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LONGITUDINAL STABILITY OF A COASTING BEAM IN A RESISTIVE VACUUM CHAMBER WITH CYLINDRICAL RESONANT CAVITY
PART I
SELF-CONSISTENT FIELD EQUATIONS AND A METHOD TO IMPROVE CONVERGENCE

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The effect of a passive resonant cavity on longitudinal stability of a coasting beam is studied. The model vacuum chamber has resistive walls throughout, and consists of a round tube with discontinuities in radius forming a cylindrical cavity. An improved method of computing the longitudinal coupling impedance is described, which overcomes restrictions on geometrical parameters encountered in earlier studies. A closed expression for the impedance is obtained that is exact for a deep cavity and surprisingly accurate for a cavity of moderate depth. Corrections to deep-cavity results are obtained by a convenient perturbation procedure, which allows one to compute the impedance for a wide range of parameters, without solving large systems of equations. Stability limits and rise times of instabilities are studied by means of linearized Maxwell-Vlasov equations with a Laplace transform in time. Special features of the resonant situation at high current are discussed, and a case of anomalous stability is explored. Part I is concerned with derivation of self-consistent equations for field mode amplitudes. The equations entail a slowly convergent series, which is transformed by the Watson-Sommerfeld method to improve the rate of convergence. Part II gives numerical and analytical results for the impedance, examples of rise-time calculations for high-current non-relativistic heavy-ion beams, and physical interpretations of the formalism.

1.1 INTRODUCTION

A beam in an accelerator or storage ring may be strongly affected by variations in the vacuum-chamber cross section. Regions of substantial widening can function as high-Q resonant cavities. As was shown by Laslett, Neil and Sessler[1] in 1961, longitudinal instability of a coasting beam may occur when an harmonic of the particle revolution frequency is nearly equal to the resonant frequency of such a cavity. If a longitudinal perturbation of charge density has a Fourier component with frequency near the cavity resonance ω_r, then there will be a charge-density wave with wave number $k = n/R$ (R = ring radius) having phase velocity ω_r/k close to the average particle velocity βc. The cavity sees this wave as a source oscillating near its resonant frequency, which excites a large resonant field that can act on the

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beam with the proper phase to cause a longitudinal instability. The latter may be manifested as spontaneous modulation of charge density in an initially uniform beam. For minimum rise time of the instability, ω_r/k is equal to βc plus a small shift that depends on the plasma frequency, the propagation characteristics of the beam tube, and the unperturbed velocity distribution. Because of periodicity, the mode number n is an integer, and the condition $\omega_r/k \approx \beta c$ is the statement that ω_r be near the n-th harmonic of the revolution frequency: $\omega_r \approx n\Omega = n\beta c/R$. For a cylindrical cavity of radius d, the fundamental resonance is at $\omega_r = j_{01}c/d$ where $j_{01} \approx 2.4$ is the first zero of the Bessel function J_0. In typical cases the harmonic n is quite high, perhaps $n = j_{01}R/d\beta \sim 10^3$, and the resonant frequency $\omega_r/2\pi$ is $10^8 - 10^9$ Hz.

Unstable behavior is favored by a high Q factor of the cavities, as well as by high current density and small velocity spread. In addition, the rise time of an instability depends on the transit-time factor, which is a sensitive function of geometric parameters. Because the unstable mode number n is so high, a small change in average velocity βc can change the mode number by one unit; namely, $\Delta\beta/\beta \approx 1/(2n)$. If the cavity resonances are very narrow, as is the case for metallic cavity walls, one might expect a rapid variation of the rise time of an instability as βc is varied by that amount. Such a variation does occur in the idealized model studied in the following.

An important problem in accelerator design is to delineate the range of parameters for which the growth time of such instabilities will be long enough to be acceptable. The set of parameters includes geometric dimensions, wall conductivity, beam current, the momentum distribution, and the machine parameter η relating changes in momentum to changes in revolution frequency. The problem was studied carefully by Keil and Zotter[2,3] for the particular model of the vacuum chamber shown in Fig. 1, namely, a straight, infinite, cylindrical pipe of radius b, which widens abruptly to a cylinder of radius d and length $2g$ (length g in the notation of Ref. 2). The widenings appear with period $2\pi R$ in the longitudinal distance z, so that the picture can be viewed as an approximation to a circular accelerator ring with large ring radius R, having just one widened segment of mean arc length $2g$. The model has resistive cylindrical walls, but perfectly conducting cavity end walls. Keil and Zotter computed the longitudinal coupling impedance, which summarizes the effect of the conductors surrounding the beam, and is the quantity required for computation of the rise time of an unstable perturbation, or parameter limits for stability, through solution of the plasma dispersion relation.

For the case of a resonant cavity (not the only case treated in Ref. 2), the computations of Keil and Zotter depend on the solution of an infinite system of linear algebraic equations which is truncated to a finite system and solved numerically [Eq. (3.4) of Ref. 2]. The unknowns of the system are certain quantities $\bar{X}_s'$ ($s = 0, 1, 2,$

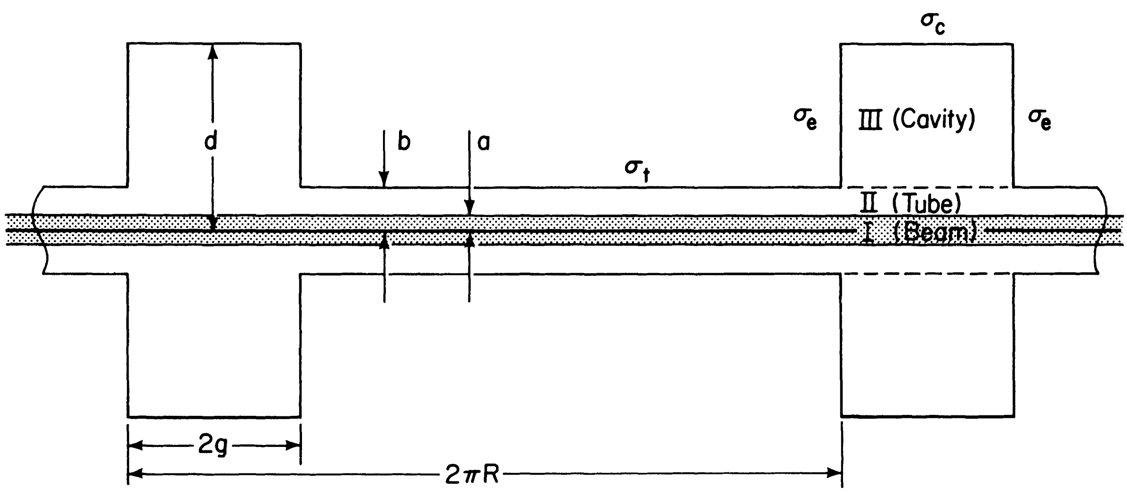

FIGURE 1 Cross section of model vacuum chamber. The quantities σ_t, σ_e, σ_c are conductivities of various portions of the walls.

. . .) that are closely related to the Fourier coefficients of the electric field in the cavity region III of Fig. 1, due to a charge-current perturbation in a particular Fourier mode of the beam region I. A solution vector $\bar{X}'$ determines the coupling impedance and also the resonant frequency and Q factor of the tube with cavity. Numerical values of those quantities are given for a range of machine parameters.

Although the method of Keil and Zotter is straightforward and seemingly tractable, it has the following features that limit its usefulness. (i) The simple physics of the resonant cavity is invisible in the formalism. Since the Fourier modes of the cavity become the normal modes of the system in the limit of small tube radius b (at constant cavity radius d), one would expect the equations to separate and simplify in that limit. Surprisingly, the form of the equations is such that the limit is hard to extract, and that prevents the derivation of explicit analytic formulas that could be useful at small but non-zero b. (ii) The matrix of coefficients of the infinite set of equations, the "kernel" M_{st}, is itself given by an infinite sum in Bessel functions. The summand has a maximum as a function of the summation index m, which occurs at a large m. Furthermore, the maximum moves to larger m as the ring radius R or the mode numbers (s, t) increase. Since one must sum far beyound the maximum to get an accurate value of the kernel, the number of terms required becomes unmanageable at large R or large (s, t). Accordingly, the calculations of Ref. 2 are restricted to values of R that are usually unrealistic ($0.4 < R < 4$ meters). (iii) The justification for the truncation to a small finite system of equations is obscure, even though some testing of accuracy was done by increasing the number of modes retained.

In this and following papers, we give methods to overcome these technical difficulties, and obtain numerical and analytical results for impedances, resonant frequencies, and Q values in parameter ranges of interest. Our model of the vacuum chamber is the same as that of Keil and Zotter, but extended to allow resistive cavity end walls. The equations we employ are literally equations for the cavity-region III Fourier amplitudes (the $\bar{D}_s$ of Ref. 2), and are somewhat simpler and more directly interpretable than those for the auxiliary amplitudes $\bar{X}_s'$. There are also equations for the beam-region I amplitudes (the $\bar{A}_n$ of Ref. 2), which are appropriate in treating very shallow cavities; for the present we deemphasize that case. In addition to studying the purely electromagnetic questions, we treat the beam dynamics by linearized Maxwell-Vlasov equations with a Laplace transform in time. We look carefully at the derivation of dispersion relations, stability criteria, and rise times, under resonant conditions with high beam current. We find a case of anomalous stability at high current, in which two different tube modes participate. The Laplace transform gives an initial-value formalism, which could be used to follow the detailed time evolution of an unstable beam.

It turns out that a solution to problem (ii) above is a key to dealing with the other problems. We apply a so-called Watson-Sommerfeld transformation to the series for the kernel, to improve its convergence. The sum is written as a contour integral around the poles of an auxiliary analytic function, the residues of the poles being the terms of the original sum. Expansion of the contour to infinity then gives a new expression for the kernel, which consists of a series with monotonically decreasing summand plus a large term occurring only on the diagonal of the kernel. The new series is easy to compute numerically, and it makes a relatively small contribution when b is small. In fact, only the diagonal piece, given in closed form as a ratio of Bessel functions, survives in the limit of vanishing b. The expected diagonalization of the cavity mode equations for small b is thus made explicit through the Watson-Sommerfeld transformation. By retaining only the diagonal term, we get an analytic expression for the impedance which is exact for $b \rightarrow 0$, but which contains a factor accounting for effects of the tube at non-zero b. Comparison with exact calculations shows that the expression is accurate for moderately large values of b/d.

Another good result of the Watson-Sommerfeld transformation is that it prepares the way for treatment of the equations by Fredholm theory. As will be shown in a paper published elsewhere, the cavity mode equation is equivalent to a certain Fredholm equation in a Banach space of sequences. That result provides a theoretical justification for truncating the infinite system, and also gives information on the behavior of the solution at large mode number.

Having transformed the kernel, we propose a further rearrangement of the equations in which the resonant mode is eliminated in favor of all the other modes. The resulting equations are better suited to numerical solution. Except in the case of a very shallow cavity, they may be solved by a rapidly convergent perturbation series which avoids costly direct solution of linear equations and gives control of error in the reduction to a finite number of dimensions. The series may be described as a way to generate corrections to the deep-cavity results (small b/d), for larger b/d; it accounts for the presence of the tube and the attendant coupling of cavity modes.

Computation of the perturbation series amounts to computing powers of an infinite matrix. We do not throw away the tails of the infinite sums involved, but instead approximate sums without truncation through a mapping and spline interpolation technique. Each infinite sum is effectively reduced to a short finite sum, by taking advantage of the smooth variation of summands. We use a similar method for non-perturbative solution of the cavity mode equations (with resonant mode eliminated). The known smooth dependence of the solution at large s is used in a spline interpolation to approximate the infinite system by a small finite system. The $\bar{D}_s$ at large s are all represented by a few values of $\bar{D}_s$ at non-integer spline knots $s = s_i$, which are taken as unknowns. By this method we gain substantial control over truncation error.

In order to allow resistivity on the cavity end walls, we use a non-harmonic Fourier series in the cavity region III. This gives the extra freedom needed to meet the resistive wall boundary condition, with little cost in computational effort. The added resistivity has an important quantitative effect, larger than that indicated by a proposed rule-of-thumb (3rd paragraph of Ref. 2).

Our discussion is based on the Maxwell-Vlasov equations linearized about the non-stationary state corresponding to a uniform beam subject to the full boundary conditions for the tube with cavity. In complete analogy to Landau's original theory of plasma oscillations,[4] we make a Laplace transform in time, rather than the Fourier transform which is often used (actually without justification) in stability studies. This clarifies the derivation of the dispersion relation and allows us to prove that the rise time of a sufficiently small unstable perturbation is independent of its initial form. We hope to consider practical applications of the Laplace initial-value formalism in later work, one possibility being to study beam evolution in an induction linac.

The Maxwell-Vlasov equations lead to "self-consistent" equations for the perturbed field coefficients, with the unperturbed momentum distribution appearing parametrically. We derive self-consistent equations for the $\bar{D}_s$ as well as for the $\bar{A}_n$. The former are presumably novel for treating longitudinal stability, and we find them to be very convenient. They give a full account of stability without explicit reference to axial fields or coupling impedance. The general form of the plasma dispersion relation governing stability is that the determinant of a self-consistent system of equations be zero: $\det(1 - K) = 0$, where K is the kernel of either the $\bar{A}_n$ or the $\bar{D}_s$ equation. Because of the coupling of tube modes induced by the cavity (in a complementary view, the coupling of cavity modes induced by the tube), all modes appear in the exact dispersion relation. Nevertheless, the usual dispersion relation involving only one tube mode is normally an excellent approximation to the full determinantal equation, as we show in a detailed discussion. There is one exception, in which two tube modes participate. The anomaly occurs only for a limited range of parameters (for instance, if the current is sufficiently high and within certain narrow intervals) but it enhances stability. We treat the anomalous stability at length in the tractable case of a deep cavity.

