Particle Accelerators
1973, Vol. 5, pp. 61–65

EMITTANCE, ENTROPY AND INFORMATION

J. D. LAWSON
Rutherford Laboratory, Chilton, Berkshire, England

P. M. LAPOSTOLLE
Centre National d'Études des Télécommunications, Issy-les-Moulineaux, France

and

R. L. GLUCKSTERN
Department of Physics and Astronomy, University of Massachusetts, Amherst, Mass. USA

The emittance of a charged particle beam can be considered as a measure of the disorder in the transverse motion of the particles. The relation of this concept to the entropy of the distribution in transverse phase space is explored, and a thermodynamic description of a beam as a 'two dimensional gas' is developed. The connection with information in beams which can be focused to form an image is noted.

1. INTRODUCTION

The concept of emittance has been widely used to define a figure of merit for the quality of charged particle beams. Although it can be given a precise meaning in connection with certain idealized types of beam, its interpretation is not always unambiguous in many situations of practical importance. It is indeed possible to invent definitions which give a unique value for irregular or filamented 'emittance plots', but none of these is wholly satisfactory. We suggest here that the closely related concept of entropy provides a very appropriate measure of the disorder in a charged particle beam, which is more subtle than specifying the emittance. It enables the increased disorder arising from filamentation to be precisely specified for example, and also assumes an appropriately low value for a beam which contains information by virtue of a structure which can be focused to form an image. Although it is not suggested that changes be made in the way that emittance is defined and used in practical situations, the conceptual clarification is perhaps helpful.

The emittance concept since its introduction by Sigurgeirsson[1] has become thoroughly familiar in the accelerator literature; it is not generally employed in the field of microwave tubes, though in electron optics the closely related concept of 'brightness' is used. In these disciplines, where non-laminar flow arising from thermal velocity distributions has been extensively studied, the entropy is a quantity which may be linked naturally to concepts such as the transverse temperature and pressure of the beam. Again, in microscopy where images are produced, the entropy of the beam is closely related to the information contained in the images. Aberrations cause increase of entropy, or alternatively, loss of information.

2. THE EMITTANCE CONCEPT

Before calculating the entropy we review the meaning of emittance as applied to particle beams. It will be assumed throughout that trajectories make a small angle with the axis, and that there is no coupling between longitudinal and transverse motion. This assumption implies that longitudinal and both transverse degrees of freedom may be considered independently; our concern will be entirely with transverse motion. Steady state conditions are assumed (or average states in a noisy or fluctuating beam).

The optical properties of the beam at a particular value of z, the distance along the axis, can be represented by a density distribution of points in transverse phase space. For simplicity we consider a system with axial symmetry, and confine attention to the projection of the four dimensional x, p_x, y, p_y distribution on the x, p_x plane. The momentum p_x is equal to $\beta\gamma m_0 c x'$, and the area occupied by the points in the x, x' plane (multiplied by $1/\pi$) is known as the 'emittance' ε of the beam. In general, the points are dense near the origin, and the iso-density contours form a set of closed curves, often roughly elliptical in shape. Typically the area occupied by 90 per cent of the points may be several times that occupied by half of them, and only half that occupied by the total number. Under these circumstances the area is not well defined, and it has been suggested for example that instead of a single figure, a plot of area against fraction of points enclosed should be specified.[2]

An alternative definition is the rms emittance proposed by Lapostolle[3] and Sacherer.[4] This may be written

$$\bar{\varepsilon} = 4(\langle x^2\rangle\langle x'^2\rangle - \langle xx'\rangle^2)^{1/2} \tag{1}$$

and has properties useful in the study of beams in which space charge forces are important. Even if ε is zero, $\bar{\varepsilon}$ may be finite. For an emittance plot consisting of a straight line symmetrically placed through the origin, ε and $\bar{\varepsilon}$ are zero. For an S-shaped line however $\varepsilon = 0$ but $\bar{\varepsilon}$ is finite.

An important feature of density distributions in x, x' space, or 'emittance plots', is that the density of points in the neighbourhood of a particular point is invariant unless p_z varies, in which case it is proportional to p_z. This follows from Liouville's Theorem, and is true in the presence of large scale self fields, but not interparticle collisions.

Although $\varepsilon_n = \beta\gamma\varepsilon$ is strictly invariant, if measured in terms of the area within a given contour, the *effective* emittance can appear to increase in the presence of aberrations, as a consequence of the phenomenon of 'filamentation'.

