E(0)$ for $U_k(0)=0.1$, $G_N=1$ (——), 0.1 (– – –), 0.0316 (– · –), and 0.01 (· · ·).



(c)

FIGURE 6(c)

we can express Eq. (18) as

$$\frac{\sigma_E(t)}{\sigma_E(0)}=\left(\frac{[1+U_k(0)](2-G_N)}{2[1+U_k(0)]-G_N}\exp\left\{-\left[4G_N-\frac{2G_N^2}{1+U_k(0)}\right]\frac{t}{\tau_{i\,\min}}\right\}+\frac{G_N U_k(0)}{2[1+U_k(0)]-G_N}\right)^{1/2}, \tag{82}$$

with

$$\frac{\sigma_E(\infty)}{\sigma_E(0)}=\left\{\frac{G_N U_k(0)}{2[1+U_k(0)]-G_N}\right\}^{1/2}, \tag{83}$$

$$\tau_i=\frac{\tau_{i\,\min}}{G_N(2-G_N)}. \tag{84}$$

When $U_k(0)\gg 1$, Eq. (82) is simplified as

$$\frac{\sigma_E(t)}{\sigma_E(0)}=\left[\left(1-\frac{G_N}{2}\right)\exp\left(-\frac{4G_N t}{\tau_{i\,\min}}\right)+\frac{G_N}{2}\right]^{1/2}, \tag{85}$$

with

$$\frac{\sigma_E(\infty)}{\sigma_E(0)}=\left(\frac{G_N}{2}\right)^{1/2}. \tag{86}$$

Figures 6a through 6c show profiles of $\sigma_E(t)/\sigma_E(0)$ given by Eq. (82) for values of $U_k(0)=10$, 1, and 0.1. As is apparent in the figures, the system gain must be set much smaller than $G_{\max}$ to attain a small final energy spread, e.g., $\sigma_E(\infty)/\sigma_E(0)\lesssim 0.1$, especially in the case of $U_k(0)>1$. As a result, it takes much longer than $\tau_{i\,\min}$.

The system gain is optimized so that a desired final energy spread is attained in as short a time as possible, and so that the electric power applied to the kicker, in practice the output power of the final power amplifier,

$$P = G_N^2 P_{\max}, \tag{87}$$

is realistic.

7. ACCELERATION RATE

Removing the notch filter from the cooling system, we can accelerate or decelerate the beam: A particle is kicked by its own signal provided that the signal's transmission time is set nearly equal to the time-of-flight of the particle.

Setting $H(f) = 1$ in Eq. (38), we have the acceleration rate per turn:

$$\Delta E_a = \frac{(qe)^2}{A} f_0 \left(1 + \kappa \frac{E}{E_0}\right) \left\{ \sum_{n=-\infty}^{\infty} G_{\text{amp}}(nf_0) \, |Z_p(nf_0) f_k(nf_0)| \right.$$

$$\left. \times [\cos \alpha(nf_0) \cos n\xi + \sin \alpha(nf_0) \sin n\xi] \right\}, \tag{88}$$

where ξ and α are defined by Eqs. (39) and (46), respectively. With an assumption that Z_p and f_k are independent of frequency and that G_{amp} is constant over the bandpass, Eq. (88) leads to

$$\Delta E_a = 2 \frac{(qe)^2}{A} f_0 \left(1 + \kappa \frac{E}{E_0}\right) G_{\text{amp}} \, |Z_p f_k| \left(\cos \alpha \sum_{n=n_1}^{n_2} \cos n\xi + \sin \alpha \sum_{n=n_1}^{n_2} \sin n\xi \right). \tag{89}$$

To measure the acceleration rate, the momentum spread of the beam is set small to observe easily the shift of the Schottky signal. In this case, we can set $E = 0$, and we have

$$\Delta E_a = 2 \frac{(qe)^2}{A} f_0 G_{\text{amp}} \, |Z_p f_k| \, F_a(n_1, n_2, \alpha, \zeta), \tag{90}$$

where

$$\zeta \equiv 2\pi f_0 \Delta T_D, \tag{91}$$

$$F_a(n_1, n_2, \alpha, \zeta) \equiv \cos \alpha \sum_{n=n_1}^{n_2} \cos n\zeta + \sin \alpha \sum_{n=n_1}^{n_2} \sin n\zeta. \tag{92}$$