The dispersion relation is normally used to establish sufficient conditions for stability, which are not influenced by the frequency dependence of the coupling impedance. Whatever the variation with ω, it is sufficient (but actually not quite necessary) for stability that the impedance evaluated on the real ω-axis stay within a certain region of the complex plane which depends on the current, the unperturbed momentum distribution, etc. On the other hand, when the dispersion relation is used to find rise times, the frequency variation of the impedance must be accounted for, especially in the resonant case. We compute the rise time under resonant conditions by a method which uses the solutions of the smooth-tube dispersion relation as a starting point. Given those solutions, it is a trivial matter to find the rise time, and to explore its variation with changing parameters.

Our quantitive examples of stability limits and rise times will be for high-current nonrelativistic heavy-ion beams such as are contemplated in designs for ion-beam fusion drivers. Our impedance calculations are of course more general, being relativistic and independent of particle dynamics.

We believe that some of the methods employed will be useful in a variety of problems in accelerator theory, and for that reason give fairly complete details. A reader more interested in results than derivations should begin with Part II.

A short summary of our formalism and a numerical example are given in Ref. 5.

1.2 RELATED WORK

Month and Peierls[6] also reexamined the equations of Keil and Zotter, using a Watson-Sommerfeld transformation. Their use of the transformation is completely different from ours, however, in that they transform the solution of the equations for field coefficients, rather than the kernel. Consequently, they have to assume analyticity properties of an unknown function, and that makes the results difficult to evaluate. In our case the function is known and the transformation is rigorously justified.

In an interesting paper Keil and Messerschmid[7] studied nonlinear effects in the longitudinal stability of a coasting beam by means of numerical simulation. They find that the linear theory gives a good first approximation, but find interesting behavior of the velocity spread in the nonlinear saturation of instability.

Measurements on destabilizing effects of vacuum chamber cross-section variations have been performed at the ISR.[8,9] A special experimental cavity placed around the ISR beam was used to study longitudinal stability of a coasting beam;[9] theoretical estimates of thresholds for instability were found to be valid. Stability of bunched beams has been the topic of many experimental and theoretical investigations. For a recent review emphasizing design considerations see Hofmann.[10]

Calculations of coupling impedance for models different from that of Ref. 2 have been done by several authors. Hahn and Zatz[11] treat single and double step discontinuities of cross section in a circular tube, without periodicity. Hereward[12] considered a single step in a rectangular tube. Kriegler, Mills, and van Bladel,[13] and also Trickett,[14] studied a reentrant cavity (annulus coupled to the main tube through a slot). Chatard-Moulin and Papiernik[15] treated an arbitrary small periodic modulation of tube radius. Their method was applied by Krinsky[16] and by Cooper and Morton,[17] and was reformulated by Krinsky and Gluckstern.[18] Sessler[19] gave a general review of the effects of beam surroundings on stability, listing further references. Related problems of wave propagation in corrugated wave guides have received much attention in the engineering literature.[20]

1.3 CONTENTS OF PART I

Section 2 is concerned with linearization of the Vlasov equation, and Laplace and Fourier transforms of the linear Maxwell-Vlasov system. We advise some care in the interpretation of the linearized system, because one must linearize about a state that is not the stationary solution of the nonlinear system. The variables of our Vlasov equation are the position and relativistic momentum of rectilinear motion, rather than the angle-action variables that are often employed. For a coasting beam, the latter are superfluous as long as one computes the coupling impedance for a straight vacuum chamber with a centered beam.

Section 3 contains the derivation of equations for Laplace-Fourier coefficients of field perturbations due to a prescribed charge-current perturbation. The discussion parallels that of Ref. 2, and uses a similar notation, but has been generalized to allow arbitrary charge-current perturbations, resistive cavity end walls, and initial-value terms. Most of the details on initial-value terms are in two appendices. We first find coupled equations for the tube modes $\bar{A}$ and cavity modes $\bar{D}$, Eqs. (3.66) and (3.67), which correspond to (1.23) of Ref. 2. We then elminate $\bar{A}$ to get Eq. (3.68) for $\bar{D}$ alone. There is no corresponding equation in Ref. 2., but (3.4) of that paper has a similar kernel and is applied in a similar way. Equations (3.67) and (3.74) give $\bar{A}$ in terms of $\bar{D}$ and vice versa.

In Section 4, the Vlasov equation is combined with the electromagnetic equations of Section 3 to give "self-consistent" equations in which the perturbed charge-current does not appear, namely Eqs. (4.12) for tube modes and (4.29) for cavity modes. The usual dispersion relation governing stability, Eq. (4.18), is derived as an approximation to (4.12). The quantitative justification of the approximation is treated in Part II. Exact expressions for the impedance are given by Eqs. (4.23) and (4.24) or (4.26). Instead of direct numerical evaluation of these formulas, we recommend the methods developed in Part II.

Section 5 explains the Watson-Sommerfeld transformation of the kernel of the $\bar{D}$ equation (3.68). The final form of the kernel, which will be applied in numerical work of Part II, is given in (5.18). The result (5.18) has a very simple structure, which the reader may better appreciate by putting the small resistivity parameters (κ, η_t) equal to zero.

Topics mentioned in the Introduction but not treated in Part I will be covered in Part II, except for the Fredholm theory of the $\bar{D}$ equation, which is to appear in a third paper.

2. LAPLACE AND FOURIER TRANSFORMS OF MAXWELL-VLASOV EQUATIONS

We take the axis of the tube to be the z-axis with the origin at a cavity centroid; the ends of the cavities then lie in the planes $z = 2\pi nR \pm g$, where n is an integer. The particle distribution function, $u(\mathbf{r}, \mathbf{p}, t)$, is presumed to obey the relativistic Vlasov equation,

$$\frac{\partial u}{\partial t} + \mathbf{v} \cdot \frac{\partial u}{\partial \mathbf{r}} + q\,(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \frac{\partial u}{\partial \mathbf{p}} = 0, \qquad (2.1)$$

where $\mathbf{p} = M\gamma\mathbf{v}$ is the momentum. We suppose that u has the form[21]

$$u(\mathbf{r}, \mathbf{p}, t) = \theta(a - r)\delta(p_x)\delta(p_y)[f_0(p_z) + f_1(z, p_z, t)], \qquad (2.2)$$

where θ is the unit step function, and δ is understood as a smooth but sharply peaked even function approximating the Dirac delta function. Thus, charge and current are spatially uniform over a cross section of the beam, within the beam radius a. On the average, particles move only in the z direction. If we substitute (2.2) in (2.1) and evaluate the equation at $\mathbf{r} = (0, 0, z)$, $\mathbf{v} = (0, 0, v)$, the result is

$$\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial z}\right) f_1(z, p_z, t) + q\, E_z(z, t) \times \frac{\partial}{\partial p_z}[f_0(p_z) + f_1(z, p_z, t)] = 0. \qquad (2.3)$$

We wish to linearize (2.3) about the configuration corresponding to f_0. Accordingly we write the electric field as

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_0(\mathbf{r}) + \mathbf{E}_1(\mathbf{r}, t), \qquad (2.4)$$

where $\mathbf{E}_0$ and $\mathbf{E}_1$ correspond to charge-current densities (ρ_0, J_0) and (ρ_1, J_1) defined as

$$\rho_0(\mathbf{r}) + \rho_1(\mathbf{r}, t) = \theta(a - r)q \times \int_{-\infty}^{\infty} [f_0(p_z) + f_1(z, p_z, t)]dp_z \qquad (2.5)$$

$$J_0(\mathbf{r}) + J_1(\mathbf{r}, t) = \theta(a - r)q \times \int_{-\infty}^{\infty} v[f_0(p_z) + f_1(z, p_z, t)]dp_z \qquad (2.6)$$

We emphasize that both $\mathbf{E}_0$ and $\mathbf{E}_1$ correspond to the same boundary conditions, those for the corrugated tube with resistive walls.

At points on the axis sufficiently far from the cavity ends the field lines of $\mathbf{E}_0$ leave the axis almost exactly in the radial direction, so that E_{0z} on the axis is negligible. At points nearly adjacent to the cavity ends, E_{0z} on the axis will be nonzero but small; the field lines must bend around to meet the cavity ends at nearly normal incidence. For the linearization, we treat E_{0z} on axis as a first-order quantity, even though it is formally of order zero. The accuracy of this procedure could be judged by solving the boundary-value problem that determines E_{0z}, using methods like those developed in the following. When second-order quantities are dropped, Eq. (2.3) takes the form

$$\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial z}\right) f_1(z, v, t) + q\, E_{z1}(z, t)\, f_0{}'(p_z) = -\, q\, E_{z0}(z) f_0{}'(p_z). \qquad (2.7)$$

This equation is to be solved together with the Maxwell equations and boundary conditions for $\mathbf{E}_1$ and $\mathbf{B}_1$, with sources (ρ_1, J_1) given by (2.5) and (2.6). Thus we have a linear system to determine (f_1, $\mathbf{E}_1$, $\mathbf{B}_1$) in terms of f_0, E_{z0}, and the initial values $(f_1, \mathbf{E}_1, \mathbf{B}_1)_{t=0}$. To calculate the rise time of an unstable perturbation we need not actually specify E_{z0} and the initial values, since the rise time turns out not to involve those time-independent quantities. Of course, we ignore completely the implications of the Vlasov equation (2.1) at points off the z-axis, as befits the approximation of one-dimensional motion.

Before proceeding to the Maxwell equations, some comments are in order. Generalizing the theory of Landau damping,[4] we begin at $t = 0$ with an arbitrary small perturbation, $(f_1, \mathbf{E}_1, \mathbf{B}_1)_{t=0}$, of the uniform beam configuration, $(f_0, \mathbf{E}_0)$, and ask how that perturbation evolves in time. We can never expect that the perturbation will decay to zero, even if parameters are such that the beam is actually stable, because $(f_0, \mathbf{E}_0)$ is not an exact steady-state solution of the nonlinear Maxwell-Vlasov system. Indeed, if $f_1 = 0$ and $\mathbf{E}_1 = 0$, then (2.3) reduces to the false equation $E_{z0}(z)\, f_0'(p_z) = 0$. Clearly, a steady-state solution must have some variation of charge density in the z-direction to account for the presence of the cavities. A rigorous discussion of stability would have to proceed by first finding an exact steady-state solution of the nonlinear system. Linearization about that solution would then decide the question of stability.

As time passes, the perturbed field $\mathbf{E}_1$, as computed from the linearized equations above, will either (i) increase in magnitude indefinitely or (ii) tend to a constant. (There is also the mathematical possibility that $\mathbf{E}_1$ could oscillate indefinitely without approaching a limit, but that would not seem to make sense physically.) We interpret case (i) as instability and (ii) as stability, while emphasizing that the interpretation is plausible but not rigorously justified. Since the configuration $(f_0, \mathbf{E}_0)$ about which we linearize is already a perturbation of the true stationary configuration $(f^{(0)}, \mathbf{E}^{(0)})$ (supposed to exist), the true initial perturbation of the stationary state is

$$(f_0 + f_1 - f^{(0)}, \mathbf{E}_0 + \mathbf{E}_1 - \mathbf{E}^{(0)}, \mathbf{B}_1)_{t=0}. \quad (2.8)$$

Our interpretation of an increasing $\mathbf{E}_1$ as instability could be wrong if the beam were actually stable to a *sufficiently small* perturbation, but not to one as large as the "minimum" perturbation that we are able to treat theoretically, the latter being

$$(f_0 - f^{(0)}, \mathbf{E}_0 - \mathbf{E}^{(0)}, 0). \quad (2.9)$$

The distribution function is to be normalized so that

$$\pi a^2 \int_{z_0}^{z_0+2\pi R} dz \int_{-\infty}^{\infty} dp_z f(z, p_z, t) = N, \quad (2.10)$$

where N is the average number of particles in the interval $z_0 < z < z_0 + 2\pi R$. We shall impose periodicity of f in z through a Fourier development, so that N will be independent of z_0. To maintain (2.10) in the linearized formalism, we first take $f_0(p_z)$ to satisfy (2.10) by itself:

$$(\pi a^2)(2\pi R) \int_{-\infty}^{\infty} dp_z f_0(p_z) = N. \quad (2.11)$$

Next, we choose the initial value of f_1, which is arbitrary, so that

$$\int_{z_0}^{z_0+2\pi R} dz \int_{-\infty}^{\infty} dp_z f_1(z, p_z, 0) = 0. \quad (2.12)$$

For instance, condition (2.12) is satisfied if there is no constant term in the Fourier series for f_1 at time 0:

$$f_1(z, p_z, 0) = \sum_{n\neq 0} \hat{f}_{1n}(p_z, 0) \exp(in\, z/R). \quad (2.13)$$

Now $f = f_0 + f_1$ satisfies (2.10) at time 0, and the linearized equation (2.7) then implies that f satisfies (2.10) for all time. In addition, the linearized equation implies that the continuity equation holds for all z and t,

$$\frac{\partial \rho}{\partial t} + \frac{\partial J}{\partial z} = 0. \quad (2.14)$$

We next state the Maxwell equations for perturbed fields $\mathbf{E}_1$ and $\mathbf{B}_1 = \mu_0 \mathbf{H}_1$ with sources ρ_1, J_1 given by (2.5) and (2.6). We take cylindrical coordinates (z, r, ϕ), and look only for solutions independent of ϕ. Higher modes depending on ϕ are believed to have relatively little effect on stability, but perhaps should be investigated at a later stage. Henceforth we suppress the subscript 1 denoting perturbations, and write the axially symmetric equations for perturbed fields of the transverse magnetic (TM) mode as

$$\frac{1}{r}\frac{\partial}{\partial r}(r E_r) + \frac{\partial E_z}{\partial z} = \frac{\rho}{\epsilon_0} \quad (2.15)$$

$$\frac{\partial E_r}{\partial z} - \frac{\partial E_z}{\partial r} = -\mu_0 \frac{\partial H_\phi}{\partial t} \quad (2.16)$$

$$\frac{\partial H_\phi}{\partial z} = -\epsilon_0 \frac{\partial E_r}{\partial t} \quad (2.17)$$

$$\frac{1}{r}\frac{\partial}{\partial r}(r H_\phi) = J + \epsilon_0 \frac{\partial E_z}{\partial t}. \quad (2.18)$$

The corresponding equations for the transverse electric (TE) mode are obtained from (2.15)–(2.18) by dropping ρ and J and making the replacements $E \leftrightarrow H$, $\mu_0 \leftrightarrow -\epsilon_0$. The boundary conditions at the resistive walls do not mix the TM fields (E_r, E_z, H_ϕ) with the TE fields (H_r, H_z, E_ϕ), at least in the standard approximate treatment of boundary conditions (Ref. 22 and Appendix B). Since only the TM fields affect the longitudinal particle distribution through the Vlasov equation (2.7), we may then ignore the TE fields entirely; [under assumption (2.2) the TE fields have no sources, and in fact vanish at all times if they vanish at $t = 0$].

