Particle Accelerators
1979, Vol. 9, pp. 213–222

FURTHER THEORETICAL STUDIES OF THE BEAM BREAKUP INSTABILITY†

V. K. NEIL and L. S. HALL

Lawrence Livermore Laboratory, University of California, Livermore, California 94550, U.S.A.

and

R. K. COOPER

Los Alamos Scientific Laboratory, University of California, Los Alamos, New Mexico 87545, U.S.A.

(Received August 8, 1978)

The beam breakup instability in linear accelerators is investigated theoretically. A somewhat restricted model of the beam-transport system is used. With this model, it is possible to treat the instability analytically and obtain an expression for the amplitude of tranverse coherent oscillations as a function of time at any position in the machine, taking external focusing into account exactly. The theory is directly applicable to linear induction accelerators in which a solenoidal beam-transport system is employed. It may also be used to obtain approximate analytic results for an rf linac consisting of side-coupled cavities and utilizing quadruple focusing.

1 INTRODUCTION

With the growing interest in high-current ($\gtrsim 1$ kA) beams of electrons with energy of the order of 10 to 100 MeV, it seems appropriate to re-examine the beam breakup instability in electron linear accelerators.[1] This instability was first observed in the SLAC 3-km electron accelerator, and is presently an effect that limits the current in that machine.

The linear induction accelerator is particularly well suited to the task of accelerating multi-kiloampere beams of electrons providing the pulse duration is around or less than 100 nsec. This work is primarily devoted to such devices, but the results are applicable to certain types of rf accelerators.

We treat a somewhat idealized model of the beam-transport system. With this model we are able to take into account the focusing provided by the transport system, treat the instability analytically, and obtain an expression for the transverse displacement of the beam as a function of time at any position in the accelerator. The transport system is such that the matrix for transverse motion of the beam centroid from one unit to the next is independent of position. That is, the phase advance μ from one unit to the next is independent of distance down the machine, and the betatron function β_f at each unit is proportional to the particle momentum p at that unit. We further assume that the units are identical. Although a transport system with these characteristics may not be desirable in practice, a method of realizing these conditions in a solenoidal transport system is given in Section 2.

At the nth accelerator unit, the transverse displacement of the beam as a function of time is given by a series consisting of n terms. If the number of units is not too large the series may be used directly, and it is found that the amplitude of transverse oscillations grows algebraically in time at any unit. The series representation becomes unwieldy if the number of units is large. From the series we derive an approximate expression that is valid for large n and predicts that the amplitude grows exponentially with the exponent proportional to $(\beta_f nt/p)^{1/2}$.

† This work was performed by the Lawrence Livermore Laboratory and is jointly supported by the U.S. Department of Energy under Contract No. W-7405-Eng-48 and the Department of the Navy under Contract N00014-78-F0012.

The matrix formalism is developed in Section 2. In Section 3 we describe the instability mechanism and discuss the transverse coupling impedance. Section 4 contains the derivation of the series representation. The expression valid for large n is derived in Section 5 which also contains a calculation of the maximum amplitude as limited by the value of the quality factor Q of the units.

In applying the theory to linear induction accelerators, one must consider that accelerating units in these machines are of two basic types. The first type uses soft iron for the induction core. Since soft iron is a good conductor, the structures indeed have resonant modes with the electromagnetic field pattern that drives the instability. Haimson[2] has measured the resonant frequencies, coupling impedance, and quality factor in an existing induction accelerator unit at the National Bureau of Standards. The second type of accelerating unit uses ferrite for the induction core. Since ferrite is an insulator for high frequencies, this second type of unit may be relatively free of the modes that can drive the instability. For any unit, measurements of the pertinent parameters can be performed and the result of these measurements used in our theory to determine whether or not the instability will be troublesome in an accelerator employing the unit.

At the end of Section 5 we give a numerical example of a 2-kA, 100-MeV electron accelerator employing units with characteristics similar to those given in Ref. 2. We find that instability is indeed a problem for a pulse duration of 100 nsec.

Another example treats a side-coupled cavity rf proton linac. The theory is not directly applicable to such a device for reasons given in Section 5, but it can give some indication of whether or not the instability may occur. The example shows that such a device with a final energy of 800 MeV is susceptible to the instability if the beam current is of the order of 100 mA.

2 TRANSVERSE PARTICLE MOTION

The linear accelerator consists of accelerating units spaced a distance L apart. Between the accelerating gaps are focusing elements. We primarily consider solenoidal focusing with an axial magnetic field B. The formalism is also valid for a quadrupole focusing system with a complete cell between units.