The summations of the circular functions can be found in Appendix B. For the case of TARN, $n_1 = 18$ and $n_2 = 84$, the factor $F_a(n_1, n_2, \alpha, \zeta)$ for $\alpha = 0$, $\pi/4$, $\pi/2$, and $3\pi/4$ are shown in Fig. 7. As is apparent in the figure, $|F_a|$ takes a maximum value of about $n_2 - n_1$, independently of α. Therefore, we obtain the maximum acceleration rate with an acceptable signal transmission time,

$$\Delta E_{a,\max} \cong 2 \frac{(qe)^2}{A} W \, |G_{\text{amp}} Z_p f_k|, \tag{93}$$

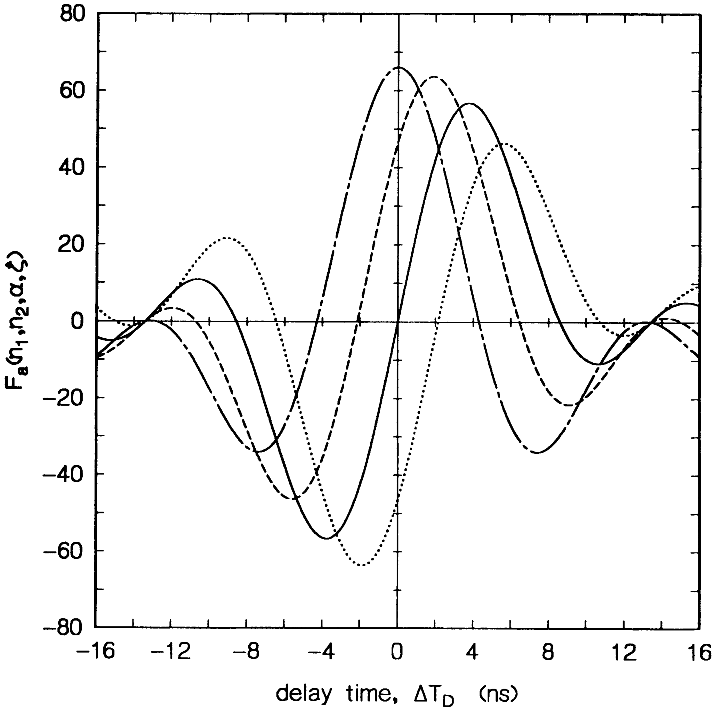


FIGURE 7 Profiles of $F_a(18, 84, \alpha, \zeta)$ for $\alpha = 0$ (– · –), $\pi/4$ (– – –), $\pi/2$ (——), and $3\pi/4$ (· · ·). The maximum value of $|F_a|$ is nearly $n_2 - n_1 = 66$, independent of α.

with

$$\Delta T_{D,\mathrm{opt}} = \frac{\zeta_{\mathrm{opt}}}{2\pi f_0}, \tag{94}$$

where ζ_{opt} is the value at which $|F_a(n_1, n_2, \alpha, \zeta_{\mathrm{opt}})|$ takes its maximum value. It should be noted that the value of ΔT_D providing the maximum acceleration rate differs from the one providing the maximum cooling rate. For instance, at $\alpha = \pi/2$, ΔT_D should be 4 ns for acceleration, but 0 ns for cooling.

As is apparent in Fig. 7, F_a is nearly zero at $\Delta T_D = \Delta T_{D,\mathrm{opt}} \pm \frac{1}{2}(f_{\min} + f_{\max})^{-1}$; this is also derived from the following approximation for $\zeta \ll 1$:

$$F_a(n_1, n_2, \alpha, \zeta) \cong \frac{2}{\zeta} \cos\left(\frac{n_2 + n_1}{2}\zeta - \alpha\right) \sin\frac{n_2 - n_1}{2}\zeta. \tag{95}$$

Therefore, $(f_{\min} + f_{\max})^{-1}$ gives the pulse width of a single-particle signal. A setting error of the signal transmission time within a few tenths of this pulse width has little effect on acceleration rate. This is the required setting accuracy of the transmission time for a given bandpass.

8. CONCLUDING REMARKS

The Fokker–Planck equation presented here was developed to analyze the experimental results of stochastic momentum cooling with a notch filter at TARN, where protons and α particles of 7 MeV/u are cooled. In the formulation, the coherent energy correction ΔE_c and the acceleration rate ΔE_a are expressed in terms of the parameter ΔT_D, denoting the difference between the particle's time-of-flight and the signal transmission time from the pickup to the kicker. This formulation provides a required setting accuracy of the signal transmission time for a given momentum spread and bandpass.

