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NONLINEAR BETATRON OSCILLATIONS†‡

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An analytic study is made of single particle motion in two dimensions for a nearly linear, nearly periodic system by considering a set of perturbed Hill's equations. It is shown how the Krylov-Bogoliubov averaging method can be applied to a nearly periodic system which may be near resonances to all orders in the perturbation parameter. The particular case of a single resonance in the first approximation is treated in detail. The fourth order system of differential equations is reduced to a second order system which describes the amplitude behavior for either the sum or the corresponding difference resonance. Approximate solutions are obtained for the betatron amplitude as a function of the independent variable for the cases in which the operating point is fixed, moves parallel to a resonance line, crosses a resonance line, or passes close by a resonance line. As a practical example, the particle losses in a storage ring due to an arbitrary nonlinear resonance are calculated in terms of the error fields.

1. INTRODUCTION

We will consider in this paper single-particle motion in two dimensions for a nearly linear, nearly periodic system such as an accelerator or a storage ring. The mathematical problem involves the solution of a set of perturbed Hill's equations

$$\frac{d^2x_i}{d\theta^2} + K_i(\theta)x_i = \varepsilon F_i(x_1,x_2;x_1',x_2';\theta) \quad (1.1)$$

$$i = 1,2$$

where the K_i and F_i are, in general, nearly periodic in θ and where ε is a small parameter describing the strength of the nonlinearity. We will be discussing betatron oscillations so that the coordinates x_i will refer to the transverse displacement of the particle from some reference curve. The set of equations (1.1) can represent many other types of nonlinear oscillations and our results will be applicable. Furthermore, the case of more than two dimensions can be treated by the same methods that will be presented here.

The theory of Courant and Snyder[1] considered the problem of stability in the linear, periodic case. This work was motivated by the discovery[2] of the alternating gradient principle by these authors along with Livingston and independently by Christofilos.[3] To a large extent the Courant-Snyder theory is an extension of the simple linearized azimuthal symmetry situation to include a periodic variation of the focusing forces.

Our purpose here is to extend the consideration of stability of betatron motion to the nonlinear, nonperiodic situation. We will do this with a greater emphasis on the nonperiodic aspect of the problem than was present in the previous works such as those of Sturrock[4] or Hagedorn and Schoch.[5,6] Also, we will consider the definition of stability in its weakest sense: That large (not necessarily infinite) relative changes in the betatron amplitudes from their initial values are considered unstable. Thus difference resonances will not, as usually is done, be assumed stable and therefore will be investigated. In fact, the methods to be presented are better suited for difference rather than sum resonances.

In this paper we will first look at the nonlinear, nonperiodic problem in as general a way as possible Then we will attempt to produce useful formulas for the betatron oscillation amplitudes in terms of simple known functions of time (approximate analytic solutions). Specifically, the plan is as follows. The equations of motion in two dimensions will be presented and transformed to action-angle variables of the linearized system. Asymptotic perturbation theory will be discussed and a particular form, the Krylov-Bogoliubov averaging method, will be studied in detail. Although this form of perturbation theory has been widely used, its validity has only recently been demonstrated to all

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orders in the perturbation parameter. We will show how this method can be applied to our problem, overcoming the difficulties of nonperiodicity and resonance. Then we will specialize our discussion to the first approximation and derive the approximate Hamiltonian for the resonant case (only at resonance does a small nonlinearity produce significant effects on the motion). The resulting equations of motion will be transformed to a simpler form. The amplitude will be expressed in terms of a certain phase function producing an expression which is valid for either the sum or corresponding difference resonance. Finally, after making a certain approximation, the actual solutions are presented. What is calculated is the variation in the betatron amplitudes as the betatron frequencies are varied throughout the course of the motion in various ways. An application of these results is then presented: The number of particles lost to betatron resonances induced by field errors is calculated for an arbitrary resonance.

A note on the terminology and notation: 'Static' means parameters fixed in time, hence a periodic system. 'Dynamic' means parameters (in particular, the period) varying, hence a nonperiodic system. Multidimensional quantities without subscripts will refer to the whole set of quantities as, for example, $x = (x_1, x_2)$.

2. EQUATIONS OF MOTION FOR TWO DIMENSIONS

2.1. *General Equations and Linearized Solution*

We will use the standard formulation of the two dimensional problem[7] in which the arc length s along a reference curve assumes the role of independent variable. The two transverse coordinates are x and z. If there is no explicit time dependence of the electromagnetic fields then the time t is a cyclic (ignorable) dynamic variable and the Hamiltonian is that of a two degree of freedom system

$$H = H(x,z; p_x,p_z; s). \tag{2.1}$$

For a circular machine with static fields the dependence of H on s will be periodic provided we choose the reference curve to be closed. For a linear machine the motion can often be represented by a periodic structure with parameters that vary slowly in time. We can consider the effect of acceleration which breaks the periodicity of the Hamiltonian (2.1) without reintroducing a third degree of freedom by requiring this Hamiltonian to depend on parameters that are slowly varying functions of the independent variable.

Let us further specify our coordinate system by choosing the reference curve (closed for a circular machine) such that in the Hamiltonian terms of first degree in the variables vanish. This is the usual definition of the equilibrium orbit. We will also require that the quadratic part of the Hamiltonian be made diagonal (by a rotation in x,p_x, z,p_z space). Let us finally also transform the scale of the independent variable by introducing the generalized azimuthal angle θ

$$\theta = 2\pi \frac{s}{L_0} \tag{2.2}$$

where L_0 is the period of the system. When we speak of 'azimuthal symmetry' we will mean the Hamiltonian contains no explicit dependence on θ (or s) and need not correspond to a circular machine.

The Hamiltonian in the variables $x_1,x_2;p_1,p_2;\theta$ will be written as

$$H_\lambda(x_1,x_2;p_1,p_2;\theta) = H_\lambda^{(0)} + \varepsilon H_\lambda^{(1)} \tag{2.3}$$

where

$$H_\lambda^{(0)} = \frac{{p_1}^2}{2} + \frac{{p_2}^2}{2} + K_{1\lambda}(\theta)\frac{{x_1}^2}{2} + K_{2\lambda}(\theta)\frac{{x_2}^2}{2}. \tag{2.4}$$

In the above K_1, K_2, and $H^{(1)}$ are periodic in θ with period 2π. The Hamiltonian of the linearized motion is $H^{(0)}$ and $\varepsilon H^{(1)}$ is a nonlinear perturbation whose strength is determined by the small parameter ε. The subscripts λ, which will soon be dropped, refer to dependence on one or more parameters $\lambda(\theta) = (\lambda_1(\theta),\lambda_2(\theta), \ldots)$ which are slowly varying functions of θ.

Before attacking the full problem let us study the periodic linearized case, that is, the case for which $\varepsilon = 0$ and $\lambda(\theta) = \text{const}$. The Hamiltonian then yields an uncoupled set of equations of motion with periodic coefficients known as Hill's equations

$$x_k'' + K_k(\theta)x_k = 0, \; k = 1,2. \tag{2.5}$$

It is a consequence of Floquet's theorem[8,9] that solutions to (2.5) can be written as

$$x_k = \tfrac{1}{2}a_k f_k(\theta)e^{i\nu_k\theta} + \text{c.c.} \tag{2.6}$$

The functions f_k are periodic with period 2π and are called Floquet functions. The a_k and ν_k are constants where the latter will be restricted to real nonintegral values so that the system is stable in the linear approximation. We call ν_k the betatron frequencies. We can write (2.6) in real form as

$$x_k = A_k |f_k(\theta)| \cos(\nu_k\theta + \phi_k(\theta) + a_k) \tag{2.7}$$

where

$$a_k = A_k e^{ia_k}$$
$$f_k(\theta) = |f_k(\theta)| e^{i\phi_k(\theta)}. \tag{2.8}$$

If we substitute (2.6) into the linearized equations of motion (2.5) we see that the equations will be satisfied whenever we have

$$\phi_k'' |f_k| + 2(\nu_k + \phi_k') |f_k|' = 0 \tag{2.9a}$$
$$-(\nu_k + \phi_k')^2 |f_k| + |f_k|'' + K_k |f_k| = 0. \tag{2.9b}$$

The second of these gives the relation between the periodic functions $K_k(\theta)$ appearing in Hill's equations and the Floquet functions. The first relation may be integrated to give

$$(\nu_k + \phi_k') |f_k|^2 = \nu_k \tag{2.10}$$

in which the integration constant on the right hand side is chosen to be the appropriate betatron frequency. This normalizes the Floquet functions.[10] Note that for azimuthal symmetry we have

$$f_k(\theta) = 1.$$

Thus all our expressions will agree with harmonic oscillator results simply by dropping the $|f_k|$ and ϕ_k from the more general expressions.

2.2. *Transformation to a,I Variables*

In order to solve the general nonlinear problem we must transform our variables in a manner suitable for perturbation theory. We will make two successive canonical transformations.

$$x_1, x_2; p_1, p_2 \to Q_1, Q_2; I_1, I_2 \to a_1, a_2; I_1, I_2. \tag{2.11}$$

These are effected by the following generating functions

$$S_1(x, Q, \theta) = \sum_{k=1}^{2} \left[\frac{\nu_k}{2|f_k|^2} \cot(Q_k + \phi_k) + \frac{|f_k|'}{2|f_k|} \right] x_k^2, \tag{2.12}$$

$$S_2(Q, I, \theta) = \sum_{k=1}^{2} (Q_k - \nu_k\theta - \pi/2) I_k. \tag{2.13}$$

Following the standard procedure as outlined in Appendix A, we obtain from the above the transformation equations for $x,p \to a,I$:

$$x_k = \left(\frac{2I_k}{\nu_k}\right)^{1/2} |f_k| \cos(\nu_k\theta + \phi_k + a_k) \tag{2.14a}$$

$$p_k = \left(\frac{2I_k}{\nu_k}\right)^{1/2} |f_k| (\phi_k' + \nu_k) \sin(\nu_k\theta + \phi_k + a_k) + \left(\frac{2I_k}{\nu_k}\right)^{1/2} |f_k|' \cos(\nu_k\theta + \phi_k + a_k).) \tag{2.14b}$$

The Hamiltonian in the x,p variables in the form (2.3) will be transformed to the following

$$H(a, I, \theta) = H^{(1)}(x(a, I, \theta), p(a, I, \theta), \theta) \tag{2.15}$$

and the equations of motion in the a,I variables will be

$$\frac{da_k}{d\theta} = \varepsilon \frac{\partial H^{(1)}}{\partial I_k}$$
$$\frac{dI_k}{d\theta} = -\varepsilon \frac{\partial H^{(1)}}{\partial a_k}. \tag{2.16}$$

The procedure in working with the a,I variables is therefore the following: The Hamiltonian in the new variables is obtained by taking just the nonlinear perturbation $\varepsilon H^{(1)}$ and writing it in terms of a,I by means of the transformation equations (2.14).

If there is no nonlinearity, $\varepsilon = 0$, then the variables a and I are constant in θ. This holds true even if the parameters are slowly varying. The I are, in fact, the usual adiabatic invariants for the linear system defined by the action integrals

$$\frac{1}{2\pi} \oint p_k dx_k = \frac{1}{2\pi} \oint I_k da_k = I_k \tag{2.17}$$

where the integration is over the finite motion consistent with a fixed value of the Hamiltonian. Comparing (2.14a) with (2.7) gives the adiabatic invariants in terms of the familiar amplitude variables A.

$$I_k = \frac{\nu_k A_k^2}{2}, \quad k = 1,2 \tag{2.18}$$

The fact that the product of the frequency with the square of the amplitude remains unchanged as the frequency is slowly changed is well known for the case of the harmonic oscillator. Preserving this

form (2.18) in the more general nonperiodic case is the reason for the particular normalization (2.10) chosen for the Floquet functions.

