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THEORY OF ELECTRON COOLING WITH MAGNETIC FIELD AND SPACE CHARGE

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In this paper we present a compact theory of electron cooling. In the frame in which the electron beam is at rest, we estimate the friction coefficients and the diffusion tensor for cases with and without a solenoidal field and the space charge of the electron beam. Cooling rates are given by the friction coefficients and equilibrium beam sizes by the elements of the diffusion tensor. Scaling that considers the cooling-region length relative to the periodic lattice structure and transformation to the laboratory frame are not discussed.

I. INTRODUCTION

Electron cooling uses the energy transfer from the hot heavy particles to the cold electrons in Coulomb collisions. The process was investigated many years ago by Spitzer[1] to derive the momentum and temperature relaxation time between the two kinds of charged particles. Thompson and Hubbard[2,3] obtained the diffusion and friction coefficients of the Fokker-Planck equation without magnetic field by statistically averaging the trajectory of a test particle in fluctuating fields. When the Fokker-Planck equation is integrated over velocity space, the moment equations express the momentum and energy relaxation and give the relaxation time of Spitzer. The Thompson-Hubbard method has the advantage that it is easy to understand and to apply in various cases. Moreover, the method is an alternative one to obtain the Balescu-Lenard collision term,[4,5] which generally determines the time evolution of the velocity distribution function.

Using the Thompson-Hubbard method, Ichimaru and Rosenbluth[6] calculated the Fokker-Planck coefficients of a plasma in a uniform magnetic field in terms of the spectral function of fluctuating fields and the plasma dielectric response function. The relaxation process in plasmas with the magnetic field has also been studied by many other authors.[7-10] They usually derive an expression for the kinetic equation, which self-consistently includes the Coulomb interaction, and the Fokker-Planck coefficients, which are still described by the troublesome integration of the spectral function of fluctuating fields.

Budker proposed the electron cooling method for heavy particles. The cooling of protons by cold electrons has been studied at Novosibirsk to observe the damping of betatron oscillations.

The theory of electron cooling in an accelerator was given by Budker et al[11] and Derbenev and Skrinsky[12,13] on the basis of the Landau collision integral, which is an excellent approximation of the Balescu-Lenard collision term.[4,5] The drag force on the heavy ion obtained agrees with the result of Spitzer or Thompson and Hubbard without magnetic field. The energy cooling time is also the same as the Spitzer's temperature relaxation time for an isotropic velocity distribution.

The drag force in a solenoidal magnetic field was given successively by Derbenev and Skrinsky,[13] Rosenbluth,[14] and Bell.[15] In that case, since the electrons are trapped by the magnetic-field lines and the motion transverse to the magnetic field is prohibited in a range larger than the electron Larmor radius, the transverse spread of electron velocity has no effect on the drag force. As a result, the cooling time with magnetic field is shorter than that without magnetic field if the spread of parallel electron velocity is much narrower than the transverse spread. The effect of the space charge of electron beam on electron cooling, which was first studied by Dikansky et al.[16] is also important because it might give a limitation of the electron cooling.

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The electron cooling in accelerators has some features different from the usual relaxation process between two kinds of particles in a plasma. Accelerator physicists want to make the beam emittance small by electron cooling. Since the emittance depends on both velocity space and position space, it is usually difficult to calculate analytically the variation of emittance in fluctuation fields. On the other hand, the momentum, energy and temperature relaxation processes discussed by plasma physicists are connected only with the variation in velocity space. Therefore, strictly speaking, the energy relaxation in plasmas should be distinguished from the variation of emittance, but, if we are interested in only the variation of momentum and energy, the calculation of the relaxation process in plasmas can be applied to the beams in the accelerator. If the emittance is practically determined only from velocity space and is almost independent of the spread in position space caused by fluctuating fields, the decrease of emittance will agree well with the energy relaxation result. Otherwise, the damping of emittance in heavy ions is often slower than that of energy.

In this paper, we will derive the friction coefficient and diffusion tensor of heavy ions in velocity space, with a solenoidal magnetic field and space-charge effects on the electron beam also taken into account.

II. DERIVATION OF THE FRICTION COEFFICIENT AND DIFFUSION TENSOR

In order to study the kinetics of the electron cooling, we will calculate the friction coefficient and the diffusion tensor in velocity space of the heavy ion. We adopt the Thompson-Hubbard approach[6,7] which calculates them from a statistical analysis of the motion of a test particle in the background fluctuating fields. The calculations are done in the rest frame of the electrons.

1. *Particle Motion in Fluctuating Fields*

The electron and ion charge densities are assumed to be given by

$$\tilde{\rho}_e = q \sum_s \delta(\mathbf{r} - \mathbf{r}_s) \tag{1a}$$

$$\tilde{\rho}_p = q_p \delta(\mathbf{r} - \mathbf{r}_p), \tag{1b}$$

where $q = -e$ and $q_p = Ze$. The vectors $\mathbf{r}_s = \mathbf{r}_s(t)$ and $\mathbf{r}_p = \mathbf{r}_p(t)$ are, respectively, the positions of the s-th electron and the ion. It is sufficient to calculate only the electric field corresponding to the scalar potential, because the velocity deviation of each particle from the streaming velocity of the beam is much smaller than the velocity of light. Hence, we can use Poisson's equation

$$\nabla^2\phi = -4\pi\tilde{\rho}, \quad \tilde{\rho} = \tilde{\rho}_e + \tilde{\rho}_p. \tag{2}$$

In this case, Eq. (1) can be rewritten by using the Fourier expansion

$$\tilde{\rho}_e = \frac{q}{(2\pi)^3} \sum_s \int e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}_s)}\, d\mathbf{k}$$

$$\tilde{\rho}_p = \frac{q_p}{(2\pi)^3} \int e^{i\mathbf{k}_1\cdot(\mathbf{r}-\mathbf{r}_p)}\, d\mathbf{k}_1 .$$