Following Landau,[4] we perform a Fourier transformation of the equations with respect to z and a Laplace transform with respect to t. (Some authors use a Fourier transform in t for stabilty studies, but are thereby led to logical inconsistencies that were already noted by Landau. In proving that the resulting prescriptions for computing growth times are correct, one is in fact led back to the Laplace transform.) We first treat the "tube region," $r < b$, in which the series in z will have period $2\pi R$. Later we use a nonharmonic series in the "cavity region," $b < r < d$. In the tube region, the Fourier-Laplace transform of a function $\phi(z, t)$ is

$$\hat{\phi}_n(p) = \frac{1}{2\pi R}\int_0^\infty e^{-pt}dt \int_{-g}^{-g+2\pi R} e^{ik_n z}\phi(z, t)dz, \tag{2.19}$$

$$k_n = n/R, \quad n = 0, \pm 1, \pm 2, \ldots . \tag{2.20}$$

The Fourier transform (without Laplace transformation), evaluated at $t = 0$, is denoted by $\phi_n(0)$. Through integration by parts,

$$\left(\frac{\partial \phi_n}{\partial t}\right)^{\wedge} = p\hat{\phi}_n(p) - \phi_n(0). \tag{2.21}$$

A reader committed to thinking in terms of frequencies may put $p = -i\omega$ in in the following equations.

Tranformation of the linearized Vlasov equation (2.8) yields

$$(p + ik_n v)\hat{f}_n(p_z, p) + q\,\hat{E}_{zn}(r, p)\,|_{r=0} f_0'(p_z)$$
$$= f_n(p_z, t)\,|_{t=0} - q\,E_{zon}\frac{1}{p} f_0'(p_z). \tag{2.22}$$

In stating the transformed Maxwell equations, we simplify notation by suppressing arguments r, p, n; thus $\hat{E}_{zn}(r, p) = \hat{E}_z$, and for initial values we write $E_{zn}(r, 0) = E_z(0)$. Then transformation of (2.15)–(2.18) yields

$$\frac{1}{r}\frac{\partial}{\partial r}(r\hat{E}_r) + ik\,\hat{E}_z = \frac{1}{\epsilon_0}\hat{\rho}, \tag{2.23}$$

$$ik\,\hat{E}_r - \frac{\partial \hat{E}_z}{\partial r} = -\mu_0 p\,\hat{H}_\phi + \mu_0 H_\phi(0), \tag{2.24}$$

$$ik\,\hat{H}_\phi = -\epsilon_0 p\,\hat{E}_r + \epsilon_0 E_r(0), \tag{2.25}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(r\,\hat{H}_\phi) = \hat{J} + \epsilon_0 p\,\hat{E}_z - \epsilon_0 E_z(0). \tag{2.26}$$

In the cavity region, $b < r < d$, $-g < z < g$, we employ generalized Fourier developments,

$$\hat{E}_r(r, z) = \sum_{s=0}^\infty [\hat{E}^1_{rs}(r)\sin\alpha_s z + \hat{E}^2_{rs}(r)\cos\alpha_s z],$$
$$\hat{E}_z(r, z) = \sum_{s=0}^\infty [\hat{E}^1_{zs}(r)\cos\alpha_s z + \hat{E}^2_{zs}(r)\sin\alpha_s z],$$
$$\hat{H}_\phi(r, z) = \sum_{s=0}^\infty [\hat{H}^1_{\phi s}(r)\cos\alpha_s z + \hat{H}^2_{\phi s}(r)\sin\alpha_s z]. \tag{2.27}$$

Here and in most of the equations to follow we suppress the variable p. One must keep in mind that almost all quantities, in particular the Laplace-Fourier coefficients of fields, are functions of p. The nonharmonic wave numbers α_s are certain nonlinear functions of s and p, determined by boundary conditions on the planar end walls of the cavity. It turns out that the functions sin $\alpha_s z$ and cos $\alpha_s z$ are mutually orthogonal, so that there is a set of equations for the cavity field coefficients analogous to (2.23)–(2.26). Again suppressing arguments r, p, s, we find that for $i = 1, 2$,

$$\frac{1}{r}\frac{\partial}{\partial r}(r\hat{E}_r^{\,i}) + (-)^i\,\alpha\hat{E}_z^{\,i} = 0, \tag{2.28}$$

$$(-)^{i+1}\,\alpha\hat{E}_r^{\,i} - \frac{\partial\hat{E}_z^{\,i}}{\partial r} = -\mu_0 p\,\hat{H}_\phi^{\,i} + \mu_0\,H_\phi^{\,i}(0), \tag{2.29}$$

$$(-)^{i+1}\,\alpha\hat{H}_\phi^{\,i} = \epsilon_0 p\,\hat{E}_r^{\,i} - \epsilon_0\,E_r^{\,i}(0), \tag{2.30}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(r\,\hat{H}_\phi^{\,i}) = \epsilon_0 p\,\hat{E}_z^{\,i} - \epsilon_0\,E_z^{\,i}(0). \tag{2.31}$$

The Eqs. (2.23)–(2.26) and (2.28)–(2.31) are to be solved subject to continuity conditions at the tube-cavity interface ($r = b,\ -g < z < g$) and boundary conditions relating tangential electric and magnetic fields at the walls. The solution is in terms of charge-current densities and initial values of fields. Expressing charge and current in terms of the distribution function, and evaluating the latter by (2.22), one finally obtains the field coefficients in terms of initial values of the fields and distribution function. For instance, one can obtain the electric-field perturbation on the axis, $\hat{E}_{zn}(r, p)\,|_{r=0}$, in terms of the initial perturbation of the distribution function, $f(z, p_z, 0)$, and two independent initial field perturbations, say $E_z(r, z, 0)$ and $H_\phi(r, z, 0)$. By an inverse Laplace transformation, one may then calculate the rise time τ of an unstable perturbation as $\tau = 1/\mathrm{Re}p_*$, where p_* is the right-most singularity of $\hat{E}_{zn}(0, p)$ in the complex p-plane. This method of solution is quite analogous to that of Landau's theory, but more complicated because of the boundary conditions and the initial-value problem.

In analogy to Landau's case, we are able to prove that the particular initial values of fields and the distribution function do not affect the rise time. We work out the role of arbitrary initial-value terms in detail, under the restriction that the end walls of the cavity have infinite conductivity; (there are technical difficulties in allowing finite conductivity with initial-value terms). We report the calculations in Appendix A, and drop the initial-value terms in most of the main text.

The resistive-wall boundary condition is usually stated under the assumption of exponential time dependence of the fields,[22] which is equivalent to stating it for the Fourier transform of the fields in time. We need a statement for the Laplace transform of the fields, which is derived in Appendix B.

3. EQUATIONS FOR LAPLACE-FOURIER COEFFICIENTS OF FIELDS

In this section, we find equations for the Laplace-Fourier coefficients of fields which satisfy (2.23)–(2.26), (2.28)–(2.31), and the required continuity and boundary conditions. We define three regions of the vacuum chamber:

I: $0 < r < a$

II: $a < r < b$

III: $b < r < d,\ -g < z < g.$

In Regions I and II, the relevant Maxwell equations are (2.23)–(2.26), whereas (2.28)–(2.31) apply to Region III. To solve (2.23)–(2.26) we first find a second-order equation for $\hat{E}_z$ alone. The system degenerates when $k = 0$, so that it is important to eliminate $\hat{E}_r$ and $\hat{H}_\phi$ in such a way that the resulting equation holds even at $k = 0$. We take $\hat{E}_r$ from (2.25) and substitute in (2.24); then solve (2.24) for $\hat{H}_\phi$, and substitute the latter in (2.26) to obtain

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\hat{E}_z}{\partial r}\right) - \chi^2\hat{E}_z = \frac{\chi^2}{\epsilon_0 p}(\hat{J} - \epsilon_0 E_z(0)) - \frac{1}{r}\frac{\partial}{\partial r}\left\{r\left[\mu_0 H_\phi(0) - \frac{ik}{p}E_r(0)\right]\right\}, \tag{3.1}$$

where

$$\chi^2 = k^2 + (p/c)^2. \tag{3.2}$$

One of the three initial fields may be eliminated from (3.1). We make use of the continuity equation

$$ik\hat{J} + p\hat{\rho} = \rho(0), \tag{3.3}$$

and the Poisson equation (2.15), after Fourier transformation but before Laplace transformation, evaluated at $t = 0$:

$$\frac{1}{r}\frac{\partial}{\partial r}(r\,E_r(0)) + ik\,E_z(0) = \frac{1}{\epsilon_0}\rho(0). \tag{3.4}$$

From (3.1)–(3.4), one finds the necessary condition on $\hat{E}_z$,

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \hat{E}_z}{\partial r}\right) - \chi^2 \hat{E}_z = \mu_0 p\,\hat{J} + \frac{ik}{\epsilon_0}\hat{\rho} - \frac{p}{c^2}E_z(0) - \frac{\mu_0}{r}\frac{\partial}{\partial r}(r\,H_\phi(0)). \quad (3.5)$$

The linear combination of $\hat{J}$ and $\hat{\rho}$ that appears in (3.5) is in some sense the natural analog of the charge density $\hat{\rho}$ that appears in Landau's electrostatic problem, and will occur repeatedly in the following.[23] We select $[f(0), E_z(0), H_\phi(0)]$ as the set of independent initial values to be chosen arbitrarily (within the restriction that they be small). We notice that the remaining initial value $E_r(0)$ may be expressed in terms of the others by integrating (3.4) and applying the requirement that $E_r(r, 0)$ be finite at $r = 0$. Thus

$$E_r(r, 0) = \frac{1}{r}\int_0^r udu\left[-ikE_z(u, 0) + \frac{q}{\epsilon_0}\theta(a-u)\int_{-\infty}^{\infty} dp_z f(p_z, 0)\right]. \quad (3.6)$$

In the remainder of this section we put all three of the initial values $[f(0), E_z(0), H_\phi(0)]$ equal to zero; hence $E_r(0) = 0$ as well. Then the only inhomogeneity in the system of linear equations is provided by the term from E_{z0} in the Vlasov equation (2.22). In other words, we are beginning with a charge and field configuration which differs from a stationary-state solution ($f^{(0)}$, $\mathbf{E}^{(0)}$) of the nonlinear Maxwell-Vlasov system by the amount (2.9). The field E_{z0} due to wall corrugations may drive an initially uniform beam to instability.