A simple example is provided by the development of a beam launched from a point source in a focusing channel uniform with z, in which the restoring force is a non-linear function of amplitude. Suppose particles are emitted uniformly over a small angle $\pm\alpha$ from a point source on the axis at $z=0$. Then the 'emittance plot' at $z = 0$ is a straight line from $x' = -\alpha$ to $x' = +\alpha$. If the channel provides a *linear* focusing force the line rotates in a clockwise direction as z increases, making one revolution per focusing wavelength. Its ends trace out an ellipse. After an odd number of quarter wavelengths the emittance line lies along the x axis, representing the situation where all the particles move parallel to the axis. The line has zero area, and so the beam has zero emittance.

If now the focusing channel is slightly non-linear, the wavelength for particles with large α differs from that of particles which remain near the axis. This means that points on the rotating line in the emittance diagram rotate with different angular velocities. After $\frac{1}{4}$ wavelength it becomes slightly S-shaped; after many wavelengths it becomes a spiral bounded by the oval curve traced out by the points at the end of the line. If now the beam emittance is measured with apparatus of poor resolution, it appears to occupy the whole of the enveloping oval; 'emittance growth' has occurred. The phenomenon is essentially that of a set of oscillators with frequency slightly dependent on amplitude which are in phase at $t=0$. After a long time, all phases are present, and on superficial inspection the system 'looks more disordered'.

3. CALCULATION OF THE ENTROPY

We now examine the entropy associated with a distribution of points in the x, x' plane. To do this, it is necessary to divide it into cells of area $A = \delta x\delta x'$, of sufficient size that each contains a large number of points. The entropy of the distribution is then by definition[5]

$$S = k \log W \tag{2}$$

where k is Boltzmann's constant and W is the number of ways in which the points can be assigned to the cells to produce the given distribution. If N is the total number of points, n_i the number in the ith cell, and M the number of cells then

$$W = \frac{N!}{n_1!\,n_2!\ldots n_M!} \tag{3}$$

For large N, n, Stirling's formula may be used to give

$$\log W = N \log N - \sum_{i=1}^{M} n_i \log n_i. \tag{4}$$

If A is sufficiently small, the summation may be replaced by an integral to give

$$S/kN = S_0 = \log N - \frac{1}{N}\int \rho \log A\,\rho \,\mathrm{d}x\,\mathrm{d}x' \tag{5}$$

where $\rho = n/A$ is the density of points in x, x' space. The quantity S_0, the normalized entropy, has been introduced for simplicity.

Equation (5) will now be applied to two special distributions, both of which have been extensively studied. The first of these is the Kapchinskij–Vladimirskij or 'normal' distribution[6] in which the density is uniform, and bounded by an ellipse. The area of the ellipse is $\pi\varepsilon$.. For this distribution Eq. (5) simplifies to

$$\begin{aligned} S_0 &= \log N - \log AN/\pi\varepsilon \\ &= \log \pi\varepsilon - \log A. \end{aligned} \tag{6}$$

Another well-known distribution is the thermal beam emitted from a hot cathode. Such a beam can be generated (ideally) by placing a grid a small distance z_1 from a hot planar cathode of radius r_c. If the potential $-q\phi$ on the grid greatly exceeds kT/q where kT is the thermionic emission temperature of the cathode, and $z_1 \ll r_c$ then a beam is produced with density distribution.

$$\left.\begin{aligned} \rho(x) &= \int 2\rho(x')\,\mathrm{d}x'(r_c{}^2 - x^2)^{1/2}/\pi r_c{}^2 \\ \rho(x') &= \int \rho(x)\,\mathrm{d}x\left(\frac{mv_z{}^2}{2\pi kT}\right)^{1/2} \exp\{-x'^2 mv_z{}^2/2kT\} \end{aligned}\right\} \tag{7}$$

where $\int \rho\,\mathrm{d}x\,\mathrm{d}x' = N$, $\frac{1}{2}mv_z{}^2 = -q\phi$.

To find S_0 this is inserted into Eq. (5); after some straightforward algebra and integration we obtain

$$S_0 = \log\left\{\pi^{3/2} r_c \left(\frac{2kT}{mv_z{}^2}\right)^{1/2}\right\} - \log A. \tag{8}$$

This may be more conveniently expressed in terms of the rms emittance $\bar{\varepsilon}$ defined in Eq. (1). For this distribution $\bar{\varepsilon} = 2r_c(kT/mv_z{}^2)^{1/2}$ so that

$$S_0 = \log \pi\bar{\varepsilon} + \tfrac{1}{2}\log 2\pi - \log A. \tag{9}$$

Equation (9) contains two additive constants. The first of these is associated with the form of the distribution function, and is larger for a Maxwellian than for a uniform distribution as might be expected. The second, related to the cell size, is more fundamental. It would seem reasonable to relate it to the resolution of the apparatus used to measure the emittance.