Moreover, we can write the electric field as

$$\mathbf{E} = \mathbf{E}_e + \mathbf{E}_p \tag{3a}$$

$$\mathbf{E}_e = -\frac{q_i}{2\pi^2} \sum_s \int \frac{\mathbf{k}}{k^2} e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}_s)}\, d\mathbf{k} \tag{3b}$$

$$\mathbf{E}_p = -\frac{q_{pi}}{2\pi^2} \int \frac{\mathbf{k}_1}{k_1{}^2} e^{i\mathbf{k}_1\cdot(\mathbf{r}-\mathbf{r}_p)}\, d\mathbf{k}_1 . \tag{3c}$$

When there is a uniform magnetic field B in the z-direction of the main motion and a radial electric field $E_r = -2\pi ner$ due to the space charge of the uniform density n of the electron beam as the zeroth-order fields, the equation of motion of the s-th electron can be written in the form

$$\dot{\mathbf{r}}_s = \mathbf{v}_s \tag{4a}$$

$$\dot{\mathbf{v}}_s = \omega_p{}^2\mathbf{r}_\perp + \Omega\mathbf{v}_s \times \mathbf{z} + \frac{q}{m}\mathbf{E} \tag{4b}$$

and that of the ion can be written as

$$\dot{\mathbf{r}}_p = \mathbf{v}_p \tag{5a}$$

$$\dot{\mathbf{v}}_p = \omega_{pi}^2\mathbf{r}_\perp + \Omega_p\mathbf{v}_p \times \hat{\mathbf{z}} + \frac{q_p}{m_p}\mathbf{E} \cong \frac{q_p}{m_p}\mathbf{E}, \tag{5b}$$

where $\dot{\mathbf{r}}_s$ is the time derivative $d\mathbf{r}_s/dt$, $\mathbf{r}_\perp = (x, y, 0)$ and $\hat{\mathbf{z}}$ is the unit vector in the z direction. Then $\omega_p{}^2 = -2\pi eqn/m = 2\pi e^2 n/m$, $\omega_{pi}^2 = -2\pi eq_p n/m_p$, $\Omega = qB/mc = -eB/mc$, $\Omega_p = q_p B/m_p c$ and $q_p = Ze$ are defined, where $Z = 1$

is used for the proton and $Z = -1$ for the antiproton. The last equation in Eq. (5b) is valid for $|\Omega_p \tau_t| \ll 1$ and $|\omega_{pi}\tau_t| \ll 1$, and that condition is well satisfied in the experiments,[17] where τ_t is the transit time of the ion through the electron cooling region.

2. *Solution of the Equation of Motion*

2.1. *No Space Charge*

We may calculate the trajectory of the s-th electron by integrating Eq. (4) for $\omega_p^2 = 0$.

$$\mathbf{r}_s(t) = \mathbf{r}_s(0) + H(t)\cdot\mathbf{v}_s(0) + \frac{q}{m}\int_0^t dt' H(t-t')\cdot\mathbf{E}[\mathbf{r}_s(t'), t']$$

$$\mathbf{v}_s(t) = G(t)\cdot\mathbf{v}_s(0) + \frac{q}{m}\int_0^t dt' G(t-t')\cdot\mathbf{E}[\mathbf{r}_s(t'), t'],$$

where $\mathbf{r}_s(0)$ is the initial position and $\mathbf{v}_s(0)$ the initial velocity. Putting $t - t' = \tau$, we can write

$$\mathbf{r}_s(t) = \mathbf{r}_s(0) + H(t)\cdot\mathbf{v}_s(0) + \frac{q}{m}\int_0^t d\tau H(\tau)\cdot\mathbf{E}[\mathbf{r}_s(t-\tau), t-\tau] \equiv \mathbf{R}_s + \Delta\mathbf{r}_s, \tag{6a}$$

$$\mathbf{v}_s(t) = G(t)\cdot\mathbf{v}_s(0) + \frac{q}{m}\int_0^t d\tau G(\tau)\cdot\mathbf{E}[\mathbf{r}_s(t-\tau), t-\tau] \equiv \mathbf{v}_{s0} + \Delta\mathbf{v}_s, \tag{6b}$$

where $\Delta\mathbf{r}_s$ and $\Delta\mathbf{v}_s$ are the terms containing integrals and where

$$H(t) = \frac{1}{\Omega}\begin{bmatrix} \sin\Omega t & 1-\cos\Omega t & 0 \\ -(1-\cos\Omega t) & \sin\Omega t & 0 \\ 0 & 0 & \Omega t \end{bmatrix}$$

$$G(t) = \frac{dH(t)}{dt} = \begin{bmatrix} \cos\Omega t & \sin\Omega t & 0 \\ -\sin\Omega t & \cos\Omega t & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Since the magnetic field makes only a small contribution to the trajectory of the ion, we may neglect its effect for ions and put $\Omega_p = 0$.

$$\mathbf{r}_p(t) = \mathbf{r}_p(0) + t\mathbf{v}_p(0) + \frac{q_p}{m_p}\int_0^t d\tau\tau\mathbf{E}[r_p(t-\tau), t-\tau] \equiv \mathbf{R}_p + \Delta\mathbf{r}_p \tag{7a}$$

$$\mathbf{v}_p(t) = \mathbf{v}_p(0) + \frac{q_p}{m_p}\int_0^t d\tau\mathbf{E}[r_p(t-\tau), t-\tau] \equiv \mathbf{v}_{p0} + \Delta\mathbf{v}_p, \tag{7b}$$

where again the terms $\Delta\mathbf{r}_p$ and $\Delta\mathbf{v}_p$ are the terms containing integrals.