The transverse motion of particles in the beam is characterized by a distance x from the axis of the solenoid and q, the momentum of the particles in the radial (x) direction. In the absence of any electromagnetic self-forces, a particle in a solenoidal focusing system undergoes a rotation θ in the azimuthal direction as it progresses down the machine. The angle θ is given by

$$\Theta(z) = \frac{z}{2\rho}, \tag{2.1}$$

in which z is the axial coordinate and ρ is the radius of gyration a particle would have in the field B if its velocity vector were perpendicular to the field. This radius is given by

$$\rho = \frac{pc}{eB}, \tag{2.2}$$

in which c is the speed of light, e is the particle's charge, and p is the particle's total momentum.

If the solenoid extends the entire distance L between the gaps, the particle will rotate through an angle $\theta \equiv \Theta(L) = L/2\rho$. We consider a coordinate system that rotates in this manner so that the x coordinate measures the distance of the particle to the axis of the solenoid. In this rotating coordinate system, the quantities $x_n^{(-)}$ and $q_n^{(-)}$ of a particle entering the nth gap are related to the quantities $x_{n-1}^{(+)}$ and $q_{n-1}^{(+)}$ leaving the $n-1$st gap by the matrix M_f. We have

$$\begin{pmatrix} x \\ q \end{pmatrix}_n^{(-)} = \begin{pmatrix} \cos\theta & 2\left(\frac{\rho}{p}\right)\sin\theta \\ -\left(\frac{p}{2\rho}\right)\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ q \end{pmatrix}_{(n-1)}^{(+)}. \tag{2.3}$$

In this idealized situation, the phase advance $\mu \equiv \theta$ and $\beta_f = 2\rho$. In the following analysis it is essential that the matrix M_f be independent of n. From Eq. (2.2) we see that β_f is proportional to p if B has the same value for all z. We may then keep θ constant by letting L increase in proportion to p.

In practice, the effective length D of the solenoid will always be less than L. If the solenoid is centrally located between the gaps, it can be shown that

$$\cos\mu = \cos\theta' - \left(\frac{l}{2\rho}\right)\sin\theta', \tag{2.4}$$

and

$$\beta_f = [4\rho l \cot\theta' + 4\rho^2 - l^2]^{1/2}, \tag{2.5}$$

in which $\theta' = D/2\rho$ and $2l = L - D$. We shall employ the form of M_f given by Eq. (2.3), keeping in mind such practical details.

It should be emphasized that Eq. (2.3) does not describe the motion of single particles within the beam (incoherent motion) if electromagnetic self-forces (i.e., forces generated by the coherent electromagnetic fields of the beam charge and current) are present. This work investigates the self-force generated by the interaction of the beam current with the accelerating units, and the effect of this self-force on the motion of the beam centroid (i.e., coherent transverse motion). Equation (2.3) is appropriate for this treatment if we interpret x as the radial position of the beam centroid and q as the average transverse momentum of beam particles.

In addition to the matrix M_f that describes transverse motion from one gap to the next, there exists a matrix M_g describing the effect of the gap on the transverse motion. If we assume that the position x of the beam centroid does not change in crossing the gap but that the value of q does change, we may write the matrix M_g in its most simple form, namely

$$M_g = \begin{pmatrix} 1 & 0 \\ \mathscr{R} & 1 \end{pmatrix}. \tag{2.6}$$

The phenomenon treated in this work is manifest by the quantity $\mathscr{R}$, which is an integral operator and will be treated in Section 3. There are other focusing or defocusing effects at the gap that arise from the detailed configuration as well as the time variation of the accelerating field. We do not take these effects into account in this work. In a linear induction accelerator, the time variation is negligible in any case.

3 THE INSTABILITY MECHANISM

The basic assumption of this work is that the accelerating units have a characteristic electromagnetic mode with a field configuration similar to the TM_{110} mode of a pill-box cavity. In a pill-box cavity the mode may be described by a vector potential in the z direction of the form (in cylindrical coordinates r, ϕ, z)

$$A_z \propto J_1(vr)\cos\phi, \tag{3.1}$$

in which J_1 is the ordinary Bessel function of order one, and the constant is chosen such that $J_1(vb) = 0$, where b is the radius of the cavity. Near the origin we have $J_1(vr) \approx vr/2$. The electric field of the mode has only an axial component E_z that is proportional to $r\cos\phi \equiv x$ near the origin. Furthermore, we have $B_y = -\partial A_z/\partial x$, which is uniform near the axis.

