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# IMPROVING THE ENERGY RESOLUTION AND DUTY FACTOR OF ISOCHRONOUS CYCLOTRONS†

M. M. GORDON

*Cyclotron Laboratory, Michigan State University, East Lansing, Michigan, USA*

A separated turn isochronous cyclotron can produce beams having a precise energy resolution together with a substantial duty factor if the effective voltage wave form is 'flat-topped'. Optimum flat-topping results are presented for five different harmonic combinations: $n = 1$ and 2; $n = 1, 2,$ and 3; $n = 1, 2, 3,$ and 4; $n = 1$ and 3; $n = 1, 3,$ and 5. For a given energy resolution, the improvement in duty factor with each added harmonic is quite impressive. The success of this technique is limited by certain practical problems which are examined.

## 1. INTRODUCTION

If an isochronous cyclotron could be equipped with an rf system which provided a square wave voltage on the dees, then each ion pulse could extend for nearly half of the rf period, while the energy distribution within this pulse would remain quite homogeneous throughout the acceleration process. Assuming a small radial beam width, optimum conditions would then prevail for 'separated turn' operation with 100 per cent beam extraction.[1] The external beam would then possess exceptionally fine energy resolution together with a duty factor approaching 50 per cent. In addition, the transverse emittance of this beam would be directly correlated to that of the ion source or injector. In a certain sense, this cyclotron would operate like a pulsed dc accelerator.

The advantages of a square wave voltage for a classical cyclotron were first recognized by Rossi who devised a method for superimposing a third harmonic voltage on the dees, and who showed that when this voltage is one-ninth that of the main harmonic, the resultant flat-topped wave form partially fulfills the function of a square wave voltage.[2] This third harmonic flat-topping method was also investigated by Goodman as a means for improving both classical and isochronous cyclotrons.[3] Welton[4] and Blosser[5] considered voltage flat-topping as an important element in an isochronous cyclotron designed to yield superlative beam characteristics including a substantial duty factor.

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The desirable effects of a flat-topped voltage wave form can be achieved by another method under certain circumstances. Since the energy gained at successive electric gap crossings is cumulative, and since only the resultant energy is significant, the voltages corresponding to different harmonics can be applied to separate sets of dees. This procedure is particularly suitable for ring cyclotrons where the ion injection energy is substantially greater than the energy gained at any one gap crossing.[1] Moreover, as pointed out by Rickey,[6] this procedure permits the utilization of even as well as odd harmonic voltages and such combinations always produce a superior flat-topping effect, and may produce duty factors exceeding 50 per cent. These considerations formed the basis for the design of the rf system of the Indiana Cyclotron now under construction.[7] Several other cyclotrons have been proposed recently which plan to utilize the flat-topping effect of either second or third harmonic voltages.[8,9]

The present paper explores the degree of energy homogeneity which can be achieved as a function of duty factor through the admixture of a small number of harmonic voltages, and discusses some of the practical problems which may limit the success of this technique. In a subsequent paper, we intend to discuss the limitation of energy resolution imposed by the longitudinal space-charge force in cyclotrons with a substantial duty factor.

## 2. FORMULATION

Using the azimuth $\theta$ as the independent variable, we replace the time $t(\theta)$ by the phase variable $\phi(\theta)$ defined by:

$$\phi(\theta) = \omega_1 t(\theta) - h\theta, \quad (1)$$

where $\omega_1/2\pi$ is the lowest (main) rf frequency of the system, and $h$ is its integral harmonic ratio; that is, $\omega_1 = 2\pi h v_0$, where $v_0$ is the constant rotation frequency of the ions under conditions of perfect isochronism. If $v$ is the ion's speed at any point, then $\phi(\theta)$ satisfies the differential equation:

$$d\phi/d\theta = (\omega_1/v)[r^2 + (r')^2 + (z')^2]^{\frac{1}{2}} - h, \quad (2)$$

with $r' = (dr/d\theta)$ and $z' = (dz/d\theta)$.

