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THE EFFECT OF RANDOM RF VOLTAGE FLUCTUATIONS ON A BUNCHED BEAM†

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The noise present in the driving current of the rf system for a storage ring will cause an increase in the average phase angle of protons stored in a bunched configuration. The purpose of this work is to calculate the rate of increase in the phase angle. We consider only small amplitude oscillations about the synchronous phase angle. Results are given in terms of the signal-to-noise ratio (in power terms) of the cavity voltage, and indicate that the noise may be a stringent limitation on storage times unless the rf system is designed with care.

1. INTRODUCTION

If protons are kept bunched by an rf system in a storage ring, random fluctuation in the rf voltage may cause a diffusion of particles out of stable phase. The purpose of this work is to calculate a rate of increase of the phase angle of a particle subjected to random fluctuations of the rf voltage. Results are expressed in terms of the signal-to-noise ratio (in power terms) of the rf voltage. Alternatively, this ratio may be found from the spectral density of the current driving the rf cavity.

This is a preliminary report, and the various possible sources of noise are not investigated. Future work (if performed) will include an investigation of these sources as well as a solution of the Fokker-Planck equation for the distribution of particles in stable phase.

The methods used in this work are conventional methods for the mathematical treatment of random phenomena.[1] A reader unfamiliar with these methods may at times feel he is being hoodwinked. Let him be assured that these methods, which may appear to be black magic to the uninitiated, have been proven in countless applications.

2. SPECTRAL DENSITY AND AUTO-CORRELATION FUNCTION OF CAVITY VOLTAGE

We consider only one mode of the rf cavity to be excited. The cavity has a very high quality factor Q, and operates on a harmonic number h. That is, the angular frequency of the cavity is ω_r and the angular circulation frequency of the particles is ω_c, with $\omega_r = h\omega_c$. The particular cavity mode in question can be represented by a series L-C-R circuit. We take the external driving mechanism to be such that the cavity wall current I satisfies the equation

$$L\frac{\mathrm{d}^2 I}{\mathrm{d}t^2} + R\frac{\mathrm{d}I}{\mathrm{d}t} + \frac{1}{C}I = M\frac{\mathrm{d}^2 I_d}{\mathrm{d}t^2}, \tag{2.1}$$

in which L, C, and R are respectively the inductance, capacitance, and resistance of the mode; M is the coupling factor with units of inductance; and I_d is the driving current. This coupling mechanism is convenient, but is not fundamental to the following treatment. A driving current varying as $\mathrm{e}^{-i\Omega t}$ will produce a voltage $V_d = \int I\,\mathrm{d}t/C$ given by

$$V_d = \frac{M\omega_r^2\,\Omega I_d}{[(\Omega R/L) + i(\omega_r^2 - \Omega^2)]}, \tag{2.2}$$

in which $\omega_r^2 = (LC)^{-1}$. We need not assume in this work that the $\Omega = \omega_r$. It is usually advantageous to drive a cavity slightly off resonance to compensate for the beam-induced voltage.

The source of noise we wish to consider is a random fluctuation of the driving current. We augment our driving current with an additional current $I(t)(\ll I_d)$, which is considered to be a stationary random quantity with a Fourier transform $\tilde{I}(\omega)$. (Actually, $\tilde{I}(\omega)$ is the Fourier transform of a function which is identical to $I(t)$ within a long time interval T and zero outside this interval. The Fourier transform of a random variable does not exist.) Referring to Eq. (2.2) we see that the

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Fourier transform $\tilde{V}(\omega)$ of the cavity voltage produced by $I(t)$ is of the form

$$\tilde{V}(\omega) = \frac{M\omega_r^2\omega\tilde{I}(\omega)}{[(\omega R/L)+i(\omega_r^2-\omega^2)]}. \quad (2.3)$$

Noting that R/L may be identified with ω_r/Q, we see that $\tilde{V}(\omega)$ is sharply peaked at $\omega = \omega_r$.