The experimental results at TARN, i.e., the initial cooling time, final momentum spread, and acceleration rate, have been compared with this theory, assuming that the pickup's coupling impedance, the kicker's efficiency, and the system gain are constant over the bandpass. Despite this assumption, good agreement with the experimental results has been obtained.[4] These results will be discussed in more detail in a forthcoming publication, which will also treat the matter of signal suppression.

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APPENDIX A

Coherent Energy Correction with a Notch Filter with a Sawtooth Phase Response

In Section 4, we assume the phase response of a notch filter is a square wave defined by Eq. (41). In the case of TARN, however, the phase response of the notch filter is a sawtooth, as in Fig. A1. Here we investigate the coherent energy correction with such a notch filter.

The amplitude and the phase characteristics are expressed as follows:

$$|H(f)| = 2g_{\text{pole}}\left|\frac{f-nf_0}{f_0/2}\right| \quad [(n-\tfrac{1}{2})f_0 \le f \le (n+\tfrac{1}{2})f_0], \tag{A-1}$$

$$\phi_{nf}(f) = \frac{\pi}{2} - \frac{\pi}{f_0}(f-nf_0) \quad [nf_0 \le f < (n+1)f_0], \tag{A-2}$$

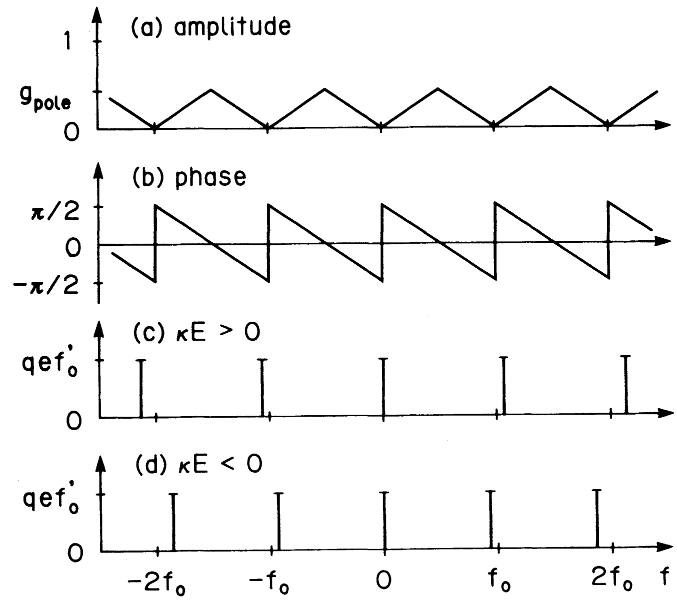

FIGURE A-1 Amplitude and phase responses of the notch filter with a sawlike phase response [(a) and (b)]; and spectra for single particle current for $\kappa E > 0$ (c), and $\kappa E < 0$ (d).

