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LONGITUDINAL INSTABILITIES OF BUNCHED BEAMS SUBJECT TO A NON-HARMONIC *RF* POTENTIAL†

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We consider the longitudinal instabilities of a bunched beam subject to a non-harmonic *rf* potential. Assuming the unperturbed bunch to be described by a Maxwell–Boltzmann distribution, our treatment is based upon the linearized Vlasov equation. The formalism developed is exact, and in particular, correctly describes the effect of the dependence on amplitude of the synchrotron oscillation frequency. We discuss the fast blowup limit, and extend Wang and Pellegrini's treatment of the microwave instability to include the case of a non-Gaussian bunch. Next, within the short-bunch approximation, we determine the Landau damping of coupled-bunch oscillations that results from the use of a higher-harmonic (Landau) cavity.

I. INTRODUCTION

We consider a beam bunched by a non-harmonic *rf* potential and discuss the longitudinal coherent instabilities[1,2] resulting from the interaction of the beam with the impedance of the storage ring. Assuming the unperturbed bunch to be described by a Maxwell–Boltzmann distribution,[3] we base our calculations upon the linearized Vlasov equation. The amplitude dependence of the synchrotron-oscillation frequency is taken into account by the use of action-angle variables,[4] and we derive an infinite set of linear homogeneous equations describing the coherent oscillations. These equations are used to extend Wang and Pellegrini's[5] treatment of the microwave instability to include non-Gaussian bunches. The Boussard criterion[6] for stability is derived, having the form of a coasting-beam stability condition, except that the average current of the coasting beam is replaced by the peak current of the bunch.

In general, the equations describing the coherent oscillations cannot be solved analytically, but these equations do become tractable in certain asymptotic limits. In the treatment of the microwave instability, we consider modes whose growth times are short compared with the synchrotron-oscillation period (fast blowup). The wavelengths of the perturbing electromagnetic fields are assumed to be short compared to the bunch length and the high frequency impedance is supposed to have a bandwidth large compared with the inverse bunch length. Eigenmodes are found that correspond to line-charge density modulations taking place within a small portion of the bunch, having dimension much shorter than the bunch length.

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The growth rate of an eigenmode is proportional to the line-charge density of the bunch at the location where the coherent oscillation corresponding to the mode is centered. Thus, the fastest-growing modes have growth rates proportional to the peak current of the bunch.

In contrast to the case just discussed, a resonant impedance having narrow bandwidth can result in coupled-bunch instabilities with growth rates proportional to the average current in the ring. As a soluble example, we consider the idealized case of a multibunch mode driven by the impedance Z_n, only for $n = n_0$. In this case, the dispersion relation describing the coherent oscillations is determined from a single diagonal matrix element, corresponding to the revolution mode n_0, rather than from an infinite determinant. Analytic results are obtainable yielding insight into the behavior of instabilities due to more general resonant impedances.

Assuming the impedance Z_n to be negligible for $n > 1/L$, where L is the bunch length in radians, a perturbation expansion[7] can be developed in the limit when the bunch length is short compared with the wavelength of the perturbing electromagnetic fields. Keeping terms only up to a given order in the small parameter, bunch length divided by wavelength, the infinite set of linear equations describing the coherent oscillations becomes of finite rank. Therefore, the infinite set can be replaced by a finite set of linear equations and the dispersion relation is derived from the vanishing of a determinant of finite dimensions. An interesting application of this short-bunch approximation is the derivation of the dispersion relation describing the Landau damping of coupled-bunch instabilities resulting from the use of a higher-harmonic (Landau) cavity.[8]

Our paper is organized as follows: In Section II, we discuss the nonlinear synchrotron oscillations resulting from a non-harmonic *rf* potential in the absence of the ring impedance. Then, in Section III, we take the impedance into account by using the linearized Vlasov equation, and we derive Eqs. (3.25)–(3.27) which describe the coherent oscillations. In the remainder of the paper, these equations are studied in special limits amenable to analytic approximation.

In Section IV, we show how our equations simplify in the fast-blowup limit when the growth rate is faster than the synchrotron-oscillation frequency and in Section V, we discuss the microwave instability. At the end of this section, we comment on the work of Messerschmid and Month.[9] The idealized case for which Z_n is negligible except for $n = \pm n_0$ is considered in Section VI, and in Section VII we derive the short-bunch expansion and discuss its relation to the synchrotron mode expansion.

In the final two sections, we illustrate the general formalism by studying two concrete examples. In Section VIII, we present an overview of the longitudinal instabilities of Gaussian bunches subject to a harmonic *rf* potential. Our emphasis is on the behavior of long bunches having lengths greater than the wavelengths of the perturbing electromagnetic fields. We exhibit the crossover between the dominance of the synchrotron modes and the coasting-beam-like distortions of the bunch distribution, which occurs as the real or imaginary part of the coherent oscillation frequency becomes large compared with the synchrotron-oscillation frequency.

In Section IX, we consider a potential comprised of the sum of a harmonic and a quartic term. Our discussion allows us to determine the Landau damping resulting from the use of a Landau cavity.

II. EQUATIONS OF MOTION FOR SYNCHROTRON OSCILLATIONS

Consider a storage ring having a circumference $2\pi R$. To simplify our notation, we shall assume that the energy of a circulating particle is large compared with its rest mass, so its velocity is very close to the speed of light c. A particle is called synchronous if the energy it gains at the *rf* cavity is equal to the energy it loses during one revolution. We denote the energy of a synchronous particle by E_0 and its angular velocity by $\omega_0 = c/R$. The azimuthal angular position of a circulating particle relative to a stationary observer is denoted θ, where $\dot{\theta}$ is the instantaneous value of the angular velocity. In writing the equations of motion, we measure the azimuthal position ϕ and energy ε relative to a synchronous particle, i.e.

$$\phi = \theta - \omega_0 t \tag{2.1}$$

and

$$\varepsilon = E - E_0. \tag{2.2}$$

The equations of motion describing the synchrotron oscillations are

$$\dot{\phi} = -\frac{\alpha\omega_0}{E_0}\varepsilon, \tag{2.3}$$

$$\dot{\varepsilon} = \frac{e\omega_0}{2\pi}(V_{rf}(\phi) + V_i(\phi, t)), \tag{2.4}$$

where e is the electric charge of the particle, α is the momentum compaction, $V_{rf}(\phi)$ is the *rf* potential and $V_i(\phi, t)$ is the induced potential resulting from the impedance of the storage ring.

The induced potential is due to a collective effect involving all the circulating particles. In order to describe mathematically the induced potential, we introduce the distribution function $\psi(\dot{\phi}, \phi, t)$, normalized by

$$\int d\dot{\phi}\, d\phi\, \psi(\dot{\phi}, \phi, t) = Ne, \tag{2.5}$$

where N is the *total* number of particles in the storage ring. The line charge density $\lambda(\phi, t)$ is related to ψ by

$$\lambda(\phi, t) = \int d\dot{\phi}\, \psi(\dot{\phi}, \phi, t). \tag{2.6}$$

Of particular interest to us is the determination of the conditions under which there exists a coherent oscillation with frequency Ω. To be specific, we shall study the conditions under which the line-charge density has the form

$$\lambda(\phi, t) = \rho_0(\phi) + \rho(\phi)e^{-i\Omega t}, \tag{2.7a}$$

giving rise to an induced potential

$$V_i(\phi, t) = V_0(\phi) + V_i(\phi)e^{-i\Omega t}. \tag{2.7b}$$

Introducing the Fourier transform of $\rho(\phi)$ via

$$\rho_n = \int \frac{d\phi}{2\pi} e^{-in\phi}\rho(\phi), \tag{2.8}$$

we express $V_i(\phi)$ in terms of the impedance $Z_n(\omega)$ of the storage ring,

$$V_i(\phi) = -\omega_0 \sum_n \rho_n Z_n(n\omega_0 + \Omega) e^{in\phi}. \tag{2.9}$$

By defining the Fourier transform $\rho_{0,n}$ of $\rho_0(\phi)$ in a manner analogous to Eq. (2.8), the time-independent piece of the induced potential is

$$V_0(\phi) = -\omega_0 \sum_n \rho_{0,n} Z_n(n\omega_0) e^{in\phi}. \tag{2.10}$$

From Eq. (2.4), it is clear that $V_0(\phi)$ corresponds to a distortion of the *rf* potential $V_{rf}(\phi)$, and consequently, gives rise to a change in the equilibrium bunch shape. In this paper we shall ignore $V_0(\phi)$, and we shall suppose the equilibrium bunch shape to be that given by the *rf* potential. Our attention shall be focused on the coherent instabilities which can arise as a result of the induced potential $V_i(\phi)\exp(-i\Omega t)$. Upon ignoring $V_0(\phi)$, the equations of motion given in Eqs. (2.3) and (2.4) can be combined to yield

$$\ddot{\phi} = -\frac{\alpha\omega_0}{E_0}\frac{e\omega_0}{2\pi}(V_{rf}(\phi) + V_i(\phi)e^{-i\Omega t}). \tag{2.11}$$

Neglecting the effect of $V_0(\phi)$ results in an over-estimate of the real frequency shift.

It is useful to consider this equation of motion as being derived from a Hamiltonian

$$H = \tfrac{1}{2}p^2 + U(\phi, t), \tag{2.12}$$

where p is the dynamical variable conjugate to ϕ. Since,

$$\dot{\phi} = \frac{\partial H}{\partial p} = p, \tag{2.13a}$$

$$\dot{p} = -\frac{\partial H}{\partial \phi} = -\frac{\partial U}{\partial \phi}, \tag{2.13b}$$

and

$$\ddot{\phi} = -\frac{\partial U}{\partial \phi}, \tag{2.13c}$$

we see that,

$$U(\phi, t) = U_0(\phi) + U_i(\phi)\exp(-i\Omega t), \tag{2.14a}$$

with

$$U_0(\phi) = \frac{\alpha\omega_0}{E_0}\frac{e\omega_0}{2\pi}\int^{\phi} V_{rf} \tag{2.14b}$$

and

$$U_i(\phi)=\frac{\alpha\omega_0}{E_0}\frac{e\omega_0}{2\pi}\int^{\phi} V_i. \tag{2.14c}$$

For a harmonic *rf* potential, $V_{rf}(\phi)=V\phi$, hence

$$U_0(\phi)=\tfrac{1}{2}\omega_s^2\phi^2, \tag{2.15}$$

with the harmonic synchrotron frequency ω_s given by

$$\omega_s^2=\frac{\alpha\omega_0}{E_0}\frac{e\omega_0}{2\pi}V. \tag{2.16}$$

Note that if the harmonic potential is to be thought of as an approximation to the sinusoidal potential, $\hat{V}\sin(h\phi+\phi_s)$, then in Eq. (2.16), we have $V=h\hat{V}\cos\phi_s$, where ϕ_s is the synchronous phase.