We now introduce the spectral density $G(\omega)$ of the random voltage $V(t)$ by the standard definition

$$G(\omega) = K\,|\,\tilde{V}(\omega)\,|^2, \quad (2.4)$$

where K is a constant with dimensions of inverse seconds. It can be shown by means of the Wiener–Khintchine theorem that $G(\omega)$ is also given by[1]

$$G(\omega) = \int_{-\infty}^{\infty} \langle V(t)V(t+u)\rangle \cos\omega u \, \mathrm{d}u, \quad (2.5a)$$

with the inversion relation

$$\langle V(t)V(t+u)\rangle = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega)\cos\omega u \, \mathrm{d}\omega. \quad (2.5b)$$

The quantity $\langle V(t)V(t+u)\rangle$ is the autocorrelation function of the random variable $V(t)$. This quantity is independent of t for a stationary random variable, and is an even function of the variable u.

We use Eq. (2.2) and write Eq. (2.3) in the form

$$\frac{\tilde{V}(\omega)}{V_d} = \frac{\omega}{\Omega}\frac{\tilde{I}(\omega)}{I_d}\frac{[(\Omega\omega_r/Q)+i(\omega_r^2-\Omega^2)]}{[(\omega\omega_r/Q)+i(\omega_r^2-\omega^2)]}. \quad (2.6)$$

We thus obtain from Eq. (2.4)

$$\frac{G(\omega)}{V_d^2} = \frac{G_I(\omega)}{I_d^2}\left(\frac{\omega}{\Omega}\right)^2\left[\frac{(\Omega\omega_r/Q)^2+(\omega_r^2-\Omega^2)^2}{(\omega\omega_r/Q)^2+(\omega_r^2-\omega^2)^2}\right], \quad (2.7)$$

in which $G_I(\omega)$ is the spectral density of the random current $I(t)$.

We can use Eq. (2.7) in Eq. (2.5b) to calculate the autocorrelation function of the random voltage V. Provided $G_I(\omega)$ is not a wildly varying function, the main contribution to the integral arises from values of ω near $\pm\omega_r$. We assume that $G_I(\omega)$ is constant in the neighborhood of $\omega = \omega_r$, and replace $\omega^2 G_I(\omega)$ by $\omega_r^2 G_I(\omega_r)$ in Eq. (2.7). We further replace the factor $(\omega_r/\Omega)^2$ by unity. The integral in Eq. (2.5b) is then readily evaluated by contour integration with the result

$$\frac{\langle V(t)V(t+u)\rangle}{V_d^2} = \frac{G_I(\omega_r)}{2\omega_r^3}\frac{Q}{I_d^2}$$
$$\times\,[(\omega_r\Omega/Q)^2+(\omega_r^2-\Omega^2)^2]\,\mathrm{e}^{-\omega_r u/2Q}\cos\omega_r u, \quad (2.8)$$

in which we have neglected terms of order Q^{-1} compared to unity. The correlation time of the random variable V is equal to $2Q/\omega_r$. Physically this time is a measure of how long the voltage produced by a fluctuation in I will persist in the cavity.

It is important to note that a particle goes around the machine in a time $2\pi/\omega_c = 2\pi h/\omega_r$. If $Q \gtrsim h$, then the random changes of energy experienced by the particle on successive turns are correlated.

3. INCREASE IN PHASE ANGLE ARISING FROM RANDOM VOLTAGE FLUCTUATIONS

We consider the orbit of the particle in the accelerator to be a circle. The azimuthal position of the particle is $\theta(t)$. The cavity is located at $\theta = 0$ and has a negligible extent along the orbit of the particle. The total voltage on the cavity $V_t(t)$ is of the form

$$V_t(t) = -V_d\sin\omega_r t + V(t). \quad (3.1)$$

We assume throughout this work that $V(t) \ll V_d$. The angular velocity of the particle is given by