2.2. *With Space Charge*

We introduce the space-charge effect of a uniform electron beam by a radial electric field $Er = -2\pi ner$. In this case we can obtain as the solution of Eq. (4)

$$\mathbf{v}_s(t) = K_1\mathbf{v}_s(0) + K_2\mathbf{r}_s(0) + \frac{q}{m}\int_0^\tau d\tau K_1(\tau)\cdot\mathbf{E}[\mathbf{r}_s(t-\tau), t-\tau] \equiv \mathbf{v}_{s0} + \Delta\mathbf{v}_s \tag{8a}$$

$$\mathbf{r}_s(t) = K_3\mathbf{v}_s(0) + K_4\mathbf{r}_s(0) + \frac{q}{m}\int_0^\tau d\tau K_3(\tau)\cdot\mathbf{E}[r_s(t-\tau), t-\tau] \equiv \mathbf{R}_s + \Delta\mathbf{r}_s \tag{8b}$$

Here we define

$$K_1(t) = \frac{1}{(\omega_1-\omega_2)}\begin{bmatrix} a_1 & a_2 & 0 \\ -a_2 & a_1 & 0 \\ 0 & 0 & \omega_1-\omega_2 \end{bmatrix}$$

$$K_2(t) = \frac{\omega_1\omega_2}{(\omega_1-\omega_2)}\begin{bmatrix} a_3 & -a_4 & 0 \\ a_4 & a_3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$K_3(t) = \frac{1}{(\omega_1-\omega_2)}\begin{bmatrix} a_3 & -a_4 & 0 \\ a_4 & a_3 & 0 \\ 0 & 0 & (\omega_1-\omega_2)t \end{bmatrix}$$

$$K_4(t) = \frac{1}{(\omega_1-\omega_2)}\begin{bmatrix} a_5 & -a_6 & 0 \\ a_6 & a_5 & 0 \\ 0 & 0 & \omega_1-\omega_2 \end{bmatrix}$$

where $a_1 = \omega_1 \cos \omega_1 t - \omega_2 \cos \omega_2 t$, $a_2 = \omega_1 \sin \omega_1 t - \omega_2 \sin \omega_2 t$, $a_3 = \sin \omega_1 t - \sin \omega_2 t$, $a_4 = \cos \omega_1 t - \cos \omega_2 t$, $a_5 = -\omega_2 \cos \omega_1 t + \omega_1 \cos \omega_2 t$, and $a_6 = \omega_2 \sin \omega_1 t - \omega_1 \sin \omega_2 t$. In these equations,

$$\omega_1 \equiv \tfrac{1}{2}[-\Omega + (\Omega^2 - 4\omega_p{}^2)^{1/2}] \cong -\Omega\ (>0)$$

and

$$\omega_2 \equiv \tfrac{1}{2}[-\Omega - (\Omega^2 - 4\omega_p{}^2)^{1/2}] \cong -\omega_p{}^2/\Omega,$$

where the approximate expressions are valid for $|\Omega| \gg |\omega_p|$. This condition is usually satisfied in the experiments.[17] In the case with space charge, the trajectory of the ion is still well described by Eq. (7).

3. *Diffusion Tensor in Velocity Space*

We will calculate the velocity-space diffusion tensor of the ion by using the solution of the equation of motion.

3.1. *Diffusion Tensor Without Magnetic Field and Space Charge*

The diffusion tensor of the ion is defined as

$$D_p \equiv \frac{1}{2}\left\langle \frac{\Delta \mathbf{v}_p \Delta \mathbf{v}_p}{\tau_c} \right\rangle = \frac{1}{2}\left\langle \frac{d}{dt} \Delta \mathbf{v}_p \Delta \mathbf{v}_p \right\rangle = \left\langle \Delta \mathbf{v}_p \frac{d\mathbf{v}_p}{dt} \right\rangle, \quad (9)$$

where τ_c is the correlation time of the fluctuation field. Since the self-field of the ions is to be excluded, we use $\mathbf{E} = \mathbf{E}_e$. Hence we can obtain from Eqs. (5b) and (7b)

$$D_p = \frac{q_p{}^2}{m_p{}^2} \int_0^t d\tau \times \langle \mathbf{E}_e[\mathbf{r}_p(t), t] \mathbf{E}_e[\mathbf{r}_p(t-\tau), t-\tau] \rangle.$$

Since the quantity $\langle \mathbf{E}_e \cdot \mathbf{E}_e \rangle$ is assumed not to depend explicitly on t, we may extend the region of integration with respect to τ to infinity. Thus we have

$$D_p = \frac{q_p{}^2}{m_p{}^2} \int_0^\infty d\tau \times \langle \mathbf{E}_e[\mathbf{r}_p(t), t] \mathbf{E}_e[\mathbf{r}_p(t-\tau), t-\tau] \rangle. \quad (10)$$

On the other hand, we can calculate approximately the fluctuating electric field for $|\mathbf{k}\cdot\Delta\mathbf{r}_p| \ll |\mathbf{k}\cdot\Delta\mathbf{r}_s| \ll 1$ from Eqs. (3b) and (6a).

$$\begin{aligned}\mathbf{E}_e[\mathbf{r}_p] &= -\frac{qi}{2\pi^2} \sum_s \int d\mathbf{k} \frac{\mathbf{k}}{k^2} e^{i\mathbf{k}\cdot(\mathbf{r}_p - \mathbf{r}_s)} \\ &= -\frac{qi}{2\pi^2} \sum_s \int d\mathbf{k} \frac{\mathbf{k}}{k^2} e^{i\mathbf{k}\cdot(\mathbf{R}_p - \mathbf{R}_s)} \times [1 + i\mathbf{k}\cdot(\Delta\mathbf{r}_p - \Delta\mathbf{r}_s)] \quad (11a) \\ &\cong -\frac{qi}{2\pi^2} \sum_s \int d\mathbf{k} \frac{\mathbf{k}}{k^2} e^{i\mathbf{k}\cdot(\mathbf{R}_p - \mathbf{R}_s)} \times [1 - i\mathbf{k}\cdot\Delta\mathbf{r}_s]\end{aligned}$$

and similarly

$$\mathbf{E}_p[\mathbf{r}_s] \cong -\frac{q_{pi}}{2\pi^2} \int d\mathbf{k}_1 \frac{\mathbf{k}_1}{k_1{}^2} e^{i\mathbf{k}_1\cdot(\mathbf{R}_s - \mathbf{R}_p)} \times [1 + i\mathbf{k}_1\cdot\Delta\mathbf{r}_s], \quad (11b)$$

