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CALCULATION OF FIELDS IN A SUPERCONDUCTING CYCLOTRON ASSUMING UNIFORM MAGNETIZATION OF THE POLE TIPS*

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Calculation of the median-plane field in a superconducting cyclotron is greatly facilitated by assuming that all pole pieces not having axial symmetry are uniformly magnetized in the vertical direction. Using this model, a detailed analysis is presented in terms of surface currents flowing in horizontal loops around the sides of each pole tip, which then leads to a field formula involving a single line integral around the closed-current contour. This formula is simpler than the one previously obtained using a surface-charge representation of the field sources. A straightforward computation technique is described and explicit formulas are presented for circular and rectangular pole geometries. The latter is used, for example, in field calculations for the focusing elements designed for the beam-extraction system.

I. INTRODUCTION

A 500-MeV superconducting cyclotron is under construction at Michigan State University for use as a heavy-ion accelerator.[1] This machine will eventually become the injector for a larger superconducting cyclotron having $K = 800$ MeV.[2]

Calculation of the magnetic field in the superconducting cyclotron is divided into several parts, which are then combined through superposition. In the first part, the program TRIM (or its successor, POISSON) is used to calculate the dominant axially symmetric portion of the field that is produced by the superconducting coils, the circular yoke and core, together with a distribution of circular iron rings designed to represent the average effect of the non-symmetric pole pieces.[3]

In the second part, which will concern us here, a simple model is used to calculate the contribution from the iron sectors (pole tips), which produce the main azimuthal variation in the field, plus the contributions from the other non-symmetric elements such as the vertical holes required for the dee stems. In addition, as part of the superposition process, this model is also used to calculate the contribution from the set of circular iron rings used in the first part to represent these elements.

In the model used to calculate the field produced by the pole tips and other such elements, it is assumed for simplicity that the iron is uniformly magnetized in the vertical direction. This assumption allows one to replace the field sources by an equivalent surface-charge distribution on the top and bottom surfaces of the pole pieces. Such a representation was used by Blosser and Johnson in the early design calculations for a superconducting cyclotron.[4] This representation is also being used at Oak Ridge National Laboratory, according to a recent report by McNeilly.[5] Since the summer of 1976, however, we have been using an improved method, which is the subject of this note.

Uniform magnetization permits an alternative representation of the field sources in terms of a surface-current distribution flowing in horizontal loops around the sides of the pole tips. This representation produces simpler formulas for the magnetic-field components because they involve only a single line integral around the periphery of the pole tips instead of two-dimensional area integrals. These formulas lead to a significant improvement in the speed of the field computations.

In Section 2 below, we outline the derivation of the formulas for the field components using the surface-current representation. Then in Section 3 we specialize these formulas to the median-

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plane field, which is the case of primary interest. Finally, in two subsections, we present specific formulas appropriate to circular and rectangular geometries. The latter are directly applicable to the focusing bars used in our extraction system and were used to calculate the focusing-bar field data, which were presented in our annual reports covering 1976–78, and in a paper describing the extraction system.[6]

2. DERIVATION OF FORMULAS

Consider a cylindrical piece of iron having horizontal surfaces $z = z_1$ at bottom and $z = z_2$ at top, and vertical sides. Let the horizontal cross section be arbitrary, but independent of z. We then assume that the pole tips under consideration can be subdivided into a finite number of such pieces. In addition, we assume that the iron is uniformly magnetized so that $\mathbf{M}$ has constant magnitude and points in the $+z$ direction.

The theory of the equivalent-current distribution can be found in many text books. [See, e.g., Smythe (sec. 12.07) or Jackson (sec. 5.10).[7]] Using *mks* units, the so-called magnetization current density is given by $\mathbf{J}_m$ = curl $\mathbf{M}$.