We assume that the electric field produced by the entire rf system in the region occupied by the beam can be derived from a scalar potential $\Phi(r, \theta, z, t)$ given by:

$$\Phi(r, \theta, z, t) = \sum W_{nj}(r, \theta, z) \sin(n\omega_1 t + \alpha_{nj}), \quad (3)$$

where $n$ specifies the harmonic number of a particular rf frequency, where $j = (1, 2, \ldots, J)$ designates one of the set of $J$ dees operating at the same frequency, and where the sum extends over these $j$ values and over whichever $n$ values are present (including $n = 1$). For a given $n$ value, the function $W_{nj}(r, \theta, z)$ gives the spatial dependence of the potential produced by a specific dee, while $\alpha_{nj}$ gives the relative phase of this potential. Note that the same formula applies even if more than one frequency is supplied to a given dee.

After making use of the foregoing definitions, we obtain the following differential equation for the kinetic energy $E(\theta)$ of the ion:

$$dE/d\theta = -q\sum[(\partial W/\partial\theta) + r'(\partial W/\partial r) + z'(\partial W/\partial z)]_{nj} \sin(nh\theta + n\phi + \alpha_{nj}), \quad (4)$$

where $q$ is the ion's charge. To obtain precise values of $E$ and $\phi$, this equation and (2) above must be integrated simultaneously together with the appropriate orbit equations for obtaining $r(\theta)$ and $z(\theta)$.

In order to simplify the analysis, the Eqs. (2, 4) are customarily replaced by those obtained from the 'averaging' method of approximation.[10] The radial and axial oscillations of the ions about a given reference orbit (usually the closed equilibrium orbit) are assumed to average to zero, or else to be negligible. Conversely, it is assumed that the discarded fluctuations in $E$ and $\phi$ have negligible effect on the radial and axial motion. To validate these assumptions, the acceleration should be adiabatic and the beam should occupy only a small area of the radial and axial phase space.

Within the confines of these approximations, Eqs. (2, 4) can be reduced to:

$$d\phi/d\tau = \omega_1 T(E) - 2\pi h, \quad (5a)$$

$$dE/d\tau = q\sum_n V_n(E) \cos[n\phi + \psi_n(E)], \quad (5b)$$

where $\tau = \theta/2\pi$ is the 'turn number' and $T(E)$ is the ion rotation period in the reference orbit. If the field is perfectly isochronous, $T(E) = 2\pi h/\omega_1 = 1/v_0$, and the value of $\phi$ is then simply a constant independent of $E$.

The functions $V_n(E)$ and $\psi_n(E)$ can be obtained from the expression:

$$V_n(E) \exp(i\psi_n) = \sum_j X_{nj}(E) \exp(i\alpha_{nj}), \quad (6a)$$

where $X_{nj}(E)$ is given by (after partial integration):

$$X_{nj}(E) = (nh) \oint W_{nj}(E, \theta) \exp(inh\theta)\, d\theta, \quad (6b)$$

and where $W_{nj}(E, \theta)$ is obtained by evaluating $W_{nj}(r, \theta, z = 0)$ along the reference orbit for the given energy $E$. In most cases of practical importance, it is possible to choose the voltage phases $\alpha_{nj}$ of (3) such that $\psi_n(E) = 0$, that is, such that the vector sum (6a) reduces to a real number.

## 2.1. AMPLITUDES AND PHASES

Let us consider specifically the case where the $J$ dees are identical and uniformly spaced apart by $\Delta\theta = 2\pi/J$, and let us assume that $W_{nj}$ is symmetric about the radial line $\theta = \Theta_{nj}$. For this case, it then follows from (6b) that:

$$X_{nj}(E) = Y_n(E) \exp(inh\Theta_{nj}), \quad (7a)$$

where $Y_n(E)$ is real and independent of $j$; it also follows from symmetry that:

$$\Theta_{n,j+1} - \Theta_{nj} = 2\pi/J. \quad (7b)$$

In order to maximize the effectiveness of each dee, and to satisfy $\psi_n = 0$, we should require that the voltage phases $\alpha_{nj}$ satisfy:

$$\alpha_{nj} + (nh)\Theta_{nj} = 2\pi(\text{integer}), \quad (8)$$

for every $n$ value and $j$ value; in this case,

$V_n(E) = JY_n(E)$ as follows from (6a, 7a). (Note that $J$ need not have the same value for different $n$ values.)