where we neglect $i\mathbf{k}\cdot\Delta\mathbf{r}_p$ because $|\Delta\mathbf{r}_p/\Delta\mathbf{r}_s| \doteqdot m/m_p \ll 1$. Using only the first term in Eq. (11a),

$$\begin{aligned}&\langle \mathbf{E}_e[\mathbf{r}_p(t), t] \mathbf{E}_e[\mathbf{r}_p(t-\tau), t-\tau] \rangle \\ &= -\frac{q^2}{4\pi^4} \left\langle \left(\sum_s \int d\mathbf{k} \frac{\mathbf{k}}{k^2} e^{i\mathbf{k}\cdot(\mathbf{R}_p - \mathbf{R}_s)_t} \right) \times \left(\sum_s \int dk_1 \frac{\mathbf{k}_1}{k^2} e^{i\mathbf{k}_1\cdot(\mathbf{R}_p - \mathbf{R}_s)_{t-\tau}} \right) \right\rangle \quad (12) \\ &= \frac{2q^2}{\pi} \sum_s \int d\mathbf{k} \frac{\mathbf{kk}}{k^4} \exp\{i\mathbf{k}\cdot[(\mathbf{R}_p - \mathbf{R}_s)_t - (\mathbf{R}_p - \mathbf{R}_s)_{t-\tau}]\}\end{aligned}$$

where $(\mathbf{R}_p - \mathbf{R}_s)_t$, for instance denotes the quantity at t.

Moreover, we replace Σ_s with $\int d\mathbf{r}\, d\mathbf{v}\, g(\mathbf{v}) = n \int d\mathbf{v}\, g(\mathbf{v})$, where n and $g(\mathbf{v})$ are, respectively the uniform number density and the velocity distribution function of the electrons. We can now write the diffusion tensor as

$$D_p = \frac{2q_p{}^2 q^2 n}{m_p{}^2 \pi} \int d\mathbf{v} \int_0^\infty d\tau \int d\mathbf{k} \frac{\mathbf{k}\cdot\mathbf{k}}{k^4} e^\theta g(\mathbf{v}) \quad (13a)$$

$$\theta \equiv i\mathbf{k}\cdot[(\mathbf{R}_p - \mathbf{R}_s)_t - (\mathbf{R}_p - \mathbf{R}_s)_{t-\tau}]. \quad (13b)$$

In the case without magnetic field and space charge, we can write θ from Eqs. (6) and (7) as

$$\theta = i\mathbf{k}\cdot\mathbf{u}\tau,$$

where $\mathbf{u} \equiv \mathbf{v}_p - \mathbf{v}$. Hence we obtain the familiar diffusion tensor,

$$D_p = \frac{2q_p{}^2q^2n}{m_p{}^2}\int d\mathbf{v}\int d\mathbf{k}\frac{\mathbf{kk}}{k^4}\delta(\mathbf{k}\cdot\mathbf{u})g(\mathbf{v})$$
$$= \frac{2\pi q_p{}^2q^2n\ln\Lambda}{m_p{}^2}\int d\mathbf{v}\frac{u^2\delta_{ij} - u_iu_j}{u^3}g(\mathbf{v}), \qquad (14)$$

where we take the Coulomb logarithm, $\ln\Lambda \equiv \ln(k_{\max}/k_{\min})$ out of the integral because its velocity dependency is weak. Here $k_{\max} = k_D$ and $k_{\min} = 1/b_I$ are valid, where k_D is the reciprocal of the electron Debye length and b_I the minimum impact parameter.

3.2. *Diffusion Tensor With Magnetic Field and Without Space Charge*

In this case, the diffusion tensor of the ions is also given by Eq. (13). Therefore, we calculate Eq. (13b) by using Eqs. (6) and (7).

$$\theta = i(\mathbf{k}\cdot\mathbf{v}_p - k_zv_z)\tau + i\frac{k_\perp v_\perp}{\Omega}$$
$$\times \{\sin[\Omega(t-\tau) - \varphi_v] - \sin[\Omega t - \varphi_v]\}$$

where $k_\perp = (k_x{}^2 + k_y{}^2)^{1/2}$, $v_\perp = (v_x{}^2 + v_y{}^2)^{1/2}$, $\varphi_v = \psi_v - \psi_k$, $\psi_v = \tan^{-1}(v_y/v_x)$, and $\psi_k = \tan^{-1}(k_y/k_x)$. Thus we can calculate the diffusion tensor from Eq. (13) by using the Bessel function expansion formula

$$D_p = \frac{2q_p{}^2q^2n}{m_p{}^2\pi}\int d\mathbf{v}\int_0^\infty d\tau\int d\mathbf{k}\frac{\mathbf{kk}}{k^4}$$
$$\times [\sum_\nu J_\nu{}^2(\zeta)e^{i(\mathbf{k}\cdot\mathbf{v}_p - k_zv_z - \nu\Omega)\tau}]$$
$$\times g(\mathbf{v})$$
$$= \frac{2q_p{}^2q^2n}{m_p{}^2}\sum_\nu\int d\mathbf{v}\int d\mathbf{k}\frac{\mathbf{kk}}{k^4}J_\nu{}^2(\zeta)$$
$$\times \delta(\mathbf{k}\cdot\mathbf{v}_p - k_zv_z - \nu\Omega)g(\mathbf{v}) \qquad (15)$$

where $\zeta = k_\perp v_\perp/|\Omega|$ and J_ν is the Bessel function of the ν-th order.

If the magnetic field is very strong, the condition $\zeta \ll 1$ becomes valid. In this case, the diffusion tensor is simply written as

$$D_p = \frac{2q_p{}^2q^2n}{m_p{}^2}\int d\mathbf{v}\int d\mathbf{k}\frac{\mathbf{kk}}{k^4}$$
$$\times \delta(\mathbf{k}\cdot\mathbf{v}_p - k_zv_z)g(\mathbf{v})$$
$$= \frac{2\pi q_p{}^2q^2n\ln\Lambda_2}{m_p{}^2}\int d\mathbf{v}\frac{u^2\delta_{ij} - u_iu_j}{u^3}g(\mathbf{v}) \qquad (16)$$

for $k_D < k < k_\rho$, where $\mathbf{u} = \mathbf{v}_p - \mathbf{v}_z$, $\mathbf{v}_z = v_z\hat{\mathbf{z}}$, $\Lambda_2 \equiv k_\rho/k_D$, and k_ρ is the reciprocal of the Larmor radius.