With initial fields absent, the general solution of (3.5) is given in terms of modified Bessel functions as

$$\hat{E}_z = c_1 I_0(\chi r) + c_2 K_0(\chi r) + \theta(a-r)F, \quad (3.7)$$

where

$$F = -\frac{q}{\chi^2}\int_{-\infty}^{\infty} dp_z \hat{f}(p_z)\left[\mu_0 p v + \frac{ik}{\epsilon_0}\right]. \quad (3.8)$$

The irregular Bessel function K_0 is disallowed in Region I because of its singularity at $r = 0$. Thus, the Fourier developments of $\hat{E}_z$ in Regions I and II must have the forms

$$\hat{E}_z^{I}(r, z) = \sum_{m=-\infty}^{\infty} [A_m I_0(\chi_m r) + F_m] e^{ik_m z}, \quad (3.9)$$

$$\hat{E}_z^{II}(r, z) = \sum_{m=-\infty}^{\infty} [B_m I_0(\chi_m r) + C_m K_0(\chi_m r)] e^{ik_m z}. \quad (3.10)$$

We next find corresponding expressions for $\hat{H}_\phi$ by a method that is compatible with the discussion of Appendix A, i.e., that brings in only $E_z(0)$ and $H_\phi(0)$ when initial-value terms are included. The method is to solve (2.26) for $\hat{H}_\phi$, with $\hat{E}_z$ given by (3.9), (3.10). The general solution of the corresponding homogeneous equation is $\hat{H}_\phi = \gamma/r$, where γ is a constant. Since $I_1 = I_0'$ and $K_1 = -K_0'$, a particular solution of the inhomogeneous equation is

$$\frac{1}{r}\int_0^r u[\hat{J}(u) + \theta(a-u)\epsilon_0 pF]\,du + \frac{\epsilon_0 p}{\chi}\begin{cases} A\,I_0(\chi r), & r < a, \\ B\,I_1(\chi r) - CK_1(\chi r), & r > a. \end{cases} \quad (3.11)$$

The first term of (3.11) may be expressed in terms of the initial value of the distribution function, as is seen by applying the continuity equation (3.3):

$$\hat{J}(u) + \theta(a-u)\,\epsilon_0 p\,F = \frac{-ik}{\chi^2}\rho(0) = \frac{-ik}{\chi^2}\theta(a-u)\,q\int_{-\infty}^{\infty} dp_z f(p_z, 0). \quad (3.12)$$

For the present discussion this quantity is zero. Furthermore, one can rule out a term γ/r in $\hat{H}_\phi$ by demanding that both (2.23) and (2.24) be satisfied; consequently,

$$\hat{H}_\phi^{I}(r, z) = \epsilon_0 p \sum_{m=-\infty}^{\infty} \chi_m^{-1} A_m I_0(\chi_m r) e^{ik_m z}, \quad (3.13)$$

$$\hat{H}_\phi^{II}(r, z) = \epsilon_0 p \sum_{m=-\infty}^{\infty} \chi_m^{-1}[B_m I_0(\chi_m r) - C_m K_0(\chi_m r)] e^{ik_m z}. \quad (3.14)$$

Corresponding expressions for $\hat{E}_r$ are obtained from (2.24), (2.25), given the results for $\hat{E}_z$ and $\hat{H}_\phi$. One can then verify that the fields satisfy all four Maxwell equations. If the $\hat{E}_z$ and $\hat{H}_\phi$ fields for Region I are matched to those for Region II at $r = a$, then $\hat{E}_r$ will automatically be matched as well, by (2.25).

In Region III the non-harmonic Fourier developments (2.27) and the Maxwell equations (2.28)–(2.31) imply an equation analogous to (3.5)

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \hat{E}_z^i}{\partial r}\right) - \Gamma^2 \hat{E}_z^i = -\frac{p}{c^2}E_z^i(0) - \frac{\mu_0}{r}\frac{\partial}{\partial r}(r\,H_\phi^i(0)), \quad (3.15)$$

$$\Gamma^2 = \alpha^2 + (p/c)^2. \quad (3.16)$$

It is convenient to state solutions in Region III in terms of certain linear combinations of $I_0(\Gamma r)$ and $K_0(\Gamma r)$ that have simple expressions on the cylindrical cavity wall, $r = d$. Accordingly we define, for $j = 0, 1$,

$$R_j(x, y) = K_0(y)I_j(x) + (-)^{j+1}I_0(y)K_j(x), \quad (3.17)$$

$$S_j(x, y) = -K_1(y)I_j(x) + (-)^{j+1}I_1(y)K_j(x). \quad (3.18)$$

We notice that

$$R_1 = \partial R_0/\partial x, \quad S_1 = \partial S_0/\partial x, \quad (3.19)$$

and, by a standard Wronskian identity,

$$R_0(x, x) = 0, \quad S_1(x, x) = 0, \quad (3.20)$$

$$R_1(x, x) = 1/x, \quad S_0(x, x) = -1/x. \quad (3.21)$$

With initial value terms neglected, the general solution of equations (2.28)–(2.31) in Region III may be written

$$\hat{E}_z^i = D^i R_0(\Gamma r, \Gamma d) + D'^i S_0(\Gamma r, \Gamma d), \quad (3.22)$$

$$\hat{H}_\phi^i = \frac{\epsilon_0 p}{\Gamma}[D^i R_1(\Gamma r, \Gamma d) + D'^i S_1(\Gamma r, \Gamma d)], \quad (3.23)$$

$$\hat{E}_r^i = (-)^{i+1}\frac{\alpha}{\Gamma}[D^i R_1(\Gamma r, \Gamma d) + D'^i S_1(\Gamma r, \Gamma d)]. \quad (3.24)$$

We are now in a position to apply continuity and boundary conditions. Continuity in r at $r = a$ determines the coefficients B_m and C_m of the Region II expansions in terms of the coefficients A_m of Region I. Using the Wronskian identity (3.21) and the definition (3.18), we can then write the Region II expansions as

$$\hat{E}_z^{II}(r, z) = \sum_{m=-\infty}^{\infty}[A_m I_0(\chi_m r) - \chi_m a\, S_0(\chi_m r, \chi_m a)F_m]e^{ik_m z}, \quad (3.25)$$

$$\hat{H}_\phi^{II}(r, z) = \epsilon_0 p \sum_{m=-\infty}^{\infty}[\chi_m^{-1}A_m I_1(\chi_m r) - a\, S_1(\chi_m r, \chi_m a)F_m]e^{ik_m z}. \quad (3.26)$$

The boundary condition (B.12) applied at the cylindrical cavity wall determines the primed coefficients of (3.22)–(3.24) in terms of the unprimed ones. After Fourier transformation in z the boundary condition at $r = d$ is

$$\hat{E}_{zs}^i(d) = -\left(\frac{\mu p}{\sigma}\right)_c^{1/2}\hat{H}_{\phi s}^i(d). \quad (3.27)$$

The subscript c (for "cavity") indicates that parameters μ, σ are those for the cylindrical cavity wall; we also write subscripts e and t ("ends" and "tube") for the cavity end walls and the cylindrical tube surface ($r = b$), respectively. By (3.22), (3.23) and identities (3.20), (3.21), we find

$$D'^i = -\frac{\eta_c}{\Gamma b}D^i, \quad (3.28)$$

where the dimensionless parameter η_c is defined as in Ref. 2

$$\eta_c = -\left(\frac{\mu p}{\sigma}\right)_c^{1/2}\epsilon_0 p b. \quad (3.29)$$

According to (3.28), the combination of Bessel functions that now appears in Region III fields is

$$\Phi^j(r) = R_j(\Gamma r, \Gamma d) - \frac{\eta_c}{\Gamma b}S_j(\Gamma r, \Gamma d), \quad (3.30)$$

and the field expansions take the form

$$\hat{E}_z^{III}(r, z) = \sum_{s=0}^{\infty} \Phi_s^0(r)[D_s^1 \cos \alpha_s z + D_s^2 \sin \alpha_s z], \tag{3.31}$$

$$\hat{H}_\phi^{III}(r, z) = \epsilon_0 p \sum_{s=0}^{\infty} \Gamma_s^{-1}\Phi_s^1(r)[D_s^1 \cos \alpha_s z + D_s^2 \sin \alpha_s z], \tag{3.32}$$

$$\hat{E}_r^{III}(r, z) = \sum_{s=0}^{\infty} \alpha_s\Gamma_s^{-1}\Phi_s^1(r)[D_s^1 \sin \alpha_s z - D_s^2 \cos \alpha_s z]. \tag{3.33}$$

The boundary condition at the cavity end walls ($z = \mp g$, $b < r < d$), *before* Fourier transformation in z, is

$$\hat{E}_r(r, \mp g) = \mp \left(\frac{\mu p}{\sigma}\right)_e^{1/2} \hat{H}_\phi(r, \mp g). \tag{3.34}$$

By choosing the non-harmonic wave numbers α_s appropriately, we may satisfy (3.34) term-by-term in the expansions (3.32), (3.33). It is sufficient that

$$[\alpha_s \sin \alpha_s g + (\eta_e/b) \cos \alpha_s g]D_s^1 + [\alpha_s \cos \alpha_s g - (\eta_e/b) \sin \alpha_s g]D_s^2 = 0, \tag{3.35}$$

$$[\alpha_s \sin \alpha_s g + (\eta_e/b) \cos \alpha_s g]D_s^1 - [\alpha_s \cos \alpha_s g - (\eta_e/b) \sin \alpha_s g]D_s^2 = 0. \tag{3.36}$$

There are two ways for these equations to be satisfied:

$$x_s \sin x_s + \kappa \cos x_s = 0,\ D_s^2 = 0,\ D_s^1 \text{ arbitrary}, \tag{3.37}$$

$$x_s \cos x_s - \kappa \sin x_s = 0,\ D_s^1 = 0,\ D_s^2 \text{ arbitary}, \tag{3.38}$$

where

$$x_s = \alpha_s g, \quad \kappa = \eta_e g/b = -\left(\frac{\mu p}{\sigma}\right)_e^{1/2} \epsilon_0 p\, g. \tag{3.39}$$

To find solutions of the nonlinear equations (3.37), (3.38), we take advantage of the circumstance that the dimensionless parameter κ is typically small compared with 1 for the values of p of interest. The values of p of interest, those involved in computing the rise times of unstable perturbations, are close to the points $p = \mp i\omega_r$, where $\omega_r = j_{01}c/d$ is the fundamental frequency of a cylindrical cavity of radius d; (see the heuristic argument of the introduction, or the calculations of Part II). Then the order of magnitude of $|\kappa|$ is

$$|\kappa| \sim \left(\frac{\mu_0\omega_r}{\sigma_e}\right)^{1/2} \epsilon_0\omega_r g = j_{01}^{3/2}(\sigma_e d Z_0)^{-1/2}(g/d), \tag{3.40}$$

where $Z_0 = (\mu_0/\epsilon_0)^{1/2} = (120\pi \text{ ohm})$ is the impedance of free space. For the conductivity of stainless steel, $\sigma_e \approx 10^6$ (ohm-meter)$^{-1}$, and $d \approx g \approx 1$ meter, we then have $|\kappa| \sim 2 \times 10^{-4}$. The parameters η_t, η_c, η_e are typically of a similar order. In the following, we shall expand various quantities in powers of κ or the η's and retain only the lowest powers, keeping in mind that the resulting approximations are not good at large $|p|$. Large values of $|p|$ enter the problem only if one wishes to compute the full time dependence of the fields, not just the asymptotic time dependence that is the sole concern of stability studies.

An expansion of the solution x_s of (3.37) or (3.38) in powers of κ gives

$$x_s = s\pi/2 - 2\kappa/s\pi - 8\kappa^2/(s\pi)^3 + \dots, \quad s = 1,2,\dots \tag{3.41}$$

where x_s is a solution of (3.37) for even s and of (3.38) for odd s. There is also a solution of (3.37) close to zero, which may be expanded in powers of $\kappa^{1/2}$ as

$$x_0 = -i\kappa^{1/2}(1 + \kappa/6 + \dots). \tag{3.42}$$

The negative of each of the solutions (3.41), (3.42) is also a solution, but by a redefinition of coefficients D_s^i one sees that it would be redundant to include a corresponding term in the expansions (3.31)–(3.33). There is also the extraneous solution $x_s = 0$ of (3.38), but it makes no contribution to the field expansions. The implicit-function theorem for analytic functions[24] may be invoked to show that the series (3.41) and (3.42) converge for sufficiently small $|\kappa|$. In addition, the solutions we have found are unique if $|\kappa| < \pi^2/16$, as may be seen by applying Rouché's theorem;[25] all the solutions are close to those of $\sin 2x = 0$.

The functions $\cos \alpha_s z$ and $\sin \alpha_s z$ form an orthogonal set on the interval $[-g, g]$. Accordingly, we define the orthonormal functions

$$f_s(z) = [1 - \kappa/(x_s^2 + \kappa^2)]^{-1/2} \times \begin{cases} (-)^{s/2} \cos \alpha_s z, & s \text{ even}, \\ (-)^{(s+1)/2} \sin \alpha_s z, & s \text{ odd}. \end{cases} \quad (3.43)$$

The peculiar sign factors are introduced so that f_s takes the convenient form $(1 + \delta_{s0})^{-1/2} \cos \alpha_s(z + g)$ at $\kappa = 0$, a form that is used in Appendix A. The f_s are orthonormal in the sense

$$\frac{1}{g} \int_{-g}^{g} f_s(z) f_t(z) dz = \delta_{st}. \quad (3.44)$$

The result (3.44) follows from the definition of $x_s = \alpha_s g$ as a solution of (3.37) (s even) or (3.38) (s odd).

The set $\{f_s(z)\}$ is complete as well as orthogonal. For $|\kappa| < \pi^2/16$ the wave numbers α_s give rise to a Riesz basis for the space $L^2[-g, g]$ of square-integrable functions, and the f_s form a complete set in that space.[26] Completeness in the space of continuous functions $C(-g, g)$ may be established by considering, for $u, v \in (-g, g)$, the integral

$$\int_\gamma d\lambda \left[\frac{\lambda \sin(\lambda u/g) \sin(\lambda v/g)}{\sin \lambda(\lambda \cos \lambda - \kappa \sin \lambda)} + \frac{\kappa \cos(\lambda u/g) \cos(\lambda v/g)}{\sin \lambda\, (\lambda \sin \lambda + \kappa \cos \lambda)} \right] = 0. \quad (3.45)$$

Here γ is any closed contour such that the zeros of the denominators all lie outside γ. By deforming the contour to infinity and noting that the integrand vanishes exponentially as $\mathrm{Im}\, \lambda \to \pm\infty$, we deduce from the residue theorem that

$$\sum_{s=0}^{\infty} f_s(u) f_s(v) = \frac{1}{2} + \sum_{s=1}^{\infty} [\cos(s\pi u/g) \cos(s\pi v/g) + \sin(s\pi u/g) \sin(s\pi v/g)] = g\delta(u - v). \quad (3.46)$$

Of course, this formal statement of completeness is to be interpreted through term-by-term integration of the product of (3.46) with a continuous function.