It is not necessary to assume a configuration of fields for the mode in the accelerating units. We merely state that the mode is characterized by a vector potential $\mathbf{A}_l(\mathbf{r})$ that obeys the Helmholtz equation

$$\nabla^2\mathbf{A}_l + \left(\frac{\omega}{c}\right)^2\mathbf{A}_l = 0, \tag{3.2}$$

in which ω is the characteristic frequency of the mode. The actual time-dependent vector potential $\mathbf{A}(\mathbf{r}, t)$ for the fields excited by the beam current $\mathbf{j}$ obeys the equation

$$\nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = -\frac{4\pi\mathbf{j}}{c}. \tag{3.3}$$

We assume

$$\mathbf{A}(\mathbf{r}, t) = \sigma(t)\mathbf{A}_l(r). \tag{3.4}$$

Inserting Eq. (3.4) into (3.3) and employing (3.2), we obtain

$$(\ddot{\sigma} + \omega^2\sigma)\mathbf{A}_l(r) = 4\pi c\mathbf{j}, \tag{3.5}$$

in which the dot indicates the time derivative. Taking the scalar product of this equation with $\mathbf{A}_l$ and integrating over the volume occupied by the fields of the mode, we have

$$\ddot{\sigma} + \omega^2\sigma = 4\omega c\left[\int \mathbf{j}\cdot\mathbf{A}_l\,\mathrm{d}v\right]\left[\int A_l^2\,\mathrm{d}v\right]^{-1}. \tag{3.6}$$

Implicit in the derivation of Eq. (3.6) is the assumption that the mode is excited by the beam only. That is, no information is transferred from one unit to the next except by the beam itself. This is a valid assumption if the frequency of the mode is below the cut-off frequency of the conducting pipe between units.

We take the displacement of the beam centroid to be in the x direction in the rotating coordinate system and now designate this displacement as ξ. We assume that ξ does not vary with z in the region enclosed by the integral $\int \mathbf{j}\cdot\mathbf{A}_l\,\mathrm{d}v$. If A_{lz} varies linearly with the coordinate x over the beam cross section, to a good approximation we may write

$$\int \mathbf{j}\cdot\mathbf{A}_l\,\mathrm{d}v = I\xi\int\frac{\partial A_{lz}}{\partial x}\,\mathrm{d}z. \tag{3.7}$$

The other integral in Eq. (3.6), $\int A_l^2\,\mathrm{d}v$, is related to the stored energy U in the mode. If the mode is driven at resonance with a $\sin \omega t$ or $\cos \omega t$ time dependence, it can be shown that

$$U = \frac{1}{8\pi}\int (E^2 + B^2)\,\mathrm{d}v = \frac{\omega^2}{8\pi c^2}\int A_l^2\,\mathrm{d}v. \tag{3.8}$$

Some vector identities, as well as Eq. (3.2), must be used to derive Eq. (3.8). It is necessary that the tangential component of either $\mathbf{A}_l$ or $\mathbf{B}_l$ be zero on all surfaces surrounding the volume.

In passing through the gap, a particle will encounter the field B_y and undergo a change in q given by

$$\Delta q = \frac{e\sigma}{c}\int\left(\frac{\partial A_{lz}}{\partial x}\right)\mathrm{d}z. \tag{3.9}$$

Even though there may be a contribution $\partial A_{lx}/\partial z$ to B_y, this contribution will not contribute to the integral in Eq. (3.9). We multiply Eq. (3.6) by $(e/c)\int(\partial A_{lz}/\partial x)\,\mathrm{d}z$ and employ Eqs. (3.7), (3.8) and (3.9) to obtain

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}t^2} + \omega^2\right]\Delta q = \frac{eI\xi\omega^2}{2c^2U}\left[\int\left(\frac{\partial A_{lz}}{\partial x}\right)\mathrm{d}z\right]^2. \tag{3.10}$$

We now introduce the quantity k with units of inverse length by the definition

$$k \equiv \frac{\left[\int\left(\frac{\partial A_{lz}}{\partial x}\right)\mathrm{d}z\right]^2}{2U}. \tag{3.11}$$

All of the pertinent characteristics of the mode (except for the quality factor Q) are contained in this one quantity that may be determined experimentally. It is related to the so-called "transverse impedance" $Z_\perp$ by the relation

$$\frac{Z_\perp}{Q} = \frac{k}{\omega}. \tag{3.12}$$

(In these units an impedance has the dimensions of time/length. To obtain the value of $Z_\perp$ in ohms, multiply the value of sec/cm by 9×10^{11}.)

A finite quality factor Q may be included in our treatment by adding a term $(\omega/Q)\,\mathrm{d}(\Delta q)/\mathrm{d}t$ to the lefthand side of Eq. (3.10). We employ the notation $\alpha \equiv \omega/2Q$. For each accelerating unit, there are no fields present until such time as a displaced beam arrives at the unit. We measure time from that instant; thus Eq. (3.10) is to be solved with the initial conditions $\Delta q = 0$, $\mathrm{d}(\Delta q)/\mathrm{d}t = 0$ at $t = 0$. The desired solution relating Δq_n at the nth unit to ξ_n at the nth unit is given by

$$\Delta q_n = \frac{e\omega k}{c^2}\int e^{-\alpha(t-t')}I(t')\xi_n(t')\sin\omega(t-t')\,\mathrm{d}t'. \tag{3.13}$$

