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THE INFLUENCE OF SPACE CHARGE ON THE PARAMETRIC RESONANCE OF THE BETATRON OSCILLATIONS

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The influence of space-charge forces on the parametric resonance of betatron oscillations is considered. The conditions under which the dimensions of the beam do not change appreciably are found. The passage through resonance is analyzed. The results are applied for choosing allowed n-trajectories in the compressors of collective accelerators.

The influence of the self charge of a beam on the parametric resonance of the noncoherent betatron oscillations has been studied in detail in connection with the determination of the limiting currents in cyclic accelerators.[1–6] The following main results were obtained:

1) Solution of the linearized Vladimirsky–Kapchinsky equation[7] for parametric resonance of matched beams gives the resonance frequency shift for the envelope of the betatron oscillations (due to the self charge of the beam) which is approximately equal to Q. (Q is the Coulomb correction to the incoherent betatron oscillation frequency ν in units of the cyclic frequency[1–6].)

2) As mismatching of the charged beam grows, the frequency of the free oscillations of the envelope approaches 2ν.[2,6] Hence for nonmatched beams the resonance conditions do not depend on the self-fields.†

3) The oscillation amplitude is bounded and does not differ significantly from the initial one for big Q's when one chooses the working point at the shifted resonance frequency.[1] The self charge of the beam prevents the resonance growth of the amplitude.[1–6]

4) Results obtained in the frame of the Vladimirsky–Kapchinsky model,[1,2,6] can be used as long as the frequency spread of the betatron oscillations in the beam does not exceed the Coulomb shift[6] (coherent resonance).

5) Passage through the parametric coherent resonance becomes easier due to the self charge when the betatron oscillation frequency is increasing and becomes more difficult in the opposite case.[3,8,9]

We have turned to the problem of self charge in parametric resonance when we analyzed the dynamics of electron rings in adhesators (compressors) of collective accelerators.[9] The specific feature of the adhesators is that in order to hold the space-charge in the initial stage of ring compression, high values of the external (weakly-focusing) field fall-off index n are needed. At the end of compression it is convenient to have the value of n sufficiently small. Then the n-trajectory of the ring usually intersects the resonance value a few times (the intersection of the resonance is due to the use of a small number of compression stages[10–13]). The characteristic compression time in Ref. 13 is 10^{-3} sec, so that resonances pass slowly if there is no special time-correction of the field.

The following questions are of interest in connection with ring compression: the influence of the beam's space-charge on the resonance for a wide range of deviations from the resonance frequency; correction of the point 1) in order to find an n-trajectory (taking Q into account) such that the parametric resonance would not be passed at all; the quantitative formulation of the point 5) (if the resonance is being passed). These questions are considered in this paper.

† One comes to the same conclusion considering the limiting case of large amplitudes of oscillations in the averaged equations (7), (8) of Ref. 1.

1) Let us take the Vladimirsky–Kapchinsky equations for the envelope of the beam in radial $= a_r$, and axial $= a_z$, directions, considering, to be specific, z-motion as being a resonance one†:

$$\frac{d^2 a_z}{d\theta^2} + (\tfrac{1}{2} + \delta)^2 a_z - \frac{F_z^2}{a_z^3} - \frac{Q(a_{r_0} + a_{z_0})a_{z_0}}{a_r + a_z} = \varepsilon a_z \cos\theta \quad (1)$$

$$\frac{d^2 a_r}{d\theta^2} + [1 - (\tfrac{1}{2} + \delta)^2] a_r - \frac{F_r^2}{a_r^3} - \frac{Q(a_{r_0} + a_{z_0})a_{z_0}}{a_r + a_z} = -\varepsilon a_r \cos\theta. \quad (2)$$

Notations in Eqs. (1) and (2) are the following: $\delta = \nu - \frac{1}{2}$ = detuning of the axial oscillation frequency without consideration of the self-field influence; F_z, F_r = phase volumes of z and r oscillations, correspondingly; a_{z_0} and a_{r_0} = dimensions of the matched beam; quantity $Q = 4\nu_e r_0^2/\beta^2\gamma^3 a_{z_0}(a_{z_0} + a_{r_0})$; r_0 = average radius of the ring; $\nu_e = r_e n_e$ = Budker parameter; n_e = number of electrons per unit length; r_e = classical electron radius; ε = relative error in the fall-off index of the external field; γ = relativistic factor of electrons rotating in the ring with the velocity βc.