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \omega_c\left(1-\eta\frac{\mathrm{d}p}{p_c}\right) \quad (3.2)$$

in which p_c is the momentum of the synchronous particle (that particle with $\dot{\theta} = \omega_c$) and $\mathrm{d}p$ is the deviation in momentum. Only extreme relativistic energies are considered. The quantity η is given by

$$\eta = \frac{1}{\gamma_t^2} - \frac{1}{\gamma^2}, \quad (3.3)$$

in which γ is the energy of the synchronous particle and γ_t is the transition energy of the accelerator. We go to a rotating coordinate system and introduce the phase angle ϕ by the definition

$$\phi = (\omega_r/\omega_c)(\theta-\omega_c t) \equiv h(\theta-\omega_c t). \quad (3.4)$$

At $\theta = 0$, $\phi = -\omega_r t$. The synchronous particle has $\phi = 0$ and therefore passes through the cavity when the driven voltage is zero. From Eq. (3.2) we have (for any particle)

$$\frac{d\phi}{dt} = -h\eta\omega_c \frac{dp}{p_c}. \tag{3.5}$$

The rate of change of ϕ is constant during one turn around the machine. A particle with a value of $\phi = \phi_n$ as it passes through the cavity for the nth time will have a value ϕ_{n+1} as it passes through the cavity the next time. On the nth passage through the cavity (at time $t = t_n$) the particle experiences a change in momentum $\delta p_n \equiv dp_n - dp_{n-1}$ given by

$$\delta p_n = (e/c)[V_d \sin\phi_n + V(t_n)]. \tag{3.6}$$

From Eqs. (3.5) and (3.6) one may derive the phase equation,

$$\phi_{n+1} + \phi_{n-1} - 2\phi_n = -k^2(\sin\phi_n + v_n), \tag{3.7}$$

in which we have introduced the quantities k^2 and v_n by the definitions

$$k^2 = 2\pi h\eta e V_d/p_c c, \tag{3.8}$$

$$v_n = V(t_n)/V_d. \tag{3.9}$$

In the absence of the random quantity v_n, the finite difference Eq. (3.7) is traditionally approximated by the differential equation[2]

$$\frac{d^2\phi}{dt^2} + \omega_s^2 \sin\phi = 0, \tag{3.10}$$

with the so-called synchrotron frequency ω_s defined by

$$\omega_s^2 = (k\omega_c/2\pi)^2 \equiv h\eta e V_d \omega_c^2/2\pi p_c c. \tag{3.11}$$

We shall not use this approximation at this stage, *but we shall limit our treatment to small values of* ϕ *so that we may approximate* $\sin\phi_n$ *by* ϕ_n *in Eq. (3.7).* In practice the quantity k^2 is quite generally very much less than unity. With the approximation $\sin\phi_n = \phi_n$, the solution of Eq. (3.6) is[3]

$$\phi_n = a\sin(\mu n + \alpha) - \frac{k^2}{\sin\mu}\sum_{l=1}^{n} v_l \sin\mu(n-l), \tag{3.12}$$

in which the constants a and α are determined from the initial conditions, and $\mu \equiv 2\sin^{-1} k/2$.

We now square both sides of Eq. (3.12) and take an ensemble average, which we indicate with $\langle\,\rangle$. Making use of the property of random variables that $\langle v\rangle = 0$ at any time, we have

$$\langle\phi_n^2\rangle = a^2\sin^2(\mu n + \alpha) + \frac{k^4}{\sin^2\mu}\sum_{l=1}^{n}\sum_{m=1}^{n}\langle v_l v_m\rangle \times \sin\mu(n-l)\sin\mu(n-m). \tag{3.13}$$

Clearly the first term on the right does not contribute to an increase of ϕ with increasing n, and will be ignored. The quantity $\langle v_l v_m\rangle$ is the autocorrelation function of the random variable $V(t)/V_d$ introduced in the last section. It is a function only of $u \equiv |t_l - t_m|$ and is given by Eq. (2.8).