3. ASYMPTOTIC PERTURBATION THEORY

3.1. *General Remarks*

The equations of motion (2.16) are in such a form that the dependent variables a and I are constant in θ for a vanishing value of the perturbation parameter ε. We wish to find approximate solutions for small but nonzero values of ε. The most successful way to deal with such problems is to use asymptotic perturbation theory which produces approximations that are asymptotic representations of the exact solution to any desired order in ε. Such an approach is useful in those cases for which ε is so small that a and I undergo little change over an interval in θ comparable to the betatron oscillation periods $2\pi/\nu_k$. We can produce approximate solutions as accurate as desired provided ε is small enough. This does not tell us, however, what the error will be for a given value of ε. A convergent perturbation theory produces, in contrast, approximations as accurate as desired for a fixed range of ε values provided a high enough order approximation is taken. The advantage of the asymptotic method in our case is that low order approximations, even the first order, provide good results.

We will use the following definition. A function $F_n(\varepsilon)$ is said to be an Nth order asymptotic representation of the function $F(\varepsilon)$ if we have

$$\lim_{\varepsilon\to 0} \frac{|F(\varepsilon)-F_N(\varepsilon)|}{\varepsilon^N} = 0. \tag{3.1}$$

If $F(\varepsilon)$ is a multidimensioned function then the absolute value sign above is to be regarded as a norm. It is clear that asymptotic representations are not unique. And, as is often pointed out, asymptotic representations may be, and usually are, divergent. That is, the limit

$$\lim_{N\to\infty} F_N(\varepsilon) \tag{3.2}$$

may not exist. Hence, for a fixed $\varepsilon > 0$, the error in describing a function by its asymptotic representation may increase without limit as higher order representations are considered. Thus higher order approximations are not necessarily better than those of lower order and the best approximation results from a certain finite, and usually unknown, order.

For an approximate solution to a differential equation to be useful it is not enough that it be simply an asymptotic representation of the true solution. As the error decreases as $\varepsilon \to 0$, the effect of the perturbation itself decreases also. We would like the error to approach zero faster than the size of the effect of the perturbation. Since the perturbed solution departs significantly from the unperturbed solution by the time θ satisfies $\varepsilon\theta = L$ for some finite $L > 0$, we would like an approximation $z_N(\theta,\varepsilon)$ to the true solution $z(\theta,\varepsilon)$ having the property that given $L > 0$ and $C > 0$, there exists an $\varepsilon_0 > 0$ such that for $\varepsilon < \varepsilon_0$ we have

$$\max_{0\le\theta\le L/\varepsilon} \frac{|z(\theta,\varepsilon)-z_N(\theta,\varepsilon)|}{\varepsilon^N} < C. \tag{3.3}$$

An approximation having this property will be said to be an Nth order asymptotic representation uniformly valid on the interval $0 \le \theta \le L/\varepsilon$ for the above statement can also be expressed as

$$\lim_{\varepsilon\to 0} \frac{|z(T/\varepsilon,\varepsilon)-z_N(T/\varepsilon,\varepsilon)|}{\varepsilon^N} = 0 \tag{3.4}$$

uniformly on $0 \le T \le L$.

3.2. *The Averaging Method*

We will now describe an asymptotic perturbation technique, having the desired properties outlined above, which was introduced by Krylov and Bogoliubov.[11] Their procedure, called the averaging method, was expanded upon in the book by Bogoliubov and Mitropolsky[12] and has subsequently been refined and put on a rigorous basis by several authors. We will use the statement of the method due to Perko.[13] We must point out that the averaging method has been widely used in accelerator physics and elsewhere, especially in the first approximation. But only in the more recent works[13,14] has the method been stated for all orders and limits to its validity fully worked out.

To use the averaging method, the equations of motion must be in the form

$$\frac{dx}{d\theta} = \varepsilon f(\theta,x) \tag{3.5}$$

with initial conditions

$$x(0,\varepsilon) = \xi_0. \tag{3.6}$$

This represents a set of n coupled first order differential equations where $x = (x_1, x_2, \ldots x_n)$ and $f = (f_1, f_2, \ldots f_n)$ are n-dimensional quantities.

The Nth order averaging approximation is given for $N \geq 1$ by

$$x_N(\theta,\varepsilon) = y_N(\theta,\varepsilon) + \sum_{j=1}^{n-1} \varepsilon^j \phi_j(\theta, y_N(\theta,\varepsilon)) \tag{3.7}$$

where $y_N(\theta,\varepsilon)$ is the solution to the system

$$\frac{dy}{d\theta} = \varepsilon Y_1(y) + \varepsilon^2 Y_2(y) + \ldots + \varepsilon^N Y_N(y)$$
$$y(0,\varepsilon) = \eta_N(\varepsilon). \tag{3.8}$$

Note that the right hand side of this equation does not contain the independent variable. This equation is easier to solve than the original equation (3.5). This is the reason the averaging method is useful in a practical sense. The initial condition $\eta_N(\varepsilon)$ for the new equation is defined implicitly by

$$\eta_N(\varepsilon) = \xi_0 - \sum_{j=1}^{N-1} \varepsilon^j \phi_j(0, \eta_N) + 0(\varepsilon^N). \tag{3.9}$$

The order symbol 0 used above is defined by

$$f(\varepsilon) = 0(g(\varepsilon)) \longleftrightarrow \lim_{\varepsilon \to 0} \frac{f(\varepsilon)}{g(\varepsilon)} = c, |c| < \infty. \tag{3.10}$$

The functions $Y_j(y)$ and $\phi_j(\theta,y)$ are defined for $j = 1,2 \ldots N$ by

$$Y_j(y) = \langle F_j(\theta,y) \rangle$$
$$\partial \phi_j(\theta,y)/\partial\theta = F_j(\theta,y) - Y_j(y), \langle \phi_j(\theta,y) \rangle = 0. \tag{3.11}$$

The angular brackets signify an average over the independent variable θ, i.e.

$$\langle g(\theta,x) \rangle = \lim_{T \to \infty} \frac{1}{T} \int_0^T g(\theta,x) d\theta. \tag{3.12}$$

The functions $F_j(\theta,y)$ are given by

$$F_1(\theta,y) = f(\theta,y)$$
$$F_j(\theta,y) = \sum_{k=1}^{j-1} \left[\frac{1}{k} D^k{}_y f(\theta,y) \sum_{i_1 + \ldots i_k = j-1} \phi_{i_1} \ldots \phi_{i_k}(\theta,y) - D_y \phi_k(\theta,y) Y_{j-k}(y) \right], j = 2,3, \ldots N. \tag{3.13}$$

The derivative $D_y{}^k$ appearing above is defined by

$$D_x{}^k f(x) g_1 \ldots g_k = \sum_{j_1, j_2, \ldots j_k = 1}^{n} \frac{\partial^k f(x)}{\partial x^{(j_1)} \ldots \partial x^{(j_k)}} g_1{}^{(j_1)} \ldots g_k{}^{(j_k)} \tag{3.14}$$

in which the superscripts in parentheses are component indices of n-dimensional vectors.

The utility of the averaging method may be stated by the following theorem.

Theorem. Assume $f(\theta,x)$ satisfies conditions to be later stated. If given some $L > 0$, $N \geq 1$, G a convex region in E^n, ξ_0 in G and there exists a solution to

$$\frac{dy}{d\tau} = Y_1(y), y(0) = \xi_0 \tag{3.15}$$

which remains in G for $0 \leq \tau \leq L$, then for a small enough ε the original equation (3.5) has a unique solution $x(\theta,\varepsilon)$ in G for $0 \leq \theta \leq L/\varepsilon$ for which the Nth order averaging approximation $x_N(\theta,\varepsilon)$ is an $(N-1)$th order asymptotic representation uniformly valid on $0 \leq \theta \leq L/\varepsilon$.

The above is essentially a statement of Theorem 5 of Perko.[13] Quite general conditions on the $f(\theta,x)$ were shown by Zabreiko and Ledovskaja.[14] They state that $f(\theta,x)$ defined on $0 \leq \tau \leq \infty$ with $x \varepsilon G$ must satisfy the following:

(a) $f(\theta,x)$ and derivatives to order $N-1$ are bounded
(b) $\phi_i(\theta, =x)$, $i = 1,2, \ldots N-1$ are bounded
(c) $\langle F_N(\theta,x) \rangle$ exists.

Perko pointed out that it would be desirable to state the requirements on $f(\theta,x)$ without referring to the quantities $\phi_i(\theta,x)$ which may be difficult to calculate. Such a set of conditions, more restrictive than the above, is the following:

(a′) $f(\theta,x)$ has N continuous derivatives with respect to x
(b′) $f(\theta,x)$ is real analytic and quasiperiodic in θ. This means it can be expressed as a function F

$$f(\theta,x) = F(\omega_1\theta, \omega_2\theta, \ldots \omega_m\theta; x) \tag{3.16}$$

which is 2π-periodic in the first m variables. Thus $f(\theta,x)$ can be written as a Fourier series

$$f(\theta,x) = \sum_k a_k(x) e^{i\Lambda_k \theta} \tag{3.17}$$

where the frequencies Λ_k are combinations of the basic frequencies $\omega = (\omega_1, \omega_2, \ldots \omega_m)$

$$\Lambda_k = (k,\omega) = k_1\omega_1 + k_2\omega_2 + \ldots + k_m\omega_m \tag{3.18}$$

The sum in (3.17) extends over all vectors $k = (k_1, k_2, \ldots k_m)$ with integer components.

(c′) $f(\theta,x)$ satisfies a nonresonance condition, an example of which is

$$\limsup_{l\to\infty}\left[\sum_{|k|=l}\max_{x\varepsilon\Gamma}\left|\left|\frac{D^j a_k(x)}{(\omega,k)}\right|\right|\right]^{1/l} < 1 \tag{3.19}$$

for $j = 0,1 \ldots N$ and each compact $\Gamma \subset G$.

The reason condition (c′) is imposed is to ensure the existence of the quantities $\phi_i(\theta,x)$ and $Y_i(x)$ and $N-i+1$ of their derivatives with respect to x, a condition necessary for the averaging approximation to make sense. To see this consider

$$D^j{}_x\phi_1(\theta,x) = D^j{}_x\sum_{k\neq 0}\frac{a_k(x)e^{i\Lambda_k\theta}}{i\Lambda_k} \tag{3.20}$$

$$= \sum_{l=1}^{\infty}\left(\sum_{|k|=l}\frac{D_x{}^j a_k(x)}{i(k,\omega)}\right).$$

Condition (3.19) ensures the uniform convergence on Γ of the infinite series and thus that ϕ_1 is N-times continuously differentiable with respect to x. Together with the fact that $Y_1 = \langle f \rangle$ is also N-times continuously differentiable, the desired result can be proved by induction for ϕ_i and Y_i, $i = 2,3 \ldots N$.

3.3. *Applying the Averaging Method*

The averaging method as stated above cannot be directly applied to our problem. The difficulty is that slowly varying parameters and resonances are excluded.

In order that the rate of change of parameters remain small in the asymptotic limit $\varepsilon \to 0$ this rate must be of the same order as the perturbation parameter ε. We can study such a situation by requiring the right hand sides of the equations of motion (3.5) to also depend on the quantity $\varepsilon\theta$. But this destroys the quasiperiodicity of these functions in θ. We can overcome this difficulty by the common procedure of introducing an $(n+1)$th dependent variable.

$$x_{n+1}(\theta) = \varepsilon\theta \tag{3.21}$$

so that the equations of motion become

$$\frac{dx_i}{d\theta} = \varepsilon f_i(\theta; x_1, x_2, \ldots x_n; x_{n+1}) \tag{3.22}$$

$$i = 1,2, \ldots n$$

$$\frac{dx_{n+1}}{d\theta} = \varepsilon.$$

Thus a system depending on slowly varying parameters will satisfy the requirements of the averaging method provided it does so whenever the parameters are fixed and that f is also N-times continuously differentiable with respect to the new variable $x_{n+1} = \varepsilon\theta$.