The system (1) and (2) cannot be studied analytically, so the model equation is used for its analysis, which can be obtained by substitution $a_r = a_z$ in (1) (at any instant of time).[1] The solution of this equation when $\varepsilon = 0$, $Q = 0$ has the form:

$$\left(\frac{a_z}{a_{z_0}}\right)^2 = \sqrt{1 + A^2} + A\ \sin(2\nu\theta + \varphi), \quad (3)$$

where A and φ are arbitrary constants.

Instead of (3) one can use as a zero approximation an expression which takes the space charge into account when $\varepsilon = 0$. One only need to substitute F_z^2 by $F_z^{*2} = F_z^2 - Qa_{z_0}^4$ to do so. The results of the following calculations coincide with results of Ref. 1. In the first order of smallness in ε in the expression (3), A and $W = \delta\theta + \varphi/2$ can be considered as slowly varying functions, which satisfy Smith's equation (Ref. 1):

$$\frac{dA}{d\theta} = 2G\sqrt{1 + A^2} \cos 2W \quad (4)$$

$$\frac{dW}{d\theta} = \delta - G\ \frac{\sqrt{1 + A^2}}{A} \sin 2W - \frac{Q}{1 + \sqrt{1 + A^2}}, \quad (5)$$

where $G = \varepsilon/2$.

Stationary solutions of the system (4)–(5) are of special interest. Functions $UM = a_{z\,\max}/a_{z_0} - 1$ ($a_{z\,\max}$ = maximal value of the envelope), calculated from the stationary amplitudes for the different values of the relative detuning $D = \delta/G$ and Coulomb shift Q/G are shown in Figures 1 and 2.

The stationary curve constructed by Smith,[1] for which the resonance condition of the linear

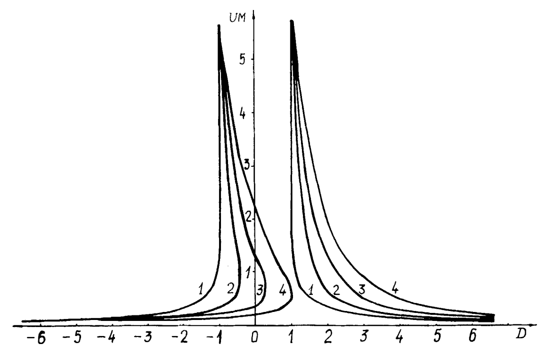

FIGURE 1 Dependence of the maximal amplitude of the envelope's oscillations on relative detuning.

$1: \frac{Q}{G} = 0, \quad 2: \frac{Q}{G} = 2, \quad 3: \frac{Q}{G} = 4, \quad 4: \frac{Q}{G} = 6.$

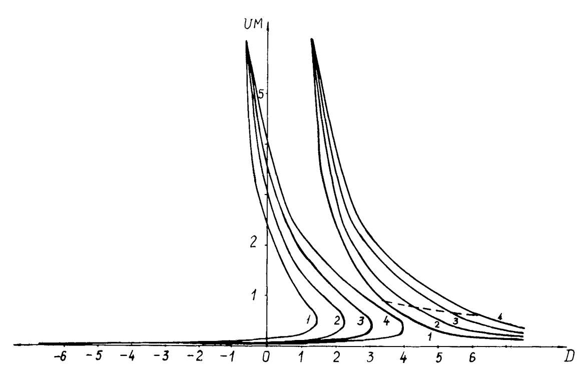

FIGURE 2 Dependence of the maximal amplitude of the envelope's oscillations on the relative detuning.