Let us first examine terms with $m = l$ in the double summation in Eq. (3.13). The quantity $\langle v_l^2\rangle$ is the expectation value of v^2, and is independent of time. Denoting the contribution from the $m = l$ terms as $\langle\phi_n^2\rangle_0$ we have

$$\langle\phi_n^2\rangle_0 = \frac{k^4\langle v^2\rangle}{\sin^2\mu}\sum_{l=1}^{n}\sin^2\mu(n-l). \tag{3.14}$$

If $\pi h/Q \gg 1$, the quantities $\langle v_l v_m\rangle$ for $m \neq l$ may be ignored and there is no correlation in the values of $V(t)$ experienced by the particle on successive passages through the cavity. In this limit Eq. (3.14) is a good approximation to Eq. (3.13).

We next consider terms for which $m = l \pm j$. There are $2(n-j)$ such terms. Denoting the contribution from these terms as $\langle\phi_n^2\rangle_j$, we have

$$\langle\phi_n^2\rangle_j = \frac{k^4}{\sin^2\mu}\sum_{l=1}^{n-j}\langle v_l v_{l+j}\rangle \sin\mu(n-l-j) \times \sin\mu(n-l) + \frac{k^4}{\sin^2\mu}\sum_{l=j+1}^{n}\langle v_l v_{l-j}\rangle \times \sin\mu(n-l+j)\sin\mu(n-l). \tag{3.15}$$

But the autocorrelation function depends only on the absolute value of the time difference, thus $\langle v_1 v_2\rangle = \langle v_2 v_1\rangle$, etc. so that we may combine the two sums and obtain

$$\langle\phi_n^2\rangle_j = \frac{2k^4}{\sin^2\mu}\sum_{l=1}^{n-j}\langle v_l v_{l+j}\rangle \sin\mu(n-l) \times \sin\mu(n-l-j). \tag{3.16}$$

The quantity $\langle v_l v_{l+j}\rangle$ is only approximately independent of l. It is a function of $t_{l+j}-t_l$, which is given by

$$t_{l+j}-t_l = j\tau-(\phi_{l+j}-\phi_l)/\omega_r, \qquad (3.17)$$

in which $\tau \equiv 2\pi/\omega_c$. The only dependence of $\langle \phi_n^2\rangle$ on the amplitude of ϕ is contained in this time difference. As long as we are restricted to small enough values of ϕ such that $\sin\phi \approx \phi$, this dependence is a weak one indeed. We shall simply set $t_{l+j}-t_l = j\tau$ in the following. This approximation is equivalent to neglecting the change in ϕ over the number of periods for which the random voltage is correlated.

Introducing the notation $\langle vv_j\rangle \equiv \langle v(t_l)v(t_l+j\tau),\rangle$ we rewrite Eq. (3.16) in the form

$$\langle \phi_n^2\rangle_j = \frac{2k^4\langle vv_j\rangle}{\sin^2\mu}\sum_{l=1}^{n-j}[\sin^2\mu(n-l)\cos j\mu - \sin j\mu\sin\mu(n-l)\cos\mu(n-l)].$$

We take n sufficiently large so that the sum extends over at least a few periods of $\mu(n-l)$. The portion of the sum that increases with increasing n may be found by replacing $\sin^2\mu(n-l)$ by its average value of 1/2 and neglecting the second term which has an average value of zero. Thus

$$\langle \phi_n^2\rangle_j = (n-j)\frac{k^4}{\sin^2\mu}\langle vv_j\rangle\cos j\mu. \qquad (3.18)$$

Replacing $\sin^2\mu(n-l)$ by 1/2 in Eq. (3.14), we may now express $\langle \phi_n^2\rangle$ as

$$\langle \phi_n^2\rangle = \frac{nk^4}{2}\frac{\langle v^2\rangle}{\sin^2\mu}+\frac{k^4}{\sin^2\mu}\sum_{j=1}^{n}(n-j)\cos j\mu\langle vv_j\rangle. \qquad (3.19)$$