It is well known that if the basic frequencies of a system are rationally dependent, or nearly so, then the behavior of the system is markedly different from that in which such a condition is not satisfied. In order to preserve this rationality or resonance condition in the asymptotic limit $\varepsilon \to 0$ we must make one or more of the basic frequencies functions of ε. It is useful to make them functions of $\varepsilon\theta$ also in order to consider the slow variation of the frequencies throughout the course of the motion. Thus we require

$$\omega_i = \omega_i(\varepsilon, \varepsilon\theta) \tag{3.23a}$$

and consequently

$$\Lambda_k = k_1\omega_1 + \ldots k_m\omega_m = \Lambda_k(\varepsilon, \varepsilon\theta). \tag{3.23b}$$

In order to consider a resonance, characterized by the set of integers k, we will require that the resonant combination of frequencies Λ_k be $0(\varepsilon)$ when $\varepsilon\theta$ is held fixed. Let us consider

$$\Lambda_k(\varepsilon, \varepsilon\theta) = \varepsilon\Lambda_k{}^{(R)}(\varepsilon\theta). \tag{3.24}$$

Now if all the Λ_k are either of this form or independent of ε, the basic system of equations can be written as

$$\frac{dx_i}{d\theta} = \varepsilon\sum_{\Lambda_k = 0(1)}(a_k(x))_i \exp(i\Lambda_k\theta)$$

$$+\varepsilon\sum_{\Lambda_k = 0(\varepsilon)}(a_k(x))_i \exp(ix_{n+1}\Lambda_k{}^{(R)}(x_{n+1}))$$

$$\frac{dx_{n+1}}{d\theta} = \varepsilon. \tag{3.25}$$

If the right hand sides of the above system have N continuous derivatives with respect to x, then

requirements (a′) and (b′) are satisfied. If also, as is usually assumed in physical problems, the function $f(\theta,x)$ can be accurately represented by a finite series then condition (c') is trivially satisfied. Thus if we write the resonant combination of frequencies in the form (3.24) the method will work to all orders N for most cases.

If we require that the basic frequencies be of the form (3.23) then some of the frequency combinations may be of the form

$$\Lambda_k(\varepsilon,\theta) = \Lambda_k^{(NR)}(\varepsilon\theta). \tag{3.26}$$

These nonresonant combinations will introduce terms like

$$(a_k(x))_i \exp(i\Lambda_k^{(NR)}(x_{n+1})\theta)$$

on the right hand side of the system (3.25). We cannot apply Perko's conditions in such a circumstance but by working out the first approximation we see that conditions (a), (b), and (c) are satisfied provided $\Lambda_k^{(NR)}$ is differentiable but that condition (c) is violated for $N > 1$. Thus this situation can be studied only in the first approximation.

As was pointed out previously, higher order approximations are not necessarily more accurate than lower order approximations. Because of this and the fact that high order approximations are difficult to obtain, it is reasonable to consider only the lowest order approximation whose contribution to the unperturbed solution does not vanish. For example, the first approximation produces a nontrivial solution in the resonance case. Thus while it may pay to consider the second order approximation in the nonresonance case, to do so in the resonance case would be advisable only if special features of the solution, such as higher order Fourier components, are being sought.

Let us consider the canonical nature of the problem. If we apply the first approximation to the equations of motion (2.16) we obtain the averaged equations

$$\frac{da_i}{d\theta} = \varepsilon \frac{\partial}{\partial I_i}\langle H^{(1)}\rangle \tag{3.27}$$

$$\frac{dI_i}{d\theta} = -\varepsilon \frac{\partial}{\partial a_i}\langle H^{(1)}\rangle.$$

We have only to average the Hamiltonian $\varepsilon H^{(1)}$ with respect to θ and then Hamilton's equations resulting from this averaged Hamiltonian are the desired approximate equations. Thus for an n-dimensional system we have only to average one expression rather than $2n$ of them.

If we have a noncanonical system, that is, one not derivable from a Hamiltonian, then we must apply the averaging method directly on the equations of motion

$$x_1'' + K_1(\theta)x_1 = \varepsilon F_1(x_1,x_2;x_1',x_2';\theta) \tag{3.28}$$
$$x_2'' + K_2(\theta)x_2 = \varepsilon F_2(x_1,x_2;x_1',x_2';\theta)$$

Such a system may be transformed to variables appropriate to the averaging method by the transformation $x_1,x_2;x_1',x_2' \to a_1,a_2;I_1,I_2$ defined by (2.14) in which we regard the momenta p_i as x_i'. The resulting equations are

$$\frac{dI_i}{d\theta} = -\varepsilon\left(\frac{2I_i}{\nu_i}\right)^{1/2} F_i|f_i|\sin(\nu_i\theta + a_i + \phi_i). \tag{3.29}$$

$$\frac{da_i}{d\theta} = -\varepsilon(2\nu_i I_i)^{-1/2} F_i|f_i|\cos(\nu_i\theta + a_i + \phi_i).$$

In these equations, the perturbations F_i must be expressed in terms of a,I according to (2.14) where, again, we regard the p_i as x_i'.

4. THE FIRST APPROXIMATION FOR THE RESONANCE CASE

We will now specialize our discussion to consider an important situation: the first order approximation to a nonlinear resonance in the case that only a single resonance predominates. Let us expand the nonlinear perturbation $\varepsilon H^{(1)}(x,p,\theta)$ in a power series in x and p and a Fourier series in θ:

$$\varepsilon H^{(1)} = \sum_{\substack{i,j,m,n,l\geq 0 \\ (i+j+m+n\geq 2)}} \varepsilon_{ijml} x_1^{\,i} p_1^{\,j} \times x_2^{\,m} p_2^{\,n} \cos(l\theta + \delta_{ijmnl}). \tag{4.1}$$

The Hamiltonian in the a,I variables is given by transforming the above according to (2.14). When the transformation is carried out and the result averaged over θ we obtain

$$\langle H(a,I,\theta)\rangle = \sum_{ijmnl} \varepsilon_{ijmnl} \times \left(\frac{2I_1}{\nu_1}\right)^{(i+j)/2}\left(\frac{2I_2}{\nu_2}\right)^{(m+n)/2}\langle E_{ijml}(\theta)\rangle. \tag{4.2}$$

The function $E_{ijmnl}(\theta)$ is a quasiperiodic function with basic frequencies ν_1, ν_2, and unity and depends on the Floquet functions and the phases a_1 and a_2. The average $\langle E_{ijmnl}\rangle$ will consist of two types of terms. The first will be nonresonant terms which do not depend on the values of the frequencies ν_1 and ν_2. The second type of term will contain resonant combinations of frequencies $n_1\nu_1+n_2\nu_2-k$ which should be written in the form (3.24). We will write the averaged Hamiltonian (4.2) as a sum of these two types of terms

$$\varepsilon\langle H^{(1)}\rangle = \langle H\rangle_{\text{Res}}+\langle H\rangle_{N.R.}. \tag{4.3}$$

If the Floquet functions do not depart too much from a constant value, the predominant terms will be

$$\langle H\rangle_{\text{Res}} = \varepsilon_1\left(\frac{2I_1}{\nu_1}\right)^{N1}\left(\frac{2I_2}{\nu_2}\right)^{N2}\cos\psi \tag{4.4}$$

$$\langle H\rangle_{N.R.} = d_1\left(\frac{2I_1}{\nu_1}\right)^2+\mathrm{d}_2\left(\frac{2I_2}{\nu_2}\right)^2+d_3\left(\frac{2I_1}{\nu_1}\right)\left(\frac{2I_2}{\nu_2}\right)$$

where

$$\psi = (n_1\nu_1+n_2\nu_2-k)\theta+n_1a_1+n_2a_2+\delta,$$
$$N_1 = |n_1|/2,\ N_2 = |n_2|/2. \tag{4.5}$$

In the above, δ is a constant and ε_1, d_1, d_2, and d_3 are parameters proportional to the original perturbation parameter ε defined by (4.3). The fourth order system of equations deriving from the Hamiltonian (4.4) can be effectively reduced to a second order system. To see this, consider the canonical transformation

$$a_1,a_2;I_1,I_2 \to \psi,q;D,p \tag{4.6}$$

generated by

$$S(a_1,a_2;D,p) = [(n_1\nu_1+n_2\nu_2-k)\theta+n_1a_1 +n_2a_2+\delta]D-n_2a_2. \tag{4.7}$$

For now, we restrict ourselves to a coupling resonance $n_1n_2\neq 0$. We will take the quantity $\mu=[\psi,D]_{a,I}$ to be

$$\mu = \frac{n_1}{\left(1-\frac{n_1\gamma}{n_2}\right)I_1(0)} \tag{4.8}$$

in which γ is the ratio of the two amplitudes at a reference (initial) point

$$\gamma = \frac{I_2(0)}{I_1(0)}. \tag{4.9}$$

With the given generating function and value of μ, the transformation equations are

$$D = \frac{1}{\left(1-\frac{n_1}{n_2}\gamma\right)I_1(0)}I_1 \tag{4.10a}$$

$$I_2 = \frac{n_2}{n_1}I_1-\frac{n_2}{n_1}\left(1-\frac{n_1}{n_2}\gamma\right)I_1(0) \tag{4.10b}$$

$$\psi = (n_1\nu_1+n_2\nu_2-k)\theta+n_1a_1+n_2a_2+\delta \tag{4.10c}$$

$$q = 0. \tag{4.10d}$$

If we transform the Hamiltonian given by (4.3) and (4.4) to these new variables we obtain

$$H(\psi,D,\theta) = \bar{\varepsilon}D^{N_1}(1-D)^{N_2}\cos\psi +\alpha D+c_2D^2/2 \tag{4.11}$$

where

$$\alpha = \mathrm{d}/\mathrm{d}\theta[(n_1\nu_1+n_2\nu_2-k)\theta]+c_1. \tag{4.12}$$

The three parameters appearing above are

$$\bar{\varepsilon} = \varepsilon_1n_1\left(\frac{2}{\nu_1}\right)^{N_1}\left(\frac{2}{\nu_2}\right)^{N_2}\left(-\frac{n_1}{n_2}\right)^{N_1-1} \times\left[I_1(0)\left(\gamma-\frac{n_2}{n_1}\right)\right]^{N_1+N_2-1},$$

$$c_1 = \left(\frac{4}{\nu_2}\right)I_1(0)\left(\gamma-\frac{n_2}{n_1}\right)\left[\frac{n_1d_3}{\nu_1}+\frac{2n_2d_2}{\nu_2}\right], \tag{4.13}$$

$$c_2 = -2I_1(0)\left(\gamma-\frac{n_2}{n_1}\right)\left[\left(\frac{2}{\nu_1}\right)^2d_1\frac{n_1^2}{n_2} +\left(\frac{2}{\nu_1}\right)\left(\frac{2}{\nu_2}\right)n_1d_3+\left(\frac{2}{\nu_2}\right)^2n_2d_2\right].$$

We see that the new Hamiltonian (4.11) is that of a one degree of freedom system. We have reduced the fourth order system for a_1,a_2,I_1,I_2 to a second order system for ψ,D by a singular transformation in which some phase information is lost. (If the equations of motion are solved, the functions $a_1(\theta)$ and $a_2(\theta)$ are not determined, just the combination $\psi(\theta)$.) Also note that this procedure makes the parameters depend on initial conditions.

Equation (4.10b) shows the well-known relation between the two amplitudes. The new amplitude D is simply proportional to the amplitude I_1. When $D(\theta)$ is known, then both $I_1(\theta)$ and $I_2(\theta)$ are known because of relation (4.10b). This behavior is shown in Fig. 1 for both a sum ($n_1 n_2 > 0$) and a difference ($n_1 n_2 < 0$) resonance.

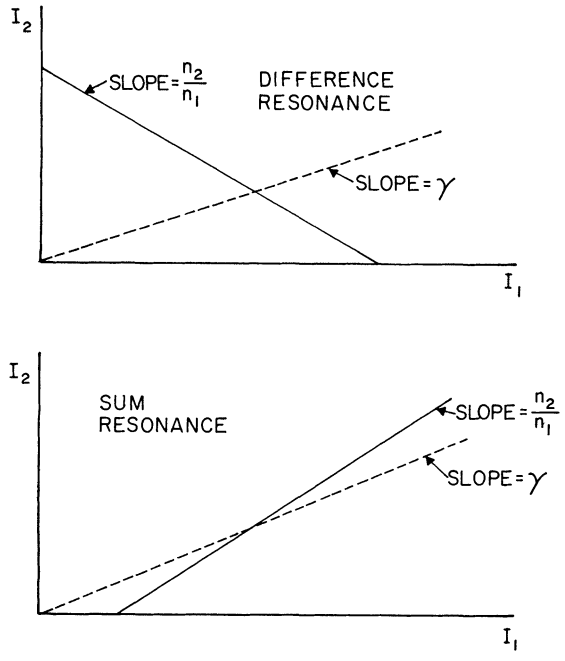

FIG. 1. Near resonance, the amplitudes I_1 and I_2 move along the solid lines. The initial point is at the intersection of the solid and dashed lines.