$1: \frac{Q}{G} = 7, \quad 2: \frac{Q}{G} = 9, \quad 3: \frac{Q}{G} = 11, \quad 4: \frac{Q}{G} = 13.$

† These equations are obtained for linear beams. One can use them, however, when considering changes in time of the ring's cross section rotating with the cyclotron frequency. Mathematically this means a change of variables in the Vladimirsky–Kapchinsky equation. The dependence on the initial azimuth of the cross section can be taken into account by choosing the appropriate initial phase in the disturbing force.

theory ($\delta = Q/2$) is satisfied, is shown on Figure 2 as a dotted line. It can be seen from Figures 1 and 2 that for any Coulomb shift the stationary curve consists of two branches. The left branch is a two-valued function of D in the region from -1 to some $D = D_{\max}$. The analysis of the solution's stability shows that in the two-valued region the upper part of the branch is unstable, while the lower part is stable.

The right branch is defined in the region $1 < D < \infty$; it grows infinitely when $D \to 1$, and is stable everywhere else.

If we will agree to understand by the resonance band that band D does not contain stationary solutions of the system (4)–(5), then the width of the resonance band is $P = 1 - D_{\max}$ if $P > 0$. The P vs Q/G dependence is shown in Figure 3.

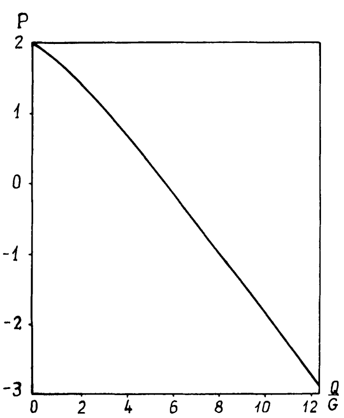

FIGURE 3 Dependence of $P = 1 - D_{\max}$ on the parameter Q/G.

When $Q > 5.8G$ the resonance band disappears and there are two stable stationary solutions in the region $1 < D < D_{\max}$ for large Coulomb shifts. In this case oscillations of the envelope of the matched beams (without taking perturbation into account) should take place near the lower stationary curve (left branch). Note that the matching requirements are very strict—small deviations of the beam parameters from the matched ones lead to envelope oscillations with large amplitude. This fact is illustrated in Figure 4, where integral curves of the system (4)–(5) for the parameters $D = 2$, $Q/G = 12$ are shown. They are constructed in the polar coordinate system with $a_{z\,\max}(A)/a_{z_0}$ as radius and W as azimuth.

The condition of the insignificant growth of the envelope's amplitude in the parametric resonance of the matched beams with dense space charge can

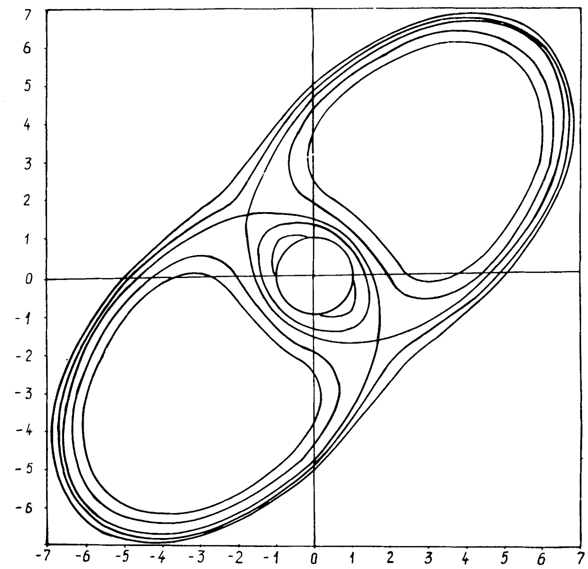

FIGURE 4 Integral curves of the system (4)–(5). The coordinate axes correspond to:

$$\frac{a_{z\,\max}}{a_{z_0}} \cos W \quad \text{and} \quad \frac{a_{z\,\max}}{a_{z_0}} \sin W.$$

be expressed by the inequality

$$\frac{Q}{G} \gg 6. \tag{6}$$

2) In order to check results obtained with the model equation,[1] the system of equations (1)–(2) was solved numerically on the computer. The maximal values of the envelope $= a_{z\,\max}$ are shown in Figure 5 as a function of the relative detuning D.