3.3. *Diffusion Tensor With Magnetic Field and Space Charge*

In this case, we can merely calculate θ by using Eqs. (6) and (7).

$$\theta = i(\mathbf{k}\cdot\mathbf{v}_p - k_zv_z)\tau$$
$$+ i\zeta_1\{\sin[\omega_1(t-\tau) - \varphi_v]$$
$$- \sin[\omega_{1t} - \varphi_v]$$
$$- \sin[\omega_2(t-\tau) - \varphi_v]$$
$$+ \sin[\omega_2 t - \varphi_v]\}$$
$$+ i\zeta_2\{-\cos[\omega_1(t-\tau) - \varphi]$$
$$+ \cos[\omega_1 t - \varphi]\}$$
$$+ i\zeta_3\{\cos[\omega_2(t-\tau) - \varphi]$$
$$- \cos[\omega_2 t - \varphi]\},$$

where we define $\varphi_v = \psi_v - \psi_k$, $\varphi = \psi - \psi_k$, $r = (x^2 + y^2)^{1/2}$, $\psi = \tan^{-1}(y/x)$,

$$\zeta_1 = \frac{k_\perp v_\perp}{\omega_1 - \omega_2} \doteqdot \frac{k_\perp v_\perp}{|\Omega|}, \quad \zeta_2 = \frac{\omega_2 k_\perp r}{\omega_1 - \omega_2} \doteqdot \frac{\omega_p{}^2}{\Omega^2}k_\perp r,$$

and

$$\zeta_3 = \frac{\omega_1 k_\perp r}{\omega_1 - \omega_2} \doteqdot k_\perp r.$$

Consequently we can calculate the diffusion tensor from Eq. (13) by using the Bessel function expansion.

$$D_p = \frac{2q_p{}^2q^2n}{m_p{}^2}\int d\mathbf{v}\int d\mathbf{k}\,\frac{\mathbf{kk}}{k^4}I_1I_2I_3I_4\delta(\Omega_r)g(\mathbf{v}), \tag{17}$$

where $\Omega_r \equiv \mathbf{k}\cdot\mathbf{v}_p - k_zv_z - \nu_1\omega_1 + \nu_2\omega_2 + \nu_3\omega_1 - \nu_4\omega_2$ and

$$I_1 \equiv \sum_{\nu_1} J^2_{\nu_1}(\zeta_1) \tag{18a}$$

$$I_2 \equiv \sum_{\nu_2} J^2_{\nu_2}(\zeta_1) \tag{18b}$$

$$I_3 \equiv \sum_{\nu_3} J^2_{\nu_3}(\zeta_2) \tag{18c}$$

$$I_4 \equiv \sum_{\nu_4} J^2_{\nu_4}(\zeta_3). \tag{18d}$$

If there is no space charge, $\omega_1 = -\Omega$, $\omega_2 = 0$, $\zeta_1 = k_\perp v_\perp/|\Omega|$, and $\zeta_2 = 0$ are valid from $\omega_p = 0$. Therefore, $I_2 = I_3 = I_4 = 1$ is given and Eq. (17) reduces to Eq. (15) which is previously given for no space charge.

If the magnetic field is strong, we can rewrite Eq. (17) for $|\zeta_1| \ll 1$ as

$$D_p = \frac{2q_p{}^2q^2n}{m_p{}^2}\int d\mathbf{v}\int d\mathbf{k}\,\frac{\mathbf{kk}}{k^4}I_3I_4\delta(\Omega_r)g(\mathbf{v}), \tag{19a}$$

where $\Omega_r = \mathbf{k}\cdot\mathbf{v}_p - k_zv_z + \nu_3\omega_1 - \nu_4\omega_2$. When the magnetic field is even stronger, the condition $|\zeta_2| \ll 1$ is also valid. Then, since $I_3 = 1$ is satisfied, the diffusion tensor can be written as

$$D_p = \frac{2q_p{}^2q^2n}{m_p{}^2}\sum_\nu\int d\mathbf{v}\int d\mathbf{k}\,\frac{\mathbf{kk}}{k^4}J_\nu{}^2(\zeta_3) \times \delta(\mathbf{k}\cdot\mathbf{v}_p - k_zv_z - \nu\omega_2)g(\mathbf{v}) \tag{19b}$$

Here it should be noted that $\zeta_3 \cong k_\perp r$ is often much larger than unity in the experimental condition.[17] In this case, the integration of Eq. (19b) can only be done by numerical calculations.

Next, we rewrite the diffusion tensor in a form that can be analytically integrated, under the conditions $|\zeta_1| \ll 1$, $|\zeta_2| \ll 1$ and $|\zeta_3| \gg 1$ because these conditions will be approximately satisfied in the experiments. In this case,

$$\begin{aligned}\theta &\doteqdot i(\mathbf{k}\cdot\mathbf{v}_p - k_zv_z - \zeta_3\omega_2\sin\varphi)\tau \\ &\equiv i(\mathbf{k}\cdot\mathbf{u} + k_\perp w\sin\varphi)\tau \\ &\equiv i\alpha_r\tau\end{aligned}$$

is given, where $\alpha_r = k_\perp(u_\perp\cos\varphi_u + w\sin\varphi) + k_zu_z$, $\mathbf{u} = \mathbf{v}_p - \mathbf{v}_z$ and

$$w \equiv -\frac{\zeta_3\omega_2}{k_\perp} = -\frac{\omega_1\omega_2 r}{\omega_1 - \omega_2} = \frac{\omega_1 v_\theta}{\omega_1 - \omega_2} \doteqdot v_\theta.$$