The two amplitudes I_1 and I_2 vary throughout the course of the motion in such a manner that they stay on the solid straight lines shown in Fig. 1. This demonstrates the well-known fact that motion is always bounded for the difference resonance case but may become unbounded in the case of a sum resonance.

The equations of motion deriving from the Hamiltonian (4.11) are

$$\frac{dD}{d\theta} = \bar{\varepsilon} D^{N_1}(1-D)^{N_2} \sin\psi \tag{4.14a}$$

$$\frac{d\psi}{d\theta} = \alpha + \frac{\bar{\varepsilon}}{2}\left[|n_1|(1-D) - |n_2| D\right] \times D^{N_1-1}(1-D)^{N_2-1}\cos\psi + c_2 D. \tag{4.14b}$$

Because of the particular value of μ chosen previously, the normalization of D is such that D will assume values in the range $0 < D < 1$ in the case of a difference resonance. For a sum resonance we have $1 < D < \infty$ if $\gamma < n_2/n_1$ and $-\infty < D < 0$ if $\gamma > n_2/n_1$. We have but one equation (and therefore solution) for both the sum and difference resonances $|n_1|\nu_1 \pm |n_2|\nu_2 - k = 0$; only the initial conditions and parameters are different. Note that the parameter $\bar{\varepsilon}$ may be imaginary for some sum resonances.

We can integrate (4.14a) to get the relation

$$F_{n_1,n_2}(D(\theta)) - F_{n_1,n_2}(D(0)) = \int_0^\theta d\theta' \bar{\varepsilon} \sin\psi(\theta'). \tag{4.15}$$

The function $F(D)$ which depends only on the absolute values of n_1 and n_2 is

$$F_{n_1,n_2}(D) = \int \frac{dD}{D^{N_1}(1-D)^{N_2}}. \tag{4.16}$$

This is an elementary integral and can be evaluated for all possible values of n_1 and n_2. For the lower order resonances the inverse of this function, F^{-1}, can also be determined. In such cases we can write the amplitude D as an explicit function of θ in the form

$$D(\theta) = F_{n_1,n_2}{}^{-1}\left[\int_0^\theta d\theta' \bar{\varepsilon} \sin\psi(\theta') + F_{n_1 n_2}(D(0))\right]. \tag{4.17}$$

The function F and its inverse F^{-1} are tabulated for several nonlinear coupling resonances in Table I. By definition (4.16) we see that

$$F_{n_1,n_2}(D) = -F_{n_2,n_1}(1-D) \tag{4.18}$$

TABLE I

The function F and its inverse for several nonlinear coupling resonances $n_1\nu_1 + n_2\nu_2 - k = 0$

$\lvert n_1\rvert$	$\lvert n_2\rvert$	$F_{n_1,n_2}(x)$	$F_{n_1,n_2}{}^{-1}(x)$
1	2	$\ln\frac{x^{1/2}+1}{x^{1/2}-1}$	$\left(\frac{1+e^x}{1-e^x}\right)^2$
2	1	$\ln\frac{(1-x)^{1/2}-1}{(1-x)^{1/2}+1}$	$1-\left(\frac{1+e^x}{1-e^x}\right)^2$
1	3	$2\left(\frac{2}{1-x}\right)^{1/2}$	$\left[1+\left(\frac{2}{x}\right)^2\right]^{-1}$
3	1	$-2\left(\frac{1-x}{x}\right)^{1/2}$	$1-\left[1+\left(\frac{2}{x}\right)^2\right]^{-1}$
2	2	$\ln\frac{-x}{1-x}$	$(1-e^{-x})^{-1}$

so that Table I has two redundant entries. Because of this symmetry and the fact that a given F describes both a sum and a difference resonance, only three different functions (and their inverses) are needed to describe all nonlinear coupling resonances up through third order ($|n_1| + |n_2| - 1 \leq 3$).

Let us examine the sum resonance near $\gamma = n_2/n_1$. At this point the normalization for D given by (4.10a) appears unsuitable because of the zero denominator. It can be shown that writing the solution (4.17) in terms of I_1 by using (4.10a) and (4.13) we get an expression which, in the limit $\gamma \to n_2/n_1$, satisfies the equation of motion (4.14). Thus the solution $I_1(\theta)$, redefined at $\gamma = n_2/n_1$ is continuous in γ at that point. This means that the solution provides a good description of a sum resonance near $\gamma = n_2/n_1$.

Now look at the one dimensional resonances. We will set $n_2 = 0$ and consider the resonance $n_1\nu_1 - k = 0$. It can be seen that with the generating function

$$S = [(n_1\nu_1 - k)\theta + n_1 a_1 + \delta]\, D + n_1 \gamma a_2 \quad (4.19)$$

and with

$$\mu = n_1/I_1(0) \quad (4.20)$$

we will obtain the same equations of motion (4.14) with $n_2 = 0$. Only the normalization for D

$$D(\theta) = I_1(\theta)/I_1(0) \quad (4.21)$$

and the three parameters

$$\begin{aligned} \bar{\varepsilon} &= \varepsilon_1 n_1 \left(\frac{2}{\nu_1}\right)^{N_1} I_1(0)^{N_1 - 1} \\ c_1 &= \left(\frac{2}{\nu_1}\right)\left(\frac{2}{\nu_2}\right) n_1 d_3 \gamma I_1(0) \\ c_2 &= n_1 \left(\frac{2}{\nu_1}\right)^2 d_1 I_1(0) \end{aligned} \quad (4.22)$$

are different. The function $F(D)$ for the one dimensional case is given simply by

$$F_{n_1,0}(x) = \frac{x^{1-N_1}}{1-N_1} \quad (4.23)$$

and its inverse is

$$F_{n_1}{}^{-1}{}_{,0}(x) = [(1-N_1)x]^{.(1-N_1)-1} \quad (4.24)$$

5. APPROXIMATE SOLUTIONS

5.1. *The Basic Approximation*

The equations of motion (4.14) in the ψ, D variables describe a system near the $n_1\nu_1 + n_2\nu_2 - k = 0$ resonance. This resonance will be denoted by (n_1, n_2, k) or simply (n_1, n_2) whenever the value of k is not important (k does not enter in the equations, it only denotes the Fourier component contributing to the perturbation). The parameter α will be called the detuning as it describes how far the system is from exact resonance. The detuning contains a contribution c_1 from the nonresonant terms in the averaged Hamiltonian (4.4). These nonresonant terms also provide a nonlinear frequency shift described by c_2. (Actually, c_1 constitutes a nonlinear frequency shift in the fourth order system a_1, a_2, I_1, I_2 but it appears as a linear shift in the second order system ψ, D. A nonlinear frequency shift is an effect that changes the frequency of the oscillation according to the value of the amplitude.) The parameter $\bar{\varepsilon}$ describes the strength of the nonlinear resonant perturbation. It is normalized to depend on the initial values of I_1 and I_2 so that a small value of $\bar{\varepsilon}$ means either that the perturbation ε_1 is small or that small initial amplitudes are involved. All three parameters $\bar{\varepsilon}$, α, and c_2 will depend on θ, even if the unnormalized perturbation parameter ε_1 is fixed, if the betatron frequencies vary with θ. We will classify the various cases to be studied by the manner in which the detuning α varies with θ.

In order to obtain the solution $D(\theta)$ by means of equation (4.17) we will have to obtain an approximate expression for the phase $\psi(\theta)$. The basic approximation we will use is to neglect the phase dependent term in the phase equation (4.14b). This procedure will be justified if the value of $\bar{\varepsilon}$ is not too large for then the phase dependent term will not contribute, on the average, to the phase $\psi(\theta)$. If we neglect also, for now, the linear and nonlinear frequency shifts ($c_1 = c_2 = 0$), the phase equation (4.14b) can be integrated to yield

$$\psi(\theta) = \psi(0) + (n_1\nu_1 + n_2\nu_2 - k)\theta. \quad (5.1)$$

This expression, substituted into (4.17), will then complete the problem. In order that the requirements of perturbation theory be satisfied and our result useful, the resonant combination of

frequencies above must be in the form

$$n_1\nu_1+n_2\nu_2-k=\varepsilon\Lambda^R(\varepsilon\theta) \tag{5.2}$$

for some function Λ^R.

5.2. *Solutions for Four Cases*

5.2a. *Static resonance.* If all parameters are independent of θ then the Hamiltonian (4.11) will not explicitly contain θ and therefore will be a first integral or invariant

$$H(\psi,D)=\text{const.} \tag{5.3}$$

Using this fact, we can solve (4.11) for $\cos\psi$ and plot $\cos\psi$ as a function of D in the manner of Sturrock's configuration diagrams[15] in order to obtain qualitative information on the behavior of the system in this case

$$\cos\psi=\frac{H-\alpha D-c_2D^2/2}{\bar{\varepsilon}D^{N_1}(1-D)^{N_2}}. \tag{5.4}$$

The denominator in this expression is always real and we can assume the phase of $\bar{\varepsilon}$ is such that this denominator is positive. We can also take α to be positive without any loss of generality.

The case of resonances with $|n_1|>2$, $|n_2|>2$ with $c_2=0$ is shown in Fig. 2. The physically

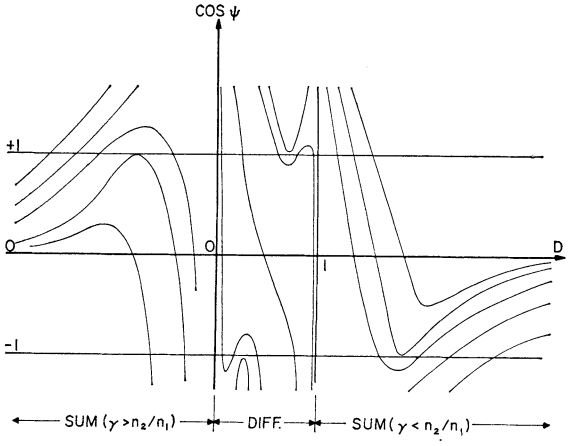

FIG. 2. Configuration diagram for $n_1\geq 3$, $n_2\geq 3$ with no nonlinear frequency shift.

allowed region is within the strip $-1\leq\cos\psi\leq 1$. From this diagram we also see how unbounded motion can occur for sum resonances for large enough initial amplitudes.

Let us look at the sum resonances. As shown in Fig. 1, possible motion in the I_1,I_2 plane occurs along lines of slope n_2/n_1. The ratio I_2/I_1 is either always greater or always smaller than the quantity n_2/n_1, leading to a separation of the configuration diagram into two regions for the sum resonance. It follows that for motion in the $I_2/I_1<n_2/n_1$ region, only the value of n_2 is important and for $I_2/I_1>n_2/n_1$, only n_1 determines the qualitative behavior. Thus, for example, the (2,1) and (2,2) resonances behave similarly for those cases in which I_2 is large compared to I_1. We therefore need to discuss only a single region, say $D<0$.