Before treating the full Hamiltonian of Eq. (2.12), let us consider the Hamiltonian H_0 describing the synchrotron oscillations in the absence of the impedance,

$$H_0=\tfrac{1}{2}p^2+U_0(\phi). \tag{2.17}$$

Consider a Hamilton–Jacobi transformation from ϕ, p to the new canonical variables Q, P. The generating function $W(\phi, P)$ is determined by

$$\frac{1}{2}\left(\frac{\partial W}{\partial\phi}\right)^2+U_0(\phi)=P, \tag{2.18}$$

and

$$p=\frac{\partial W}{\partial\phi}=(2(P-U_0(\phi)))^{1/2} \tag{2.19}$$

$$Q=\frac{\partial W}{\partial P}=\int^{\phi}\frac{d\phi'}{(2(P-U_0(\phi')))^{1/2}}. \tag{2.20}$$

The time dependence of the new variables Q, P result from the transformed Hamiltonian $\tilde{H}_0=P$, hence

$$\dot{P}=-\frac{\partial\tilde{H}_0}{\partial Q}=0, \tag{2.20a}$$

and

$$\dot{Q}=\frac{\partial\tilde{H}_0}{\partial P}=1, \qquad Q=t+Q_0. \tag{2.20b}$$

Let us suppose the *rf* potential is such that the synchrotron motion is periodic with period $T_s(P)$, and angular frequency

$$\omega_s(P)=2\pi/T_s(P). \tag{2.21}$$

In this case of periodic motion, it is often useful to introduce the action-angle variables[10] θ, J. We define

$$J=\frac{1}{2\pi}\oint p\,d\phi, \tag{2.22}$$

where the integral is over one period of the motion. Using Eqs. (2.19) and (2.13a) in (2.22), we see that

$$\frac{dJ}{dP}=\frac{1}{\omega_s(P)}. \tag{2.23}$$

In terms of the action-angle variables the transformed Hamiltonian is $\tilde{H}_0=P(J)$, and

$$\dot{J}=-\frac{\partial \tilde{H}_0}{\partial \theta}=0. \tag{2.24a}$$

$$\dot{\theta}=\frac{\partial \tilde{H}_0}{\partial J}=\omega_s(J), \qquad \theta=\omega_s(J)t+\theta_0. \tag{2.24b}$$

From Eq. (2.24b), it follows that the change of θ in one period is 2π, independent of J. This is in contrast to the behavior of the variable Q, which changes by $T_s(P)$ in one period. Because the increment in θ in one period is independent of J, action-angle variables offer a significant advantage over the conjugate pair Q, P.

The solution of the equations of motion of the synchrotron oscillations, corresponding to the Hamiltonian H_0 of Eq. (2.17), can be written

$$\phi=\phi_0(J,\theta), \tag{2.25}$$

where $\phi_0(J,\theta+2\pi)=\phi_0(J,\theta)$. Equation (2.25) will be of great use to us in the following sections of this paper, when we study the effect of the induced potential.

Let us include here for later reference, some results valid for a symmetric potential, $U_0(\phi)=U_0(-\phi)$. Denote the amplitude of the synchrotron oscillation by

$$r=\phi_{\max}, \tag{2.26}$$

so $H_0=U_0(r)$. Then,

$$J=\frac{2}{\pi}\int_0^r d\phi(2(U_0(r)-U_0(\phi)))^{1/2}, \tag{2.27}$$

and

$$T_s(J)=\frac{2\pi}{\omega_s(J)}=4\int_0^r \frac{d\phi}{(2(U_0(r)-U_0(\phi)))^{1/2}} \tag{2.28}$$

The function ϕ_0 of Eq. (2.25) is implicitly determined by

$$\theta=\omega_s(J)\int^{\phi}\frac{d\phi'}{(2(U_0(r)-U_0(\phi')))^{1/2}} \tag{2.29}$$

where $r=r(J)$ is found as a function of J from Eq. (2.27).

III. VLASOV EQUATION

The Vlasov equation for the distribution $\psi(p,\phi,t)$ can be written as

$$\frac{\partial \psi}{\partial t}+(\psi, H)=0, \tag{3.1}$$

where the Poisson–Bracket is defined by

$$(\psi, H) = \frac{\partial\psi}{\partial\phi}\frac{\partial H}{\partial p} - \frac{\partial\psi}{\partial p}\frac{\partial H}{\partial\phi} \tag{3.2a}$$

$$= \frac{\partial\psi}{\partial\theta}\frac{\partial H}{\partial J} - \frac{\partial\psi}{\partial J}\frac{\partial H}{\partial\theta}. \tag{3.2b}$$

The equality between the expressions of (3.2a) and (3.2b) follows because the Poisson–Bracket is invariant under canonical transformations. Moreover, since ϕ and p are conjugate variables, their Poisson–Bracket is unity, so

$$1 = (\phi, p) = \frac{\partial\phi}{\partial\theta}\frac{\partial p}{\partial J} - \frac{\partial\phi}{\partial J}\frac{\partial p}{\partial\theta}. \tag{3.3}$$

The right-hand side of Eq. (3.3) is the Jacobian $\partial(\phi, p)/\partial(\theta, J)$; therefore

$$d\phi\, dp = d\theta\, dJ. \tag{3.4}$$

We shall first consider a single bunch, but at the end of this section we shall show how the results carry over to the case of M equally spaced bunches all having the same number of particles. Using action-angle variables, the full Hamiltonian given in Eq. (2.12) can be written,

$$H = \tilde{H}_0(J) + U_i(\phi_0(J, \theta))e^{-i\Omega t}, \tag{3.5}$$

where $\phi_0(J, \theta)$, defined in Eq. (2.25), describes the unperturbed motion existing in the absence of the induced potential. We look for a solution of the Vlasov equation (3.1) having the form

$$\psi = \psi_0(J) + \psi_i(J, \theta)e^{-i\Omega t}. \tag{3.6}$$

Here, $\psi_0(J)$ is the equilibrium bunch distribution, and $\psi_i(J, \theta)\exp(-i\Omega t)$ corresponds to a coherent oscillation with frequency Ω.

As discussed in Section II, following Eq. (2.10), we have neglected the potential-well distortion which would modify the time-independent part of the distribution. The work in later sections of this paper will be based upon the assumption that the equilibrium distribution has the Maxwell–Boltzmann[2] form

$$\psi_0(J) = Ae^{-\tilde{H}_0(J)/\sigma^2}. \tag{3.7}$$

In terms of the variables ϕ and p, this distribution can be written as

$$\psi_0(p, \phi) = Ae^{-p^2/2\sigma^2}e^{-U_0(\phi)/\sigma^2}. \tag{3.8}$$

From Eq. (3.8) it is evident that σ represents one standard deviation of the distribution in $p = \dot{\phi}$, i.e. σ is the spread in revolution frequency among particles in the bunch. Since from Eq. (2.3), $\dot{\phi} = -\alpha\omega_0\varepsilon/E_0$, it is evident that σ is related to the energy spread σ_ε by

$$\sigma = \frac{\alpha\omega_0}{E_0}\sigma_\varepsilon. \tag{3.9}$$

The constant A is determined from the normalization condition of Eq. (2.5),

$$A\sigma\sqrt{2\pi}\int d\phi e^{-U_0(\phi)/\sigma^2} = Ne. \tag{3.10}$$

To proceed with our study of the coherent instabilities induced by the beam interacting with the impedance of the storage ring, we insert Eqs. (3.5) and (3.6) into the Vlasov equation (3.1), and linearize the result by dropping terms of second order in ψ_i. In this manner, we derive the linearized Vlasov equation

$$-i\Omega\psi_i+\omega_s(J)\frac{\partial\psi_i}{\partial\theta}-\psi_0'(J)\frac{\partial U_i(\phi_0(J,\theta))}{\partial\theta}=0, \tag{3.11}$$

where we have defined

$$\psi_0'(J)=d\psi_0/dJ. \tag{3.12}$$

Using the definition of U_i given in Eq. (2.14b), we obtain

$$\frac{\partial U_i(\phi_0(J,\theta))}{\partial\theta}=\frac{dU_i}{d\phi_0}\frac{\partial\phi_0}{\partial\theta}=\frac{\alpha\omega_0}{E_0}\frac{e\omega_0}{2\pi}V_i(\phi_0(J,\theta))\frac{\partial\phi_0}{\partial\theta}. \tag{3.13}$$

Now expressing V_i in terms of the impedance via Eq. (2.9), and defining

$$2\pi\kappa=\frac{\alpha\omega_0}{E_0}\frac{e\omega_0}{2\pi}\omega_0, \tag{3.14}$$

it follows that

$$\frac{\partial U_i(\phi_0(J,\theta))}{\partial\theta}=-2\pi\kappa\frac{\partial\phi_0}{\partial\theta}\sum_n\rho_nZ_n(n\omega_0+\Omega)e^{in\phi_0(J,\theta)}. \tag{3.15}$$

By using Eq. (3.15), we can rewrite the linearized Vlasov equation of Eq. (3.11) as

$$-i\Omega\psi_i+\omega_s(J)\frac{\partial\psi_i}{\partial\theta}=\chi(\theta), \tag{3.16}$$

where $\chi(\theta)=\chi(\theta+2\pi)$ is defined by

$$\chi(\theta)=-2\pi\kappa\psi_0'(J)\frac{\partial\phi_0(J,\theta)}{\partial\theta}\sum_n\rho_nZ_n(n\omega_0+\Omega)e^{in\phi_0(J,\theta)}. \tag{3.17}$$

The first-order differential equation of Eq. (3.16) has the periodic solution

$$\psi_i(J,\theta)=\frac{e^{iQ(J)\theta}}{\omega_s(J)(1-e^{2\pi iQ(J)})}\int_{\theta-2\pi}^{\theta}d\theta' e^{-iQ(J)\theta'}\chi(\theta'), \tag{3.18}$$

where

$$Q(J)=\Omega/\omega_s(J). \tag{3.19}$$

Note that $Q(J)$ defined here is different from the quantity Q used in Section III. Inserting the defining expression (3.17) for $\chi(\theta)$ into (3.18) and making a change of integration variable, we obtain

$$\psi_i(J,\theta)=\frac{-2\pi\kappa}{\omega_s(J)}\frac{\psi_0'(J)}{1-e^{2\pi iQ(J)}}\sum_n\rho_nZ_n\int_{-2\pi}^{0}d\theta'\frac{\partial\phi_0(J,\theta+\theta')}{\partial\theta}e^{in\phi_0(J,\theta+\theta')}e^{-iQ(J)\theta'} \tag{3.20}$$

where we use the shorthand notation,

$$Z_n = Z_n(n\omega_0 + \Omega). \tag{3.21}$$

Equation (3.20) has the important property that it expresses the perturbed distribution $\psi_i(J, \theta)$ in terms of the Fourier coefficients ρ_n of the perturbed line-charge density. On the other hand, $\rho(\phi)$ is determined from the distribution ψ_i by

$$\rho(\phi) = \int dp\, \psi_i(p, \phi), \tag{3.22}$$

as seen from Eqs. (2.6), (2.7a) and (3.6). The Fourier components are given by

$$\rho_n = \int \frac{d\phi}{2\pi} e^{-in\phi} \int dp\, \psi_i(p, \phi), \tag{3.23}$$

which can be rewritten as an integral over action-angle variables using Eq. (3.4)

$$\rho_n = \frac{1}{2\pi} \int_{-2\pi}^{0} d\theta \int_0^{\infty} dJ\, \psi_i(J, \theta) e^{-in\phi_0(J,\theta)}. \tag{3.24}$$

Equation (3.24) expresses ρ_n as a functional of $\psi_i(J, \theta)$, hence using (3.24) and (3.20), we can derive an infinite matrix equation determining the Fourier components

$$\rho_m = \sum_{n=-\infty}^{\infty} T_{mn} \rho_n, \tag{3.25}$$

where

$$T_{mn} = -\kappa Z_n F_{mn}, \tag{3.26}$$

and

$$F_{mn} = \int_0^{\infty} dJ \frac{\psi_0'(J)}{\omega_s(J)(1 - e^{2\pi i Q(J)})} \int_{-2\pi}^{0} d\theta \int_{-2\pi}^{0} d\theta' e^{-iQ(J)\theta'}$$
$$* e^{i(n\phi_0(J,\theta+\theta') - m\phi_0(J,\theta))} \frac{\partial \phi_0(J, \theta + \theta')}{\partial \theta}. \tag{3.27}$$

After solving Eq. (3.25) for the Fourier coefficients ρ_n, we can determine the distribution $\psi_i(J, \theta)$ from Eq. (3.20).