The use of nonharmonic Fourier series has a long history.[26–29] The functions $f_s(z)$ are well-known in the theory of heat conduction.[27,28]

Redefining the Fourier coefficients, we write (3.31), (3.32) as

$$\hat{E}_z^{III}(r, z) = \sum_{s=0}^{\infty} D_s \Phi_s^0(r) f_s(z), \quad (3.47)$$

$$\hat{H}_\phi^{III}(r, z) = \epsilon_0 p \sum_{s=0}^{\infty} \Gamma_s^{-1} D_s \Phi_s^1(r) f_s(z). \quad (3.48)$$

The remaining continuity and boundary conditions to be satisfied are at $r = b$:

$$\hat{E}_z^{II}(b, z) = \hat{E}_z^{III}(b, z), \qquad -g < z < g, \quad (3.49)$$

$$\hat{E}_z^{II}(b, z) = -\left(\frac{\mu p}{\sigma}\right)_t^{1/2} \hat{H}_\phi^{II}(b, z), \qquad g < z < 2\pi R - g, \quad (3.50)$$

$$\hat{H}_\phi^{II}(b, z) = \hat{H}_\phi^{III}(b, z), \qquad -g < z < g. \quad (3.51)$$

To state these conditions in terms of Fourier coefficients, we use the orthogonality (3.44), as well as the additional overlap integrals,

$$\frac{1}{2\pi R} \int_{-g}^{-g+2\pi R} \exp[i(k_n - k_m)z] dz = \delta_{nm} \quad (3.52)$$

$$\frac{1}{2\pi R} \int_{-g}^{g} \exp[-ik_n z] f_s(z) dz = \alpha f_s(g) N_{ns}, \quad (3.53)$$

$$\frac{1}{2\pi R} \int_{g}^{-g+2\pi R} \exp[i(k_n - k_m)z] dz = \delta_{nm} - \alpha V_{nm}. \quad (3.54)$$

The quantities appearing here are

$$N_{ns} = \frac{1}{2i}\frac{(-)^s}{(k_n g)^2 - x_s^2}[(k_n g + i\kappa) \times \exp(ik_n g) - (-)^s(k_n g - i\kappa)\exp(-ik_n g)], \tag{3.55}$$

$$\alpha = \frac{g}{\pi R}, \quad V_{nm} = \frac{\sin(k_n - k_m)g}{(k_n - k_m)g}. \tag{3.56}$$

By (3.37), (3.38) it is possible to write $f_s(g)$ as

$$f_s(g) = (-)^s x_s(x_s^2 - \kappa + \kappa^2)^{-1/2}. \tag{3.57}$$

Now we multiply (3.50) and (3.51) by exp $(-ik_n z)$ and integrate over $[-g, -g + 2\pi R]$ and multiply (3.49) by $f_s(z)$ and integrate over $[-g, g]$. Noting the expansions (3.25), (3.26), (3.47), (3.48) and the overlap integrals (3.44), (3.52), (3.53), (3.54), we obtain two sets of linear equations for the A_m and D_s. Adopting a notation similar to that of Ref. 2, we state the equations in terms of the functions

$$\bar{A}_n = A_n I_0(\chi_n b), \quad \bar{D}_s = D_s \Phi_s^0(b) f_s(g), \tag{3.58}$$

$$\bar{B}_n = -\chi_n a S_0(\chi_n b, \chi_n a)F_n,$$
$$\bar{C}_n = -(a/b)S_1(\chi_n b, \chi_n a)F_n, \tag{3.59}$$

$$R_s = 2f_s^2(g)\Phi_s^0(b)\Gamma_s b/\Phi_s^1(b),$$
$$I_n = I_1(\chi_n b)/\chi_n b I_0(\chi_n b). \tag{3.60}$$

The equations, analogous to Eqs. (1.23) of Ref. 2, are

$$\bar{A}_m = -\bar{B}_m + \alpha\sum_{s=0}^{\infty} N_{ms}\bar{D}_s + \eta_t \sum_{n=-\infty}^{\infty} (\delta_{mn} - \alpha V_{mn})(I_n\bar{A}_n + C_n) \tag{3.61}$$

$$\bar{D}_s = R_s \sum_{m=-\infty}^{\infty} N_{-ms}(I_m\bar{A}_m + \bar{C}_m). \tag{3.62}$$

To solve (3.61), (3.62), a convenient first step is to eliminate $\bar{A}_n$ in favor of $\bar{D}_s$, or vice versa. To that end we use matrix notation, and define a matrix $\tilde{N}$ with elements

$$\tilde{N}_{sn} = N_{-ns}. \tag{3.63}$$

If the cavity end walls have infinite conductivity ($\kappa = 0$), $\tilde{N}$ is equal to $N\dagger$, the Hermitian adjoint of N. We shall prove that $V = \{V_{mn}\}$ has the representation

$$V = Nv\tilde{N}, \tag{3.64}$$

$$v = \{v_{st}\} = \{2f_s^2(g)\delta_{st}\}. \tag{3.65}$$

Defining diagonal matrices $R = \{R_s\delta_{st}\}$, $I = \{I_m\delta_{mn}\}$, we may then write Eqs. (3.61), (3.62) in matrix notation as

$$\bar{A} = -\bar{B} + \alpha N\bar{D} + \eta_t(1 - \alpha Nv\tilde{N})(I\bar{A} + \bar{C}), \tag{3.66}$$

$$\bar{D} = R\tilde{N}(I\bar{A} + \bar{C}). \tag{3.67}$$

The unit matrix is always written as a numeral 1.

To eliminate $\bar{A}$, we first rearrange (3.66): we bring the term $\eta_t I\bar{A}$ to the left side, and then multiply the equation on the left by $\tilde{N}I(1 - \eta_t I)^{-1}$. The equation then involves $\bar{A}$ only in the product $\tilde{N}I\bar{A}$, which may be expressed in terms of $\bar{D}$ by (3.67). The resulting equation for $\bar{D}$ alone takes the form

$$\bar{D} = RE\bar{D} + R\tilde{N}(1 - \eta_t I)^{-1}\bar{Y}, \tag{3.68}$$

where the kernel matrix E is

$$E = \alpha\tilde{N}I(1 - \eta_t I)^{-1}N(1 - \eta_t vR^{-1}). \tag{3.69}$$

The source term of (3.68), linear in the charge-current vector $F = \{F_n\}$, entails the vector

$$\bar{Y} = \bar{C} - I\bar{B}. \tag{3.70}$$

By (3.18), (3.21), (3.59), and (3.60), the components of $\bar{Y}$ may be written as

$$\bar{Y}_n = -\frac{1}{\chi_n b}\frac{a}{b}\frac{I_1(\chi_n a)}{I_0(\chi_n b)}F_n. \tag{3.71}$$

Elimination of $\bar{D}$ in favor of $\bar{A}$ leads from (3.66), (3.67) to the equation

$$\bar{A} = GI\bar{A} + G\bar{C} - \bar{B}, \tag{3.72}$$

with kernel matrix

$$G = \eta_t + \alpha N(R - \eta_t v)\tilde{N}. \tag{3.73}$$

Another useful equation is that which expresses $\bar{A}$ in terms of $\bar{D}$. In (3.66) we take $\eta_t I\bar{A}$ to the left side, and then multiply on the left by

$(1 - \eta_t I)^{-1}$. We express $\tilde{N}I\bar{A}$ in terms of $\bar{D}$ by (3.67), and then eliminate $\bar{C}$ in favor of $\bar{Y}$ by (3.70). The result is

$$\bar{A} = \alpha(1 - \eta_t I)^{-1}N(1 - \eta_t vR^{-1})\bar{D} + \eta_t(1 - \eta_t I)^{-1}\bar{Y} - \bar{B}. \quad (3.74)$$

The companion equation giving $\bar{D}$ in terms of $\bar{A}$ is (3.67).

To demonstrate (3.64), we expand $\exp(ik_n z)$ in the orthonormal set $\{f_t(z)\}$ or $[-g, g]$ to obtain

$$\exp(ik_n z) = 2\sum_{t=0}^{\infty} f_t(g)N_{-nt}f_t(z). \quad (3.75)$$

Now we multiply by $\exp(-ik_m z)$ and integrate over $[-g, g]$. The resulting equation is exactly (3.64).

For a given charge-current distribution, specified through the function F_n of (3.8), the electromagnetic fields may be determined either by solving (3.68) for $\bar{D}$ or by solving (3.72) for $\bar{A}$. For deep cavities, $d \gg b$, the Region III field coefficients $\bar{D}_s$ are close to being the normal mode amplitudes of the system, and (3.68) is the appropriate equation. The kernel E is nearly diagonal when $d \gg b$, so that the various cavity modes almost decouple, and the equation is easy to solve numerically. In the case of shallow cavities, $b \approx d$, the $\bar{A}_n$ are approximately normal-mode amplitudes, and Eq. (3.72) is the more tractable one. In some cases of interest, the range of d/b may be of the order $2 < d/b < 5$; then (3.68) is strongly preferred.

A remark on the relation of (3.68) to the equation employed by Keil and Zotter[2] may be helpful. Their equation (3.4) is most easily compared with our (3.68) in the case where all wall conductivities are infinite. The unknown of their equation, $\bar{X}'$, is then simply related to $\bar{D}$; in fact, $\bar{X}' = \bar{D} - RN\dagger\bar{Y}$. The equations for $\bar{X}'$ and $\bar{D}$ consequently have the same kernel, but different source terms;

$$\bar{X} = \alpha RN\dagger IN\bar{X} + \alpha RN\dagger INRN\dagger\bar{Y}. \quad (3.76)$$

$$\bar{D} = \alpha RN\dagger IN\bar{D} + RN\dagger\bar{Y}. \quad (3.77)$$

With finite conductivity, the situation is essentially the same. The equation of $\bar{X}'$ has mathematical properties similar to that for $\bar{D}$, but has a more complicated form. We prefer the $\bar{D}$ equation for its simplicity and its more direct physical interpretation as the equation for cavity modes.

4. EQUATIONS FOR FIELD COEFFICIENTS WITH VLASOV SELF-CONSISTENCY

To compute rise times of unstable perturbations the Vlasov equation (2.22) must be combined with the electromagnetic equation, either (3.68) or (3.72). The charge-current density, expressed through F_n of (3.8), is to be eliminated in favor of field coefficients. By (3.9) and (3.58),

$$\hat{E}^I_{zn}(r)\big|_{r=0} = A_n + F_n = \bar{A}_n/I_0(\chi_n b) + F_n. \quad (4.1)$$

Following the viewpoint of Section 3 we put the initial-value term $f_n(p_z, 0)$ equal to zero, so that (4.1) and (2.22) give

$$\hat{f}_n(p_z, p) = -q f_0'(p_z)(p + ik_n v)^{-1} \times [A_n(p) + F_n(p) + E_{zon}/p]. \quad (4.2)$$

By the definition (3.8) of F_n and (4.2) we can integrate on p_z to get

$$F_n(p) = q^2\mu_0 \int_{-\infty}^{\infty} dp_z \frac{f_0'(p_z)}{\chi_n^2}\frac{pv + ik_n c^2}{p + ik_n v} \times [A_n(p) + F_n(p) + E_{zon}/p], \quad (4.3)$$

where $v = [M^2 + (p_z/c)^2]^{-1/2}p_z$. The integral may be cast into a form familiar in plasma theory. For $n \neq 0$ the definition (3.2) of χ_n^2 gives

$$\frac{pv + ik_n c^2}{p + ik_n v} = \frac{p}{ik_n} - \frac{c^2\chi_n^2}{ik_n}\frac{1}{p + ik_n v}. \quad (4.4)$$

The first term on the right integrates to zero (since $f_0(\pm\infty) = 0$), and the factor in front of the bracket in (4.3) may be written as

$$\frac{-q^2}{\epsilon_0}\frac{1}{ik_n}\int_{-\infty}^{\infty}\frac{dp_z f_0'(p_z)}{p + ik_n v} = -\frac{q^2}{\epsilon_0 M}\int_{-\infty}^{\infty}\frac{dp_z f_0(p_z)}{(p + ik_n v)^2\gamma^3(v)} \quad (4.5)$$

The latter way of writing the factor is also correct for $n = 0$. Recall that by (2.11),

$$\int_{-\infty}^{\infty} dp_z f_0(p_z) = \mathcal{N}, \quad (4.6)$$

where $\mathcal{N}$ is the average particle density. Let us define a plasma dispersion function,

$$W_n(p) = \frac{\omega_p{}^2}{\mathcal{N}} \int_{-\infty}^{\infty} \frac{dp_z f_0(p_z)}{(p + ik_n v)^2 \gamma^3(v)}$$
$$= \frac{\omega_p{}^2 M}{\mathcal{N}} \begin{cases} \dfrac{1}{ik_n} \displaystyle\int_{-\infty}^{\infty} \frac{dp_z f_0'(p_z)}{p + ik_n v}, & n \neq 0, \\ -\dfrac{1}{p^2} \displaystyle\int_{-\infty}^{\infty} dp_z f_0'(p_z) v, & n = 0. \end{cases} \tag{4.7}$$

where $\omega_p{}^2$ is the squared plasma frequency,

$$\omega_p{}^2 = \frac{q^2 \mathcal{N}}{\epsilon_0 M}. \tag{4.8}$$

Then (4.3) may be written as

$$F_n = -W_n[\hat{E}_{zn} + E_{zon}/p], \tag{4.9}$$

where $\hat{E}_{zn}$ represents the left side of (4.1). Equivalently,

$$F_n = \frac{-W_n}{1 + W_n} \left[\frac{\bar{A}_n}{I_0(\chi_n b)} + \frac{E_{zon}}{p} \right]. \tag{4.10}$$

According to (3.72), (3.59), and (4.1), the coefficients of the electric field on the axis, $\hat{E}_{zm}$, are homogeneous linear functions of the F_n, which we write as

$$\hat{E}_{zm} = \sum_{n=-\infty}^{\infty} \zeta_{mn} F_n. \tag{4.11}$$

In view of (4.9) we then have an equation for the $\hat{E}_{zn}$ alone,

$$\hat{E}_{zm}(p) + \sum_{n=-\infty}^{\infty} \zeta_{mn}(p) W_n(p) \hat{E}_{zn}(p) = E_{zm}^{(0)}(p) \tag{4.12}$$

where

$$E_{zm}^{(0)}(p) = -\frac{1}{p} \sum_{n=-\infty}^{\infty} \zeta_{mn}(p) W_n(p) E_{zon}. \tag{4.13}$$

If initial-value terms were included, the right side of (4.12) would have additional terms depending on initial values $[f(0), E_z(0), H_\phi(0)]$, but the left side would be exactly the same; see Appendix A.