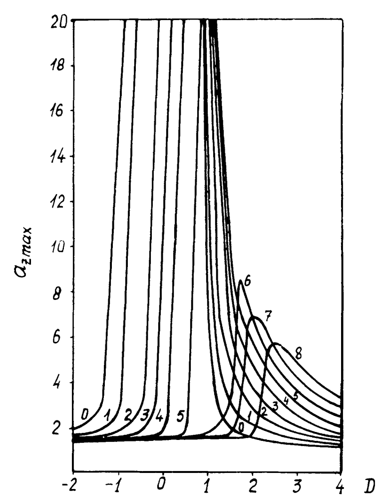

FIGURE 5 The maximal values of the envelope [the result of the numerical solution of the system (1)–(2)].

The initial conditions correspond to the matched beams with self charge. Phase volumes are taken to be $F_z = 0.5$, $F_r = 0.8$, the quantity $G = 0.01$. Different curves in Figure 5 are constructed for the values of Q/G in the interval from 0 to 8 with step 1. The curves are labeled by increasing Q/G values. The cutoff on the maximum amplitude $a_{z\,\max} \leq 20$ was used in the calculation.

The curves from Figure 5 are in a good agreement with the model curves in Figures 1, 2, 3. The amplitude of the envelope shows an infinite growth when $Q/G \lesssim 5$. When $Q/G > 5$ the values of D close to $D_{\max}$ correspond to the spasmodic change of the maximum amplitudes. The highest quantities $a_{z\,\max}$ in Figure 5 approximately coincide with the stationary amplitudes in Figures 1 and 2 taken on the right branches in the points $D = D_{\max}$.

Note that the highest amplitudes are reached when $\delta \cong D_{\max} G$, which corresponds to the transition of the stationary stable points in Figures 1 and 2 from the right branch to the lower left one. It does not happen at $\delta = Q/2$, as it would in linear theory.

Hence when the Coulomb shift is large, nonlinear corrections to the frequency of the envelope oscillations ω become important. The frequency ω can be found from Eq. (5)

$$\omega = 1 + 2\delta - Q + \tfrac{1}{8}QA^2. \tag{7}$$

3) When the matched beam is passing slowly through the parametric resonance the amplitude of the envelope's oscillations approximately follows stationary stable branches of the curves from Figures 1 and 2. So the passage of the resonance