Equations (3.25)–(3.27) provide the basis for the discussions presented in the rest of this paper. The coherent frequency Ω is fixed by the dispersion relation

$$\det(\delta_{mn} - T_{mn}(\Omega)) = 0, \tag{3.28}$$

where δ_{mn} represents the unit matrix. In general, analytic results cannot be found, and one would have to proceed numerically. However, in the following sections, we shall consider some special limiting cases for which analytic solutions are obtainable.

Before going on to consider solutions of the equations, we first want to conclude the present discussion by briefly noting how the results thus far obtained carry over to the case of M equally spaced bunches each containing the same number of particles. Let us label the bunches by $l = 0, 1, 2, \ldots, M-1$, and specify

ϕ to be the phase relative to the center of the $l=0$ bunch. Denoting the symmetric multibunch modes by $s=0,1,2,\ldots,M-1$, the distribution function for the lth bunch in the sth mode is

$$\psi^{ls}(\dot{\phi},\phi,t)=\psi_0\left(\phi-\frac{2\pi l}{M},\dot{\phi}\right)+e^{2\pi isl/M}\psi_i^s\left(\phi-\frac{2\pi l}{M},\dot{\phi}\right)e^{-i\Omega t}. \tag{3.29}$$

The full distribution,

$$\Psi^s=\sum_{l=0}^{M-1}\psi^{ls}, \tag{3.30}$$

is normalized by Eq. (2.5) to the total number of particles in the ring. Therefore, the distribution ψ_0 in Eq. (3.29) is normalized to the number of particles in one bunch. The linearized Vlasov equation for the $l=0$ bunch in the sth mode is

$$-i\Omega\psi_i^s+\dot{\phi}\frac{\partial\psi_i^s}{\partial\phi}-U_0'(\phi)\frac{\partial\psi_i^s}{\partial\dot{\phi}}-\frac{\alpha\omega_0}{E_0}\frac{e\omega_0}{2\pi}MV_i^s(\phi)\frac{\partial\psi_0}{\partial\dot{\phi}}=0. \tag{3.31}$$

where we have suppressed the superscript "0" referring to the zeroth bunch. The induced potential MV_i^s is defined by

$$MV_i^s(\phi)=\sum_{k=0}^{M-1}e^{2\pi isl/M}v\left(\phi-\frac{2\pi k}{M}\right), \tag{3.32}$$

where

$$v(\phi)=-\omega_0\sum_{n=-\infty}^{\infty}\rho_nZ_ne^{in\phi}. \tag{3.33}$$

Together, Eqs. (3.32) and (3.33) show that

$$V_i^s(\phi)=-\omega_0\sum_{j=-\infty}^{\infty}\rho_nZ_ne^{in\phi}\quad(n=Mj+s) \tag{3.34}$$

The linearized Vlasov equation of Eq. (3.31) is seen to be the same as that for a single bunch, except the sum in Eq. (3.34) relating V_i^s to the impedance is restricted to $n=Mj+s$ $(j=-\infty,\ldots,\infty)$, and there appears a multiplicative factor M in the last term in Eq. (3.31). Since ψ_0 is normalized to the number of particles in one bunch, the last term in Eq. (3.31), the impedance term, is proportional to the total number of particles in all bunches.

In conclusion, all results derived earlier in this section for a single bunch carry over to the case of M equally spaced bunches each containing N/M particles. One merely normalizes the equilibrium distribution ψ_0 appearing in Eq. (3.27) to the total number of particles in the ring, N, and restricts the values of the indices $m,n=Mj+s$ $(j=-\infty,\ldots,\infty)$, for fixed mode number $s=0,1,2,\ldots,M-1$.

IV. FAST-BLOWUP LIMIT

When the growth rate $\mathrm{Im}\,\Omega$ of the coherent oscillation is large compared with the synchrotron oscillation frequency, a useful asymptotic expression for the matrix element F_{mn} (Eq. (3.27)) can be obtained. Taking the limit $\mathrm{Im}\,Q(J)\rightarrow+\infty$ in the integrand of Eq. (3.27), and performing a partial integration with respect to the variable θ, we find

$$F_{mn} \approx \frac{m}{n}\int_0^\infty dJ \frac{\psi_0'(J)}{\omega_s(J)} \int_{-2\pi}^0 d\theta e^{-im\phi_0(J,\theta)} \frac{\partial \phi_0(J,\theta)}{\partial \theta} * \int_{-2\pi}^0 d\theta' e^{-iQ(J)\theta'} e^{in\phi_0(J,\theta+\theta')}. \tag{4.1}$$

Now since Im $Q(J)$ is very large, we can Taylor expand $\phi_0(J, \theta+\theta')$ about $\theta'=0$ in the last integral of Eq. (4.1), then keeping only up to second order in θ',

$$F_{mn} \approx \frac{m}{n}\int_0^\infty dJ \frac{\psi_0'(J)}{\omega_s(J)} \int_{-2\pi}^0 d\theta e^{-im\phi_0(J,\theta)} \frac{\partial \phi_0(J,\theta)}{\partial \theta} * \int_{-\infty}^0 d\theta' e^{-iQ(J)\theta'} \exp\left[in\left(\phi_0(J,\theta) + \frac{\partial \phi_0}{\partial \theta}\theta' + \frac{1}{2}\frac{\partial^2\phi_0}{\partial\theta^2}\theta'^2\right)\right] \tag{4.2}$$

It is helpful to rewrite Eq. (4.2) by introducing the time derivatives of ϕ_0:

$$\dot{\phi}_0(J,\theta) = \frac{\partial \phi_0(J,\theta)}{\partial \theta}\,\omega_s(J) \tag{4.3a}$$

and

$$\ddot{\phi}_0(J,\theta) = \frac{\partial^2 \phi_0(J,\theta)}{\partial \theta^2}\,\omega_s^2(J). \tag{4.3b}$$

Also, from Eq. (3.7), we note that

$$\psi_0'(J) = -\frac{1}{\sigma^2}\,\omega_s(J)\psi_0(J). \tag{4.4}$$

Now changing the integration variable in Eq. (4.2) from θ' to $\xi = \theta'/\omega_s(J)$, we obtain

$$F_{mn} \approx -\frac{1}{\sigma^2}\frac{m}{n}\int_0^\infty dJ\,\psi_0(J) \int_{-2\pi}^0 d\theta e^{i(n-m)\phi_0(J,\theta)} \dot{\phi}_0(J,\theta) * \int_{-\infty}^0 d\xi e^{-i\Omega\xi} e^{in\dot{\phi}_0(J,\theta)\xi} e^{in\ddot{\phi}_0(J,\theta)\xi^2/2}. \tag{4.5}$$

Further simplification of the expression for F_{mn} results upon changing the integration variables J, θ to ϕ, $\dot{\phi}$. This is accomplished by using Eq. (3.4), which states that $dJ\,d\theta = d\dot{\phi}\,d\phi$, and the result is

$$F_{mn} \approx -\frac{1}{\sigma^2}\frac{m}{n} A \int_{-\infty}^0 d\xi e^{-i\Omega\xi} \int \dot{\phi}\,d\dot{\phi} e^{-\dot{\phi}^2/2\sigma^2} e^{in\dot{\phi}\xi} * \int d\phi e^{-U_0(\phi)/\sigma^2} e^{i(n-m)\phi} e^{-inU_0'(\phi)\xi^2/2}. \tag{4.6}$$

In Eq. (4.6), we have used the expression of Eq. (3.8) for $\psi_0(\dot{\phi}, \phi)$, and we have replaced $\ddot{\phi}_0(J,\theta)$ by

$$\ddot{\phi}_0(J,\theta) = -dU_0/d\phi \equiv -U_0'(\phi). \tag{4.7}$$

The Gaussian integral over $\dot{\phi}$ in Eq. (4.6) can be performed and we obtain the desired form of the asymptotic expression for the matrix element F_{mn}:

$$F_{mn} \approx -im\sigma\sqrt{2\pi}\, A \int_{-\infty}^{0} \xi\, d\xi e^{-i\Omega\xi} e^{-n^2\sigma^2\xi^2/2}$$

$$* \int_{-\infty}^{\infty} d\phi e^{i(n-m)\phi} e^{-U_0(\phi)/\sigma^2} e^{-inU_0'(\phi)\xi^2/2}, \tag{4.8}$$

where the normalization constant A is explicitly given in Eq. (3.10).

In the special case of a harmonic potential, $U_0 = \omega_s^2\phi^2/2$, the integral over ϕ in Eq. (4.8) can be performed, yielding for $\operatorname{Im}\Omega \to +\infty$ and $mn > 0$,

$$F_{mn} \approx \frac{-2\pi im\sigma^2}{\omega_s} e^{-(n-m)^2L^2/2} A \int_{-\infty}^{0} \xi\, d\xi e^{-i\Omega\xi} e^{-mn\sigma^2\xi^2/2}, \tag{4.9}$$

where the bunch length L in radians is related to the spread σ in rotation frequency by

$$\sigma = \omega_s L. \tag{4.10}$$

When $mn < 0$, Eq. (4.8) is only good enough to give the leading behavior for $\operatorname{Im}\Omega \gg \sqrt{|mn|}\,\sigma$,

$$F_{mn} \approx \frac{-2\pi im\sigma^2}{\omega_s} e^{-(n-m)^2L^2/2} A \frac{1}{\Omega^2}. \tag{4.11}$$

This matrix element is seen to be very small when n and m are large in magnitude, but of opposite sign. A more detailed discussion of the harmonic potential is given in Section VIII.

The asymptotic analysis presented above leading from Eq. (4.8) to Eqs. (4.9) and (4.11) is not sufficient to estimate the magnitude of neglected terms. However, that the obtained results are correct for the harmonic potential can be seen from the matrix element T_{mn} derived in Section VIII and given in Eqs. (8.16) and (8.17). Recalling the definition of F_{mn} in Eq. (3.26), the results of Section VIII show that

$$F_{mn} = \frac{im}{\omega_s^2} \frac{2\pi\sigma^2}{\omega_s} A \frac{e^{-(m^2+n^2)L^2/2}}{1-e^{2\pi iQ}} \int_0^{2\pi} d\theta \sin\theta e^{iQ\theta} e^{mnL^2\cos\theta},$$

where A is given below in Eq. (4.12), $Q = \Omega/\omega_s$ and $\sigma = \omega_s L$. In the fast-blowup limit, $\operatorname{Im} Q \to +\infty$, so $\exp(2\pi iQ)$ vanishes and the integral in the above expression for F_{mn} is seen to be dominated by the region near $\theta = 0$, since $|\cos\theta| \leq 1$. For $mn > 0$, the integral can be approximated by

$$\int_0^{\infty} \theta\, d\theta e^{iQ\theta} e^{mnL^2(1-\theta^2/2)},$$

so defining $\xi = \theta/\omega_s$, we obtain the asymptotic expression for F_{mn} given in Eq. (4.9). When $mn < 0$, the leading behavior of the integral in the expression for F_{mn} is given by

$$\int_0^{\infty} \theta\, d\theta e^{iQ\theta} e^{mnL^2} = -\frac{1}{Q^2} e^{mnL^2},$$

which yields the result given in Eq. (4.11).