In our case, $\mathbf{J}_m$ is evidently zero inside the iron. However, application of Stokes' theorem shows that a surface current flows in horizontal loops around the sides of the iron with a corresponding current density given by $d\mathbf{I}/dz = \mathbf{M} \times \mathbf{n}$, where $\mathbf{n}$ is a unit vector along the outward normal to the surface.

Figure 1 shows a top view and a side view of the iron geometry under consideration. Included here is $\mathbf{n}$ at some point on the side surface, together with $d\mathbf{s}$, the horizontal line element directed along the current flow at this point.

The current dI flowing around a horizontal band of height dz' is therefore given by $dI = M\,dz'$. Moreover, if $B_0 = \mu_0 M$ is the corresponding magnetization field, then $dI = (B_0/\mu_0)dz'$. Because of the high fields produced by the superconducting coils, we usually take B_0 to be the saturation value, $b_0 = 2.14\ T$.

This current distribution corresponds exactly to a uniformly wound solenoid, and the resultant field $\mathbf{B}$ can be calculated from the Biot-Savart law. We choose instead an equivalent, but simpler procedure which makes use of the vector potential $\mathbf{A}$, defined by $\mathbf{B}$ = curl $\mathbf{A}$.

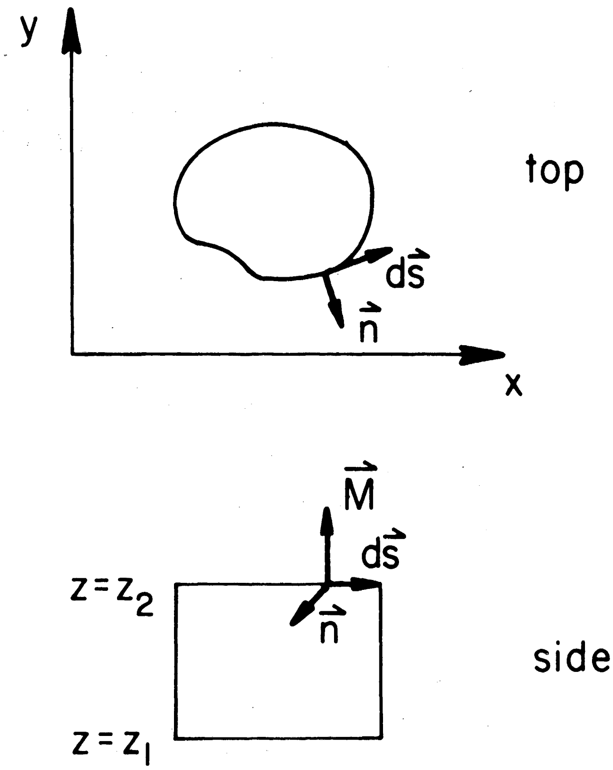

FIGURE 1 Top view and side view of a uniformly magnetized iron cylinder which forms the basic geometry used in the field calculations. Here, $\mathbf{M}$ indicates the magnetization vector, while $\mathbf{n}$ is a unit vector along the outward normal to the side surface. The horizontal line element ds is directed along the surface current flow at the given point.

In our situation, this potential is given by

$$\mathbf{A} = \frac{B_0}{4\pi}\oint ds \int \frac{dz'}{R}, \tag{1}$$

where $R = |\mathbf{r} - \mathbf{r}'|$ is the distance from the source point $\mathbf{r}'$ to the field point $\mathbf{r}$. That is,

$$R = ((x - x')^2 + (y - y')^2 + (z - z')^2)^{1/2}, \tag{2}$$

where $\mathbf{r} = (x, y, z)$ is the point at which the field is to be evaluated, and $\mathbf{r}' = (x', y', z')$ gives the location of a particular current element.

The primed coordinates are therefore the integration variables. Thus, the components of $d\mathbf{s}$ are $(dx', dy', 0)$. As a consequence, we note that $A_z = 0$.