4. THE BEAM AS A TWO DIMENSIONAL GAS

If we consider the x–y projection of the four dimensional phase space distribution, the points represent the particles of a two dimensional gas. We examine now the thermodynamic properties of this gas for a matched beam of circular cross section. If the beam is so weak that self forces are negligible, then a matched beam represents an equilibrium state which is not thermal equilibrium though, as we see below, a temperature can be assigned to it. When self forces are present the self-consistency of the self fields and particle motion must be ensured; when the self fields are sufficiently large compared with the external fields the distribution may not be stable.[7]

Parameters of the beam are assigned in the following way. The (two dimensional) volume and temperature are defined as

$$\begin{aligned} V &= \pi(\langle x^2\rangle + \langle y^2\rangle) = \pi\langle r^2\rangle \\ kT &= \tfrac{1}{2}m\beta^2c^2(\langle x'^2\rangle + \langle y'^2\rangle) = m\langle r'^2\rangle\beta^2c^2. \end{aligned} \tag{10}$$

where the z-axis is the beam axis and βc is the z-velocity of the beam. In the first instance we confine attention to a beam with a Kapchinskij–Vladimirskij distribution. Such a beam has uniform density over the cross-section with a sharp edge at radius $r = a$. The internal energy of the beam is taken as the sum of the *transverse* kinetic energy of the particles, and the electric and magnetic energies associated with the charge and current. This is legitimate provided that $\dot{z} \gg \dot{x}$. The internal energy is given by

$$U = NkT + \tfrac{1}{4}\int (E^2 + B^2) r\,\mathrm{d}r \tag{11}$$

where N is the number of particles per unit length of beam. E_r and B_0 are given by

$$\left.\begin{aligned} E(r<a) &= 2Nqr/a^2, \quad & E(r>a) &= 2Nq/r \\ B(r<a) &= 2Nq\beta r/a^2, \quad & B(r>a) &= 2Nq\beta/r \end{aligned}\right\} \tag{12}$$

so that from Eq. (11)

$$\begin{aligned} U &= NkT - N^2q^2(1+\beta^2)\log a + \text{const.} \\ &= NkT - \tfrac{1}{2}N^2q^2(1+\beta^2)\log V \end{aligned} \tag{13}$$

where the constant term has been omitted. Finally, the pressure is defined as

$$P = -\left(\frac{\partial U}{\partial V}\right)_s = -Nk\left(\frac{\mathrm{d}T}{\mathrm{d}V}\right)_s + \frac{\pi N^2 \varepsilon_0 q^2(1+\beta^2)}{8V} \tag{14}$$

The quantity $(\partial T/\partial V)_s$ is found from the adiabatic invariant for the transverse oscillations. For an adiabatic change, the action $J = \int p\,\mathrm{d}q$ is constant.

Since all the oscillations are simple harmonic, J is strictly proportional to $(\langle x^2\rangle\langle \dot{x}^2\rangle)^{1/2}$. From Eq. (10) therefore, since $\langle x^2\rangle = \langle y^2\rangle$ and $\langle x'^2\rangle = \langle y'^2\rangle$ we have $VT =$ constant for adiabatic changes. This implies a thermodynamic γ of 2, as expected.

Since VT is constant

$$\left(\frac{\partial T}{\partial V}\right)_s = -\frac{T}{V^2} \tag{15}$$

so that Eq. (14), the equation of state for the beam, becomes

$$PV = NkT+\tfrac{1}{2}N^2q^2(1+\beta^2). \tag{16}$$

The entropy is found from the relation

$$T\,\mathrm{d}S = \mathrm{d}U+p\,\mathrm{d}V. \tag{17}$$

From Eqs. (16) and (17)

$$\mathrm{d}S = Nk\left(\frac{\mathrm{d}T}{T}+\frac{\mathrm{d}V}{V}\right) \tag{18}$$

whence

$$S = Nk\log VT+\text{const.} \tag{19}$$

In terms of the rms emittance for a matched beam, $\bar{\varepsilon} = 4(\langle x^2\rangle\langle x'^2\rangle)^{1/2}$, and normalized entropy S_0, Eq. (19) becomes

$$S_0 = 2\log\bar{\varepsilon}+C \tag{20}$$

which, for each transverse direction x or y, gives the entropy term $\log\bar{\varepsilon}$ as in Eq. (6).