The quantity, $v_\theta \equiv -\omega_2 r$ is the rotation velocity of the electrons due to the solenoid magnetic field and the radial electric field of the uniform space charge. Hence we obtain the diffusion tensor.

$$D_p = \frac{2q_p{}^2q^2n}{m_p{}^2}\int d\mathbf{v}\int d\mathbf{k}\,\frac{\mathbf{kk}}{k^4}\delta(\alpha_r)g(\mathbf{v}).$$

Integrating this equation with respect to $\mathbf{k}$, the following expression is obtained.

$$D_p = \frac{2\pi q_p{}^2q^2n\ln\Lambda_2}{m_p{}^2}\int d\mathbf{v}\,\frac{u^2\delta_{ij} - u_iu_j}{u^3}g(\mathbf{v}), \tag{20}$$

where $\mathbf{u} = \mathbf{v}_p - \mathbf{v}_z$ is used for $|u_\perp\cos\varphi_u| \gg |w\sin\varphi|$; on the contrary, $\mathbf{u} = \mathbf{v}_\varphi + \mathbf{v}_{pz} - \mathbf{v}_z$ should be used for $|u_\perp\cos\varphi_u| \ll |w\sin\varphi|$ and $\mathbf{v}_\varphi \equiv (w\sin\varphi, -w\cos\varphi, 0)$. Then, the diffusion tensor given by Eq. (20) can be approximately applied to either case with $\mathbf{u} = \mathbf{v}_p + \mathbf{v}_\varphi - \mathbf{v}_z$.

4. *Friction Coefficient in Velocity Space*

We will calculate the friction coefficient of the ion in velocity space. Although only the variation of momentum of the ion is calculated, it gives the correct friction coefficient because we can neglect the displacement on the ion.

4.1. *Friction Coefficient Without Magnetic Field and Space Charge*

The variation of the momentum of the ion or the friction coefficient is defined as

$$A_p \equiv \left\langle\frac{\Delta\mathbf{v}_p}{\tau_c}\right\rangle = \left\langle\frac{d\mathbf{v}_p}{dt}\right\rangle. \tag{21}$$

Since the self-field of the ion can be ignored, we obtain from Eqs. (5b) and (11a).

$$\begin{aligned}\frac{d\mathbf{v}_p}{dt} &= \frac{q_p}{m_p}\mathbf{E}_e[\mathbf{r}_p(t)] \\ &= -\frac{q_pq_i}{2\pi^2m_p} \times \sum_s\int d\mathbf{k}\,\frac{\mathbf{k}}{k^2}e^{i\mathbf{k}\cdot(\mathbf{R}_p-\mathbf{R}_s)_t}[1 - i\mathbf{k}\cdot\Delta\mathbf{r}_s].\end{aligned}$$

The first term does not contribute on the average. On the other hand, we have from Eq. (6a)

$$\Delta \mathbf{r}_s = \frac{q}{m}\int_0^t d\tau H(\tau)\cdot\mathbf{E}[\mathbf{r}_s(t-\tau), t-\tau]$$

$$\doteqdot \frac{q}{m}\int_0^t d\tau\, H(\tau)\cdot\mathbf{E}_p[\mathbf{r}_s(t-\tau), t-\tau]$$

$$\doteqdot -\frac{qq_p i}{2\pi^2 m}\int_0^t d\tau\, H(\tau)$$

$$\times \int d\mathbf{k}_1 \frac{\mathbf{k}_1}{k_1^2} e^{i\mathbf{k}_1\cdot(\mathbf{R}_s-\mathbf{R}_p)_{t-\tau}}.$$

Here we neglect the contribution from the background electrons. This approximation is satisfied when the interaction length is shorter than the Debye length.

Substituting these equations into Eq. (21), we obtain

$$A_p = \left\langle \frac{d\mathbf{v}_p}{dt}\right\rangle = \frac{q_p^2 q^2 i}{m_p m 4\pi^4}\langle I\rangle$$

$$\langle I\rangle \equiv \Bigg\langle \sum_s \int d\mathbf{k}\frac{\mathbf{k}}{k^2} e^{i\mathbf{k}\cdot(\mathbf{R}_p-\mathbf{R}_s)_t}$$

$$\times \left[\mathbf{k}\cdot\int_0^t d\tau\, H(\tau)\cdot\int d\mathbf{k}_1\frac{\mathbf{k}_1}{k_1^2}\right.$$

$$\left.\times\, e^{i\mathbf{k}_1\cdot(\mathbf{R}_s-\mathbf{R}_p)_{t-\tau}}\right\rangle$$

$$= (2\pi)^3\sum_s\int_0^t d\tau\int d\mathbf{k}\frac{\mathbf{k}}{k^2}\frac{\mathbf{k}\cdot H(\tau)\cdot\mathbf{k}}{k^2} e^{\theta}.$$

Since $\theta = i\mathbf{k}\cdot[(\mathbf{R}_p-\mathbf{R}_s)_t - (\mathbf{R}_p-\mathbf{R}_s)_{t-\tau}]$ does not explicitly depend on t, we can extend the region of integration with respect to τ to infinity. Moreover, we replace Σ_s with $\int d\mathbf{r}\, d\mathbf{v}\, g(\mathbf{v}) = n\int d\mathbf{v}\, g(\mathbf{v})$. As a result, we obtain the friction coefficient

$$A_p = \frac{2q_p^2q^2ni}{m_p m\pi}\int d\mathbf{v}\int_0^\infty d\tau \times \int d\mathbf{k}\frac{\mathbf{k}}{k^2}\frac{\mathbf{k}\cdot H(\tau)\cdot\mathbf{k}}{k^2} e^{\theta} g(\mathbf{v}). \tag{22}$$