To aid in interpreting the configuration diagrams, lines of constant D and $\bar{\varepsilon}/\alpha$ in the I_1,I_2 plane are plotted in Fig. 3 for the sum resonance

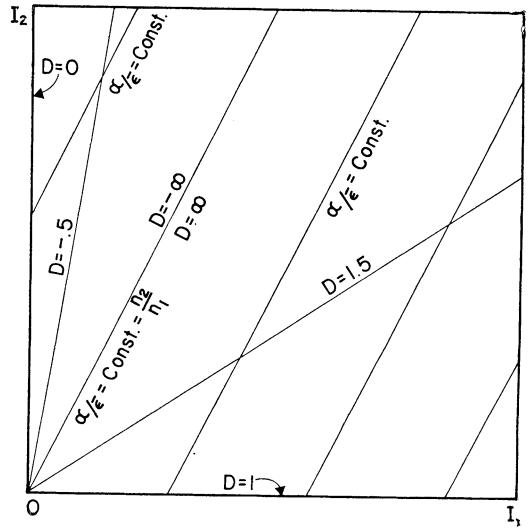

FIG. 3. Lines of constant D and $\bar{\varepsilon}/\alpha$ in the I_1, I_2 plane.

case. Since the normalized perturbation parameter $\bar{\varepsilon}$, defined by (4.13) or (4.22), depends on the initial amplitudes, we can make $|\bar{\varepsilon}/\alpha|$ as small as desired for any set of parameters by making the initial amplitudes small enough. Thus near the origin of the I_1,I_2 projection of the a_1,a_2,I_1,I_2 phase space we always have bounded orbits. Oscillations of small enough initial amplitudes will not grow indefinitely. But for $I_2\gg n_2/n_1$ (the region $D<0$), the phase $\psi(\theta)$ is severely distorted from the simple expression (5.1) for small values of I_1 for the case of $n_1=1$. The simplest such resonance is the

$(1,\pm 2)$ resonance for which the equations of motion with $c_2 = 0$, are

$$\frac{dD}{d\theta} = \bar{\varepsilon} D^{1/2} (1-D) \sin\psi \tag{5.5a}$$

$$\frac{d\psi}{d\theta} = \alpha + \frac{\bar{\varepsilon}}{2} \frac{1-3D}{D^{1/2}} \cos\psi. \tag{5.5b}$$

For a given value of $\bar{\varepsilon}/\alpha$, small enough values of D will cause the phase dependent term in (5.5b) to be comparable in magnitude to α. Depending on the initial phase, the two terms in the phase equation then either add to produce a rapidly changing phase or they subtract to produce a nearly constant value of $\cos\psi(\theta) \approx 1$. Thus for small amplitudes in the $D < 0$ region, the phase behavior depends strongly on the initial conditions but the amplitude variation is roughly the same. Similar remarks apply for $n_2 = 1$ in the $D > 1$ region. Such motion behavior is demonstrated in Fig. 4 where the $D < 0$ region of the configuration diagram is shown for $n_1 = 1$ with $|\bar{\varepsilon}/\alpha| < 1/2$.

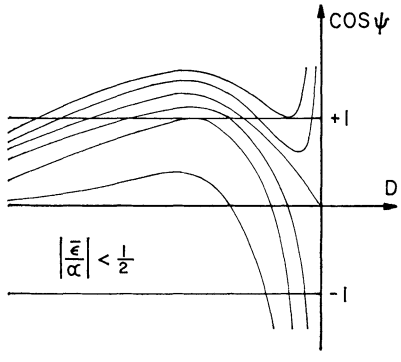

FIG. 4. Configuration diagram for $n_1 = 1$ in the $D < 0$ region (sum resonance).

Let us conclude these qualitative remarks by considering the effect of the nonlinear frequency shift, i.e. $c_2 \neq 0$. By (5.4) we see that the trajectories will go as $\cos\psi \sim D^{1/2}$ for large D for $|n_1| + |n_2| = 3$; thus all orbits are bounded. If $|n_1| + |n_2| = 4$ then $\cos\psi \to \pm |c_2/2\bar{\varepsilon}|$ which means all orbits are bounded if this limiting value is greater than unity. For $|n_1| + |n_2| > 4$ we have $\cos\psi \to 0$ for large D so the c_2 term has little effect on large amplitude motion. For such high order resonances, however, higher order frequency shifts may modify these results. We note that for difference resonances, a value of c_2 small compared to α will have no qualitative effect on the motion for the reason that D is limited to the range $0 < D < 1$.

Let us now look at the quantitative behavior of static resonances. Excluding the nonlinear frequency shift, $c_2 = 0$, the basic approximation for the phase function (5.1) yields the following approximate solution

$$D(\theta) = F^{-1}\left(-\frac{\bar{\varepsilon}}{\alpha}\left[\cos(\alpha\theta + \psi(0) - \cos\psi(0) \right] + F(D(0)) \right). \tag{5.6}$$

We see that the size of the amplitude variation depends on the ratio $\bar{\varepsilon}/\alpha$ while the frequency of the beats is simply the detuning α. Because of the requirement (5.2) and the fact that c_1 is proportional to ε, we must have that the detuning is proportional to ε. Hence, unless α is zero, we have

$$\frac{\bar{\varepsilon}}{\alpha} = 0(1). \tag{5.7}$$

Thus the range $0 \leq |\bar{\varepsilon}/\alpha| \lesssim 1$, which is the largest interval over which the simple approximation (5.6) could be valid, leads to amplitude variations in the range $0 \leq \Delta D/D \lesssim 1$. Using Table I, the approximate solution for the $(1,\pm 2)$ static resonance is

$$D(\theta) = \left[\frac{1+d(\theta)}{1-d(\theta)} \right]^2 \tag{5.8a}$$

where

$$d(\theta) = \frac{D^{1/2}(0)+1}{D^{1/2}(0)-1} \exp -\frac{\bar{\varepsilon}}{\alpha}[\cos(\alpha\theta + \psi(0)) - \cos\psi(0)]. \tag{5.8b}$$

This function is plotted in Fig. 5 for $D(0) = 0.5$ and $\psi(0) = 0$ for various values of the parameter $\bar{\varepsilon}/\alpha$ which characterizes this static resonance situation. Here we are dealing with the difference resonance region as $0 < D < 1$ and $\bar{\varepsilon}$ is real. The independent variable in Fig. 5 is taken to be $\alpha\theta/\pi$ rather than θ so that the period of the beats is fixed to be 2. In these graphs, the exact solutions to Eqs. (5.5), obtained by numerical integration, are also shown for comparison. We see that the error is significant for large beat amplitudes but that useful results are still possible with this simple approximation. By the form of the solution (5.6)

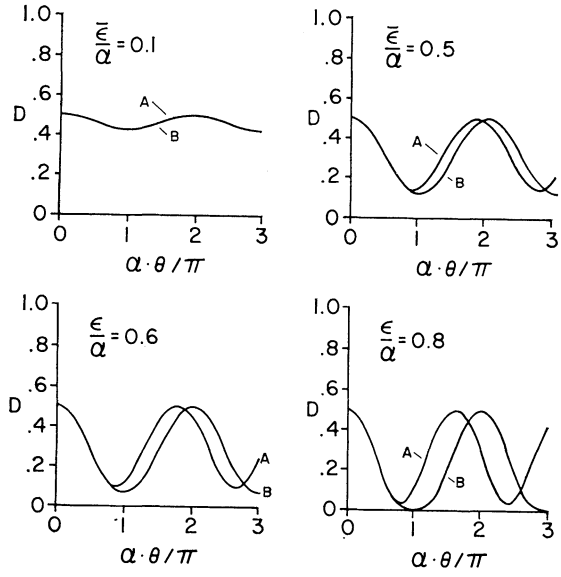

FIG. 5. Amplitude D as a function of $\alpha\theta/\pi$ for the static (1, −2) resonance with $D(0) = 0.5$, $\psi(0) = 0$, and no nonlinear frequency shift. Curve A is exact and curve B is the approximation.

we see that an increase in the beat frequency α will lead to a decrease in the beat amplitude. Thus in the examples in Fig. 5, since the beat frequency was underestimated, the beat amplitudes were overestimated. The nature of the problem is such that simply by correcting the beat frequency α, we improve the approximation. The exact solution for the phase is

$$\psi(\theta) = \bar{\alpha}\theta + h(\bar{\alpha}\theta) + \psi(0) - h(0) \qquad (5.9)$$

where $\bar{\alpha}$ is the true beat frequency and h is a periodic function with zero mean. The correction $\Delta\alpha = \bar{\alpha} - \alpha$ is simply the average of the phase dependent term in the phase equation. For this static resonance case, however, such an average is difficult to estimate.

It is clear that the approximation (5.6) is not accurate whenever the phase dependent term in the phase equation is comparable to α. In the case of an $n_1 = 1$ or $n_2 = 1$ resonance, particular difficulty may occur whenever D is near 0 or 1, respectively. For all other resonances, the requirement is that $|\varepsilon/\alpha|$ be small, preferably much smaller than unity.

In order to calculate the effect of the nonlinear frequency shift, the following substitution applied to (5.6) yields good results.

$$\alpha \to \alpha + (D_{\min} + D_{\max})c_2/2. \qquad (5.10)$$

This just replaces the $c_2 D$ term in the phase equation by a constant effective value. The maximum and minimum values of the amplitude that are required can be obtained from (5.6) by setting the argument of F^{-1} equal to its maximum and minimum values, i.e. by setting

$$\cos(\alpha\theta + \psi(0)) = \pm 1. \qquad (5.11)$$

This follows from the fact that the function F^{-1} has no critical points in the regions of interest. To see this note

$$\frac{dF^{-1}(x)}{dx} = (F^{-1})^{N_1}(1 - F^{-1})^{N_2} \qquad (5.12)$$

so writing $D = F^{-1}(x)$ we see that (5.12) is nonzero for all D in the three regions $(-\infty,0)$ $(0,1)$, and $(1,\infty)$. It was found that the substitution (5.10) yielded good results for our example, with negligible error even for large frequency shifts ($c_2 \leq 3\alpha$).

In conclusion we have seen that a linearly increasing phase function is a good approximation for the static resonance case in the region for which the beats in the amplitude are not too large. We must apply a correction to the beat frequency α to ensure good results; this correction is easy to obtain for the nonlinear frequency shift term but not for the phase dependent term.

5.2b. *Fixed distance from resonance line.* Let us consider the situation in which the betatron frequencies vary with θ in such a manner that

$$n_1\nu_1(\theta) + n_2\nu_2(\theta) - k = \text{const.} \qquad (5.13)$$

The operating point (ν_1,ν_2) thus moves along a line parallel to the resonance line, as shown by curve A in Fig. 6. With $c_1 = 0$ (or, alternatively, with betatron frequencies satisfying $\alpha = \text{const.}$ rather than (5.13)) the basic approximation yields the same phase function as for the static case. But because of the dependence of $\bar{\varepsilon}$ on the frequencies, and therefore on θ, the expression (5.6) for $D(\theta)$ is no longer valid unless we make the additional assumption that the betatron frequencies vary at a rate small compared to the beat frequency

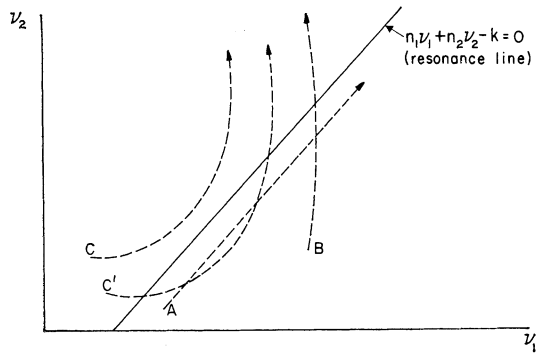

FIG. 6. Movement of operating point (ν_1,ν_2) in relation to resonance line is shown for four cases.

α. With this assumption, we see that the amplitude D depends on θ as

$$D = D(\alpha\theta,\bar{\varepsilon}(\nu_1(\theta),\nu_2(\theta))/\alpha). \tag{5.14}$$

By (4.13) or (4.22) we see that by increasing the betatron frequencies, the parameter $\bar{\varepsilon}$ is decreased. For a constant α this means the beating amplitude decreases as the frequencies are increased. This effect may be described by writing down the approximate first integral for the ψ,D system corresponding to the basic approximation for this case (which includes the static)

$$F(D)+(\bar{\varepsilon}/\alpha)\cos\psi = \text{const.} \tag{5.15}$$

This nonlinear effect produced by an adiabatic change in the frequencies is in addition to the linear effect in which the amplitude $A_1 = (2I_1/\nu_1)^{1/2}$ will vary in θ because of the variation of ν_1.