We are now in a position to relate ξ_n, q_n at the entrance to the nth gap to ξ_{n-1}, q_{n-1} at the entrance to the previous gap. We form the product $M = M_f M_g$ from Eqs. (2.3) and (2.7) and obtain

$$\begin{pmatrix}\xi\\ q\end{pmatrix}_n = \begin{pmatrix}\cos\theta + 2\left(\frac{\rho}{p}\right)\sin\theta\mathscr{R} & 2\left(\frac{\rho}{p}\right)\sin\theta \\ -\left(\frac{p}{2\rho}\right)\sin\theta + \cos\theta\mathscr{R} & \cos\theta\end{pmatrix}\begin{pmatrix}\xi\\ q\end{pmatrix}_{n-1}. \tag{3.14}$$

In this relation, $\mathscr{R}\xi_{n-1}$ is to be interpreted as Δq_{n-1} given by Eq. (3.13). There is one further assumption inherent in Eq. (3.14): the transit time from one unit to the next is the same for all portions of the beam. This is certainly true for relativistic particles. Thus, if ξ_{n-1} and q_{n-1} are functions of time measured from the time of arrival of the displaced beam at the n-1st unit, then ξ_n and q_n are functions of time measured from the time of arrival of the displaced beam at the nth unit.

It is now a simple task to investigate computationally the solution to Eq. (3.14). The computer code takes some initial values ξ_0 and q_0 and performs the indicated matrix multiplication n times to obtain the values ξ_n and q_n as a function of t at the nth gap. The matrix elements contain the particle energy, gap separation L and magnetic field B, all of which are allowed to vary from gap to gap. In addition, any time dependence of the beam current $I(t)$ may be treated. Two such computational treatments are described in Refs. 1 and 2.

4 AN ANALYTICAL SOLUTION

We consider the beam current to be independent of time and the matrix elements in Eq. (3.14) to be the same for all portions of the machine. It is necessary to take the Laplace transform of Eq.

(3.13). A tilde over a quantity devotes the Laplace transform, and we use s for the Laplace transform variable. We apply the convolution theorem to Eq. (3.13) to find

$$\Delta\tilde{q}_n = \frac{ekI}{c^2}\frac{\omega^2\tilde{\xi}_n}{[(s+\alpha)^2+\omega^2]} \equiv \tilde{\mathscr{R}}\tilde{\xi}_n. \tag{4.1}$$

Equation (3.14) may be written in terms of transformed quantities,

$$\begin{pmatrix}\tilde{\xi}\\ \tilde{q}\end{pmatrix}_n = \begin{pmatrix}\cos\theta + \left(\frac{2\rho}{p}\right)\sin\theta\tilde{\mathscr{R}} & \left(\frac{2\rho}{p}\right)\sin\theta \\ -\left(\frac{p}{2\rho}\right)\sin\theta + \cos\theta\tilde{\mathscr{R}} & \cos\theta\end{pmatrix}\begin{pmatrix}\tilde{\xi}\\ \tilde{q}\end{pmatrix}_{n-1}. \tag{4.2}$$

In matrix notation with $\mathbf{y}_n$ a vector with components $\tilde{\xi}_n$, $\tilde{q}_n$, we have

$$\mathbf{y}_n = \tilde{M}\mathbf{y}_{n-1}. \tag{4.3}$$

To obtain the vector $\mathbf{y}_n$ at the entrance of the nth gap in terms of the vector y_0 at the first ($n = 0$) gap, we iterate Eq. (4.3) to obtain

$$\mathbf{y}_n = \tilde{M}^n\mathbf{y}_0. \tag{4.4}$$

The nth power of the matrix M is found by the method of diagonalization. The eigenvalues of the matrix are found to be

$$\lambda_1 = \psi + (\psi^2 - 1)^{1/2},$$
$$\lambda_2 = \psi - (\psi^2 - 1)^{1/2},$$

in which

$$\psi = \cos\theta + \left(\frac{\rho}{p}\right)\sin\theta\tilde{\mathscr{R}}. \tag{4.5}$$

Next we find a matrix G with the determinant equal to unity and the inverse matrix G^{-1} such that

$$\tilde{M} = G^{-1}\Lambda G, \tag{4.6}$$

with

$$\Lambda = \begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}. \tag{4.7}$$