Recall that the matrix T_{mn} of Eqs. (3.25) and (3.26) is related to F_{mn} by $T_{mn} = -\kappa Z_n F_{mn}$, with κ defined in Eq. (3.14). From Eq. (3.10), we see that for the harmonic potential,

$$A = \frac{\omega_s Ne}{2\pi\sigma^2} \tag{4.12}$$

where N is the total number of electrons in the ring. It now follows from Eq. (4.9) that

$$T_{mn} = \frac{-i\alpha\omega_0^2 eI_0}{2\pi E_0\sigma^2}\frac{Z_n}{n}e^{-(n-m)^2L^2/2}h\left(\frac{\Omega}{\sigma\sqrt{mn}}\right) \tag{4.13}$$

with $I_0 = Ne\omega_0/2\pi$ being the average current in the ring and the function $h(x)$ defined by

$$h(x) = \int_0^\infty \xi\, d\xi e^{ix\xi}e^{-\xi^2/2} \tag{4.14a}$$

$$= e^{-x^2/4}D_{-2}(-ix), \tag{4.14b}$$

where $D_{-2}(\xi)$ is a parabolic cylinder function discussed in Gradshteyn and Ryzhik.[11] Equation (4.13) has been found previously in Ref. 5.

The function $h(x)$ defined in Eq. (4.14) can be expressed as a dispersion integral. Suppose $\operatorname{Im} x > 0$, and insert the Fourier integral representation of the Gaussian function

$$e^{-\xi^2/2} = \int_0^\infty \frac{dz}{\sqrt{2\pi}}e^{-iz\xi}e^{-z^2/2}, \tag{4.15}$$

into (4.14a). After changing the order of integration and performing the integral over ξ, we find $h(x)$ is equal to the dispersion integral

$$h(x) = -\int_{-\infty}^\infty \frac{dz}{\sqrt{2\pi}}\frac{e^{-z^2/2}}{(z-x)^2}, \tag{4.16}$$

which is well known from the study of the instabilities of coasting beams.

Let us close this discussion of the function $h(x)$ by noting that it is easily seen from Eq. (4.14a) that for $\operatorname{Im} x \geq 0$,

$$|h(x)| \leq 1, \tag{4.17}$$

with the equality only holding for $x = 0$, and again for $\operatorname{Im} x \geq 0$,

$$h(x) \sim \frac{-1}{x^2} \quad \text{for} \quad |x| \to \infty. \tag{4.18}$$

V. MICROWAVE INSTABILITY

We shall now discuss the microwave instability of a single bunch. This case may be described as high-frequency fast blowup. In addition to the condition that the growth rate be fast compared with the synchrotron frequency, we also require the wavelength of the perturbing electromagnetic field to be short compared with the bunch length. The expression for F_{mn} derived in Eq. (4.8), valid in the fast-blowup limit, simplifies further when we also consider

$$|n|\, L \to \infty, \tag{5.1}$$

where L is the bunch length in radians.

Let us rewrite Eq. (4.8) using the new integration variable $x = |n|\, \sigma\xi$,

$$F_{mn} \approx -i\frac{m}{n^2}\frac{\sqrt{2\pi}}{\sigma} A \int_{-\infty}^{0} x\, dx e^{-i\Omega x/|n|\sigma} e^{-x^2/2}$$
$$* \int_{-\infty}^{\infty} d\phi e^{i(n-m)\phi} e^{-U_0(\phi)/\sigma^2} \exp\left(-\frac{i}{2} U_0'(\phi)\frac{x^2}{n\sigma^2}\right) \tag{5.2}$$

The factor $\exp(-x^2/2)$ in the integrand of Eq. (5.2) restricts $|x| \lesssim 1$, and $\exp(-U_0(\phi)/\sigma^2)$ restricts $\phi \lesssim L$, where L is the bunch length in radians. Assuming $U_0(\phi)$ is smooth enough so that its derivative, $U_0'(\phi)$, is not very large for $\phi \lesssim L$, it follows that in the limit $|n| \to \infty$, we can make the approximation

$$\exp\left(-\frac{i}{2} U_0'(\phi)\frac{x^2}{n\sigma^2}\right) \simeq 1. \tag{5.3}$$

If $U_0(L) \sim L^p \sim \sigma^2$ and $U_0'(L) \sim L^{p-1}$, then $U_0'(L)/n\sigma^2 \sim 1/nL$, and Eq. (5.3) is seen to hold for $nL \gg 1$.

When the approximation of Eq. (5.3) is made in the integrand of Eq. (5.2), the double integral is seen to factor into the product of two single integrals. Evaluating the constant A using the normalization condition (3.10), we find that for mL and nL large, $(mn > 0)$

$$F_{mn} \approx \frac{i}{n}\frac{Ne}{\sigma^2} h\left(\frac{\Omega}{|n|\,\sigma}\right)\beta(m-n), \tag{5.4}$$

where the function $h(x)$ was defined in Eq. (4.14) and $\beta(m-n)$ is the normalized Fourier transform of the bunch distribution function $\exp(-U_0(\phi)/\sigma^2)$,

$$\beta(m-n) = \frac{\int_{-\infty}^{\infty} d\phi e^{-i(m-n)\phi} e^{-U_0(\phi)/\sigma^2}}{\int_{-\infty}^{\infty} d\phi e^{-U_0(\phi)/\sigma^2}}. \tag{5.5}$$

In writing Eq. (5.4), we have made the approximation $m/n^2 \simeq 1/n$, since $\beta(m-n)$ is a sharply peaked function about $m = n$, whose width is $1/L$. The matrix $T_{mn} = -\kappa Z_n F_{mn}$ of Eqs. (3.25) and (3.26) is now given by $(mn > 0)$

$$T_{mn} \approx \frac{-i\alpha\omega_0^2 e I_0}{2\pi E_0 \sigma^2}\frac{Z_n}{n} h\left(\frac{\Omega}{|n|\,\sigma}\right)\beta(m-n), \tag{5.6}$$

where $I_0 = Ne\omega_0/2\pi$ is the average current, and recall that $Z_n = Z(n\omega_0 + \Omega)$.

Let us consider the case of a wake field (Fourier transform of the impedance) whose range is short compared with the bunch length L. To be more specific, we assume that

$$Z_n \approx Z_{n_0} \quad \text{for} \quad |n - n_0| \le \Delta, \tag{5.7}$$

where Δ is of the order of the inverse range of the wake field, so $\Delta \gg 1/L$. We shall now look for a coherent frequency near $\omega = n_0\omega_0 + \Omega \approx n_0\omega_0$, where $n_0 \gg \Delta$. For $n_0 - \Delta < m$, $n < n_0 + \Delta$, we can approximate Eq. (5.6) by

$$T_{mn} \approx -\frac{ieI_0}{2\pi E_0 \alpha(\sigma_\varepsilon/E_0)^2} \frac{Z_{n_0}}{n_0} h\left(\frac{\Omega}{|n_0|\,\sigma}\right)\beta(m-n), \tag{5.8}$$

where Eq. (3.9) has been used to express σ in terms of the energy spread σ_ε.

The discussion leading to Eq. (3.25) showed that the condition for the existence of a coherent oscillation is

$$\rho_m = \sum_{n=-\infty}^{\infty} T_{mn}\rho_n, \tag{5.9}$$

where ρ_n is the Fourier coefficient of the perturbed line-charge density. There can be no solution of Eq. (5.9) as long as the maximum eigenvalue of T_{mn} is less than unity, and this condition determines a threshold current for the instability.

To proceed, let us find the approximate solutions of Eq. (5.9) for which ρ_m is negligibly small when $|m - n_0| > \Delta$. In this case we can restrict our attention to the finite set of equations

$$\rho_m = \sum_{n=n_0-\Delta}^{n_0+\Delta} T_{mn}\rho_m, \qquad |m - n_0| \le \Delta. \tag{5.11}$$

Since $\beta(m-n)$ is sharply peaked about $m = n$, the peak width being of order $1/L \ll \Delta$, we expect that the largest eigenvalues do not depend strongly on the cutoff value Δ. Therefore, they should be closely approximated by the eigenvalues of the easier problem

$$\lambda^\infty v_m^\infty = \sum_{n=-\infty}^{\infty} \beta(m-n) v_n^\infty \tag{5.12}$$

which results upon letting the cutoff approach infinity. The eigenfunctions of (5.12) are

$$v_n^\infty(\zeta) = e^{-i\zeta n}, \tag{5.13}$$

and the corresponding eigenvalues are

$$\lambda^\infty(\zeta) = \sum_{n=-\infty}^{\infty} \beta(n) e^{i\zeta n}, \tag{5.14}$$

where ζ $(0 \le \zeta < 2\pi)$ parametrizes the different eigensolutions. Substituting the defining expression (Eq. (5.5)) for $\beta(n)$ into Eq. (5.14), and using the Poisson summation formula, we rewrite Eq. (5.14) in the form

$$\lambda^\infty(\zeta) = \frac{2\pi \sum_{n=-\infty}^{\infty} e^{-U_0(\zeta+2\pi n)/\sigma^2}}{\int_{-\infty}^{\infty} d\phi\, e^{-U_0(\phi)/\sigma^2}} \tag{5.15}$$

For a Gaussian bunch $\beta(n)=\exp(-n^2L^2/2)$. Since this expression is always positive, it follows from Eq. (5.14) that the largest eigenvalue corresponds to $\zeta=0$. When $L\approx 2\pi$, $\beta(n)$ falls off sufficiently rapidly so that only one term in Eq. (5.14) needs to be retained, and

$$\lambda_{\max}^{\infty}\approx\beta(0)=1. \tag{5.16}$$

On the other hand, when $L\ll 1$, Eq. (5.15) provides the rapidly convergent representation of the eigenvalue and in this case

$$\lambda_{\max}^{\infty}\approx\frac{2\pi}{\displaystyle\int_{-\infty}^{\infty} d\phi e^{-U_0(\phi)/\sigma^2}}. \tag{5.17}$$

When $U_0(\phi)/\sigma^2=\phi^2/2L^2$,

$$\lambda_{\max}^{\infty}\approx\sqrt{2\pi}/L. \tag{5.18}$$

The Fourier transform of the bunch density, $\beta(n)$, is no longer positive definite for a general non-harmonic potential $U_0(\phi)$. Therefore, we cannot argue directly from Eq. (5.14) that the maximum eigenvalue always corresponds to $\zeta=0$. However, from Eq. (5.15), it can be seen that for sufficiently small σ, the maximum eigenvalue will indeed correspond to $\zeta=0$, since the minimum of the potential $U_0(\phi)$ is located at $\phi=0$. Taking $U_0(0)=0$, it is found that Eq. (5.17) holds also for the non-harmonic potential.