If there is no magnetic field or space charge, we get

$$H(\tau) = [\tau]$$

$$\theta = i\mathbf{k}\cdot\mathbf{u}\tau$$

$$\mathbf{u} = \mathbf{v}_p - \mathbf{v}.$$

Thus we have the friction coefficient

$$A_p = \frac{2q_p^2q^2ni}{m_p m\pi}\int d\mathbf{v}\int_0^\infty d\tau\int d\mathbf{k}\frac{\mathbf{k}}{k^4}\tau e^{\theta} g(\mathbf{v})$$

$$= \frac{2q_p^2q^2n}{m_p m}\int d\mathbf{v}\int d\mathbf{k}\frac{\mathbf{k}}{k^4}\frac{d\delta(\mathbf{k}\cdot\mathbf{u})}{d(\mathbf{k}\cdot\mathbf{u})} g(\mathbf{v})$$

$$= \frac{-4\pi q_p^2q^2n\ln\Lambda}{m_p m}\int d\mathbf{v}\frac{\mathbf{u}}{u^3} g(\mathbf{v}), \tag{23}$$

where we take $\ln\Lambda = \ln(k_{\max}/k_{\min})$ out the integral because of its weak velocity dependency, and $u = |\mathbf{u}|$.

4.2. *Friction Coefficient With Magnetic Field and Without Space Charge*

Since Eq. (22) is also valid in this case and θ was previously given, we have

$$\mathbf{k}\cdot H(\tau)\cdot\mathbf{k} = \frac{\sin\Omega\tau}{\Omega}k_\perp^2 + \tau k_z^2.$$

Hence we obtain by using the Bessel function expansion.

$$A_p = \frac{2q_p^2q^2ni}{m_p m\pi}\int d\mathbf{v}\int_0^\infty d\tau\int d\mathbf{k}\frac{\mathbf{k}}{k^4}$$

$$\times\left[\frac{\sin\Omega\tau}{\Omega}k_\perp^2 + \tau k_z^2\right] e^{\theta} g(\mathbf{v})$$

$$= \frac{2q_p^2q^2n}{m_p m}\sum_\nu\int d\mathbf{v}\int d\mathbf{k}\frac{\mathbf{k}}{k^4}$$

$$\times\left[\frac{\delta(\alpha_r+\Omega)-\delta(\alpha_r-\Omega)}{2\Omega}k_\perp^2\right.$$

$$\left. + k_z^2\frac{d\delta(\alpha_r)}{d\alpha_r}\right] J_\zeta^2(\zeta) g(\mathbf{v}), \tag{24}$$

where $\alpha_r = \mathbf{k}\cdot\mathbf{v}_p - k_z v_z - u\Omega$ and $\zeta = k_\perp v_\perp/|\Omega|$

If the magnetic field is strong, $|\zeta| \ll 1$, Eq. (24) reduces to

$$A_p = \frac{2q_p^2q^2n}{m_p m}\int d\mathbf{v}$$

$$\times \int d\mathbf{k} \frac{\mathbf{k}}{k^4} k_z^2 \frac{d\delta(\alpha)}{d\alpha} g(\mathbf{v})$$

$$= -\frac{2\pi q_p^2 q^2 n \ln \Lambda_2}{m_p m}$$

$$\times \int d\mathbf{v} \frac{(u_\perp^2 - 2u_z^2)\mathbf{u}_\perp + 3u_\perp^2 \mathbf{u}_z}{u^5} g(\mathbf{v}). \quad (25)$$

for $k_D < k < k_\rho$, where $\alpha = \mathbf{k}\cdot\mathbf{v}_p - k_z v_z$, $\mathbf{u} = \mathbf{v}_p - \mathbf{v}_z$ and $\Lambda_2 = k_\rho / k_D$.

On the other hand, if the magnetic field is weak, $|\zeta| \gg 1$ and $|\Omega\tau| \ll 1$, the friction coefficient without magnetic field given by Eq. (23) is again given from $\theta = i\mathbf{k}\cdot\mathbf{u}\tau$, $\mathbf{k}\cdot H(\tau)\cdot\mathbf{k} = \tau k^2$ and $\mathbf{u} = \mathbf{v}_p - \mathbf{v}$.

4.3. *Friction Coefficient With Magnetic Field and Space Charge*

As can be understood by comparing Eq. (6) with Eq. (8), the friction coefficient given by Eq. (22) is also valid if we replace $H(\tau)$ with $K_3(\tau)$. We therefore calculate

$$\mathbf{k}\cdot K_3(\tau)\cdot\mathbf{k} = \frac{\sin \omega_1\tau - \sin \omega_2\tau}{\omega_1 - \omega_2} + \tau k_z^2.$$

Then θ is already given. Using these expressions, we have

$$A_p = \frac{2q_p^2 q^2 n i}{m_p m \pi} \int d\mathbf{v} \int_0^\infty d\tau \int d\mathbf{k} \frac{\mathbf{k}}{k^4}$$

$$\times \left[\frac{\sin \omega_1\tau - \sin \omega_2\tau}{\omega_1 - \omega_2} k_\perp^2 + \tau k_z^2\right] e^\theta g(\mathbf{v})$$

$$= \frac{2q_p^2 q^2 n}{m_p m} \int d\mathbf{v} \int d\mathbf{k} \frac{\mathbf{k}}{k^4} I_1 I_2 I_3 I_4$$

$$\times \left[\frac{Q}{2(\omega_1 - \omega_2)} k_\perp^2 + k_z^2 \frac{d\delta(\Omega_r)}{d\Omega_r}\right] g(\mathbf{v}), \quad (26)$$

where $Q \equiv \delta(\Omega_r + \omega_1) - \delta(\Omega_r - \omega_1) - \delta(\Omega_r + \omega_2) + \delta(\Omega_r - \omega_2)$, $\Omega_r \equiv \mathbf{k}\cdot\mathbf{v}_p - k_z v_z - \nu_1\omega_1 + \nu_2\omega_2 + \nu_3\omega_1 - \nu_4\omega_2$ and I_1 to I_4 are given by Eq. (18).