The rise time τ of an unstable perturbation is to be read off from the inverse Laplace transform,

$$E_{zn}(t) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{pt} \hat{E}_{zn}(p)\, dp. \tag{4.14}$$

The integration contour in (4.14) may be moved to the left by decreasing γ until $\gamma = \mathrm{Re} p_*$, where p_* is the location of the rightmost singularity of $\hat{E}_{zn}(p)$ in the complex p-plane. Generically, two complex-conjugate simple poles in the right half p-plane are the rightmost singularities. If those poles are at $p_* = u \pm iv$, then a translation of the contour beyond the poles gives the asymptotic form in terms of pole residues,

$$E_{zn}(t) \sim e^{(u+iv)t} \operatorname{res}(\hat{E}_{zn})_{u+iv} + e^{(u-iv)t} \operatorname{res}(\hat{E}_{zn})_{u-iv}, \quad t \to \infty \tag{4.15}$$

and the rise time is defined as $\tau = 1/u$.

Since $W_n(p)$ is analytic in the right half plane, singularities of $\hat{E}_{zn}(p)$ as determined by (4.12) could come only from $\zeta_{nm}(p)$ or from zeros of the determinant of the system,

$$\det(1 + \zeta(p) W(p)) = 0. \tag{4.16}$$

We shall argue that ζ is in fact analytic in the right half plane, so that zeros of the determinant fix the rise time; see the discussion of ζ following (4.27) below.

The determinant simplifies greatly for p close to a relevant zero, so that locating the zero is not a difficult task. Following formula (4.19) below, we write $\zeta = \theta + \tilde{\zeta}$, where $\theta = \{\theta_n \delta_{mn}\}$ is a diagonal matrix. Then (4.16) is equivalent to

$$\det(1 + \tilde{\zeta}(p)[W^{-1}(p) + \theta(p)]^{-1}) = 0. \tag{4.17}$$

As the work of Part II will show, the function $W_n{}^{-1} + \theta_n$ has a pair of poles very close to the imaginary axis in the p-plane. For typical accelerator parameters, this function is extremely small unless p is close to one of those poles; consequently, the relevant zero of the determinant (4.17) is close to a pole of $W_n{}^{-1} + \theta_n$ for just one value of n. That value is approximately given by $\omega_r = n\Omega$, in accordance with the heuristic argument of the Introduction. For locating the zero, only the n-th column of the matrix $\tilde{\zeta}[W^{-1} + \theta]^{-1}$ is appreciable. Thus, evaluation of the determinant is a trivial step, and to an excellent approximation (4.17) becomes

$$1 + \zeta_{nn}(p)W_n(p) = 0. \qquad (4.18)$$

This is the familiar dispersion relation of plasma theory, and ζ_{nn} is a dimensionless form of the usual coupling impedance of accelerator theory. A more quantitative discussion of the approximation yielding (4.18) is given in Part II.

There is one anomalous case in which (4.18) is not correct. For special values of accelerator parameters, a pole of $W_n^{-1} + \theta_n$ may coincide with a pole of $W_m^{-1} + \theta_m$, for some m close to n, with the coinciding poles having residues nearly equal in magnitude and opposite in sign. Then two columns of $\tilde{\zeta}[W^{-1} + \theta]^{-1}$ must be taken into account for evaluation of (4.17). If this situation holds (or holds approximately), the beam has anomalous stability, as will be explained in Part II.

For general p, the single impedance function $\zeta_{nn}(p)$ does not fully determine the response of the axial field to an arbitrary charge-current perturbation. Owing to coupling of the various tube Fourier modes induced by the cavity, the entire matrix $\{\zeta_{mn}\}$ (equivalent to the response function of plasma theory) is required to find the full effect of a perturbation. Nevertheless, only the element ζ_{nn} has quantitative importance in the stability question, with the one exception noted above.

To derive the relation between ζ_{nn} and the coupling impedance Z_n we recall the definition of the latter,

$$-2\pi R\hat{E}_{zn} = Z_n\hat{I}_n. \qquad (4.19)$$

Here $\hat{I}_n$ is the n-th Fourier coefficient of a perturbation in beam current, and $\hat{E}_{zn}$ the coefficient of the resulting axial field perturbation. The coefficients are for Fourier rather than Laplace transforms in time, so that we must put $p = -i\omega$ and drop initial-value terms to derive (4.19). By (4.11), ζ_{nn} may be calculated as the numerical value of $\hat{E}_{zn}$ when $F_m = \delta_{mn}$ (in some definite system of units). Thus

$$-2\pi R\zeta_{nn} = Z_n[\hat{I}]_{F_m=\delta_{mn}}. \qquad (4.20)$$

The relation of $\hat{I}_n$ to F_n is obtained from the definition (3.8) and the continuity equation (2.14):

$$F_n = \frac{i}{\chi_n^2}\left(\mu_0\omega\hat{J}_n - \frac{k_n}{\epsilon_0}\hat{\rho}_n\right) = \frac{1}{i\pi a^2\omega\epsilon_0}\hat{I}_n. \qquad (4.21)$$

Putting $F_n = 1$ we get the desired relation

$$Z_n = \frac{2iR}{\epsilon_0\omega a^2}\zeta_{nn}. \qquad (4.22)$$

Since $\omega = \beta ck_n$ approximates the frequency determining stability, the value of Z_n needed in the dispersion relation may be written as

$$\frac{Z_n}{n} = \frac{2iZ_0}{(k_na)^2\beta}\zeta_{nn}, \qquad (4.23)$$

where $Z_0 = (\mu_0/\epsilon_0)^{1/2}$ is the impedance of free space.

There are two ways to compute ζ_{mn}. For shallow cavities, the proper way is to solve (3.72) for $\bar{A}_m$ with $F_m = \delta_{nm}$, and then obtain $\hat{E}_{zm}$ from (4.1). For deep cavities, one should solve (3.68) for $\bar{D}_s$, then get $\bar{A}_m$ from (3.74) and $\hat{E}_{zm}$ from (4.1). The latter approach is comparable to the method of Ref. 2, and gives the expression for ζ

$$\begin{aligned}\zeta_{mn} &= \theta_n\delta_{mn} \\ &\quad - \alpha(a/b)^2[N(1 - \eta_t vR^{-1})(1 - RE)^{-1}R\tilde{N}]_{mn} \\ &\quad \times \frac{I_1(\chi_n a)}{I_0(\chi_m b)(1 - \eta_t I_m)\chi_n a I_0(\chi_n b)(1 - \eta_t I_n)},\end{aligned} \qquad (4.24)$$

where

$$\theta_n = 1 + \frac{\chi_n a}{I_0(\chi_n b)}S_0(\chi_n b, \chi_n a) - \left(\frac{a}{b}\right)^2 \frac{\eta_t}{1 - \eta_t I_n}\frac{I_1(\chi_n a)}{\chi_n a I_0^2(\chi_n b)}. \qquad (4.25)$$

The formula obtained by solving (3.72) is

$$\begin{aligned}\zeta_{mn} &= \delta_{mn} + \sum_{l=-\infty}^{\infty}[1 - H]^{-1}_{ml} \\ &\quad \times [I_0(\chi_l b) - \eta_t I_1(\chi_l b)/\chi_l b]^{-1} \\ &\quad \times \{\delta_{ln}[\chi_n a S_0(\chi_n b, \chi_n a) - \eta_t(a/b)S_1(\chi_n b, \chi_n a)] \\ &\quad - \alpha(a/b)[N(R - \eta_t v)\tilde{N}]_{ln}S_1(\chi_n b, \chi_n a)\},\end{aligned} \qquad (4.26)$$

where the matrix H is defined by

$$H_{nm} = \alpha[I_0(\chi_n b) - \eta_t I_1(\chi_n b)/\chi_n b]^{-1} \times [N(R - \eta_t v)\tilde{N}]_{nm} I_1(\chi_m b)/\chi_m b. \quad (4.27)$$

As mentioned above, we wish to show that $\zeta(p)$ has no singularity in the right half p-plane. Since ζ involves the solution of an infinite-dimensional equation, not known in explicit form, it is not easy to make a direct analytical demonstration. We can give a persuasive physical argument, however, on the basis of the definition of ζ in (4.11). If the charge and current perturbations were given time-independent functions, rather than being determined by Vlasov dynamics, then the field perturbation would also have to be time independent. Consequently the Laplace transform of the field would have no singularity in the right half p-plane, but would have a singularity on the imaginary axis. The Laplace transform of a constant function is proportional to $1/p$, so that by (3.8) the $F_n(p)$ for a time-independent charge-current distribution is analytic except for poles at $p = 0$ and $p = \pm ick_n$ (the latter from the factor χ_n^{-2}). Since $\hat{E}_{zn}(p)$ must have no singularity in the right half-plane for such an F_n, we infer that $\zeta_{nm}(p)$ must also have none; (take $\hat{f}_n(p_z, p) = \delta_{nm} f_m(p_z)/p$, to see that each element of ζ is free of singularities).

The above discussion emphasizes the electric field on the tube axis, and represents the conventional viewpoint of accelerator theory. It is more natural, however, to emphasize the field in the cavity region, if the cavities are fairly deep and resonant. The axial field and the distribution function may be eliminated in favor of the cavity mode coefficients $\bar{D}$. The rise time of an instability may be found directly from the inverse Laplace transform of $\bar{D}(p)$, and there is no need to consider the axial field. The self-consistent equation for $\bar{D}$ has an appealing form, and is easier to analyze in a precise way than the scheme described above.

To derive the equation, we take $\bar{A}_n$ from (3.74) and substitute in (4.10). We solve the resulting equation for F_n in terms of $\bar{D}_s$ to obtain

$$F_n = \frac{-W_n}{1 + \theta_n W_n}\left[\frac{\alpha}{(1 - \eta_t I_n) I_0(\chi_n b)} \times [N(1 - \eta_t v R^{-1})\bar{D}]_n + \frac{E_{zon}}{p}\right]. \quad (4.28)$$

Now we introduce this result for F_n in the source term of (3.68), using (3.71), to obtain the desired equation for $\bar{D}$ with Vlasov self-consistency,

$$\bar{D}_s = R_s\left[\sum_{t=0}^{\infty} (E_{st} + S_{st})\bar{D}_t + \bar{D}_s^{(0)}\right] \quad (4.29)$$

The letters E and S denote "electromagnetic" and "self-consistency" parts of the kernel. That is, E is the same kernel (3.69) that occurs in our previous equation (3.68) with given source term, and S is the new piece that arises from expressing the source in terms of the field itself. We have

$$E_{su} + S_{su} = \alpha \sum_{m=-\infty}^{\infty} \tilde{N}_{sm}\left[\frac{I_m}{1 - \eta_t I_m} + \frac{1}{(1 - \eta_t I_m)^2}\left(\frac{a}{b}\right)^2 \frac{I_1(\chi_m a)}{\chi_m a I_0^2(\chi_m b)} \times \frac{W_m}{1 + \theta_m W_m}\right] N_{mu}\cdot[1 - \eta_t v_u R_u^{-1}] \quad (4.30)$$

Under present assumptions, the inhomogeneous term is

$$\bar{D}_s^{(0)} = \frac{1}{p}\left(\frac{a}{b}\right)^2 \sum_{m=-\infty}^{\infty} \tilde{N}_{sm} \times \frac{1}{1 - \eta_t I_m} \frac{I_1(\chi_m a)}{\chi_m a I_0(\chi_m b)} \frac{W_m E_{zom}}{1 + \theta_m W_m}. \quad (4.31)$$

More generally, $\bar{D}^{(0)}$ contains various initial-value terms; see Appendix A.

Rise times of unstable perturbations could be obtained from zeros of the determinant of (4.29):

$$\det[1 - R(p)(E(p) + S(p))] = 0. \quad (4.32)$$

By solving (4.32) one effectively solves the electromagnetic problem and the dispersion relation simultaneously, without the intermediate step of computing an impedance. In Part II we shall find it better not to work with (4.32) as it stands. We eliminate the resonant mode from (4.29), and consider the determinant of the reduced equation.