The above relations (7b, 8) yield certain well-known rules for the voltage phases. In the case of two dees ($J = 2$), for example, if $nh$ is an odd integer, then $\alpha_{n2} = \alpha_{n1} + \pi$, while if $nh$ is even, then $\alpha_{n2} = \alpha_{n1}$. For the general case of $J$ dees, if $m = (nh$ modulo $J)$, we then obtain:

$$\alpha_{n,j+1} = \alpha_{nj} - m(2\pi/J), \tag{9}$$

where $m = 0, 1, \ldots, J-1$.

In order to obtain an explicit formula for $V_n$ from (6–8) above, let us assume that $W_{nj}(E, \theta)$ has a symmetric trapezoidal shape; that is,

$$W_{nj} = U_n, \quad \text{for} \quad |\theta - \Theta_{nj}| < A - a, \tag{10a}$$

$$W_{nj} = 0, \quad \text{for} \quad |\theta - \Theta_{nj}| > A + a, \tag{10b}$$

so that $U_n$ is the maximum voltage supplied to the dee, $\Delta\theta = 2A$ is the full width at half maximum for the trapezoidal potential, and $\Delta\theta = 2a$ is the full width of the electric field region at each edge of the dee. In this case, Eqs. (6–8) then yield:

$$V_n = 2JU_n \sin(nhA)[\sin(nha)/(nha)], \tag{11}$$

where $U_n$, $A$, and/or $a$ may depend on $E$ (that is, on radius), and where $U_n$ may be positive or negative depending on the sign desired for $V_n$. Although a trapezoidal shape for $W_{nj}$ is not very realistic, this formula is widely used because of its simplicity. The factor in brackets (which is known as the 'transit time' factor) is the Fourier transform of the electric field profile at the dee edge, and actually falls off faster with increasing $(nha)$ than this formula indicates. In general, as the value of $nh$ increases, the value of $V_n$ becomes more sensitive to the detailed shape of $W_{nj}$ or the electric field it produces.

## 3. OPTIMUM FLAT-TOPPING

Given the functions $T(E)$ and $V_n(E)$, the simplified Eqs. (5a, b) for the longitudinal motion can be integrated (numerically, at least) so as to yield $E(\tau)$ and $\phi(\tau)$ in terms of their initial values. Corresponding to a given distribution of these values for the injected beam, one can then find a set of harmonic voltage amplitudes which minimizes the final energy spread within the beam. In order to obtain results which are both concrete and useful, we shall assume here that perfect isochronism obtains so that $\phi$ is simply a constant. In addition, we shall replace $V_n(E)$ in (5b) by the value obtained from averaging this function over the energy. Equation (5b) can then be integrated directly so as to yield:

$$E(\phi) - E_i = \tau q \textstyle\sum_n V_n \cos(n\phi), \tag{12}$$

where $E_i$ is the initial energy, where $V_n = \langle V_n(E) \rangle$, and where $\psi_n = 0$ as discussed above. The sum in Eq. (12) therefore represents an 'effective' voltage wave form.