The second integral in (1) can readily be evaluated and yields

$$\int \frac{dz'}{R} = \sinh^{-1}\left(\frac{z - z_1}{p}\right) - \sinh^{-1}\left(\frac{z - z_2}{p}\right), \quad (3)$$

where p is the distance in the (x, y) plane from the source point to the field point; that is,

$$p = ((x - x')^2 + (y - y')^2)^{1/2}. \quad (4)$$

With this result, the expression for **A** reduces to a single line integral around the periphery of the iron cylinder.

It is also clear that the expression for **A** now divides into two separate terms, one involving only z_1 and the other only z_2. We may therefore treat the field as arising from two separate parts, one from the lower (negatively charged) surface at $z = z_1$, and the other from the upper (positively charged) surface at $z = z_2$. Since we need to make use of superposition eventually, we can simplify the results here by considering just the upper surface.

With this in mind, we let $z_1 \to -\infty$. That is, we consider a semi-infinite cylinder that extends from $z = -\infty$ up to z_2. From $\mathbf{B} = \text{curl } \mathbf{A}$, it then follows that

$$B_x = \frac{B_0}{4\pi} \oint \frac{dy'}{R_2}, \quad B_y = -\frac{B_0}{4\pi} \oint \frac{dx'}{R_2}, \quad (5)$$

where R_2 is the same as R in (2), but with $z' = z_2$.

The expression for B_z is more complicated, and we give here one of several possible forms:

$$B_z = \frac{B_0}{4\pi} \oint \left[1 - \frac{z - z_2}{R_2}\right] \times \frac{(y - y')dx' - (x - x')dy'}{(x - x')^2 + (y - y')^2}. \quad (6)$$

This expression has been checked in certain limiting cases and special cases where the result is known. For example, if we let $z_2 \to +\infty$, so that the cylinder becomes infinitely long, then we must obtain $B_z = B_0$ inside, and $B_z = 0$ outside. For this limiting case, the expression (6) becomes

$$B_z(\infty) = \frac{B_0}{2\pi} \oint \frac{(y - y')dx' - (x - x')dy'}{(x - x')^2 + (y - y')^2} = B_0 \delta_c, \quad (7)$$

where $\delta_c = 1$ if the point (x, y) lies inside the contour, and $\delta_c = 0$ if it lies outside. Perhaps the simplest way to prove this identity is to let $Z = (x' - x) + i(y' - y)$, and then use Cauchy's theorem.

The same analysis applies to another special case, namely, $z = z_2$. Thus we find $B_z(z_2) = \frac{1}{2} B_0 \delta_c$, which shows that the field at the surface of the iron is just one-half the limiting value. This same result may be used to evaluate the term in (6) that is independent of z, but such an evaluation produces a formula which is less suitable for a computer program.

One may ask how the results obtained above using a surface-current formulation are related to those obtained using a surface-charge formulation. Mathematically, the two results are connected by Stokes' theorem, which can be used to transform the line integral into a surface integral, or vice versa. One must recognize, however, that the surface-current description yields the B field directly, while the surface-charge description produces the H field (multiplied by μ_0), so that the two results differ inside the iron.

3. MEDIAN-PLANE FIELD

The formulas in the previous section apply to a single piece of iron and we are usually more interested in a geometry having median-plane symmetry. In this case, for each piece below the median plane, there is a matching piece symmetrically located above this plane. Here again, the general situation can be obtained by superposition from one basic geometry.

This basic geometry consists of a semi-infinite cylinder extending from $z = -\infty$ up to $-z_0$, and a matching cylinder extending from $z = +z_0$ up to $+\infty$. This geometry is equivalent to a single infinite cylinder with a piece cut out between $z = -z_0$ and $+z_0$.

We restrict ourselves here to calculating the field in the median plane $z = 0$. This median-plane field is denoted by B_{z0} and we present a formula corresponding to the one given in (6) above, namely,

$$B_{z0} = \frac{B_0}{2\pi} \oint \left[1 - \frac{z_0}{R_0}\right] \times \frac{(y - y')dx' - (x - x')dy'}{(x - x')^2 + (y - y')^2}. \quad (8)$$

Here the distance R_0 is given by

$$R_0 = ((x - x')^2 + (y - y')^2 + z_0^2)^{1/2}, \quad (9)$$

as a direct consequence of (2) above.