It is important to now note that the value of $\lambda_{\max}^{\infty}$ given in Eq. (5.17) is equal to the ratio of the peak current to the average current. To see this, recall that the unperturbed bunch distribution is

$$\psi_0(p,\phi)=Ae^{-p^2/2\sigma^2}e^{-U_0(\phi)/\sigma^2}. \tag{5.19}$$

The line-charge density is given by

$$\rho(\phi)=\int dp\psi_0(p,\phi), \tag{5.20}$$

hence its average is

$$\rho_{\mathrm{av}}=\frac{1}{2\pi}\int d\phi\rho(\phi)=\frac{A}{2\pi}\int dpe^{-p^2/2\sigma^2}\int d\phi e^{-U_0(\phi)/\sigma^2}. \tag{5.21}$$

The peak value of the line-charge density is

$$\rho_{\mathrm{peak}}=\rho(0)=A\int dpe^{-p^2/2\sigma^2}. \tag{5.22}$$

Therefore

$$\frac{\rho_{\mathrm{peak}}}{\rho_{\mathrm{av}}}=\frac{2\pi}{\displaystyle\int d\phi e^{-U_0(\phi)/\sigma^2}}. \tag{5.23}$$

Equations (5.17) and (5.23) establish that for small σ,

$$\lambda_{\max}^{\infty}\approx\rho_{\mathrm{peak}}/\rho_{\mathrm{av}}. \tag{5.24}$$

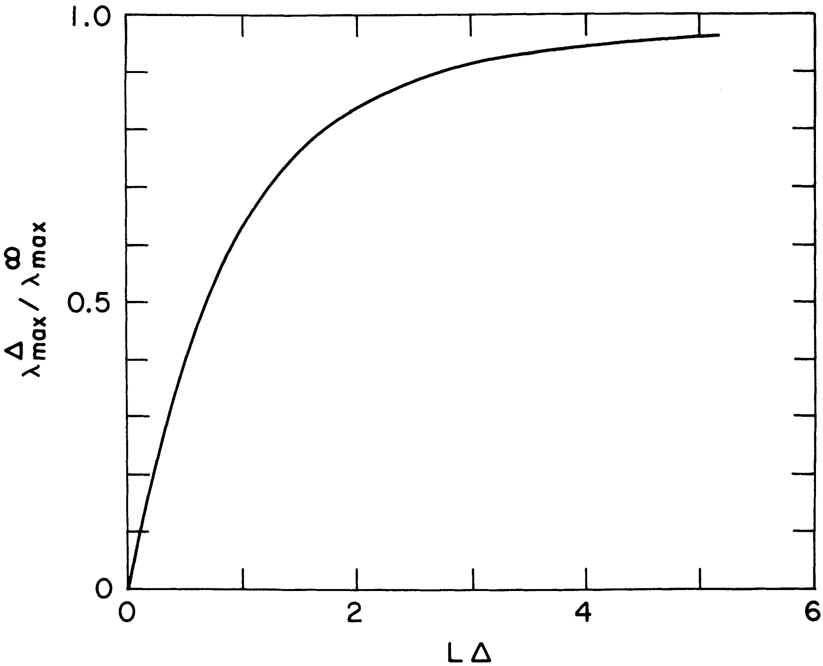

FIGURE 1 The ratio of the largest eigenvalues corresponding to finite bandwidth Δ and infinite bandwidth, plotted against $L\Delta$, where L is the bunch length in radians.

Let us now return to the eigenvalue problem of Eq. (5.11) and study the validity of the approximation just discussed, in which we let the cutoff go to infinity. To illustrate the rate of convergence, we have solved numerically for the eigenvalues of

$$\lambda^{\Delta} v_m = \sum_{n=n_0-\Delta}^{n_0+\Delta} e^{-(n-m)^2L^2/2} v_n, \tag{5.25}$$

corresponding to a harmonic potential. In Fig. 1 we plot $\lambda_{\max}^{\Delta}/\lambda_{\max}^{\infty}$, as a function of $L\Delta$. It is seen that the error is less than 10% when $L\Delta > 3$.

To gain some insight into the nature of the perturbed line-charge density, let us take as an approximation to the eigenvectors of Eq. (5.25)

$$\begin{aligned} \rho_n &= v_n^{\infty}(\zeta) \quad \text{for} \quad |n-n_0| \leq \Delta \\ &= 0 \quad \text{for} \quad |n-n_0| > \Delta \end{aligned} \tag{5.26}$$

where $v_n^{\infty}(\zeta)$ is defined in Eq. (5.13). The perturbation to the line charge density is

$$\rho(\phi) = \sum_n \rho_n e^{in\phi}, \tag{5.27}$$

and substituting (5.26) into Eq. (5.27), we obtain

$$\rho(\phi) = \sum_{n=n_0-\Delta}^{n_0+\Delta} e^{in(\phi-\zeta)} = e^{in_0(\phi-\zeta)} f_{\Delta}(\phi-\zeta), \tag{5.28}$$

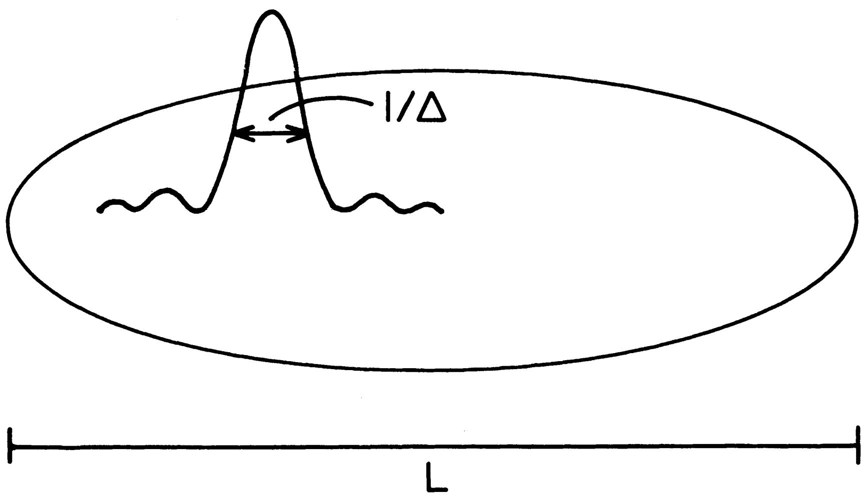

FIGURE 2 Sketch of a localized eigenmode having peak width of order 1/Δ, in a bunch of length L radians.

where

$$f_\Delta(\phi-\zeta)=\frac{\sin\left((\Delta+1/2)(\phi-\zeta)\right)}{\sin\left((\phi-\zeta)/2\right)}. \tag{5.29}$$

The perturbation $\rho(\phi)$ of the line-charge density is a plane wave modulated by the function $f_\Delta(\phi-\zeta)$. For large $\Delta, f_\Delta(\phi-\zeta)$ is sharply peaked about $\phi=\zeta$, and the peak width is of order $1/\Delta$, see Fig. 2. The detailed structure within the peak will depend upon the details of the short-distance behavior of the wake field, which has been ignored in making the approximation of Eq. (5.7), and hence is outside the scope of our discussion.

The growth rate of the mode corresponding to the maximum eigenvalue $\lambda_{\max}^{\infty}$ of Eq. (5.12) is determined by the dispersion relation

$$\frac{-ieI_0\lambda_{\max}^{\infty}(\sigma)}{2\pi E_0\alpha(\sigma_\varepsilon/E_0)^2}\,h\!\left(\frac{\Omega}{|n_0|\,\sigma}\right)\frac{Z(n_0\omega_0)}{n_0}=1. \tag{5.30}$$

We shall now define a threshold current I_{th} by

$$\frac{eI_{th}}{2\pi E_0\alpha(\sigma_\varepsilon/E_0)^2}\left|\frac{Z(n_0\omega_0)}{n_0}\right|=1. \tag{5.31}$$

From the definition of $h(x)$ in Eq. (4.14), it is seen that

$$|h(x)|\le 1 \quad \text{for} \quad \operatorname{Im} x\ge 0. \tag{5.32}$$

Therefore there will exist no coherent frequency with $\operatorname{Im}\Omega>0$ as long as

$$I_0\lambda_{\max}^{\infty}(\sigma)<I_{th}. \tag{5.33}$$

When the inequality (5.33) holds there is no microwave fast blowup. If $I_0\lambda^{\infty}_{\max}(\sigma)=I_{th}$, then there is a solution of (5.30) only for

$$\frac{\operatorname{Im}\Omega}{n_0\sigma}\rightarrow 0^+. \tag{5.34}$$

From Eq. (5.16), we see that if σ is large, then

$$\lambda^{\infty}_{\max}(\sigma)\approx 1. \tag{5.35}$$

In this case, corresponding to a bunch length comparable to the ring circumference, the threshold condition of Eq. (5.33) says simply that the average current $I_0<I_{th}$. On the other hand, when the bunch length is short compared with the ring circumference, corresponding to small σ, Eq. (5.24) shows that

$$\lambda_{\max}(\sigma)\approx I_{\text{peak}}/I_0,$$

so the condition for the absence of microwave fast blowup becomes

$$\frac{eI_{\text{peak}}}{2\pi E_0\alpha(\sigma_\varepsilon/E_0)^2}\left|\frac{Z(n_0\omega_0)}{n_0}\right|\leq 1. \tag{5.36}$$

This condition was conjectured by Boussard[6] on the basis of an intuitive physical argument. He noted that when the perturbing electromagnetic fields have wavelengths short compared with the bunch length, the bunch looks like a coasting beam having a current equal to the peak current of the bunch.

In order to emphasize the correspondence between our results and those known for coasting beams, let us consider the dispersion relation (5.30) far above threshold, so that $|\Omega|\gg|n_0|\,\sigma$. From its definition in Eq. (4.13), we see that when $\operatorname{Im}x\geq 0$,

$$h(x)\approx -x^{-2}\quad\text{for}\quad |x|\gg 1, \tag{5.37}$$

hence far above threshold, the dispersion relation becomes

$$\Omega^2=i\frac{e\alpha\omega_0^2}{2\pi E_0}\,n_0Z_{n_0}I_0\lambda^{\infty}_{\max}(\sigma), \tag{5.38}$$

which is the well-known dispersion relation for a coasting beam with current $I_0\lambda^{\infty}_{\max}(\sigma)$.

Let us conclude this section by commenting on the attempt made by Messerschmid and Month[9] to describe the microwave instability. Their approach was based upon the ansatz, $\rho(\phi)=\exp(in_0\phi)\rho_0(\phi)$, where $\rho_0(\phi)$ is the unperturbed bunch density. This has the form of a plane wave modulated by a shape function; however, the shape function is always taken to be $\rho_0(\phi)$ independent of the bandwidth Δ of the impedance. Our discussion leading to Eq. (5.28) shows that this is incorrect, and that the shape function should have a peak width of the order of $1/\Delta$, the range of the wake field. This local behavior is closely related to the peak current dependence of the coherent frequency for $\Delta\gg 1/L$. Whereas the ansatz of Messerschmid and Month is inconsistent with their results for the case of a broad-band impedance, it is more appropriate to the case of a narrow-band resonant impedance with $\Delta\ll 1/L$. Then, as discussed in the following section, the coherent frequency depends on the average current instead of the peak current (Eq. (6.8)) and the perturbed density is approximately as given in Eq. (6.9).