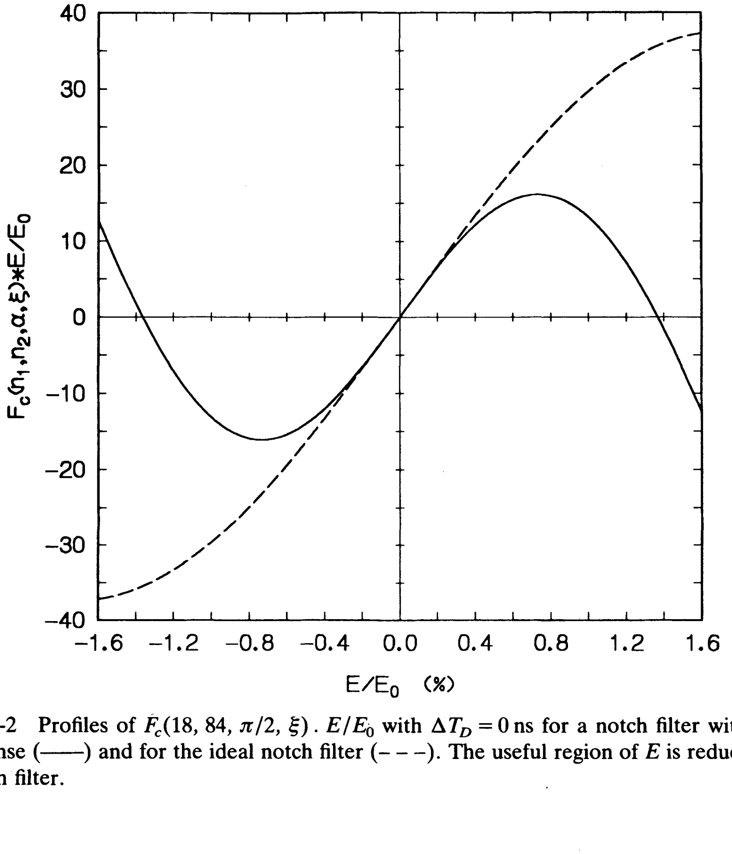

FIGURE A-2 Profiles of $F_c(18, 84, \pi/2, \xi) \cdot E/E_0$ with $\Delta T_D = 0$ ns for a notch filter with a sawlike phase response (——) and for the ideal notch filter (- - -). The useful region of E is reduced with the former notch filter.

and consequently,

$$H(nf_0') = j2g_{\text{pole}} n\kappa \frac{E}{E_0} \exp\left(-jn\pi\kappa \frac{E}{E_0}\right). \tag{A-3}$$

With constant Z_p, f_k, and G_{amp} over the bandpass, Eq. (38) leads to

$$\Delta E_c(E) = -4\frac{(qe)^2}{A} f_0 G \left|Z_p f_k\right| \kappa F_c(n_1, n_2, \alpha, \xi) \frac{E}{E_0}, \tag{A-4}$$

where

$$\xi \equiv 2\pi f_0 T_F \left[\left(1 + \frac{1}{2f_0 T_F}\right) \kappa \frac{E}{E_0} + \frac{\Delta T_D}{T_F}\right]. \tag{A-5}$$

Equation (A-4) is the same as Eq. (44) except that the definition of ξ is different from Eq. (39). Consequently, the single particle cooling time is given by Eq. (58). However, the region of E where ΔE_c is proportional to E is appreciably reduced; Fig. A-2 shows $F_c(n_1, n_2, \alpha, \xi) \cdot E/E_0$ for $\alpha = \pi/2$ and $\Delta T_D = 0$, which can be compared with the one for the ideal linear notch filter discussed in Section 4.

APPENDIX B

Formulas for the Summation of Circular Functions

$$\sum_{n=1}^{N} \cos nx = \frac{\cos\left(\frac{N+1}{2}x\right) \sin \frac{N}{2}x}{\sin \frac{x}{2}} \tag{B-1}$$

$$\sum_{n=1}^{N} \sin nx = \frac{\sin\left(\frac{N+1}{2}x\right) \sin \frac{N}{2}x}{\sin \frac{x}{2}} \tag{B-2}$$

$$\sum_{n=1}^{N} n \cos nx = \frac{1}{2\sin^2 \frac{x}{2}} \left[N \sin\left(N + \tfrac{1}{2}\right)x \sin \frac{x}{2} - \sin^2 \frac{N}{2}x\right] \tag{B-3}$$

$$\sum_{n=1}^{N} n \sin nx = \frac{1}{2\sin^2 \frac{x}{2}} \left[-N \cos\left(N + \frac{1}{2}\right)x \sin \frac{x}{2} + \frac{1}{2} \sin Nx\right] \tag{B-4}$$

For $x \ll 1$, n_1, $n_2 \gg 1$:

$$\sum_{n=n_1}^{n_2} \cos nx \cong \frac{1}{x}(\sin n_2 x - \sin n_1 x) \tag{B-5}$$

$$\sum_{n=n_1}^{n_2} \sin nx \cong -\frac{1}{x}(\cos n_2x - \cos n_1x) \tag{B-6}$$

$$\sum_{n=n_1}^{n_2} n \cos nx \cong \frac{n_2 \sin n_2x - n_1 \sin n_1x}{x} + \frac{\cos n_2x - \cos n_1x}{x^2} \tag{B-7}$$

$$\sum_{n=n_1}^{n_2} n \sin nx \cong -\frac{n_2 \cos n_2x - n_1 \cos n_1x}{x} + \frac{\sin n_2x - \sin n_1x}{x^2} \tag{B-8}$$