Raising Eq. (4.6) to the nth power, we find

$$\tilde{M}^n = G^{-1}\Lambda^n G, \tag{4.8}$$

with

$$\Lambda^n = \begin{pmatrix}\lambda_1^n & 0\\ 0 & \lambda_2^n\end{pmatrix}.$$

In terms of the quantities

$$g \equiv (\psi^2 - 1)^{1/2}, \tag{4.9}$$

and

$$h \equiv \left(\frac{\rho}{p}\right)\sin\theta\tilde{\mathscr{R}}, \tag{4.10}$$

the matrix G is given by

$$G = \begin{pmatrix} g+h & 2\left(\frac{\rho}{p}\right)\sin\theta \\ \dfrac{h-g}{4g\left(\frac{\rho}{p}\right)\sin\theta} & \dfrac{1}{2g}\end{pmatrix}. \tag{4.11}$$

$$\psi = \cos\theta + h.$$

The quantities λ_1^n and λ_2^n may be written in terms of $T_n(\psi)$ and $U_n(\psi)$, which are the Chebyshev polynomials of the first and second kind respectively. We employ the relations[3]

$$[\psi + (\psi^2 - 1)^{1/2}]^n + [\psi - (\psi^2 - 1)^{1/2}]^n = 2T_n(\psi), \tag{4.12a}$$

$$[\psi + (\psi^2 - 1)^{1/2}]^n - [\psi - (\psi^2 - 1)^{1/2}]^n = 2(\psi^2 - 1)^{1/2}U_{n-1} \tag{4.12b}$$

with the result

$$\Lambda^n = \begin{pmatrix}T_n(\psi) + gU_{n-1}(\psi) & 0\\ 0 & T_n(\psi) - gU_{n-1}(\psi)\end{pmatrix}. \tag{4.13}$$

Inserting Eqs. (4.11) and (4.13) into Eq. (4.8) we find the matrix $\tilde{M}^n$ to be given by

$$\tilde{M}^n = \begin{pmatrix} T_n(\psi) + hU_{n-1}(\psi) & 2\left(\frac{\rho}{p}\right)\sin\theta U_{n-1}(\psi) \\ \dfrac{(2h\cos\theta - \sin^2\theta)U_{n-1}(\psi)}{2\left(\frac{\rho}{p}\right)\sin\theta} & T_n(\psi) - hU_{n-1}(\psi)\end{pmatrix} \tag{4.14}$$

To perform the Laplace inversion, we isolate the Laplace variable s that occurs in $h = (\rho/p)\sin\theta\tilde{\mathscr{R}}$ and $\psi = \cos\theta + h$. A Taylor series expansion of $T_n(\psi)$ yields the expression

$$T_n(\cos\theta + h) = \sum_{l=0}^{n} \frac{h^l}{l!}\left(\frac{\mathrm{d}}{\mathrm{d}(\cos\theta)}\right)^l T_n(\cos\theta), \tag{4.15}$$

which may also be written in the form[4]

$$T_n(\cos\theta + h) = T_n(\cos\theta) + \sum_{l=1}^{n} \frac{n}{2l} C_{n-l}^{l}(\cos\theta)(2h)^l. \tag{4.16}$$

In Eq. (4.16), the functions C_{n-l}^{l} are the Gegenbauer (ultraspherical) polynomials. A similar expansion of $U_{n-1}(\psi)$ may be combined with Eq. (4.15) to yield

$$T_n(\psi) \pm hU_{n-1}(\psi) = T_n(\cos\theta) + \sum_{l=1}^{n} \frac{(n \pm l)}{2l} C_{n-l}^{l}(\cos\theta)(2h)^l. \tag{4.17}$$

We point out that $T_n(\cos\theta) \equiv \cos n\theta$.

We restrict ourselves to consideration of an initial displacement only, so that $\xi_0(t) \neq 0$, $q_0(t) = 0$. From Eqs. (4.4) and (4.13) we have

$$\tilde{\xi}_n(s) = \tilde{\xi}_0(s)[T_n(\psi) + hU_{n-1}(\psi)]. \tag{4.18}$$

We use the symbol $\mathscr{L}^{-1}$ to indicate the Laplace inversion. Employing Eq. (4.17), we write $\xi_n(t)$ in the form

$$\xi_n(t) = \xi_0(t)\cos n\theta + \sum_{l=1}^{n} \frac{(n+l)}{2l} C_{n-l}^{l}(\cos\theta) \times \mathscr{L}^{-1}\{(2h)^l\tilde{\xi}_0\}. \tag{4.19}$$

More compact notation can be achieved by introducing the "Alfven" current $I_a \equiv \gamma\beta m_0 c^3/e$, in which γ is the total energy of the particle in units of m_0c^2, and $\beta^2 = 1 - \gamma^{-2}$. For electrons, $I_a = 17\beta\gamma k\mathrm{A}$. Inserting the expression for $\tilde{\mathscr{R}}$ from Eq. (4.1) into Eq. (4.10) and noting that $p = \gamma\beta m_0 c$, we may write h in the form

$$h = \frac{k\rho\left(\dfrac{I}{I_a}\right)\omega^2 \sin\theta}{[(s+\alpha)^2 + \omega^2]}. \tag{4.20}$$

The Laplace inversion of $(2h)^l$ is readily shown to be[5]

$$\mathscr{L}^{-1}\{(2h)^l\} = \frac{2\omega\left[k\rho\sin\theta\left(\dfrac{I}{I_a}\right)\right]^l e^{-\alpha t}(\omega t)^l j_{l-1}(\omega t)}{(l-1)!}. \tag{4.21}$$

in which j_{l-1} is the spherical Bessel function of the first kind.