Inspection of Eq. (2.8) reveals that

$$\langle vv_j\rangle = \langle v^2\rangle e^{-j\omega_r\tau/2Q}\cos(j\omega_r\tau),$$

but $\omega_r r = 2\pi h$, thus Eq. (3.19) becomes

$$\langle \phi_n^2\rangle = \frac{k^4}{\sin^2\mu}\langle v^2\rangle\left[\frac{n}{2}+\sum_{j=1}^{n}(n-j)\cos j\mu\, e^{-j\pi h/Q}\right]. \qquad (3.20)$$

Writing $\cos j\mu$ as $Re\, e^{ij\mu}$ gives the summation the form of a finite geometric series and the derivative of such a series, with the result that the sum has the approximate value (neglecting 1 with respect to n and taking $e^{-n\pi h/Q}\to 0$)

$$\sum_{j=1}^{n}\approx\frac{n}{2}\frac{(\cos\mu-e^{-\pi h/Q})}{(\cosh\pi h/Q-\cos\mu)}.$$

Combining this result with the first term on the right hand side of Eq. (3.20) we have

$$\langle \phi_n^2\rangle = \frac{nk^4}{\sin^2\mu}\frac{\langle v^2\rangle}{2}\frac{\sinh(\pi h/Q)}{[\cosh(\pi h/Q)-\cos\mu]}. \qquad (3.21)$$

This expression has a maximum value at $\cosh(\pi h/Q) = \sec\mu$. To second order in k and $\pi h/Q$ this maximum occurs at $\pi h/Q = k$ and is given by

$$\langle \phi_n^2\rangle_{\max} = nk\langle v^2\rangle/2. \qquad (3.22)$$

We note that for very small Q the summation is negligible (from Eq. (3.20) or (3.21)). For very small Q the random voltage encountered by the particle on successive turns is uncorrelated. On the other hand, in the limit $Q\to\infty$, the right hand side of Eq. (3.21) vanishes. In this limit the voltage $v(t)$ is not a random variable. It is simply a constant error in the amplitude (or phase) of the driving voltage.

The quantity $\langle v^2\rangle$ may be known or measurable for a given rf system. Alternatively, we may use Eq. (2.8), which to a good approximation may be simplified to

$$\langle v^2\rangle = G_I(\omega_r)\omega_r/2QI_d^2. \qquad (3.23)$$

This equation allows a calculation of $\langle v^2\rangle$ if the spectral density $G_I(\omega_r)$ of the driving current is known.

As a numerical example, we consider the parameters of a proton storage ring as presented in Ref. 4. From these parameters we obtain $k = 0.1$ and a circulation frequency f_c of 2×10^5 cps. If only the first term on the right hand side of Eq. (3.20) is retained (turn-to-turn correlation completely neglected) we find that ϕ_n^2 has an expectation value of unity after a number of turns $n = 200/\langle v^2\rangle$. This corresponds to a time $t = n/f_c = 10^{-3}/\langle v^2\rangle$ sec. Little information is available regarding the value of $\langle v^2\rangle$ for rf systems in operation at present. A value of 10^{-6} could reasonably be expected in a conventionally designed system.[5] Using this value we find $t = 10^3$ sec. But now if $\pi h/Q = k$, Eq. (3.22) applies and the time becomes 10^2 sec. These numbers indicate that rf systems for bunched-proton storage rings must be designed very carefully if long storage times are to be achieved.[6]

Finally, we remark that particles with large amplitude of phase oscillations will experience a growth rate of ϕ_n that is slower than that given by Eq. (3.21). This fact can be discovered by a more accurate treatment of the quantities $\langle v_l v_{l+j} \rangle$, taking into account the change in phase over a correlation time.

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6. Admittedly, our treatment does not include the presence of a phase lock or other feedback control of the rf system. Such a control may have both good and bad influence on the storage time. Preliminary investigation shows that the feedback suppresses the growth of coherent phase motion, but has little effect on the incoherent motion within the rf bucket

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