5. WATSON-SOMMERFELD TRANSFORMATION

A difficulty arises in the practical computation of the sum in (4.30) that defines the kernel E. The factor $\tilde{N}_{sm}$ or N_{ms} is maximum as a function of m when its denominator is minimum, $k_m g = mg/$

$R \approx \pm x_s$. In typical cases of interest, this occurs at a value of $m = m_*$ that is large from the view point of practical computation, even for the first few values of s. Values of m far beyond m_* must be included for an accurate summation of the series, and the situation gets worse as s increases.

A Watson-Sommerfeld transformation[29,30] replaces the difference of squares in the denominators by a sum of squares, and thereby circumvents the difficulty. Furthermore, the transformation eliminates the Bessel functions in favor of easily computed Bessel-function zeros, reveals the behavior of the sum for $b/d \to 0$, and facilitates the treatment of Eq. (4.29) by Fredholm theory. There is no reason to make a corresponding transformation of the sum defining the self-consistency kernel S. As we show in Part II, only one or two terms of this sum are important (those for which $1 + \theta_m W_m \approx 0$), and in any case the sum converges exponentially.

Let us define

$$f(m, s, t) = \tilde{N}_{sm} N_{mt} = N_{-ms} N_{mt}. \quad (5.1)$$

The sum that occurs in the kernel E is

$$\Sigma_{st} = \sum_{m=-\infty}^{\infty} f(m, s, t) I_m (1 - \eta_t I_m)^{-1}. \quad (5.2)$$

For $s - t$ odd, Σ_{st} vanishes: then f is odd in m, while I_m as defined in (3.60) is even. Henceforth taking $s - t$ to be even we write

$$f = f^{(+)} + f^{(-)}, \quad (5.3)$$

$$f^{(\pm)} = \frac{1}{4} \frac{1}{(k_m g)^2 - x_s^2} \frac{1}{(k_m g)^2 - x_t^2} [(k_m g)^2 + \kappa^2 + (-)^{s+1} \times (k_m g \mp i\kappa)^2 e^{\mp 2ik_m g}] \quad (5.4)$$

The numerator of (5.4) vanishes at $k_m g = \pm x$, where x satisfies (3.37) for s even and (3.38) for s odd. Thus $f^{(\pm)}$ is bounded at $k_m g = \pm x_s, \pm x_t$ provided that $s \neq t$. If $s = t$, there is a pole with residue given by

$$\lim_{k_m g \to \pm x_s} [(m \mp x_s R/g) f^{(+)}(m, s, s)] = -\lim_{k_m g \to \pm x_s} [(m \mp x_s R/g) f^{(-)}(m, s, s)] = \frac{iR}{g} \frac{x_s^2 - \kappa + \kappa^2}{8x_s^2}. \quad (5.5)$$

Except for the poles that occur when $s = t$, $f^{(\pm)}$ has no singularities in the finite m plane and for large $|m|$ has the bound

$$|f^{(\pm)}(m)| \leq \frac{c}{|m|^2} [1 + e^{\pm(2g/R)\text{Im}(m)}], \quad (5.6)$$

where c is a positive constant.

To convert the sum to an integral we employ the functions

$$\phi^{(\pm)}(m) = J(m) f^{(\pm)}(m) e^{\pm i\pi m} / \sin \pi m, \quad (5.7)$$

where

$$J(m) = \frac{I_m}{1 - \eta_t I_m} = \frac{I_1(\chi_m b)}{\chi_m b I_0(\chi_m b) - \eta_t I_1(\chi_m b)}. \quad (5.8)$$

Now $\phi^{(\pm)}$ has poles at the integers with residue

$$\lim_{m \to n} (m - n)\phi^{(\pm)}(m) = \frac{1}{\pi} J(n) f^{(\pm)}(n), \quad (5.9)$$

and poles that arise from zeros of the denominator of J. Since η_t is small compared to one, the latter are close to the zeros of $I_0(\chi_m b)$, which is to say near the points at which $\chi_m b = \pm i j_{0i}$, where j_{0i} is the i-th zero of the ordinary Bessel function J_0. With such points as the first approximation, Newton's method locates the poles of $J(m)$ at the points $m = \pm m_i$, where

$$m_i = iR \left[\left(\frac{p}{c}\right)^2 + \left(\frac{j_{0i}}{b}\right)^2 \times \left(1 - \frac{2\eta_t}{j_{0i}^2}\right) \right]^{1/2}, \quad (5.10)$$

$$i = 1, 2, \ldots$$

Inside the square root the exact expression has been expanded to lowest order in η_t. The poles at $m = \pm m_i$ have residue

$$\lim_{m \to \pm m_i} (m \mp m_i) J(m) = \frac{\pm 1}{m_i} \left(1 + \frac{2\eta_t}{j_{0i}^2}\right) \left(\frac{R}{b}\right)^2, \quad (5.11)$$

to lowest order in η_t. Except for these poles,

$J(m)$ is analytic in the finite m-plane. The branch point of χ_m does not appear in $J(m)$, because the entire functions $I_0(z)$ and $I_1(z)/z$ contain only even powers of z.

In view of (5.6) and the fact that I_0 and I_1 have the same asymptotic behavior, the functions $\phi^{(\pm)}$ are bounded as follows at large $|m|$:

$$|\phi^{(\pm)}(m)| \leq \frac{c}{|m|^3}. \tag{5.12}$$

In fact, each of the functions $\phi^{(\pm)}$ decreases cubically in one half-plane, and exponentially in the other.

We shall integrate over a path C consisting of a rectangle with corners $-A \pm iB$, $A \pm iB$, where A lies between two positive integers, $M < A < M + 1$. We first take $s \neq t$, and choose B and p so that $J(m)$ has no poles inside C. Then by (5.3) and (5.9),

$$\int_C [\phi^{(+)}(m) + \phi^{(-)}(m)]dm = 2i \sum_{m=-M}^{M} J(m)f(m), \tag{5.13}$$

since the only singularities of the integrand inside C are poles at integers. Taking a sequence of paths C with increasing A we obtain the required sum (5.2) as an integral,

$$\Sigma_{st} = \frac{1}{2i}\left[\int_{-\infty-iB}^{\infty-iB} + \int_{\infty+iB}^{-\infty+iB}\right] \times [\phi^{(+)}(m) + \phi^{(-)}(m)]dm. \tag{5.14}$$

Now the integral on $\mathrm{Im}(m) = \pm B$ may be replaced by an integral over an infinite semi-circle in the upper (lower) half-plane, plus a contribution from the poles at the points $\pm m_i$. According to (5.12) the integrals on the semi-circles vanish, and the formula (5.11) for pole residues gives

$$\Sigma_{st} = \pi\left(\frac{R}{b}\right)^2 \sum_{i=1}^{\infty} \frac{1}{m_i}\left(1 + \frac{2\eta_t}{j_{0i}^2}\right) \times \left[\frac{1}{\sin(-\pi m_i)}[f^{(+)}(-m_i) + f^{(-)}(-m_i)] - \frac{1}{\sin\pi m_i}[f^{(+)}(m_i) + f^{(-)}(m_i)]\right] \tag{5.15}$$

We have chosen the branch of the square root in (5.10) so that m_i is in the upper half-plane. Since $f^{(+)}(m) = f^{(-)}(-m)$, the poles in the upper and lower half-planes give equal contributions. For $s = t$ there is an additional term in Σ_{st} from the poles at $m = \pm x_s R/g$ with residue (5.5). These poles are close to the real axis, and give the following addition to Σ_{st}

$$\frac{1}{2\alpha}\left[1 - \frac{\kappa(1-\kappa)}{x_s^2}\right] J(x_s R/g)\delta_{st}. \tag{5.16}$$

Since the m_i are near the imaginary axis for the values of p of interest, it is convenient to state the final form of Σ_{st} in terms of nearly real numbers μ_i defined by

$$m_i = i\mu_i. \tag{5.17}$$

Then (5.4), (5.15), and (5.16) give

$$\begin{aligned}\alpha\,\Sigma_{st} &= \alpha \sum_{m=-\infty}^{\infty} \tilde{N}_{sm} \frac{I_m}{1 - \eta_t I_m} N_{mt} \\ &= -\pi\alpha\left(\frac{g}{b}\right)^2 \sum_{i=1}^{\infty} \mu_i(1 + 2\eta_t/j_{0i}^2) \\ &\times \frac{1}{(\pi\alpha\mu_i)^2 + x_s^2} \cdot \frac{1}{(\pi\alpha\mu_i)^2 + x_t^2} \\ &\times \frac{1}{\sinh \pi\mu_i}[\cosh \pi\mu_i + (-)^{s+1} \\ &\times \cosh[\pi\mu_i(1 - 2\alpha)] \\ &+ 2(-)^{s+1}\kappa(\pi\alpha\mu_i)^{-1} \\ &\times \sinh[\pi\mu_i(1 - 2\alpha)]] + \Delta_s\delta_{st},\end{aligned} \tag{5.18}$$

$$\Delta_s = \frac{1}{2f_s^2(g)} \frac{I_1(\Gamma_s b)}{\Gamma_s b I_0(\Gamma_s b) - \eta_t I_1(\Gamma_s b)} \tag{5.19}$$

We have neglected κ^2 in comparison with $(\pi\alpha\mu_i)^2$ in the coefficients of the hyperbolic cosines. The notation of (5.18) is defined in the following equations: μ_i in (5.17), (5.10), x_s in (3.41), (3.42), η_t as in (3.29), κ in (3.39), α in (3.56), Γ_s in (3.16), $f_s(g)$ in (3.57). Note that α in (3.16) stands for α_s of (3.39).

Since μ_i and x_s are approximately real for the values of p of interest, the denominator $(\pi\alpha\mu_i)^2 + x_s^2$ is nearly a sum of squares of real numbers, rather than the troublesome difference of squares that appeared in the original expression. The series converges cubically, and is quite easy to compute numerically.

The last term in (5.18), proportional to δ_{st}, is important in the analysis of the equations for the case of fairly deep cavities, because it is the only term that survives in the limit $b/d \to 0$. It is interesting that this limit is difficult to treat without the Watson-Sommerfeld transformation. One may draw an analogy to the use of that transformation in Regge's scattering theory.[31] There the transformation gives the asymptotic behavior, for $\cos\theta \to \infty$, of the sum of a series in Legendre polynomials, $P_l(\cos\theta)$. Here we get the asymptotic behavior, for $b \to 0$, of the sum of a series in Bessel function ratios, $I_1(\chi_m b)/\chi_m b I_0(\chi_m b)$.

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APPENDIX A
INITIAL VALUE TERMS

We repeat the considerations of Section 3, allowing arbitrary initial values $[f(0), E_z(0), H_\phi(0)]$, but requiring infinite conductivity on the end walls of the cavities. Solving (3.5) by the method of variation of parameters, we find that (3.9) and (3.10) must be modified by adding to their summands the term $\exp(ik_m z)G_m$, where

$$G_m(r) = \int_0^r u\, du g_m(u) R_0(\chi_m r, \chi_m u), \quad \text{(A.1)}$$

with

$$g_m(r) = -\frac{p}{c^2} E_{zm}(r, 0) - \frac{\mu_0}{r}\frac{\partial}{\partial r}(rH_{\phi m}(r, 0)). \quad \text{(A.2)}$$

Similarly, by solving (2.31) we find that (3.13) acquires the new term

$$e^{ik_m z}(L_m(r) + K_m r/2) \quad \text{(A.3)}$$

while (3.14) is augmented by

$$e^{ik_m z}(L_m(r) + K_m a^2/2r), \quad \text{(A.4)}$$

where

$$K_m = \frac{-ik_m q}{\chi_m^2}\int_{-\infty}^{\infty} f_m(p_z, 0)\, dp_z, \quad \text{(A.5)}$$

$$L_m(r) = \frac{\epsilon_0}{\chi_m^2}\left[p\frac{\partial G_m}{\partial r} - \frac{k_m^2}{r}\int_0^r u\, du \times E_{zm}(u, 0) + \frac{p}{\epsilon_0 c^2} H_{\phi m}(r, 0)\right] \quad \text{(A.6)}$$

In solving (2.31), one first has to allow an arbitrary solution γ/r of the homogeneous equation, in Region II. The requirement that the fields satisfy all four Maxwell equations then determines γ to be zero. After an application of the continuity condition at $r = a$ the fields in Region II take the form

$$\hat{E}_z^{II}(r, z) = \sum_{m=-\infty}^{\infty} [A_m I_0(\chi_m r) - \chi_m a S_0(\chi_m r, \chi_m a) + G_m(r)] e^{ik_m z}, \quad \text{(A.7)}$$

$$\hat{H}_\phi^{II}(r, z) = \sum_{m=-\infty}^{\infty}\left[\frac{\epsilon_0 p}{\chi_m} A_m I_1(\chi_m r) - \epsilon_0 p a S_1(\chi_m r, \chi_m a) F_m + L_m(r) + K_m a^2/2r\right] e^{ik_m z}. \quad \text{(A.8)}$$

In Region III we now have $\kappa = 0$ (infinite endwall conductivity), and the functions f_s of (3.43) reduce to $(1 + \delta_{s0})^{-1/2}\cos\alpha_s(z + g)$. We solve (3.15) by variation of parameters and then solve (2.31) to obtain

$$\hat{E}_z^{III}(r, z) = \sum_{s=0}^{\infty} [D_s R_0(\Gamma_s r, \Gamma_s d) + D_s' S_0(\Gamma_s r, \Gamma_s d) + \tilde{G}_s(r)] f_s(z) \quad \text{(A.9)}$$

$$\hat{H}_\phi^{III}(r, z) = \epsilon_0 p \sum_{s=0}^{\infty} [\Gamma_s^{-1} D_s R_1(\Gamma_s r, \Gamma_s d) + \Gamma_s^{-1} D_s' S_1(\Gamma_s r, \Gamma_s d) + \tilde{L}_s(r) + \tilde{K}_s a^2/2r] f_s(z) \quad \text{(A.10)}$$

where

$$\tilde{G}_s(r) = \int_b^r u\,du g_s(u) R_0(\Gamma_s r, \Gamma_s u), \tag{A.11}$$

$$g_s(r) = -\frac{p}{c^2} E_{zs}(r,0) - \frac{\mu_0}{r}\frac{\partial}{\partial r}(r H_{\phi s}(r,0)),$$

$$\tilde{L}_s(r) = \frac{\epsilon_0}{\Gamma_s^2}\left[p\frac{\partial \tilde{G}_s}{\partial r} - \frac{\alpha_s^2}{r}\int_0^r u\,du \right. \tag{A.12}$$

$$\left. \times E_{zs}(u,0) + \frac{p}{\epsilon_0 c^2} H_{\phi s}(r,0)\right], \tag{A.13}$$

$$\tilde{K}_s = \frac{-\alpha_s q}{\Gamma_s^2}\int_{-\infty}^{\infty} f_s(p_z, 0)\, dp_z. \tag{A.14}$$

Here $f_s(p_z, 0)$ is the coefficient in a development of $f(z, p_z, 0)$ in the functions $(1 + \delta_{s0})^{-1/2} \sin \alpha_s(z + g)$. In deriving $\hat{H}_\phi^{III}$ from (2.31), one again has to allow a term γ/r, and this time γ is not zero (for the choice of particular solution of the inhomogeneous equation that we have found convenient). The requirement that the two Maxwell equations (2.29), (2.30) give the same $\hat{E}_{rs}$ determines γ. The field $E_r(r, 0)$ is eliminated through Poisson's equation, as in (3.6).