The question now arises, is the argument true for a general non-linear system in which the individual oscillators are not simple harmonic? If so, what can be said about the constant C?

Looking back over the various steps of the argument, Eqs. (10) and (11) can still be used to define V and kT, and U respectively. Equation (12) does not hold; nevertheless the internal energy can still be written as the sum of NkT and the self field term which, for a given distribution function, is a function of the volume alone. Proceeding as before, but not evaluating $(\partial T/\partial V)_s$, it turns out that the self field term disappears in the expression for the entropy. Corresponding to Eq. (18)

$$\mathrm{d}S = Nk\left\{\frac{\mathrm{d}T}{T}-\frac{\mathrm{d}V}{T}\left(\frac{\partial T}{\partial V}\right)_s\right\}. \tag{21}$$

For a general system the transverse oscillations are not simple harmonic; the potential well in which they move may change shape when the beam expands, so that it is not strictly correct to set $\int p\,\mathrm{d}q/(\langle x^2\rangle\langle x'^2\rangle)^{1/2}$ constant without a correcting factor which depends on the density distribution and hence in general on V. During the adiabatic expansion this suggests that although the entropy may remain constant the emittance can change. This corresponds to a change in the first constant noted after Eq. (9), associated with the changing form of the distribution function.

5. ENTROPY AND INFORMATION ASSOCIATED WITH IMAGES

It would be interesting to extend these studies to beams capable of producing images. At an image, the x, x' plane will have a density variation with x but not with x'. The information in the image can clearly be quantified by dividing the phase plane into strips and allowing a finite number of discrete densities, one to be associated with each strip. The more detail in which this is done, the more restricted is the number of ways the particles can be distributed, and hence the lower the entropy. The rms emittance however would not be affected greatly, and is therefore a less appropriate measure for the quality of such beams. A small amount of aberration could seriously blur an image and cause a considerable increase of entropy (loss of information).

6. CONCLUDING REMARKS

A thermodynamic description of the transverse motion of a charged particle beam acted upon by an external focusing force, taking into account the self-forces arising from the collective electric and magnetic fields associated with the charge and current, has been presented. This shows a close relation between the entropy and beam emittance, and provides a better understanding of the sources of arbitrariness in the latter concept. The connection with information is noted, but not explored in detail.

The entropy of collisionless plasmas has been considered by earlier authors. Schmidt[8] for example, briefly considered the adiabatic behaviour of two and three dimensional gases in a manner essentially the same as that of the present paper. Minardi[9] has developed a different and more sophisticated approach in which the assignment of entropy is made in a completely different way. His method yields stability criteria which should be applicable to problems such as that described in Ref. 7.

ACKNOWLEDGEMENTS

The existence of a link between emittance and entropy has been recognized for a number of years by several persons. One of us (P.M.L.) recalls discussions on the subject with C. S. Taylor in CERN about ten years ago; mention of it was also made in a report in 1968.[10] The major progress occurred however in 1971 after discussions at CERN between F. Sacherer and ourselves.

REFERENCES

1. T. Sigurgeirsson, CERN Report CERN-T/TS/2, unpublished, 1952.
2. P. M. Lapostolle, *Proc. Première Conférence Internationale sur les Sources d'Ions, Saclay, 1969* (INSTN Saclay, 1969), p. 165.
3. P. M. Lapostolle, *IEEE Trans. Nucl. Sci.* **NS–18**, No. 3, 1101 (1971).
4. F. J. Sacherer, as Ref. 3, p. 1105.
5. A. Sommerfeld, *Thermodynamics and Statistical Mechanics* (Academic Press, N.Y. and London, 1955), p. 213.
6. I. M. Kapchinskij and V. V. Vladimirskij, *Proc. Int. Conf. on High Energy Accelerators and Instrumentation, CERN, 1959*, p. 274.
7. R. L. Gluckstern, R. Chasman, and K. Crandall, *Proc. Linear Accelerator Conf., National Accelerator Laboratory, Batavia, Ill., 1970*, p. 823.
8. G. Schmidt, *Physics of High Temperature Plasmas* (Academic Press, N.Y. and London, 1966), p. 71.
9. E. Minardi, *Phys. Fluids*, **16**, 122 (1973).
10. P. M. Lapostolle, C. S. Taylor, P. Tetu, and L. Thorndahl, CERN Report 68–35 (1968).

Received 13 April 1973