A practical situation may arise in which the operating point moves parallel to the resonance line for a distance which may be large compared to the distance to the resonance line but in which the total relative changes in the frequencies are small. For this case, we have shown therefore, that the system behaves as in the static case.

5.2c. *Passage through resonance.* We would like to study the important situation in which the operating point crosses a resonance line, as shown by curve B of Fig. 6. We will use the following model in which β characterizes the rate at which the system is brought through resonance (exact resonance occurs at $\theta = 0$)

$$n_1\nu_1(\theta)+n_2\nu_2(\theta)-k = \beta\theta. \tag{5.16}$$

With $c_1 = c_2 = 0$, the basic approximation yields

$$\psi(\theta) = \beta\theta^2+\psi(0). \tag{5.17}$$

Only positive values of β need be considered for $\beta\to-\beta$ corresponds to $\psi\to\psi+\pi$ and $\theta\to-\theta$. Such a phase function yields the solution

$$D(\theta) = F^{-1}(\bar{\varepsilon}(\pi/2\beta)^{1/2}\,[\cos\psi(0)\,S((2\beta/\pi)^{1/2}\theta) + \sin\psi(0)\,C((2\beta/\pi)^{1/2}\theta)]+F(D(0))). \tag{5.18}$$

The functions S and C are the Fresnel integrals defined by[16]

$$S(x)=\int_0^x dt\,\sin\frac{\pi}{2}t^2,\quad C(x)=\int_0^x dt\,\cos\frac{\pi}{2}t^2. \tag{5.19}$$

We see that the size of the variation in amplitude depends on the combination of parameters $\bar{\varepsilon}/\beta^{1/2}$. Comparing (5.17) with (5.2) we see that β must be proportional to ε^2 so that

$$\bar{\varepsilon}/\beta^{1/2} = 0(1). \tag{5.20}$$

Let us look at the properties of the approximate solution (5.18). It is easy to see that extrema of $D(\theta)$ occur whenever we have

$$\psi(0)+\beta\theta^2 = n\pi,\ n = \text{integer}. \tag{5.21}$$

By using the property of Fresnel integrals

$$\lim_{x\to\pm\infty} S(x) = \lim_{x\to\pm\infty} C(x) = \pm\tfrac{1}{2} \tag{5.22}$$

we can obtain the limiting values of the amplitude far away from resonance, $D(\pm\infty)$. Then we see that the maximum and minimum final amplitudes occur for initial phases satisfying

$$\psi(0) = \pi/4,\ 5\pi/4 \tag{5.23}$$

where we consider the range $0\le\psi(0)\le 2\pi$. For such initial phases, the extrema of $D(\theta)$ closest to the exact resonance point of $\theta = 0$ are located at

$$\theta = \pm\left(\frac{3\pi}{4\beta}\right)^{1/2}. \tag{5.24}$$

At these values of θ, the absolute maximum and minimum values of D as a function of θ are obtained. Expression (5.24) is a good measure of the resonance region; since β is proportional to ε^2, the length of the resonance region is $0(\varepsilon^{-1})$.

When (5.18) is maximized and minimized over both θ and $\psi(0)$ we see (this can be done numerically) that the initial phases must be 0.27π and

1.27π rather than those given by (5.23). Accordingly, these extrema occur at the points $\theta = \pm(0.73\pi/\beta)^{1/2}$ rather than those given by (5.24). Since a function is slowly varying near its extrema we can disregard the distinction for practical purposes and say that the maximum (minimum) amplitude results for the same initial phase (5.23) that maximizes (minimizes) the final amplitudes $D(\pm\infty)$.

Now let us again choose the $(1,\pm 2)$ resonance as an example. The approximate solution is

$$D(\theta) = [(1+\mathrm{d}(\theta))/(1-\mathrm{d}(\theta))]^2$$

where

$$\mathrm{d}(\theta) = \frac{D^{1/2}(0)+1}{D^{1/2}(0)-1} \times \exp\left\{\bar{\varepsilon}\left(\frac{\pi}{2\beta}\right)^{1/2}\left[\cos\psi(0)S\left(\left(\frac{2\beta}{\pi}\right)^{1/2}\theta\right) + \sin\psi(0)C\left(\left(\frac{2\beta}{\pi}\right)^{1/2}\theta\right)\right]\right\}. \tag{5.25}$$

In Fig. 7 this solution is plotted for $D(0) = 0.5$ and $\bar{\varepsilon}/\beta^{1/2} = 1$ for the two initial phases that lead to the maximum and minimum final amplitudes. The independent variable for these graphs is $(2\beta/\pi)^{1/2}\theta$. The exact solution is also plotted for comparison for the $\psi(0) = 5\pi/4$ case; the error for the $\psi(0) = \pi/4$ case is negligible. Although not shown here, it was found that the maximum error occurs for phases near $\psi(0) = 3\pi/4$ at which point the variation in the amplitude is near zero.

The error in the approximate solution introduced by neglecting the phase dependent term in the phase equation may be examined by replacing this term by an equivalent constant c

$$\frac{\mathrm{d}\psi}{\mathrm{d}\theta} = 2\beta\theta + c. \tag{5.26}$$

To evaluate c we must average the phase dependent term in (4.14b). Fixing $D(\theta)$ to its value at resonance and using the basic approximation (5.17) for the phase and integrating the subsequent phase equation yields

$$\psi = \psi(0) + \beta\theta^2 + \bar{\varepsilon}\left(\frac{\mathrm{d}f(D)}{\mathrm{d}D}\right)_{D(0)}\left(\frac{\pi}{2\beta}\right)^{1/2} \times \left\{\cos\psi(0)\,C\left(\left(\frac{2\beta}{\pi}\right)^{1/2}\theta\right) - \sin\psi(0)S\left(\left(\frac{2\beta}{\pi}\right)^{1/2}\theta\right)\right\} \tag{5.27}$$

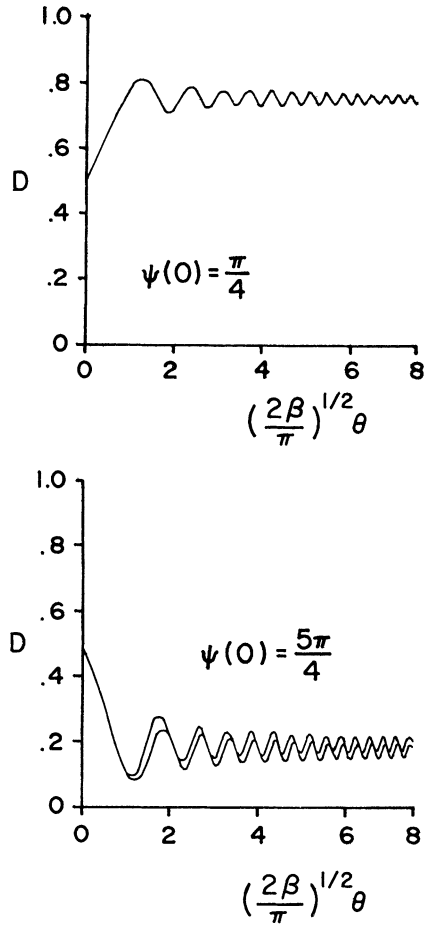

FIG. 7. Approximate solutions D as a function of $(2\beta/\pi)^{1/2}\theta$ for passage through resonance case are shown for two phases. The error in the $\psi(0) = \pi/4$ case is negligible. For $\psi(0) = 5\pi/4$ the exact solution, with a larger final amplitude than the approximation, is also shown.

where

$$f(D) = D^{N_1}(1-D)^{N_2} \tag{5.28}$$

Both of the Fresnel integrals can be approximated by

$$S(x) = C(x) = x/2^{1/2} \tag{5.29}$$

in the resonance region (the slope is chosen by taking derivatives of S and C at points midway between the origin and the first extremum). With this choice, we see that an approximate value of the constant c of Eq. (5.26) is

$$c = \bar{\varepsilon}\left(\frac{\mathrm{d}f(D)}{\mathrm{d}D}\right)_{D(0)}[\cos\psi(0) - \sin\psi(0)]/2^{1/2}. \tag{5.30}$$

With a phase equation of the form (5.26), the solution for $D(\theta)$ becomes

$$D(\theta) = F^{-1}\left\{\bar{\varepsilon}\left(\frac{\pi}{2\beta}\right)^{1/2} \times\left[\cos\left(\psi(0)-\frac{c^2}{4\beta}\right)\left(S\left(\left(\frac{2\beta}{\pi}\right)^{1/2}\left(\theta+\frac{c}{2\beta}\right)\right)-S\left(\frac{c}{2\beta}\left(\frac{2\beta}{\pi}\right)^{1/2}\right)\right)+\sin\left(\psi(0)-\frac{c^2}{4\beta}\right)\left(C\left(\left(\frac{2\beta}{\pi}\right)^{1/2}\times\left(\theta+\frac{c}{2\beta}\right)\right)-C\left(\frac{c}{2\beta}\left(\frac{2\beta}{\pi}\right)^{1/2}\right)\right)\right]+F(D(0))\right\}. \quad (5.31)$$

We note that in the most important cases $\psi(0) = \pi/4$ and $5\pi/4$ the correction constant (5.30) is zero. Let us check the validity of this prediction by obtaining, for these phases, the maximum and minimum values of $D(\theta)$ as functions of the parameter $\bar{\varepsilon}/\beta^{1/2}$. These are plotted in Fig. 8 for the $(1,-2)$ resonance along with the exact values for three initial amplitudes. We see that the error is indeed small except for those cases in which D is near zero (the same would be true for D near unity in the case of an $n_2 = 1$ resonance). Note that the error here is much smaller than that found for the static resonance case for equivalent variations in the amplitude

Since in a practical problem the situation is that a system, initially far away from resonance, is brought through a resonance, we would like to write the (corrected) solution (5.31) in terms of $D(-\infty)$ rather than $D(0)$. Using (5.22) we obtain

$$D(\theta) = F^{-1}\left\{\bar{\varepsilon}\left(\frac{\pi}{2\beta}\right)^{1/2} \times\left[\cos\left(\psi(0)-\frac{c^2}{4\beta}\right)\left(S\left(\left(\frac{2\beta}{\pi}\right)^{1/2}\left(\theta+\frac{c}{2\beta}\right)\right)+\tfrac{1}{2}\right)+\sin\left(\psi(0)-\frac{c^2}{4\beta}\right)\left(C\left(\left(\frac{2\beta}{\pi}\right)^{1/2}\left(\theta+\frac{c}{2\beta}\right)\right)+\tfrac{1}{2}\right)\right]+F(D(-\infty))\right\}. \quad (5.32)$$

After the system has passed through resonance the final amplitude will be

$$D(\infty) = F^{-1}\left\{\bar{\varepsilon}\left(\frac{\pi}{2\beta}\right)^{1/2}\left[\cos\left((0)\,\psi-\frac{c^2}{4\beta}\right)+\sin\left(\psi(0)-\frac{c^2}{4\beta}\right)\right]+F(D(-\infty))\right\}. \quad (5.33)$$

We see that the correction c has only the effect of shifting the phase. Thus the phase dependent term can be neglected to a good approximation if only the maximum and minimum values of D are sought. This is often the case in practical problems involving a uniform distribution of particles with different phases. It does not really matter which particle has which phase. And since the nonlinear frequency shift term in the phase equation can be approximated by the constant value $c_2 D(0)$, we see that this correction also has only the effect of shifting the phase. Therefore, in a practical sense,

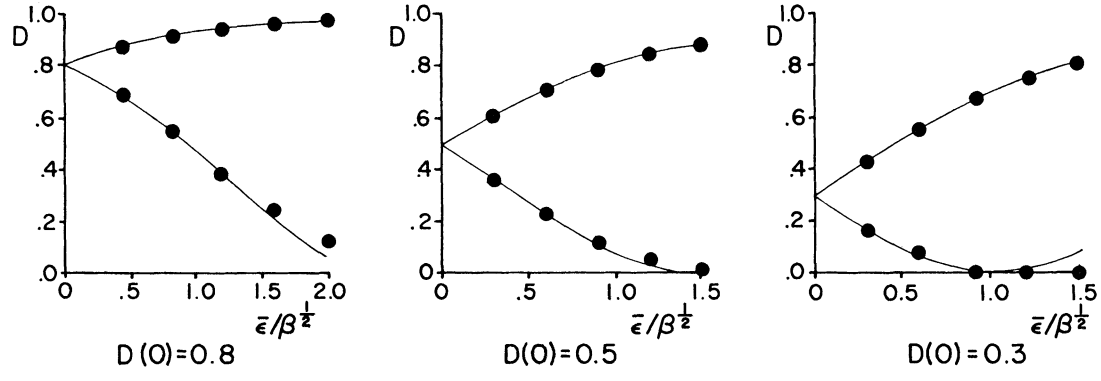

FIG. 8. The maximum and minimum amplitudes D resulting from crossing the resonance line as a function of $\bar{\varepsilon}/\beta^{1/2}$. The discrete points marked are the exact solutions.

the results for the passage through resonance case are independent of both the phase dependent and the nonlinear frequency shift terms in the phase equation.