Given the series (A.7)–(A.10), the remaining calculations for continuity and boundary conditions can be done in precise analogy to Section 3. The general form of the boundary condition is derived in Appendix B. On the cavity wall $r = d$ the boundary condition stated in terms of Fourier components is

$$\hat{E}_{zs}(d) = -\left(\frac{\mu p}{\sigma}\right)_c^{1/2} \hat{H}_{\phi s}(d) + \lambda_{sc}, \tag{A.15}$$

where the term λ_{sc} arises from the initial-value term of (B.12). Similarly, on the tube wall $r = b$,

$$\hat{E}_{zm}(b) = -\left(\frac{\mu p}{\sigma}\right)_t^{1/2} \hat{H}_{\phi m}(b) + \lambda_{mt}. \tag{A.16}$$

In place of (3.28) the condition (A.15) gives

$$D_s' = -(\eta_s/\Gamma_s b) D_s + J_s, \tag{A.17}$$

$$J_s = \Gamma_s d\left\{\left(\frac{\mu p}{\sigma}\right)_c^{1/2}\left[\tilde{L}_s(d) + \frac{a^2}{2d}\tilde{K}_s\right] + \tilde{G}_s(d) - \lambda_{sc}\right\}. \tag{A.18}$$

The continuity and boundary conditions at $r = b$ now lead to our previous equations (3.66), (3.67) for $\bar{A}$ and $\bar{D}$, specialized to $\kappa = 0$ and augmented with terms $A^{(0)}$ and $RD^{(0)}$, respectively, on their right hand sides. With the argument p of all functions indicated explicitly, the latter terms have the form

$$A_m^{(0)}(p) = \alpha \sum_{s=0}^{\infty} N_{ms}[J_s(p) S_0(\Gamma_{sp} b, \Gamma_{sp} d) + \tilde{G}_s(b,p)] - G_m(b,p) + \sum_{n=-\infty}^{\infty} (\delta_{mn} - \alpha V_{mn}) \times \left[-\left(\frac{\mu p}{\sigma}\right)_t^{1/2}\left(L_n(b,p) + \frac{a^2}{2b}K_n(p)\right) + \lambda_{nt}(p)\right], \tag{A.19}$$

$$R_s(p) D_s^{(0)}(p) = R_s(p)\left[\frac{1}{\epsilon_0 p b}\sum_{m=-\infty}^{\infty} N_{-ms}\left[L_m(b,p) + \frac{a^2}{2b}K_m(p)\right] - \frac{1}{\epsilon_0 p b}\left[\tilde{L}_s(b,p) + \frac{a^2}{2b}\tilde{K}_s(p)\right] - \frac{J_s(p)}{\Gamma_{sp} b} S_1(\Gamma_{sp} b, \Gamma_{sp} d)\right]. \tag{A.20}$$

The initial-value term in the Vlasov equation (2.22) must also be accounted for. Its effect is to modify the expression (4.10) for F_n by addition of a term

$$(1 + W_n)^{-1} F_n^{(0)}, \tag{A.21}$$

where

$$F_n^{(0)}(p) = \frac{-q\mu_0}{\chi_{np}^2}\int_{-\infty}^{\infty} dp_z f_n(p_z, 0)\frac{pv + ik_n c^2}{p + ik_n v}. \tag{A.22}$$

To see the implications of initial values for the rise time of instabilities, we must look at the p-plane analyticity properties of initial-value terms in the "self-consistent" equation [either equation (4.12) for the axial field or (4.29) for the cavity field]. The complete inhomogeneous term of (4.29) is $R_s \bar{D}_s^{(0)}$, where

$$\bar{D}_s^{(0)} = D_s^{(0)} + \sum_{m=-\infty}^{\infty} N_{-ms}\left[\frac{I_m}{1-\eta_t I_m} + \frac{1}{(1-\eta_t I_m)^2} \times \left(\frac{a}{b}\right)^2 \frac{I_1(\chi_m a)}{\chi_m a I_0^2(\chi_m b)} \frac{W_m}{1+\theta_m W_m}\right][A_m^{(0)} + \alpha\eta_t(N v D^{(0)})_m] + \sum_{m=-\infty}^{\infty} N_{-ms}\frac{1}{1-\eta_t I_m}\left(\frac{a}{b}\right)^2 \times \frac{I_1(\chi_m a)}{\chi_m a I_0(\chi_m b)}\frac{1}{1+\theta_m W_m}\left[\frac{W_m E_{z0m}}{p} - F_m^{(0)}\right] \quad (A.23)$$

Despite the complicated appearance of (A.23) it is not too difficult to be convinced that it is analytic in the right half p-plane and consequently does not affect the rise time of an instability. It does have singularities on the imaginary axis, which appear to reflect the circumstance mentioned in Section 2; namely, that one cannot expect the field perturbations to vanish in the course of time.

To check analyticity of (A.23) we note that the required analyticity of the ingredients $D_s^{(0)}$, $A_m^{(0)}$, and $F_m^{(0)}$ follows easily from their definitions. It then remains to show that $1 - \eta_t I_m$ and $1 + \theta_m W_m$ have no zeros in the right half plane. A zero of the former would, by (4.24), imply a pole of ζ_{mn}, which has already been ruled out. Analysis of $1 + \theta_m W_m$ for particular choices of $f_0(p_z)$ (see Part II) suggests that its zeros are all in $\mathrm{Re}p \leq 0$, but a general proof may be difficult. We needn't be concerned, however, since $(1 + \theta_m W_m)^{-1}$ also occurs in the kernel of (4.29). A pole of $(1 + \theta_m W_m)^{-1}$ in (A.23) would be cancelled by a similar pole from the kernel, and would not appear in the solution $\bar{D}$. The same argument applies to the factor R_s in (4.29).

APPENDIX B RESISTIVE WALL BOUNDARY CONDITIONS FOR THE LAPLACE TRANSFORM

We adapt the standard treatment[22] of resistive-wall boundary conditions to accommodate the Laplace transform. We suppose that the wall is planar, obeys $\mathbf{J} = \sigma\mathbf{E}$, $\mathbf{D} = \epsilon\mathbf{E}$, $\mathbf{B} = \mu\mathbf{H}$, and is substantially thicker than the skin depth for penetration of fields at the frequencies of interest. The unit normal pointing away from the wall is denoted by $\mathbf{n}$. We analyze fields inside the wall, supposing that they have relatively little variation in directions parallel to the wall; this is the essential assumption. Then, if ξ is the distance from the surface to a point inside the wall, the gradient acting on any field may be written as $\nabla = -\mathbf{n}\partial/\partial\xi$. Laplace transformation of the Maxwell equations involving curls yields

$$\mathbf{n} \times \frac{\partial\hat{\mathbf{E}}}{\partial\xi} = \mu p\,\hat{\mathbf{H}} - \mu\mathbf{H}(0), \quad (B.1)$$

$$\mathbf{n} \times \frac{\partial\hat{\mathbf{H}}}{\partial\xi} = -(\sigma + \epsilon p)\hat{\mathbf{E}} + \epsilon\mathbf{E}(0). \quad (B.2)$$

Elimination of $\hat{\mathbf{E}}$ gives

$$\frac{\partial^2\hat{\mathbf{H}}}{\partial\xi^2} - \mu p(\sigma + \epsilon p)\hat{\mathbf{H}} = \mathbf{g}, \quad (B.3)$$

$$\mathbf{g} = \frac{1}{p}\left(\mathbf{n}\cdot\frac{\partial^2\mathbf{H}(0)}{\partial\xi^2}\right)\mathbf{n} - \mu(\sigma + \epsilon p)\mathbf{H}(0) - \epsilon\mathbf{n} \times \frac{\partial\mathbf{E}(0)}{\partial\xi}. \quad (B.4)$$

By variation of parameters, we find the general solution of (B.3) having exponential decrease for increasing ξ. It has the form

$$\hat{\mathbf{H}}(\xi) = \mathbf{a}e^{-\xi/\delta} - \frac{\delta}{2}e^{\xi/\delta}\int_\xi^\infty \mathbf{g}(x)e^{-x/\delta}\,dx - \frac{\delta}{2}e^{-\xi/\delta}\int_0^\xi \mathbf{g}(x)e^{x/\delta}\,dx, \quad (B.5)$$

where

$$\delta = [\mu(\sigma + \epsilon p)p]^{-1/2}, \quad (B.6)$$

and $\mathbf{a}$ is an arbitrary constant vector. The branch of the square root in (B.6) is such that $\mathrm{Re}\delta^{-1} > 0$ when $\mathrm{Re}p > 0$. Equation (B.5) implies that $\hat{\mathbf{H}}$ satisfies the relation

$$\partial\hat{\mathbf{H}}/\partial\xi = -\frac{1}{\delta}\hat{\mathbf{H}} - e^{\xi/\delta}\int_\xi^\infty \mathbf{g}(x)e^{-x/\delta}\,dx. \quad (B.7)$$

Substitution of (B.7) in (B.2) and evaluation of the resulting equation at $\xi = 0$ yields the relation between $\hat{\mathbf{H}}$ and $\hat{\mathbf{E}}$ that must hold at the surface:

$$\hat{\mathbf{E}} = \left[\frac{\mu p}{\sigma + p}\right]^{1/2} \mathbf{n} \times \hat{\mathbf{H}} + \boldsymbol{\lambda}, \quad (\xi = 0), \qquad \text{(B.8)}$$

where

$$\boldsymbol{\lambda} = \frac{1}{\sigma + \epsilon p}\left[\mathbf{E}(0) + \mathbf{n} \times \int_0^\infty \mathbf{g}(x) e^{-x/\delta}\, dx\right]_{\xi=0}$$

$$= \frac{1}{\sigma + \epsilon p}\left[\epsilon \mathbf{E}(0, 0) - \mathbf{n} \times \int_0^\infty \left[\epsilon \mathbf{n} \times \frac{\partial \mathbf{E}(0, \xi)}{\partial \xi} + \mu(\sigma + \epsilon p)\mathbf{H}(0, \xi)\right] e^{-\xi/\delta}\, d\xi\right]. \qquad \text{(B.9)}$$

The component of Eq. (B.8) in the direction of $\mathbf{n}$ is of no interest, since it merely coincides with the corresponding component of (B.2). We may then write the tangential part of (B.8) as

$$\hat{\mathbf{E}}_\parallel = \left[\frac{\mu p}{\sigma + p}\right]^{1/2} \mathbf{n} \times \hat{\mathbf{H}}_\parallel + \boldsymbol{\lambda}_\parallel, \qquad \text{(B.10)}$$

where, through an integration by parts, $\boldsymbol{\lambda}_\parallel$ may be cast in the form

$$\boldsymbol{\lambda}_\parallel = \int_0^\infty \left[\frac{\alpha\epsilon}{\sigma + \epsilon p} \mathbf{E}_\parallel(0, \xi) - \mu \mathbf{n} \times \mathbf{H}_\parallel(0, \xi)\right] e^{-\xi/\delta}\, d\xi. \qquad \text{(B.11)}$$

For the values of p of interest it is an excellent approximation to drop the displacement current by putting $\epsilon = 0$. We then have the form of the boundary condition used in the foregoing work,

$$\hat{\mathbf{E}}_\parallel = \left[\frac{\mu p}{\delta}\right]^{1/2} \mathbf{n} \times \hat{\mathbf{H}}_\parallel - \mu \int_0^\infty \mathbf{n} \times \mathbf{H}_\parallel(0, \xi) e^{-\xi/\delta}\, d\xi. \qquad \text{(B.12)}$$

By using cylindrical coordinates and the appropriate Bessel functions, one can find a similar relation for a cylindrical surface. That refinement involves little extra effort; we have avoided it only to simplify notation.

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