When the magnetic field is strong, we can write Eq. (26) for $|\zeta_1| \ll 1$ as

$$A_p = \frac{2q_p^2 q^2 n}{m_p m} \int d\mathbf{v} \int d\mathbf{k} \frac{\mathbf{k}}{k^4} I_3 I_4$$

$$\times \left[\frac{Q}{2(\omega_1 - \omega_2)} k_\perp^2 + k_z^2 \frac{d\delta(\Omega_r)}{d\Omega_r}\right] g(\mathbf{v}), \quad (27a)$$

where $\Omega_r = \mathbf{k}\cdot\mathbf{v}_p - k_z v_z + \nu_3\omega_1 - \nu_4\omega_2$. When the magnetic field is even stronger, $|\zeta_2| \ll 1$, $I_3 = 1$ is also valid and Eq. (27a) reduces to

$$A_p = \frac{2q_p^2 q^2 n}{m_p m} \sum_\nu \int d\mathbf{v}$$

$$\times \int d\mathbf{k} \frac{\mathbf{k}}{k^4} \left[\frac{Q}{2(\omega_1 - \omega_2)} k_\perp^2 + k_z^2 \frac{d\delta(\Omega_r)}{d\Omega_r}\right] J_\nu^2(\zeta_3) g(\mathbf{v}), \quad (27b)$$

where $\Omega_r = \mathbf{k}\cdot\mathbf{v}_p - k_z v_z - \nu\omega_2$. The integration of Eq. (27b) requires numerical calculation because $\zeta_3 \doteqdot k_\perp r$ is often much larger than unity.

Next we rewrite the friction coefficient in another form for $|\zeta_1| \ll 1$, $|\zeta_2| \ll 1$ and $|\zeta_3| \gg 1$ by using a procedure similar to that used in the calculation of the diffusion tensor.

$$A_p = \frac{2q_p^2 q^2 n}{m_p m} \int d\mathbf{v} \int d\mathbf{k} \frac{\mathbf{k}}{k^4} k_z^2 \frac{d\delta(\alpha_r)}{d\alpha_r} g(\mathbf{v}), \quad (28)$$

where $\alpha_r = k_\perp(u_\perp \cos \varphi_u + w \sin \varphi) + k_z u_z$ and $\mathbf{u} = \mathbf{v}_p - \mathbf{v}_z$. By integration of Eq. (28) with respect to $\mathbf{k}$, the friction coefficient is found to be

$$A_p = \frac{-2\pi q_p^2 q^2 n \ln \Lambda_2}{m_p m} \int d\mathbf{v}$$

$$\times \frac{(u_\perp^2 - 2u_z^2)\mathbf{u}_\perp + 3u_\perp^2 \mathbf{u}_z}{u^5} g(\mathbf{v}), \quad (29)$$

for $k_D < k < k_\rho$, where $\mathbf{u} = \mathbf{v}_p - \mathbf{v}_z$ is used for $|u_\perp \cos \varphi_u| \gg |w \sin \varphi|$; on the contrary, $\mathbf{u} = \mathbf{v}_\varphi + \mathbf{v}_{pz} - \mathbf{v}_z$ is used for $|u_\perp \cos \varphi| \ll |w \sin \varphi|$ and $\mathbf{v}_\varphi = (w \sin \varphi, -w \cos \varphi, 0)$. As a result, the friction coefficient described by Eq. (29) can be approximately applied to either case with $\mathbf{u} = \mathbf{v}_p + \mathbf{v}_\varphi - \mathbf{v}_z$.

When we rewrite Eq. (29) in a cylindrical coordinate system by using the transformation

$$A_{pr} = A_{px} \cos \psi + A_{py} \sin \psi$$

$$A_{p\theta} = -A_{px} \sin \psi + A_{py} \cos \psi,$$

we have

$$A_{pr} = 0 \tag{30a}$$

$$A_{p\theta} = \frac{2\pi q_p{}^2 q^2 n \ln \Lambda_2}{m_p m} \int d\mathbf{v} \frac{(w^2 - 2u_z{}^2)w}{(w^2 + u_z{}^2)^{5/2}} g(\mathbf{v}) \tag{30b}$$

for $|u_\perp \cos \varphi_u| \ll |w \sin \varphi|$, where $w = v_\theta$ which is the rotation velocity of the electrons in the steady state. In this case, though A_{pz} does not materially change, $\nu_{p\perp}$ is entirely different from the usual friction coefficient because $A_{p\perp}$ does not depend on $\mathbf{u}_\perp = \mathbf{v}_{p\perp}$, $|\mathbf{v}_{p\perp}|$ does not decrease but $\mathbf{v}_{p\perp}$ approaches w. In other words, the drag force perpendicular to the magnetic field has no effect on the proton for $|w \sin \varphi| \gg |u_\perp \cos \varphi_u|$ or for $|w| \gg |\mathbf{v}_{p\perp}|$.

On the other hand, since $\mathbf{k}\cdot K_3(\tau)\cdot\mathbf{k} = k^2\tau$ and $\theta = i\mathbf{k}\cdot\mathbf{u}\tau$ are given for $|\zeta_1| \gg 1$ or for $|\omega_2\tau| \ll |\omega_1\tau| \ll 1$ and $|\omega_2 t| \ll |\omega_1 t| \ll 1$, the friction coefficient without magnetic field given by Eq. (23) again becomes valid for $k > k_\rho$.

III. CONCLUSIONS

We have shown a compact way to estimate cooling rates and diffusion coefficients for an electron-cooling experiment. We want to stress the following two major results:

(i) The presence of a strong magnetic field makes the cooling rate independent of the transverse velocity spread of the electron beam. In this case the cooling rate depends solely on the longitudinal velocity spread. An enhancement of the cooling can be expected if the latter spread is considerably smaller than the transverse. Unfortunately in some real situations,[17] it is desired to cool hadronic beams with a momentum spread comparable, if not larger, than the transverse spread. For these beams, the cooling rates are not expected to be affected by the presence of a solenoidal field.

(ii) The presence of space charge in the electron beam weakens the performance of electron cooling. This happens because the drag force is no longer directed radially with respect to the axis of the two beams but rotates, acquiring an azimuthal component that does not affect the cooling. Therefore, long electron beams are not recommended for cooling.

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