In accordance with the discussion in the previous section for the case of an infinite cylinder, we can rewrite B_{z0} as follows

$$B_{z0} = B_0 \delta_c - \frac{B_0}{2\pi} \oint \left(\frac{z_0}{R_0}\right) \times \frac{(y - y')dx' - (x - x')dy'}{(x - x')^2 + (y - y')^2}. \quad (10)$$

Although this expression appears simpler, the formula in (8) above is more suitable for a computer program because it requires no special test to determine whether the point (x, y) lies inside or outside the integration contour.

Except for the special cases described below, we generally use a simple straightforward method to evaluate numerically the contour integral in B_{z0}. First, the contour is divided into a sequence of n straight-line segments joining the points (x_j, y_j), where $j = 0, 1, 2, \ldots, n$. Here, $(x_n, y_n) = (x_0, y_0)$ is required in order that the contour be closed. Next, we replace the integral by a sum over the line segments after making the substitutions

$$x' \to \tfrac{1}{2}(x_j + x_{j-1}), \quad y' \to \tfrac{1}{2}(y_j + y_{j-1}) \quad (11)$$
$$dx' \to x_j - x_{j-1}, \qquad dy' \to y_j - y_{j-1}.$$

Since the integrand involves only simple algebraic expressions, the numerical integration is quite rapid. The accuracy can be tested and improved by systematically increasing n.

3.1 Circular Geometry

The circular geometry is important since it includes such frequently used cases as circular discs, holes, and rings. Here we take as our basic geometry a circular disc having radius a centered at (x_0, y_0).

We first define the distance r and the angle u through the relations

$$r = ((x - x_0)^2 + (y - y_0)^2)^{1/2}, \quad (12a)$$

$$p^2 = (r + a)^2 - 4ar \sin^2 u, \quad (12b)$$

$$R_0 = (p^2 + z_0^2)^{1/2}, \quad (12c)$$

where p and R_0 are given in (4) and (9). Then, the equation for B_{z0} in (8) reduces to

$$B_{z0} = \frac{B_0}{\pi} \int_0^{\pi/2} \frac{(p^2 + a^2 - r^2)du}{R_0(R_0 + z_0)}. \quad (13)$$

This formula is quite suitable for numerical calculations. For example, using the trapezoidal method with an integration step $\Delta u = 2°$, we find that the maximum error is less than $10^{-6} B_0$ in a case where $z_0/a = 0.2$. (The maximum error usually occurs at $r = a$.)

As a useful point to test, we note that for $r = 0$, the exact result is

$$B_{z0}(r = 0) = B_0(1 - z_0/(z_0^2 + a^2)^{1/2}). \quad (14)$$

3.2 Rectangular Geometry

We turn next to the rectangular geometry, whose main application is to the case of focusing bars, already mentioned in the introduction. This geometry is the only one we know that yields simple analytical solutions, not only for the median plane field B_{z0}, but also for the general field components given in (5) and (6) of Section 2. Since the latter are, however, useful only for rather special purposes, we restrict ourselves here to B_{z0} alone.

We take as our basic surface area a rectangle whose boundaries are specified by $x_1 \le x \le x_2$, and $y_1 \le y \le y_2$. Next, we define C_j and D_k as follows

$$C_j = (x_j - x)/(z_0^2 + (x_j - x)^2)^{1/2}, \quad (15)$$
$$D_k = (y_k - y)/(z_0^2 + (y_k - y)^2)^{1/2},$$

where $j, k = 1, 2$. We then find the following analytical formula for the median-plane field

$$B_{z0} = \frac{B_0}{2\pi} (\phi_{22} + \phi_{11} - \phi_{12} - \phi_{21}), \quad (16)$$

where the angle ϕ_{jk} is given by

$$\phi_{jk} = \sin^{-1}(C_j D_k), \quad (16a)$$

subject to the condition $-\pi/2 \le \phi_{jk} \le +\pi/2$. This formula can then be used directly to calculate the desired field values.