A particular functional form of $\xi_0(t)$ that is a favorite among stability theorists is $\xi_0(t) = \mathrm{d}\delta(t)/\omega$, where d is a constant with dimensions of length and $\delta(t)$ is the Dirac delta function. For this $\xi_0(t)$ we have $\xi_0(s) = \mathrm{d}/\omega$, and Eqs. (4.19) and (4.21) now yield

$$\xi_n(t) = \frac{\mathrm{d}}{\omega}\delta(t)\cos n\theta + \mathrm{d}e^{-\alpha t}\sum_{l=1}^{n}\left\{\frac{(n+l)}{l!}C_{n-l}^{l}(\cos\theta) \times \left[k\rho\sin\theta\left(\frac{I}{I_a}\right)\omega t\right]^l j_{l-1}(\omega t)\right\}. \tag{4.22}$$

Another form of $\xi_0(t)$ that is amenable to inversion is $\xi_0(t) = de^{-\alpha t}\sin\omega t$. For this form we have

$$\tilde{\xi}_0(s) = \frac{\omega\,\mathrm{d}}{[(s+\alpha)^2 + \omega^2]},$$

and the Laplace inversion is again readily performed, with the result

$$\xi_n(t) = \mathrm{d}e^{-\alpha t}\sin\omega t\cos n\theta + \mathrm{d}e^{-\alpha t}\sum_{l=1}^{n}\left\{\frac{(n+l)}{2l}C_{n-l}^{l}(\cos\theta) \times \left[k\rho\sin\theta\left(\frac{I}{I_a}\right)\omega t\right]^l \frac{\omega t j_l(\omega t)}{l!}\right\}. \tag{4.23}$$

Equation (4.23) shows that the amplitude $\xi_n(t)$ grows algebraically in time with all powers of t from 1 to n. For a modest number of accelerating units Eq. (4.23) may be evaluated numerically, the procedure being considerably easier than utilizing a computer program to solve the whole problem.

5 BEHAVIOR FOR N LARGE

We now obtain an approximate expression for ξ_n that is valid for n large. The function $w(\theta) \equiv \sin^l\theta C_{n-l}^{l}(\cos\theta)$ obeys the equation[6]

$$\frac{\mathrm{d}^2 w}{\mathrm{d}\theta^2} + \left[n^2 - \frac{l(l-1)}{\sin^2\theta}\right]w = 0. \tag{5.1}$$

Thus for $n^2 \sin^2\theta \gg l(l-1)$, w is either $\sin n\theta$ or $\cos n\theta$. We find the coefficient from the relation

$$C^l_{n-l}(\cos\theta) = \frac{1}{n2^{(l-1)}(l-1)!}\left(\frac{d}{d\cos\theta}\right)^l \cos n\theta, \tag{5.2}$$

which has previously been employed to derive Eq. (4.16). We employ the approximation $(d/d\cos\theta)^l \approx (-1)^l(\sin\theta)^{-l}(d/d\theta)^l$, so that

$$(\sin\theta)^l C^l_{n-l}(\cos\theta) \approx \frac{(-1)^l}{n2^{(l-1)}(l-1)!}\left(\frac{d}{d\theta}\right)^l \cos n\theta. \tag{5.3}$$

Furthermore, for ωt large (i.e., after a few cycles of the oscillation), the dominant term in $\omega t j_l(\omega t)$ is $\sin\omega t$ or $\cos\omega t$. We use the approximation,

$$\omega t j_l(\omega t) \approx (-1)^l\left(\frac{d}{d\omega t}\right)^l \sin\omega t. \tag{5.4}$$

Employing Eqs. (5.3) and (5.4) we may write the sum in Eq. (4.23) in the form

$$\sum_{l=1}^{n} \frac{(n+l)}{(2l)l!} C^l_{n-l}(\cos\theta)\left[k\rho\sin\theta\left(\frac{I}{I_a}\right)\omega t\right]^l \omega t j_l(\omega t)$$
$$\approx -\sin(n\theta-\omega t)\sum_{l=1}^{n} F(l)$$
$$+\sin(n\theta+\omega t)\sum_{l=1}^{n}(-1)^l F(l), \tag{5.5}$$

in which

$$F(l) = \frac{1}{2(l!)^2}\left(1+\frac{l}{n}\right)\left[\frac{nk\rho I\omega t}{2I_a}\right]^l. \tag{5.6}$$

The second term on the right-hand side of Eq. (5.5) is negligible compared to the first term because of the alternating series. The dominant contribution to the remaining series is made by terms such that $l \approx l_0$, where l_0 is the value of l for which $F(l)$ is a maximum. Furthermore the condition $l_0 \ll n$ is assumed to be satisfied, so that we may neglect l_0/n with respect to unity. The value of l_0 must be large enough to justify employing Stirling's formula, namely

$$l! \approx \sqrt{2\pi l}\exp(l\ln l - l). \tag{5.7}$$

A more stringent condition on l_0 will be encountered in the following analysis.