VI. FAST BLOWUP DUE TO A HIGH-Q RESONANCE

We suppose the storage ring contains M equally spaced bunches, each having N/M particles, interacting with a resonant element whose impedance is so sharply peaked that to a good approximation[12]

$$Z_n = Z_{n_0}\delta_{n,n_0} + Z^*_{n_0}\delta_{n,-n_0}. \tag{6.1}$$

Here δ_{n,n_0} is the Kronecker delta vanishing when $n \neq n_0$ and having value unity when $n = n_0$. Let us assume $n_0 = Mj_0 + s$, where j_0 is an integer and the multibunch mode number s is not zero. We now write the condition (Eq. (3.25)) for the existence of a coherent oscillation as

$$\rho_m = \sum_{j=-\infty}^{\infty} T_{mn}\rho_n, \tag{6.2}$$

where the indices m and n take only the values $Mj + s$. Recalling from Eq. (3.26) that $T_{mn} = -\kappa Z_n F_{mn}$, it follows that when the impedance is given by Eq. (6.1), then Eq. (6.2) becomes simply

$$\rho_m = T_{mn_0}\rho_{n_0} \tag{6.3}$$

There is no contribution from the $n = -n_0$ term of the impedance, since it corresponds to a different multibunch mode number. It is clear that in order for Eq. (6.3) to be satisfied, it is necessary that

$$T_{n_0n_0} = 1. \tag{6.4}$$

In the fast blowup regime, F_{mn} is given by Eq. (4.7). Now let us suppose that

$$\operatorname{Im}\Omega \gg |n_0|\,\sigma, \tag{6.5}$$

then the expression for F_{mn} simplifies to

$$F_{mn} = \frac{-imNe}{\Omega^2}\beta(m-n), \tag{6.6}$$

where the normalized Fourier transform β of the bunch distribution was defined in Eq. (5.5). Therefore, when the inequality (6.5) holds,

$$T_{mn_0} = \frac{im\alpha\omega_0^2}{2\pi E_0\Omega^2} I_0 Z_{n_0}\beta(m-n_0), \tag{6.7}$$

where $I_0 = Ne\omega_0/2\pi$ is the average current in the ring. The dispersion relation (6.4) giving the growth rate of the coherent oscillation is

$$\Omega^2 = \frac{i\alpha\omega_0^2}{2\pi E_0} eI_0 n_0 Z_{n_0}, \tag{6.8}$$

which is the coasting-beam dispersion relation found earlier in Eq. (5.38). In addition, the perturbed line-charge density can be determined from Eqs. (6.3) and (6.7), and up to a multiplicative constant

$$\rho(\phi) = \frac{\partial}{\partial \phi}\left(e^{in_0\phi}\rho_0(\phi)\right), \tag{6.9}$$

where $\rho_0(\phi)$ is the unperturbed line charge density

$$\rho_0(\phi) \propto e^{-U_0(\phi)/\sigma^2}. \tag{6.10}$$

As in the case of the microwave instability, we can derive an inequality assuring the absence of fast blowup. From the expression for F_{mn} given in Eq. (4.8), we see that

$$|F_{n_0n_0}| \leq n_0\sigma\sqrt{2\pi}A\int_{-\infty}^{0} \xi\, d\xi e^{-n_0^2\xi^2/2}\int_{-\infty}^{\infty} d\phi e^{-U_0(\phi)/\sigma^2}. \tag{6.11}$$

Performing the Gaussian integral, and using the normalization condition of Eq. (3.10) for A, Eq. (6.11) becomes

$$|F_{n_0n_0}| \leq \frac{Ne}{n_0\sigma^2}. \tag{6.12}$$

There can be no solution of Eq. (6.3) if $|T_{n_0n_0}|<1$, and using Eq. (3.26) for T_{mn}, we find there will be no solution of Eq. (6.3) as long as

$$\frac{eI_0}{2\pi E_0\alpha(\sigma_\varepsilon/E_0)^2}\,\frac{|Z_{n_0}|}{n_0}<1. \tag{6.13}$$

This is the usual Landau damping condition known from the study of coasting beams and differs from the Boussard criterion of Eq. (5.20) only in that the average current I_0 appears in Eq. (6.13) rather than the peak current.

VII. SHORT-BUNCH APPROXIMATION

When the bunch length is short compared to the wavelengths of the perturbing electromagnetic fields, Eqs. (3.25)–(3.27) describing coherent oscillations become amenable to solution by a perturbation expansion.[7] The function $\phi_0(J, \theta)$, defined in Eq. (2.25), is in general of the form $\phi_0(J, \theta) = r\delta_0(r, \theta)$, where $\delta_0(r, \theta)$ goes to a finite limit as the amplitude r approaches zero. As a consequence, when the bunch length is short, it is useful to expand (3.27) in a power series in ϕ_0. If we truncate the series, then the resulting infinite-dimensional matrix T_{mn} becomes one of finite rank.

Assuming $n\phi_0$ and $m\phi_0$ to be small, we expand the exponential, $\exp\,(in\phi_0(J, \theta+\theta') - im\phi_0(J, \theta))$, appearing in Eq. (3.27) in a Taylor series. Let us define the functions $F_\mu(n, J)$ and $\Phi_\mu^{(j)}(J)$ by

$$e^{in\phi_0(J,\theta)} = \sum_{\mu=-\infty}^{\infty} F_\mu(n, J)e^{i\mu\theta} \tag{7.1}$$

and

$$(\phi_0(J,\theta))^j = \sum_{\mu=-\infty}^{\infty} \Phi_\mu^{(j)}(J) e^{i\mu\theta}. \tag{7.2}$$

From Eqs. (7.1) and (7.2), we see that

$$F_\mu(n, J) = \sum_{j=0}^{\infty} \frac{(in)^j}{j!} \Phi_\mu^{(j)}(J). \tag{7.3}$$

The matrix T_{mn} as given by (3.26) and (3.27) now becomes

$$T_{mn} = -2\pi i\kappa \frac{Z_n}{n} \sum_{\mu=-\infty}^{\infty} \mu \int_0^\infty dJ \psi_0'(J) \frac{F_\mu(n,J) F_\mu^*(m,J)}{\Omega - \mu\omega_s(J)} \tag{7.4}$$

$$= -2\pi i\kappa \frac{Z_n}{n} \sum_{j,k=1}^{\infty} \sum_{\mu=-\infty}^{\infty} \mu \frac{(in)^j}{j!} \frac{(-im)^k}{k!} \int_0^\infty dJ \psi_0'(J) \frac{\Phi_\mu^{(j)}(J)\Phi_{-\mu}^{(k)}(J)}{\Omega - \mu\omega_s(J)} \tag{7.5}$$

where in going to (7.5) from (7.4), we used the fact that $\mu\Phi_\mu^{(j)}(J)\Phi_{-\mu}^{(k)}(J) = 0$ if $j=0$ or $k=0$.

From (7.2) and the fact that $\phi_0 \sim r$ as $r \sim 0$, we have $\Phi_\mu^{(j)}(J) \sim r^j$ in the same limit. For a short bunch, if we truncate the summation in j and k in (7.5) at $j=k=j_{\max}$, on the ground that $\psi_0'(J)$ is neglible unless r is small, then T_{mn} as given by (7.5) becomes a matrix of rank $\leq j_{\max}$, and the infinite dimensional secular equation (3.25) reduces to one of dimension $j_{\max}$

$$\det(\delta_{kj} - M_{kj}) = 0, \tag{7.6}$$

where M_{kj} is a $j_{\max}$-dimensional matrix

$$M_{kj}(\Omega) = -2\pi i\kappa \sum_{l=1}^{j_{\max}} (Z_{\text{eff}})_{kl} D_{lj}(\Omega) \quad (k, j = 1, \ldots, j_{\max}). \tag{7.7a}$$

The matrix M is seen to be proportional to the matrix product of an effective impedance matrix[13]

$$(Z_{\text{eff}})_{kl} = \sum_{n=-\infty}^{\infty} \frac{Z_n}{n} \frac{(in)^k(-in)^l}{k!\,l!}, \tag{7.7b}$$

and a matrix of dispersion integrals

$$D_{lj}(\Omega) = \sum_{\mu=-\infty}^{\infty} \mu \int_0^\infty dJ \psi_0'(J) \frac{\Phi_\mu^{(j)}(J)\Phi_{-\mu}^{(l)}(J)}{\Omega - \mu\omega_s(J)}. \tag{7.7c}$$

In the lowest-order approximation, $j_{\max} = 1$, Eqs. (7.6) and (7.7) become

$$1 = -4\pi i\kappa Z_{\text{eff}} \sum_{\mu=-\infty}^{\infty} \mu^2 \int_0^\infty dJ \psi_0'(J) \frac{\omega_s(J)\,|\Phi_\mu^{(1)}(J)|^2}{\Omega^2 - \mu^2\omega_s^2(J)}, \tag{7.8a}$$

where

$$Z_{\text{eff}} = \sum nZ_n. \tag{7.8b}$$

Using Eq. (4.4), we see that Eq. (7.8) is equivalent to

$$1 = \frac{4\pi i\kappa}{\sigma^2} Z_{\text{eff}} \int_0^\infty dJ \psi_0(J)\omega_s^2(J) \sum_{\mu=-\infty}^{\infty} \frac{\mu^2 |\Phi_\mu^{(1)}(J)|^2}{\Omega^2 - \mu^2\omega_s^2(J)}. \tag{7.9}$$

When there are M equally spaced bunches, each containing N/M particles, ψ_0 is normalized to N by Eq. (3.10) and the sum $\sum nZ_n$ in the definition of Z_{eff} in Eq. (7.8b) is taken only over the values $n = Mj + s$ ($j = -\infty, \ldots, \infty$). Within the same approximation, T_{mn} is given by

$$T_{mn} = \frac{4\pi i\kappa}{\sigma^2} mZ_n \int_0^\infty dJ\psi_0(J) \sum_{\mu=1}^{\infty} \frac{\mu^2\omega_s^2(J)\,|\Phi_\mu^{(1)}(J)|^2}{\Omega^2 - \mu^2\omega_s^2(J)}. \tag{7.10}$$

From (7.10) and (3.25), we have $\rho_m = im$ up to a multiplication constant, so we see that $\rho(\phi)$ is proportional to the derivative of a periodic delta function. This corresponds, of course, to the rigid oscillation of a point bunch.

We can make contact with the results of Section VI, by considering the fast blowup limit $\operatorname{Im}\Omega \to +\infty$. When $\operatorname{Im}\Omega$ is greater than all the relevant synchrotron oscillation frequency spreads, $l\omega_s(J)$, Eq. (7.3) becomes

$$T_{mn} \approx \frac{4\pi i\kappa}{\sigma^2} mZ_n \frac{1}{\Omega^2} \int_0^\infty dJ\psi_0(J)\omega_s^2(J) \sum_{\mu=1}^{\infty} \mu^2 |\Phi_\mu^{(1)}(J)|^2. \tag{7.11}$$

From the definition of $\Phi_\mu^{(1)}(J)$ in Eq. (7.2), it is clear that Eq. (7.11) can be written as

$$T_{mn} \approx i\kappa mZ_n \frac{1}{\sigma^2\Omega^2} \int_0^\infty dJ\psi_0(J) \int_0^{2\pi} d\theta\dot{\phi}_0^2(J, \theta). \tag{7.12}$$

Applying Eq. (3.4), let us change the integration variables from J, θ to $\dot{\phi}$, ϕ. Then using the representation of ψ_0 given in Eq. (3.8), we obtain

$$T_{mn} = i\kappa AmZ_n \frac{1}{\sigma^2\Omega^2} \int d\dot{\phi}\dot{\phi}^2 e^{-\dot{\phi}^2/2\sigma^2} \int d\phi e^{-U_0(\phi)/\sigma^2}. \tag{7.13}$$

Now we perform the integration over $\dot{\phi}$, use the normalization condition of Eq. (3.10) to evaluate A, and invoke the definition of κ given in Eq. (3.14) to finally get

$$T_{mn} \approx \frac{i\alpha\omega_0^2 eI_0}{2\pi E_0} \frac{mZ_n}{\Omega^2}, \tag{7.14}$$

where $I_0 = Ne\omega_0/2\pi$ is the average current in the ring.

The dispersion relation is now

$$\Omega^2 = \frac{i\alpha\omega_0^2 eI_0}{2\pi E_0} Z_{\text{eff}}. \tag{7.15}$$

It is noteworthy that in the long-wavelength limit (short-bunch limit), the dispersion relation (7.15) is just the linear superposition of the coasting-beam results for the different values of n. For a short-wavelength case, this is no longer true.