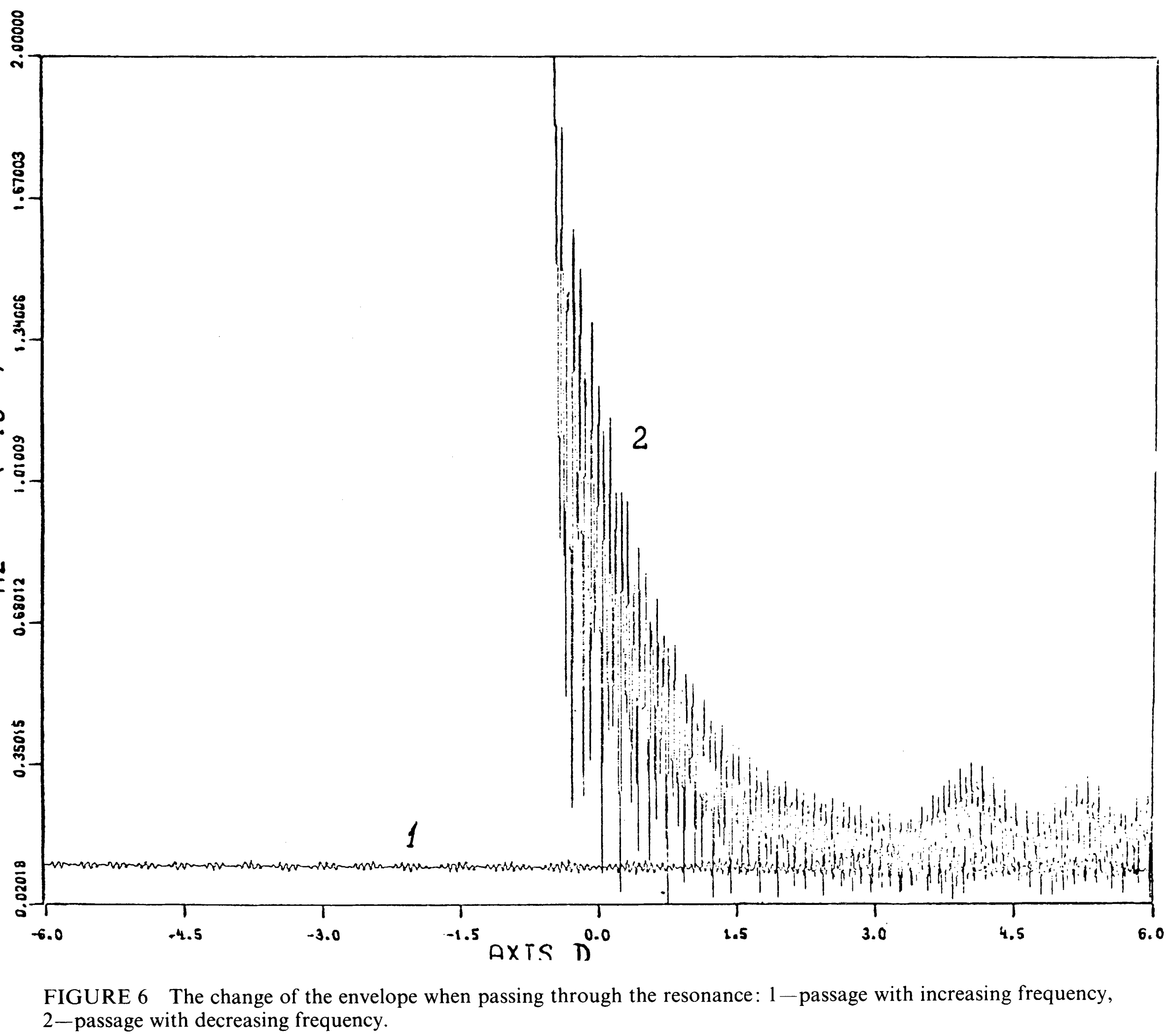

FIGURE 6 The change of the envelope when passing through the resonance: 1—passage with increasing frequency, 2—passage with decreasing frequency.

will lead to unlimited amplitude growth when the frequency gets smaller. The passage of the resonance in the reverse direction corresponds to the following of the lower branch when $D \lesssim D_{max}$ and to the transition to the right branch when $D \gtrsim D_{max}$. In this case the maximum amplitude is finite and it decreases with increase of the Coulomb correction Q.

These results were obtained in Refs. 3, 8 and 9 from qualitative arguments. As an example, Figure 6 shows the results of the numerical integration of the system of Eqs. (1)–(2) for the case of passage through the resonance with constant velocity ($\delta = \delta_0 + \delta'\theta$, $\delta_0 = \pm 6G$, $|\delta'| = 10^{-4}$). The value for the Coulomb parameter is $Q = 6G$, the beam is matched initially. The curve 1 corresponds to the passage through the resonance with increasing frequency, curve 2—with decreasing frequency. Note that the permissible region of the frequency variation of the matched beam is

$$\delta(\theta) < D_{max} G. \tag{8}$$

CONCLUSIONS

On the basis of the analysis conducted one can say that with regard to choosing the parameters for the adhesators of the collective accelerators, the effective index of the external field's fall-off should be designed to be smaller than $n_{max} = 0.25$. (The effective index takes into account corrections from the fields reflected by the chamber, but it does not include the Coulomb correction Q.) In the previous section we found the less rigid restriction (8), which approximately agrees with the results of the linear theory. However, according to Section 1, small deviations from the matched beam lead to large amplitudes of the envelope's oscillations. Hence, one should work in such a region where only one stable left branch of the stationary curve exists.

The upper bound on the fall-off index in the collective accelerator leads to the limitation of the electron ring's current. Therefore, it is necessary to provide fast passing of the parametric resonance. In particular, this remark corresponds to the system calculated in Refs. 14 and 15.

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