We use Eq. (5.7) in Eq. (5.6) and approximate the series with an integral,

$$\sum_{l=1}^{n} F(l) \approx \frac{1}{4\pi l_0}\int_{1/2}^{n} e^{f(l)}\, dl, \tag{5.8}$$

in which

$$f(l) = l\left[2 - 2\ln l + \ln\left(\frac{nk\rho\omega t I}{2I_a}\right)\right]. \tag{5.9}$$

Setting $df/dl = 0$ we find

$$l_0 = \left(\frac{nk\rho\omega t I}{2I_a}\right)^{1/2}. \tag{5.10}$$

We expand $f(l)$ in a Taylor series about l_0,

$$f(l) \approx f(l_0) + \frac{(l-l_0)^2}{2}\left.\frac{d^2 f}{dl^2}\right|_{l_0} = 2l_0 - \frac{(l-l_0)^2}{l_0}. \tag{5.11}$$

Inserting Eq. (5.10) in Eq. (5.8) we obtain

$$\sum_{i=1}^{n} F(l) \approx \frac{e^{2l_0}}{4\pi l_0}\int_{1/2}^{n}\exp\left[-\frac{(l-l_0)^2}{l_0}\right] dl.$$

If the conditions $e^{-l_0} \ll 1$ and $\exp[-(n-l_0)^2/l_0] \ll 1$ we may extend the limits of integration to $-\infty$ and ∞, with the result

$$\sum_{l=1}^{n} F(l) \approx \frac{e^{2l_0}}{4(\pi l_0)^{1/2}}. \tag{5.12}$$

We insert the expression for l_0 in Eq. (5.12) and then use Eqs. (5.12) and (5.5) in Eq. (4.23) to obtain the final expression for $\xi_n(t)$,

$$\xi_n(t) \approx de^{-\alpha t}\sin\omega t\cos\theta$$
$$-\frac{d\sin(n\theta-\omega t)\exp\left[\left(\frac{2nk\rho\omega t I}{I_a}\right)^{1/2} - \alpha t\right]}{2\sqrt{2\pi}\left(\frac{2nk\rho\omega t I}{I_a}\right)^{1/4}}. \tag{5.13}$$

It is worth noting that $\sin\theta$ does not appear explicitly in this expression, but recall that we have derived Eq. (5.13) under the conditions $n^2\sin^2\theta \gg l_0^2$, or

$$n\sin^2\theta \gg \frac{k\rho\omega t I}{2I_a}.$$

We recall that if the effective length of the solenoid is significantly less than the distance between accelerating gaps, we replace 2ρ by β_f from Eq. (2.5) and θ by μ from Eq. (2.4).

The exponential behavior exhibited in Eq. (5.13) has been obtained in a more rigorous calculation by one of the authors (LSH), who obtained an integral representation of the sum in Eq. (4.23). For the same range of parameters, a steepest descent evaluation of the integral yields a result in agreement with Eq. (5.13). The above derivation, although lacking in mathematical rigor, is presented for brevity.

Replacing α with $\omega/2Q$, we see that the exponent, Γ, in Eq. (5.13) has a maximum Γ_m given by

$$\Gamma_m = \frac{nk\rho IQ}{I_a}. \tag{5.14}$$

The maximum occurs when ωt equals ωt_m, where

$$\omega t_m = \left(\frac{2nk\rho I}{I_a}\right)Q^2. \tag{5.15}$$

The pulse duration may or may not be long enough to achieve the maximum amplitude.

As a numerical example, we consider an accelerator consisting of 225 accelerating units that accelerates electrons from 10 to 100 MeV, each unit imparting 400 keV to the electrons. The beam current is 2 kA and the pulse duration 100 nsec. We take the solenoidal magnetic field B to be 4 kG over the entire length of the machine. The relevant parameters are given in Table I. We apply Eq. (5.13) at the end of the machine ($n = 225$) and at the tail of the pulse ($\omega t = 500$). Employing these values and the parameters of Table I in Eq. (5.15) we see that Γ reaches its maximum value only if $Q < 20$. If $Q < 20$, we find from Eq. (5.14) that $\Gamma_m = 0.7Q$. If $Q \gg 20$, then we neglect the $-\omega t/2Q$ term in the exponent in Eq. (5.13) and find that the amplitude of ξ_n is $\exp(26.5)/20.6 \approx \exp(23.5)$.