VIII. HARMONIC POTENTIAL: $U_0(\phi) = \omega_s^2\phi^2/2$

It is worthwhile to discuss in some detail the special case of a harmonic potential. A distinguishing feature of harmonic motion is that the oscillation frequency is independent of amplitude. From Eqs. (2.27) and (2.29), it follows that for a harmonic potential

$$\phi_0(J, \theta) = r \cos \theta, \tag{8.1}$$

where the oscillation amplitude r is related to the action variable J by

$$r = (2J/\omega_s)^{1/2}. \tag{8.2}$$

Using Eq. (8.1), we see that

$$e^{in\phi_0(J,\theta)} = \sum_{\mu=-\infty}^{\infty} i^{\mu} J_{\mu}(nr) e^{i\mu\theta}, \tag{8.3}$$

showing that the synchrotron-mode coefficients $F_{\mu}(n, J)$ defined in Eq. (7.1) are in this case expressed as Bessel functions

$$F_{\mu}(n, J) = i^{\mu} J_{\mu}(nr). \tag{8.4}$$

The unperturbed distribution $\psi_0(J)$, as defined in Eq. (3.7), is given by

$$\psi_0(J) = Ae^{-\omega_s J/\sigma^2}, \tag{8.5}$$

with

$$A = \frac{\omega_s N e}{2\pi\sigma^2}. \tag{8.6}$$

Employing Eq. (8.4) and recalling that the synchrotron frequency is independent of J, we can write the synchrotron-mode expansion of Eq. (7.4) as

$$T_{mn} = -2\pi i\kappa \frac{Z_n}{n} \sum_{\mu=-\infty}^{\infty} \frac{\mu}{\Omega - \mu\omega_s} \int_0^{\infty} dJ \psi_0'(J) J_{\mu}(nr) J_{\mu}(mr). \tag{8.7}$$

The integral over J in Eq. (8.7) can be performed using Eq. (8.2) and

$$\int_0^{\infty} r\, dr e^{-r^2/2L^2} J_{\mu}(nr) J_{\mu}(mr) = L^2 e^{-(m^2+n^2)L^2/2} I_{\mu}(mnL^2), \tag{8.8}$$

where I_{μ} is the Bessel function of imaginary argument. We derive

$$T_{mn} = \frac{-i\alpha\omega_0^2 e I_0}{2\pi E_0 \omega_s^2 L^2} \frac{Z_n}{n} e^{-(m^2+n^2)L^2/2} \sum_{\mu=-\infty}^{\infty} \frac{\mu I_{\mu}(mnL^2)}{\mu - Q}. \tag{8.9}$$

In the above equation, $I_0 = Ne\omega_0/2\pi$ is the average current,

$$Q = \Omega/\omega_s, \tag{8.10}$$

and Eqs. (3.14) and (8.6) have been employed to evaluate κ and A, respectively. The bunch length L in radians is related to the spread in revolution frequency σ via

$$\sigma = L\omega_s. \tag{8.11}$$

In the case of the slow blowup of a synchrotron mode $\mu = \mu_0$, we have $Q \approx \mu_0$, so the sum in Eq. (8.9) is dominated by a single pole. For a short bunch (long wavelength), nL and mL are small, therefore,

$$I_{\mu_0}(mnL^2) \approx \frac{1}{\mu_0!}\left(\frac{mnL^2}{2}\right)^{\mu_0},$$

and

$$T_{mn} \approx \frac{i\alpha\omega_0^2 eI_0}{2\pi E_0\omega_s}\frac{Z_n}{n}\frac{L^{2\mu_0-2}}{2^{\mu_0}(\mu_0-1)!}\frac{m^{\mu_0}n^{\mu_0}}{\Omega-\mu_0\omega_s}. \tag{8.12}$$

Since this matrix is of rank one, the dispersion relation is given by $\sum_n T_{nn}=1$. Therefore, when the impedance Z_n is negligible for $n \gtrsim 1/L$, we obtain

$$\Omega-\mu_0\omega_s = \frac{i\alpha\omega_0^2 eI_0}{2\pi E_0\omega_s}\frac{L^{2\mu_0-2}}{2^{\mu_0}(\mu_0-1)!}\sum_n n^{2\mu_0-1}Z_n, \tag{8.13}$$

in agreement with the result of Wang.[14] Recall that in the case of M equally spaced bunches each containing N/M particles, the sum in Eq. (8.13) is only over $n=Mj+s$ ($j=-\infty,\ldots,\infty$) for fixed $s=0,1,2,\ldots,M-1$).

When there is fast blowup, $\mathrm{Im}\,\Omega \gg \omega_s$, the synchrotron-mode expansion of Eq. (8.9) is not a very useful representation for T_{mn}, because many terms in the sum over l contribute. This is in fact also true for the slow blowup of a long bunch due to a high-frequency impedance Z_n, with $nL \gg 1$. To study these cases, we shall replace the synchrotron-mode expansion by an equivalent integral representation. Upon differentiating the generating function for the I_μ Bessel function,

$$e^{z\cos\theta} = \sum_{\mu=-\infty}^{\infty} I_\mu(z)e^{i\mu\theta}, \tag{8.14}$$

with respect to θ, it is easily verified that the identity holds

$$\sum_{\mu=-\infty}^{\infty}\frac{\mu I_\mu(z)}{\mu-Q} = e^{|z|}H(z,Q), \tag{8.15}$$

with

$$H(z,Q) = \frac{z}{1-e^{2\pi iQ}}\int_0^{2\pi} d\theta\,\sin\theta\, e^{iQ\theta}e^{-|z|+z\cos\theta} \tag{8.16}$$

Using this identity in the synchrotron-mode expansion of Eq. (8.9), we derive the integral representation

$$T_{mn} = \frac{-i\alpha\omega_0^2 eI_0}{2\pi E_0\omega_s^2 L^2}\frac{Z_n}{n}\, e^{-(|m|-|n|)^2L^2/2}H(mnL^2, Q). \tag{8.17}$$

In the fast blowup limit, $\mathrm{Im}\,Q \to +\infty$, an asymptotic analysis of the integral in Eq. (8.16) shows that

$$H(z,Q) \approx h\left(\frac{Q}{\sqrt{z}}\right), \qquad z>0 \tag{8.18}$$

and

$$H(z,Q) \approx -\frac{z}{Q^2}e^{-2|z|}, \qquad z<0, \qquad \mathrm{Im}\,Q \gg \sqrt{|z|}. \tag{8.19}$$

From Eq. (8.17), these imply that for $\mathrm{Im}\, Q \to +\infty$,

$$T_{mn} \approx \frac{-i\alpha\omega_0^2 e I_0}{2\pi E_0 \omega_s^2 L^2} \frac{Z_n}{n} e^{-(m-n)^2L^2/2} h\left(\frac{Q}{\sqrt{mn}L}\right) mn > 0, \tag{8.20}$$

and

$$T_{mn} \approx \frac{i\alpha\omega_0^2 e I_0}{2\pi E_0 \Omega^2}(mZ_n)e^{-(m-n)^2L^2/2}, \qquad mn < 0, \qquad \mathrm{Im}\, Q \gg \sqrt{|mn|}L. \tag{8.21}$$

The function $h(x)$ was introduced earlier in Eq. (4.14), and Eqs. (8.20) and (8.21) are seen to agree with the results of Eqs. (4.11) and (4.13).

We can obtain additional insight by studying the behavior of the integral in Eq. (8.16) in a new asymptotic limit, relevant to the instability of a long bunch due to a high-frequency impedance Z_n, $n \gg 1/L$. The growth rate is only restricted by $\mathrm{Im}\, Q \ll nL$, so both slow blowup and fast blowup are included. To be specific, let

$$|z| \to \infty, |Q| \ll |z|, \qquad \mathrm{Im}\, Q \ll \sqrt{|z|}, \tag{8.22}$$

then one can show that the asymptotic behavior of $H(z, Q)$ is given by

$$H(z, Q) \approx h\left(\frac{Q}{\sqrt{z}}\right) - \frac{\pi}{2}(i + \cot \pi Q)\frac{Q}{\sqrt{z}} e^{-Q^2/2z}, \qquad z > 0 \tag{8.23}$$

and

$$H(z, Q) \approx \frac{-1}{\sin \pi Q}\sqrt{\frac{\pi}{2}} \frac{Q}{\sqrt{|z|}} e^{-Q^2/2|z|}, \qquad z < 0. \tag{8.24}$$

The matrix T_{mn} of Eq. (8.17) is determined by using Eqs. (8.23) and (8.24) with $z = mnL^2$.

The poles at $Q =$ integer in the function $\cot \pi Q$, in Eq. (8.23) correspond to the synchrotron-oscillation modes, which are dominant for slow blowup. When $\mathrm{Im}\, Q \gg 1$ or $\mathrm{Re}\, Q \gg \sqrt{z}$, the second term in Eq. (8.23) becomes negligible, due to the factors $(i + \cot \pi Q)$ and $\exp(-Q^2/2z)$, respectively. Then we are left with the coasting beam type of behavior described by the function $h(Q/\sqrt{z})$.

In the case of a coasting beam, different revolution modes m and n are not coupled. On the other hand, these modes are coupled for a bunched beam by the matrix T_{mn}. However, when nL and mL are large in magnitude, and the growth rate is fast ($\mathrm{Im}\, Q \gg 1$), then Eqs. (8.24) and (8.21) show that the coupling between modes with $mn < 0$ becomes negligible. That is, there is no coupling between the slow and fast waves. We also see from Eq. (8.24) that when nL and mL are large in magnitude and $\mathrm{Re}\, Q \gg \sqrt{|mn|}L$, then the slow and fast wave decouple, even if the growth rate is slow.

IX. QUARTIC POTENTIAL: $U_0(\phi) = \omega_{s0}^2\phi^2/2 + b\phi^4/4$

When a quartic term is added to the harmonic potential discussed in the last section, the equations of motion become nonlinear. Consequently, the synchrotron-oscillation frequency now varies with amplitude r. From Eq. (2.28), the amplitude dependence of the frequency is

$$\omega_s(r) = \frac{\pi}{2K}(\omega_{s0}^2 + br^2)^{1/2}, \tag{9.1}$$

where $K = K(k)$ is an elliptic integral of the first kind, with the modulus k given by

$$k^2 = \frac{br^2/2}{\omega_{s0}^2 + br^2}. \tag{9.2}$$

The action variable J can be related to the amplitude r using Eq. (2.27), and it follows that

$$J = \frac{4}{3\pi b}(\omega_{s0}^2 + br^2)^{1/2}\left(\left(\omega_{s0}^2 + \frac{b}{2}r^2\right)K(k) - \omega_{s0}^2 E(k)\right), \tag{9.3}$$

where $E(k)$ is an elliptic integral of the second kind. From Eq. (2.29), we see that the synchrotron motion is given in terms of the Jacobi elliptic function $\mathrm{cn}\,(u; k)$, and

$$\phi_0(J, \theta) = r\,\mathrm{cn}\left(\frac{2K}{\pi}\theta; k\right). \tag{9.4}$$

The Fourier expansion of ϕ_0 is

$$\phi_0(J, \theta) = \sum_{\mu=-\infty}^{\infty} \Phi_\mu(r) e^{i\mu\theta}, \tag{9.5}$$

where the Fourier coefficient vanishes for μ even, and is given for μ odd by

$$\Phi_\mu(r) = \frac{\pi r}{kK}\frac{q^{\mu/2}}{1 + q^\mu}, \tag{9.6}$$

with

$$q = \exp(-\pi K'/K) \tag{9.7}$$

where $K' = K(k') = K(\sqrt{1-k^2})$. A useful expansion for calculating q when k is small is

$$q = \frac{k^2}{16} + 8\left(\frac{k^2}{16}\right)^2 + 84\left(\frac{k^2}{16}\right)^3 + \cdots. \tag{9.8}$$

The results discussed in Section VIII for the harmonic potential are recovered in the limit $k \to 0$, or $r \to 0$. In this limit, $K \approx \pi/2$, $K' \simeq \ln(4/k)$, $K - E \approx \pi k^2/4$, and $\mathrm{cn}\,(x, k) \approx \cos(x)$.