Equation (5.32) is really the same as (5.31) as far as D as a function of $\bar{\varepsilon}/\beta^{1/2}$ is concerned. Therefore, the $D(0)$ values in Fig. 7 can be interpreted as $D(-\infty)$ values and the results presented are the maximum and minimum amplitudes as a function of $\bar{\varepsilon}/\beta^{1/2}$ for the given values of the initial amplitude, far away from resonance. Since the sum of the Fresnel integrals is maximized as

$$\begin{aligned}\overset{\max}{\underset{\min}{x}}\,(S(x)+C(x)) &= S(\pm(3/2)^{1/2})+S(\pm(3/2)^{1/2}) \\ &= \pm(1+0.17)/2 \qquad (5.34)\end{aligned}$$

the maximum values to which the amplitude can rise is 17 per cent above its final value, provided the total change is not too large. Figure 9 illustrates the amplitude behavior for two particles with the same amplitude ($D = 0.5$) before crossing the resonance but with phases of $\psi(0) = \pi/4$ and $\psi(0) = 5\pi/4$.

5.2d. *Passage by resonance line.* In this final example we will consider the situation in which the operating point, initially far away from the resonance line, passes close by the line and finally ends up again far away from the resonance line. This situation is shown in Fig. 6 for two cases: Curve C' crosses the resonance line and curve C does not. We use the following two parameter model

$$n_1\nu_1(\theta)+n_2\nu_2(\theta)-k = \beta_1\theta^2+\beta_0. \qquad (5.35)$$

Neglecting the frequency shifts $c_1 = c_2 = 0$, the basic approximation yields

$$\psi(0) = \beta_1\theta^3+\beta_0\theta+\psi(0) \qquad (5.36)$$

and the solution $D(\theta)$, in terms of $D(-\infty)$, is

$$D(\theta) = F^{-1}\left\{\bar{\varepsilon}\int_{-\infty}^{\theta}\sin[\beta_1 t^3+\beta_0 t+\psi(0)]dt + F(D(-\infty))\right\}. \qquad (5.37)$$

The integral appearing above is not a well known function except in the limit $\theta\to\infty$. Therefore let us look at the final amplitude $D(\infty)$

$$D(\infty) = F^{-1}\{2\pi\rho_1\rho_2\sin\psi(0)\,Ai(\rho_2)+F(D(-\infty))\} \qquad (5.38)$$

which is expressed in terms of the Airy function[17] Ai defined by

$$(3a)^{-1/3}\,\pi\,Ai[\pm(3a)^{-1/3}x] = \int_0^{\infty}\cos(at^3\pm xt)\mathrm{d}t. \qquad (5.39)$$

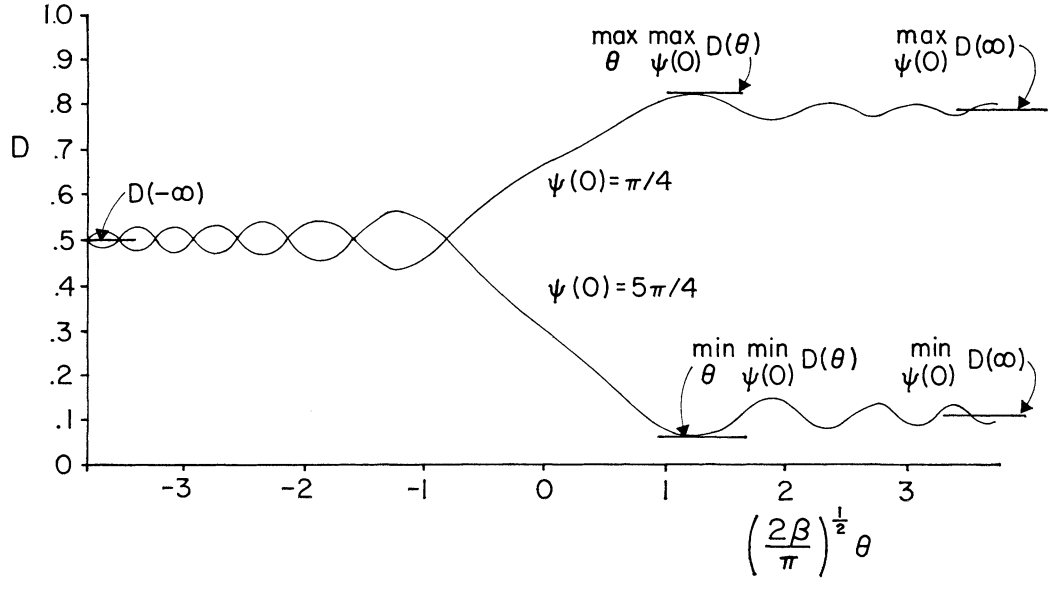

FIG. 9. Amplitude D as a function of $(2\beta/\pi)^{1/2}\theta$ for two phases with the same amplitude before resonance is crossed.

The form of the solution (5.38) is such that there are two characteristic parameters, which are

$$\rho_1 = \frac{\bar{\varepsilon}}{\beta_0}, \quad \rho_2 = \frac{\beta_0}{(3\beta_1)^{1/3}}. \tag{5.40}$$

By the condition (5.2) we know that $\beta_0 \sim \varepsilon$ and $\beta_1 \sim \varepsilon^3$ so that both ρ_1 and ρ_2 are 0(1).

The Airy function $Ai(x)$ is oscillatory for negative x with an amplitude that decreases as $(-x)^{-1/4}$ for large negative x. For positive x, $Ai(x)$ monotonically decreases with increasing x at a rate which, for large x, is faster than any power of $1/x$. From these properties of $Ai(x)$ we may draw the following conclusions:

(a) For $\rho_1\rho_2 > 0$, the resonance line is not crossed (curve C in Fig. 6). In this case, for either $\rho_2 \to \infty$ (slow passage) or $\rho_2 \to 0$ (fast passage), the final change in the amplitude D tends to zero. The maximum effect is found to occur at $\rho_2 \approx 0.9$. In any case, the size of the effect is determined also by $\rho_1 = \bar{\varepsilon}/\beta_0$. Since β_0 is the minimum detuning, this dependence on the parameter ρ_1 is similar to that found in the static case. This effect is important for the slow passage (near static) situation in which the amplitude near resonance will beat with an amplitude depending on ρ_1, although the final change in the amplitude will be small.

(b) For $\rho_1\rho_2 < 0$ the resonance line will be crossed twice (curve C'). For slow passage, the amplitude will be severely affected for any value of ρ_1 since $\rho_2 Ai(\rho_2)$ can get large for large negative ρ_2. For fast passage, the effect on the amplitude again tends to zero. We note, however, that if ρ_2 is a zero of Ai, i.e. if the rate of passage is just right, then the final change is zero. Thus crossing the resonance the second time will undo, in effect, the consequences of having crossed the first time; this effect is independent of the phase $\psi(0)$.

5.3. *Particle Losses Due to Resonances*

We consider a collection of particles with various initial conditions. A uniform distribution of amplitudes I_1 and I_2 up to maximum values I_{1m} and I_{2m}, respectively will be assumed. We will calculate the fraction of particles lost by striking the vacuum chamber walls because of nonlinear resonances. Since various initial conditions are involved, the variables I_1 and I_2 will be used although the solution will still be expressed in terms of the now-known form of $D = D(\theta)$.

5.3a. *Resonance effects.* We will calculate curves H and V in Fig. 10 which have the property that particles with initial conditions between these

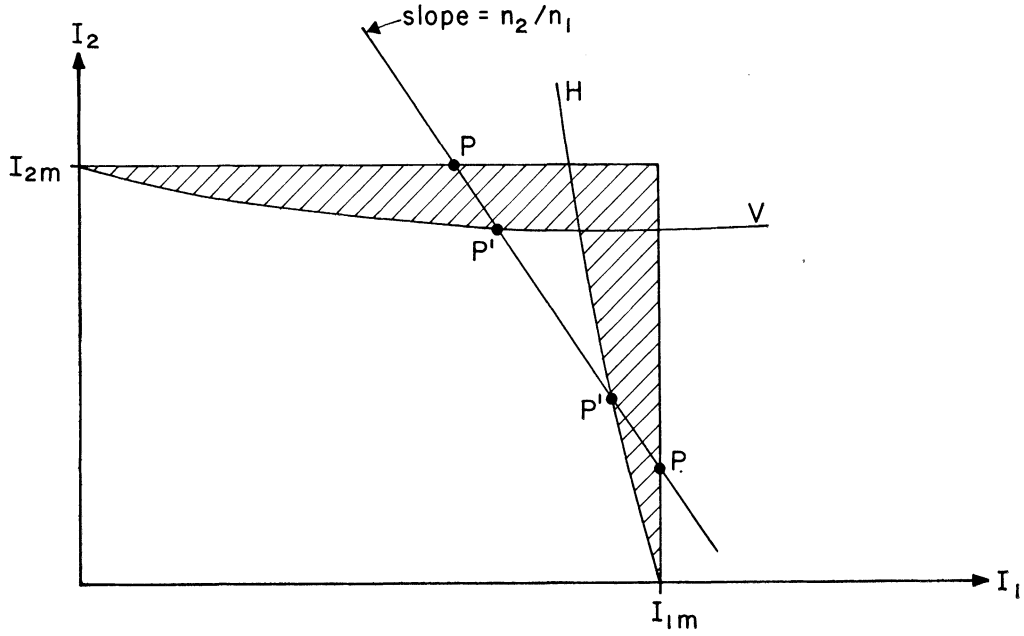

FIG. 10. The nonlinear resonance will cause the particles in the shaded areas to be lost to the walls.

curves and the boundaries $I_1 = I_{1m}$ and $I_2 = I_{2m}$ are lost to the horizontal and vertical aperture walls. A point $P' = (I_1(0), I_2(0))$ on one of the curves H or V will move, under the influence of a (n_1,n_2,k) resonant perturbation, along a straight line with slope n_2/n_1, to a point on the boundary $P = (I_1,I_2)$. The set of all points P′ that just reach the boundary give us curves H and V. Point P′ is given in terms of P by

$$I_1 = I_1(0) + \Delta I_1$$
$$I_2 = I_2(0) + (n_2/n_1)\Delta I_1. \tag{5.41}$$

By using the solution (4.17) to calculate the change in the amplitude I_1 in terms of the quantity

$$P_m = \max_{\theta,\psi(0)} \int_0^\theta \sin\psi(\theta')\mathrm{d}\theta' \tag{5.42}$$

we obtain, for small changes,

$$\Delta I_1 = n_1\varepsilon_1 (2I_1(0)/\nu_1)^{N_1} (2I_2(0)/\nu_2)^{N_2} P_m. \tag{5.43}$$

To obtain the actual losses, the quantity P_m must be divided by two because of the uniform distribution of the particle phases. Now setting $I_1(0) = I_{1m}$ gives curve H and $I_2(0) = I_{2m}$ gives curve V. Calculating the shaded area in Fig. 9 assuming the approximation that the amplitude change is small compared to both amplitudes and dividing by the total area $I_{1m}I_{2m}$ gives an approximate expression for the relative losses.