In the limit where the rectangle becomes an infinite strip, the above result can be checked against that obtained from a simple two-dimensional analysis. In both cases, one finds

$$B_{z0}(\text{strip}) = \frac{B_0}{\pi}(\sin^{-1} C_2 - \sin^{-1} C_1), \quad (17)$$

where we have let $y_2 \to +\infty$ and $y_1 \to -\infty$ in obtaining the infinite strip. Here, C_2 and C_1 are the same as those defined in (15).

In the formula (16) for B_{z0}, we note that there is one ϕ_{jk} for each corner of the rectangle and that these quantities alternate in sign as one circulates around the rectangle from one corner to the next. This sign alternation has an interesting consequence.

Suppose we need to calculate B_{z0} for some more general surface area whose shape is such that it can be subdivided into a large number of small rectangles. This situation is reminiscent of the small rectangular tiles which are cemented together to form a table top or a bathroom floor.

For this type of surface area, the resultant B_{z0} is obtained through superposition by adding up the contributions from all of the small rectangles, or rather, the corners of these rectangles. However, because of the sign alternation noted above, one finds that the contributions from all of the interior corners cancel each other out since they occur in pairs having opposite signs. Thus, the resultant B_{z0} is determined entirely from the contributions of the unpaired corners which occur only around the periphery of the surface area. Moreover, in adding up these contributions, if one traverses the periphery as a closed curve, the contributions from successive corners again alternate in sign. One can therefore recognize that the resultant B_{z0} reduces here to the evaluation of a line integral around the periphery, exactly as indicated in the general formula given in (8) above.

4. ADDED NOTES

At first glance, assuming uniform magnetization of the pole tips appears to be a somewhat dubious approximation. This assumption does, of course, provide a simple, rapid method for calculating the required fields. Moreover, the speed of these calculations is particularly important in the design of a superconducting cyclotron, where a wide variety of operating conditions must be explored before a final decision is reached on the optimum shape of the pole tips.

Fortunately, the accuracy of these calculations turns out to be reasonably satisfactory. A comparison of the calculated fields with some preliminary measurements showed that the average field values differ by about 1%, while values of the (main) three-sector field component differ by about 5%.[8] These conclusions have been confirmed for the most part by a much more comprehensive study, the results of which will soon be available.[9]

Finally, a note is in order regarding the change from a surface-integral to a line-integral formulation of a field calculation. Our attention has been called to a similar transformation carried out by Beth[10] in the calculation of a strictly two-dimensional field produced by a current density that is constant within a given contour in the xy-plane. In this case, the transformation is carried out through the use of certain theorems concerning functions of a complex variable.

Postscript (May 19, 1980). In response to receiving a preprint of this paper, Heighway[11] has kindly sent us a copy of an interesting report by Westcott[12] which is practically unknown outside of Chalk River. In this report, Westcott used the Biot-Savart law to derive an expression for the median plane field which is entirely equivalent to our Eq. (8) except for his use of polar coordinates throughout. Although he did not consider the rectangular geometry discussed above, Westcott did present a detailed analysis of the case where a portion of the bounding contour consists of a circular arc, and for this important case, he obtained an analytical expression for the resultant field involving elliptic integrals of both the first and third kinds. Most of Westcott's report deals with a computer program for carrying out field calculations, and this program was used at Chalk River in the early design[13] of a four sector magnet for the superconducting cyclotron which is now under construction there.[14] Quite inadvertently, none of the external reports of published papers arising out of this project refers explicitly to Westcott's report, so that knowledge of this work has remained quite hidden.

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