We observe that a value of Q of the order of 20 is difficult to achieve in practice, but as mentioned in Section 1, ferrite induction cores may provide very low values of Q. Certainly 23 e-folds cannot be tolerated, and other means of suppressing the instability (such as varying the frequency of the mode in different accelerating units) must be considered. There is no physical reason for making the units identical, only an economic reason—the machine costs less to design and build.

TABLE I
Example parameters of electron accelerator

Coupling impedance, $Z_\perp/Q$	10 Ω
Angular frequency, ω(800 MHz)	5×10^9 s^{-1}
Coupling factor, k^{-1}	16 cm
Beam current, I	2 kA
Pulse duration, τ	100 nsec
Solenoidal magnetic field, B	4 kG
Bending radius, ρ	0.425 $\gamma\beta$cm
Number of accelerating units	225

As a second example, we treat a proton rf linac composed of side-coupled cavities. The LAMPF accelerator is such a device, and the resonant frequency, coupling impedance, and quality factor of the cavities in that machine have been measured.[7] For several reasons, this example may be considered a guideline only. The cavities are not identical, there may be considerable coupling of the TM_{110} mode from one cavity to the next in the absence of the beam, and the transport system does not correspond with the model used in the theory.

TABLE II
Example parameters of an rf proton accelerator

Coupling impedance $Z_\perp T^2/Q$	8 Ω
Angular frequency ω(1500 MHz)	9.4×10^9 s^{-1}
Coupling factor k^{-1}	12 cm
Beam current I	10 and 100 mA
Pulse duration	500 μsec
Quality factor Q	2.5×10^4
Betatron function β_f	2×10^3 $\gamma\beta$cm
Number of accelerating units	5000

In this example we use the parameters given in Table II. Since the beam is bunched, the coupling impedance includes the transit time factor T. The value of 8 Ω in Table II is obtained by taking the value per unit length given in Ref. 7 and multiplying by the 15-cm length of each cavity. The value of k is calculated from Eq. (3.12). Presently, LAMPF employs 5000 cavities to accelerate protons and H^- ions from 100 to 800 MeV with a proton average current during the pulse of 10 mA (the H^- current is much less). We obtain an estimate of β_f from the relation between beam radius r_b and emmitance ε,

$$r_b^2 = \beta_f \varepsilon. \tag{5.16}$$

We take r_b to be independent of particle energy, and since $\varepsilon \propto p^{-1}$ we have $\beta_f \propto p$. Although ω, Z/Q and Q have been measured for these cavities, the present beam current is too small to permit accurate measurement of ε. A value of $\varepsilon = (5 \times 10^{-4}/\gamma\beta)$ cm–rad is consistent with estimates. Using this value and $r_b = 1$ cm we obtain $\beta_f = 2 \times 10^3\gamma\beta$ cm. For protons $I_a = 3.13 \times 10^7\gamma\beta$A.

For $I = 10$ mA, we employ Eq. (5.14) to obtain $\Gamma_m = 3.3$. This value occurs at $t = t_m = 17$ μsec as found from Eq. (5.15). With only three e-folds the instability would not be observed at 10 mA. However, at $I = 100$ mA, a value $\Gamma_m = 33$ is obtained at $t = t_m = 170$ μsec. Taking into account the denominator in the second term on the right-hand side of Eq. (5.13), we have about 30 e-folds, which is certainly enough to have disastrous consequences. To avert disaster at 100 mA, focusing in the machine would have to be increased to reduce β_f by an order of magnitude from the value used in this example.

REFERENCES

1. R. Helm and G. Loew, Beam Breakup, Chapter B. 1.4 in *Linear Accelerators* (Eds. P. M. Lapostolle and A. L. Septier), North Holland Book Co., Amsterdam, 1970.
2. *Resonance, Q Investigation and Beam Instability Studies of a High Current Induction Linac*, Haimson Research Corp., Report HRC-703, Burlington, Mass., 1972.
3. W. Magnus, F. Oberhettinger, and R. P. Soni, *Formulas and Theorems for the Special Functions of Mathematical Physics*, Springer-Verlag, New York, 1966, 3rd ed., p. 257.
4. Magnus, Oberhettinger, and Soni, *Formulas and Theorems for the Special Functions of Mathematical Physics*, Springer-Verlag, New York, 1966, 3rd ed., p. 258.
5. G. E. Roberts and H. Kaufman, *Table of Laplace Transforms*, (W. B. Saunders Co., Philadelphia, 1966), p. 213.
6. Magnus, Oberhettinger, and Soni, *Formulas and Theorems for the Special Functions of Mathematical Physics*, Springer-Verlag, New York, 1966, 3rd ed., p. 223.
7. J. M. Potter, Proc. of the 1966 Linear Accelerator Conf., Los Alamos Scientific Laboratory. Report LA-3609 (1966), p. 109.