$$L(\Delta r) = \left|\frac{\varepsilon_1 n_1}{4}\right| \left(\frac{2}{\nu_2}\right)^{N_1} \left(\frac{2}{\nu_2}\right)^{N_2} {I_{1m}}^{N_1-1} {I_{2m}}^{N_2-1}$$
$$\times [\,|n_1|\,I_{2m} + |n_2|\,I_{1m}]\,P_m. \tag{5.44}$$

The losses are dependent on Δr, the displacement of the equilibrium orbit from the center of the storage region. The maximum horizontal and vertical amplitudes are given by

$$I_{1m} = \nu_1(a_0 - |\Delta r|)^2/2$$
$$I_{2m} = \nu_2 {b_0}^2/2 \tag{5.45}$$

where a_0 and b_0 are the horizontal and vertical semiapertures, respectively. If we assume a uniform distribution in horizontal phase space we obtain the following distribution in Δr

$$N(\Delta r) \sim (1 - |\Delta r/a_0|)^2. \tag{5.46}$$

Integrating (5.44) over this distribution yields the total losses.

$$L = 3/2\,|\varepsilon_1 n_1|\,{a_0}^{|n_1|-2}\,{b_0}^{|n_2|-2}$$
$$\times\left[\frac{|n_1|}{|n_1|+1}\frac{{b_0}^2}{\nu_1} + \frac{|n_2|}{|n_1|+3}\frac{{a_0}^2}{\nu_2}\right] P_m. \tag{5.47}$$

5.3b. *The static case.* For the static case, P_m is simply

$$P_m = 2/\alpha, \tag{5.48}$$

where, as usual, α is the detuning $n_1\nu_1 - n_2\nu_2 - k$. Let us calculate an example. In a proposed muon $g-2$ experiment[18] a muon sample will be stored in a weak focussing superconducting storage ring operating near the $(1,-2,0)$ resonance. Since only very small losses can be tolerated by this experiment, the method presented above is particularly applicable. For the $(1,-2,0)$ resonance, the relevant terms in the perturbation Hamiltonian are[19]

$$\varepsilon H = \frac{1}{2}\left(\frac{r}{B}\frac{\partial^2 B_z}{\partial r^2}\right)_0 \left(x_1 {x_2}^2 - \frac{{x_1}^3}{3}\right)$$
$$+ \frac{1}{12}\left(\frac{r}{B}\frac{\partial^3 B_z}{\partial r^3}\right)_0 \left(3{x_1}^2{x_2}^2 - \frac{{x_1}^4}{2}\right) \tag{5.49}$$

which in a, I variables become, after averaging,

$$H(a,I) = \frac{1}{8}\left(\frac{r}{B}\frac{\partial^2 B_z}{\partial r^2}\right)_0 \left(\frac{2I_1}{\nu_1}\right)^{1/2} \left(\frac{2I_2}{\nu_2}\right) \cos\psi$$
$$+ \frac{1}{16}\left(\frac{r}{B}\frac{\partial^3 B_z}{\partial r^3}\right)_0 \left\{\left(\frac{2I_1}{\nu_1}\right)\left(\frac{2I_2}{\nu_2}\right) - \frac{1}{4}\left(\frac{2I_1}{\nu_1}\right)^2\right\}. \tag{5.50}$$

Thus the parameters by which the perturbation is described in these variables are

$$\varepsilon_1 = -\frac{1}{8}\left(\frac{r}{B}\frac{\partial^2 B_z}{\partial r^2}\right)_0$$
$$d_1 = \frac{1}{64}\left(\frac{r}{B}\frac{\partial^3 B_z}{\partial r^2}\right)$$
$$d_2 = 0 \tag{5.51}$$
$$d^3 = \frac{1}{16}\left(\frac{r}{B}\frac{\partial^3 B_z}{\partial r^3}\right)_0.$$

Neglecting the frequency shift terms, the losses for the static $(1,-2,0)$ resonance in terms of the error field are given by

$$L = \frac{3}{16}\left|\frac{\left(\frac{r}{B}\frac{\partial^2 B_z}{\partial r^2}\right)_0}{\alpha}\right| \left(\frac{a_0}{\nu_2} + \frac{{b_0}^2}{a_0\nu_1}\right). \tag{5.52}$$

In the proposed experiment, the error fields are not expected to exceed ± 10 parts per million on average over azimuth. Therefore as an estimate we can take

$$\frac{1}{2!}\left(\frac{a_0^2}{B}\frac{\partial^2 B_z}{\partial r^2}\right) = \pm 10^{-5}. \tag{5.53}$$

Taking the values $r = 120$ cm, $a_0 = 1.5$ cm, $b_0 = 2.0$ cm, and a field index value of $n = -(r/B_z)(\partial B_z/\partial r)_0 = 0.18$, which determines ν_1 and ν_2 by $\nu_1 = (1-n)^{1/2}$ and $\nu_2 = n^{1/2}$ we obtain losses of $L = 2.3$ per cent, a tolerable figure for the experiment.

For the static case, the frequency shift terms may be important. Their effect is effectively to shift the detuning by different amounts for different particles. We see by (5.52) that, for small changes, the relative change in the losses is equal to the relative change in the detuning so that we have the following estimate

$$\left|\frac{\Delta L}{L}\right| = \frac{c_1 + c_2 D(0)}{\alpha}. \tag{5.54}$$

Using (4.22) with an estimate similar to (5.53) for the cubic error term we see that this effect is at most several per cent for particles with the maximum amplitudes. Thus losses due to resonances will not be significantly affected by frequency shift effects. And by the remarks made previously, frequency shifts are even less important in non-static situations.

6. DISCUSSION

We have seen how a nearly linear, nearly periodic system in two dimensions can be treated to all orders by the Krylov-Bogoliubov averaging method. In order that slowly varying parameters be studied near resonance it was found that the resonant combination of frequencies be of a certain form (3.24). If we study the higher order ($N > 1$) approximations, then no combination of the basic frequencies should be 0(1) whenever $\varepsilon\theta$ is fixed unless this combination is independent of $\varepsilon\theta$ also. In general (i.e. for a general Hamiltonian function), we will indeed have such terms which we are unable to handle. The trouble is that the combination $n_1\nu_1 - n_2\nu_2 - k$ will appear whenever the $n_1\nu_1 + n_2\nu_2 - k = 0$ resonance is examined. Practically, the problem is overshadowed by the fact that other resonances closer to $n_1\nu_1 + n_2\nu_2 - k = 0$ are more important for typical physical systems. The nonapplicability of superposition in the nonlinear problem does not allow examination of each resonance separately. Furthermore, although a perturbation Hamiltonian which gives rise to only one resonance can be written, it will not satisfy the differentiability requirement. The perturbation method does allow us to study two or more resonances at a time but the resulting equations will be difficult to solve. For this reason only single resonances were studied in the practical calculations.

In studying the first order single resonance effects the detuning or resonant combination of frequencies $n_1\nu_1 + n_2\nu_2 - k$ was used to classify the various cases by the form of its dependence on $\varepsilon\theta$. Because only one resonance was present, it was possible to reduce the fourth order system a_1, a_2; I_1, I_2 to one of second order for ψ, D involving but three parameters: effective perturbation strength $\bar{\varepsilon}$, detuning α, and nonlinear frequency shift c_2. The parameter $\bar{\varepsilon}$ depends only on the absolute values of n_1 and n_2 so that the results are valid for both difference and sum resonances (corresponding to different regions of $\bar{\varepsilon}$ and D). In working out a particular case (static, passage through resonance, etc.) it was found that the solution depends on a certain combination of parameters which is 0(1). For example, in the case involving passage through resonance, it was found that the size of the amplitude change depends on the normalized perturbation strength divided by the square root of the rate of change of the detuning. Solving the approximate equations required the use of an approximation in addition to that implicit in perturbation theory. This causes some difficulty in the static resonance case but not in the important case of passage through resonance.

Since only very small particle losses can be tolerated in a storage ring, the problem of determining losses due to resonances is a particularly good application of these methods. It was possible to produce a formula that gives the losses for any resonance and for any type of variation of the betatron frequencies.

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APPENDIX

Canonical Transformations

A canonical transformation from a set of variables x,p to a new set X,P is a transformation that preserves the form of Hamilton's equations

$$\frac{dX_i}{d\theta} = \frac{\partial H(X,P,\theta)}{\partial P_i}, \frac{dP_i}{d\theta} = -\frac{\partial H(X,P,\theta)}{\partial X_i}, i = 1,2 \ldots n. \tag{A.1}$$

Such transformations have constant Jacobians; a more restrictive, and usual, definition of canonical transformation is one with unity Jacobian. A canonical transformation may be obtained from a generating function

$$S = S(z,Z,\theta) \tag{A.2}$$

where z are any of the old variables x, p and Z are any of the new variables X,P with the restriction that S does not depend simultaneously on a canonically conjugate pair of variables, for example both x_1 and p_1. Thus S is at most a function of $2n+1$ variables. Now if the desired Jacobian of the transformation is denoted by μ^n, a canonical transformation will be obtained by the following set of transformation equations (for each degree of freedom i choose the proper two of the four below depending on the arguments of S)

$$\mu p_i = \frac{\partial S}{\partial X_1}, X_i = \frac{\partial S}{\partial P},$$
$$\mu x_i = -\frac{\partial S}{\partial p_i}, P_i = -\frac{\partial S}{\partial X_i}. \tag{A.3}$$

The new Hamiltonian is given by

$$H \rightarrow \left[\mu H + \frac{\partial S}{\partial \theta}\right]_{X,P} \tag{A.4}$$

where the right hand side must be expressed in terms of the new variables X,P by means of the transformation equations (A.3). Note that if $\mu \neq 1$ then the fundamental Poisson brackets are not invariant under canonical transformations

$$[X_i,P_j]_{x,p} = \sum_k \frac{\partial(X_i P_j)}{\partial(x_k,p_k)} = \mu\delta_{ij}. \tag{A.5}$$

REFERENCES

1. E. D. Courant and H. S. Snyder, *Ann. Phys.*, **3**, 1 (1958).
2. E. D. Courant, M. S. Livingston, and H. S. Snyder, *Phys. Rev.*, **88**, 1190 (1952).
3. N. Christofilos, U.S. Patent No. 2,736,799 (filed 1950).
4. P. A. Sturrock, *Ann. Phys.*, **3**, 113 (1958).
5. R. Hagedorn, CERN Report 57-1 (1957).
6. R. Hagedorn and A. Schoch, CERN Report 57-14 (1957).
7. Ref. 1, Appendix B.
8. E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis* (Cambridge Univ. Press, London and New York, 1952), p. 412.
9. W. Magnus and S. Winkler, *Hill's Equation* (Interscience, New York, 1966).
10. This is not the usual normalization found in standard works such as A. A. Kolomensky and A. N. Lebedev, *Theory of Cyclic Accelerators* (North-Holland, Amsterdam, 1966), p. 65, which uses a normalization equivalent to that in Ref. 1.
11. N. Krylov and N. N. Bogoliubov, 'Introduction to Nonlinear Mechanics' in *Annals of Mathematics Studies*, No. 11 (Princeton Univ., Princeton, 1947).
12. N. N. Bogoliubov and Y. A. Mitropolski, *Asymptotic Methods in the Theory of Nonlinear Oscillations* (Hindustan Publishing Co., Delhi, India, 1961).
13. L. M. Perko, *SIAM J. Appl. Math.*, **17**, 698 (1969).
14. P. P. Zabreiko and I. B. Ledovskaja, *Soviet Math. Dokl.*, **7**, 453 (1966).
15. Ref. 4.
16. M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions* (Dover, New York, 1965), p. 300.
17. Ref. 16, p. 446 and J. C. P. Miller, *The Airy Integral*, Brit. Ass. Math. Tab. (Cambridge University Press, Cambridge, 1946), Pt.-Vol. B.
18. V. W. Hughes, private communication.
19. R. Hagedorn, in *Particle Accelerators*, ed. R. Kollath (Sir Isaac Pitman and Sons, Ltd., London, 1967), p. 202.

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