A second limiting case, which will be of particular interest to us, corresponds to $\omega_{s0} = 0$. This condition is achieved in a storage ring by using a Landau cavity.[8] A Landau cavity operates at an integral multiple of the fundamental *rf* frequency with its voltage and phase chosen such that for small-amplitude oscillations the *rf* "potential energy" becomes $U_0(\phi) = b\phi^4/4$, with $b > 0$. The use of such a cavity results in a non-Gaussian bunch density, $\rho_0(\phi) \propto \exp(-U_0(\phi)/\sigma^2)$, and an increase of the rms bunch length. Hence, the use of the Landau cavity reduces the peak current and allows the threshold (expressed in terms of the average current) of the microwave instability to be increased. In addition, because of the nonlinear restoring force, the Landau cavity produces a large spread of synchrotron-oscillation frequencies within the bunch. This provides stability via Landau damping against coupled-bunch instabilities. When $\omega_{s0}=0$,

$$k^2=1/2, \qquad K=K'=\frac{1}{4\sqrt{\pi}}\Gamma^2(1/4)\simeq 1.85, \tag{9.9a}$$

$$\omega_s(r)=\frac{\pi}{2K}\sqrt{b}\,r, \tag{9.9b}$$

$$J=\frac{2K}{3\pi}\sqrt{b}\,r^3, \tag{9.9c}$$

$$\Phi_\mu(r)=\frac{\pi r}{\sqrt{2}K\cosh(\mu-1/2)\pi}\ (\mu \text{ odd}). \tag{9.9d}$$

Recall that Eq. (9.2) determines k as a function of r, and it implies that as long as $b\geq 0$, the modulus is bounded by $k^2\leq 1/2$. Since q is an increasing function of k, it follows that q is always small, and is bounded by its value at $k^2=1/2$, i.e. $q\leq \exp(-\pi)=0.043$. This shows that to a good approximation, $\Phi_1(r)=r/2$ and $\Phi_\mu(r)$, $\mu>1$, can be neglected. Hence, we can replace Eq. (9.4) by

$$\phi_0(J,\theta)\cong r\cos\theta, \tag{9.10}$$

where r is related to J via Eq. (9.3). Within this approximation, the synchrotron-mode coefficients $F_\mu(n,J)$ of Eq. (7.1) are expressed in terms of Bessel functions as in Eq. (8.4). Hence the synchrotron-mode expansion is approximately

$$T_{mn}\cong\frac{2\pi i\kappa}{\sigma^2}\frac{Z_n}{n}\sum_{\mu=-\infty}^{\infty}\mu\int_0^\infty dr U_0'(r)\psi_0(r)\frac{J_\mu(nr)J_\mu(mr)}{\Omega-\mu\omega_s(r)}. \tag{9.11}$$

In writing this equation, we have used Eq. (4.4) to express $\psi_0'(J)=-\sigma^{-2}\omega_s(J)\psi_0(J)$, and we have transformed the integration variable from J to r using the derivative

$$\frac{dJ}{dr}=\frac{dJ}{dH_0}\frac{dH_0}{dr}=U_0'(r)/\omega_s(r). \tag{9.12}$$

The approximate expression for the matrix T_{mn} appropriate for a short bunch was discussed in Section VII, and the dispersion relation was given in Eq. (7.9). Neglecting all terms in the summation with $|\mu|>1$, we find for the particular case under consideration

$$1=\frac{4\pi i\kappa Z_{\text{eff}}}{\sigma^2}\int_0^\infty dr\frac{U_0'(r)\omega_s(r)\psi_0(r)(\Phi_1(r))^2}{\Omega^2-\omega_s^2(r)}, \tag{9.13}$$

where Z_{eff} was defined in Eq. (7.8b) to be

$$Z_{\text{eff}}=\sum_n nZ_n. \tag{9.14}$$

For equally spaced bunches, each having N/M particles, the sum in Eq. (9.14) is over $n = Mj + s$ ($j = -\infty, \ldots, \infty$) for fixed $s = 0, 1, 2, \ldots, M-1$. Upon making the further approximation, $\Phi_1(r) = r/2$, Eq. (9.13) would follow from keeping only the $\mu = \pm 1$ terms in Eq. (9.11), and using the small-argument approximation for the Bessel function, $J_1(x) \approx x/2$.

Let us now write the dispersion relation (9.13) for the special case

$$\omega_{s0} = 0, \qquad U_0(\phi) = b\phi^4/4. \tag{9.15}$$

In this case the unperturbed bunch distribution is

$$\psi_0(r) = A \exp(-r^4/r_0^4), \tag{9.16}$$

where

$$r_0 = (4\sigma^2/b)^{1/4}, \tag{9.17}$$

and the normalization constant A is determined from Eq. (3.10) to be

$$A = \frac{Ne\sqrt{2/\pi}}{\sigma r_0 \Gamma(1/4)}. \tag{9.18}$$

The dependence of oscillation frequency on the amplitude $\omega_s(r)$ is given in Eq. (9.9b). It is convenient to introduce a measure $\Delta\omega_s$ of the synchrotron-oscillation frequency spread in the bunch by

$$\Delta\omega_s \equiv \omega_s(r_0) = \frac{\pi}{2K}(4b\sigma^2)^{1/4}. \tag{9.19}$$

Employing (9.15)–(9.19) and the definition of κ given in Eq. (3.14), the dispersion relation (9.13) becomes

$$1 = \frac{-i\alpha\omega_0^2 e I_0 Z_{\text{eff}}}{2\pi E_0 (\Delta\omega_s)^2} C_4 \int_0^\infty dx \frac{x^6 e^{-x^4}}{x^2 - \dfrac{\Omega^2}{(\Delta\omega_s)^2}}, \tag{9.20}$$

with the constant

$$C_4 = \frac{16\pi\sqrt{2\pi}}{K\Gamma(1/4)} \Phi_1^2(1) \cong 4.30. \tag{9.21}$$

In the fast blowup limit, $\operatorname{Im}\Omega \gg \Delta\omega_s$, the dispersion relation of Eq. (9.20) simplifies to

$$1 = \frac{i\alpha\omega_0^2 e I_0 Z_{\text{eff}}}{2\pi E_0 \Omega^2} C_4 \int_0^\infty dx x^6 e^{-x^4}. \tag{9.22}$$

Upon noting that

$$\int_0^\infty dx x^6 e^{-x^4} = 1/4\Gamma(7/4) \cong 0.99/C_4, \tag{9.23}$$

we see that Eq. (9.22) is very close to the exact result of Eq. (7.14), i.e.

$$1 = \frac{i\alpha\omega_0^2 e I_0 Z_{\text{eff}}}{2\pi E_0 \Omega^2}. \tag{9.24}$$

Thus dropping the terms in Eq. (7.9) with $|\mu|>1$ has resulted in only a 1% error in the fast blowup limit.

Let us define the dispersion integral $G(q)$ by

$$\frac{i}{G(q)}=\int_0^{\infty} dx \frac{x^6 e^{-x^4}}{x^2-q^2}, \tag{9.25}$$

In Fig. 3, we plot Im $G(Q)$ against the Re $G(q)$ at threshold, Im $q=0+$, and above, Im $q=0.1$. It is convenient to define

$$(\delta\Omega_0)^2=\frac{\alpha\omega_0^2 e I_0 Z_{\text{eff}}}{2\pi E_0} \tag{9.26}$$

then the dispersion relation of Eq. (9.20) can be written as

$$4.3\left(\frac{\delta\Omega_0}{\Delta\omega_s}\right)^2=G\left(\frac{\Omega}{\Delta\omega_s}\right). \tag{9.27}$$

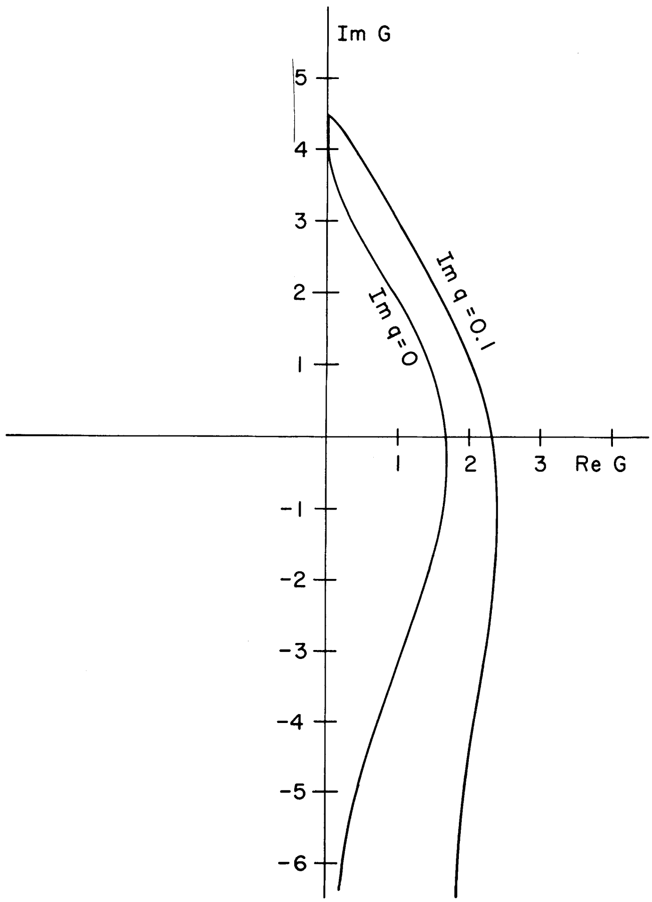

FIGURE 3 Stability boundaries.

A stability condition, in the sense of Keil and Schnell, corresponds to the minimum of $|G|$ along the threshold curve. From Fig. 3 we find $\min|G| = 1.7$, so we have stability for

$$|\delta\Omega_0| \leq 0.6\ \Delta\omega_s. \tag{9.28}$$

There have been discussions by F. Sacherer[4] and by Y. Chin[5] on the problem of longitudinal instabilities subject to a purely quartic *rf* potential, $U_0(\phi) = b\phi^4/4$. They attempted to solve the problem by ignoring the coupling between the different synchrotron modes, $\mu = 0, \pm 1, \pm 2, \ldots$. This, in our opinion, is inadequate. For a purely quartic potential, the harmonics, $\mu\omega_s(r)$, of the synchrotron frequency vanish at $r = 0$ for all μ's. This causes the different synchrotron modes to couple, no matter how small I_0Z_{eff} is, and the mode number μ ceases to be an appropriate parameter to classify the eigensolutions. In other words, these authors are using a nondegenerate perturbation method for a highly degenerate case.

Let us illustrate our point by considering the case of a very short bunch in a quartic potential. This case is simple because the contribution of the higher modes, $|\mu| > 1$, is negligible. However, as is clear from Eq. (9.20), we can not ignore the coupling between the modes $\mu = 1$ and -1 even in such a simple limit.

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13. In order to carry out the short bunch expansion discussed in this section, we must suppose the impedance to be negligible for large *n*. Again one violates casuality, but appeals to the physical intuition that the very high frequency impedance should not qualitatively change the rigid bunch oscillations.
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