We introduce the normalized wave form $v(\phi)$ defined by:

$$v(\phi) = \textstyle\sum_n v_n \cos(n\phi), \tag{13}$$

where $v_n = V_n/V_0$, and where $V_0$ is so chosen that $v_{\max} = 1$. Equation (12) can then be rewritten as follows:

$$E(\phi) - E_i = \tau q V_0 v(\phi), \tag{13a}$$

so that $qV_0$ is therefore the (averaged) peak energy gain per turn. It then follows that the maximum energy $E_m$ is given by:

$$E_m - E_i = \tau q V_0, \tag{14a}$$

and that the deviation $\Delta E$ from this maximum is given by:

$$\Delta E = E_m - E(\phi) = \tau q V_0 (1 - v(\phi)), \tag{14b}$$

assuming $E_i$ is constant.

Let us assume that each ion pulse extends over a phase range $\Delta\phi = \omega_1(\Delta t)$ so that the duty factor for this beam is:

$$D = (\Delta\phi)/2\pi. \tag{15}$$

If $v_{\min}$ is the minimum value of $v(\phi)$ over this phase range, then we define the parameter $\delta$ as:

$$\delta = 1 - v_{\min}, \tag{16a}$$

and we find that the energy resolution of the emergent beam is given by:

$$(\Delta E)/E_m = (\tau q V_0/E_m)\delta \leqq \delta, \tag{16b}$$

as follows from (14). For a particular set of harmonics, the flat-topping problem now consists of finding the set of coefficients $v_n$ in (13) such that

$\delta$ is minimized for a given $D$, or such that $D$ is maximized for a given $\delta$.

Since $v(\phi)$ is symmetric about $\phi = 0$, it follows that the optimum range of $\phi$ values is:

$$-(\Delta\phi)/2 \leqq \phi \leqq (\Delta\phi)/2, \tag{17}$$

and that only the positive half of this range need be considered. If $k$ is the number of harmonics added to the main harmonic ($n = 1$), then because of the normalization condition $v_{\max} = 1$, this $k$ gives the number of coefficients $v_n$ in (13) which are freely adjustable. Alternatively, the function $v(\phi)$ will have $k$ adjustable extrema located at $\phi = \phi_1, \phi_2, \ldots, \phi_k$, and these extrema should be positioned such that:

$$0 < \phi_1 < \phi_2 < \ldots < \phi_k < (\Delta\phi)/2, \tag{18}$$

since we wish to minimize $|dv/d\phi|$ over the specified $\phi$ range. Since $v(\phi)$ has no extrema for $\phi > \phi_k$ (except those near $\phi = \pi$, which may be ignored), it follows that $v(\phi)$ must have a maximum at $\phi = \phi_k$; it also follows that $v(\phi = 0)$ is a maximum/minimum when $k$ is an even/odd integer. For a specified set of extrema points $\phi_i$, we also conclude that the optimum result will be obtained when $\Delta\phi$ is determined from the condition:

$$v(\phi = \Delta\phi/2) = 1 - \delta = v_{\min}. \tag{19}$$

These considerations are sufficient to determine the optimum result when $k = 1$.

The analysis proceeds by introducing the variable $u = \cos\phi$, which is restricted to the range: $1 \geqq u \geqq \cos(\Delta\phi/2)$. In terms of this variable, the function $v$ reduces to a polynomial whose degree is determined by the largest $n$ value present in the sum (13). In order to save space, we shall omit the details of this analysis here. The primary conclusion is that the optimum result will be obtained when all the maxima of $v$ have the same height, and all the minima have the same depth; that is,

$$1 = v(\phi_k) = v(\phi_{k-2}) = \ldots; \tag{20a}$$

$$1 - \delta = v(\phi_{k-1}) = v(\phi_{k-3}) = \ldots. \tag{20b}$$

These conditions, together with those specified in the preceding paragraph, are sufficient to determine all of the parameters.

The resultant flat-top region of $v(\phi)$ is then characterized by a 'ripple' which, though irregular, has a constant amplitude. In the five subsections which follow, we present without commentary the formulas obtained in the most interesting cases. These formulas relate the $k+1$ values of $v_n$, the $k$ values of $\phi_i$, the energy resolution parameter $\delta$ of (16), and the phase width $\Delta\phi$. Instead of $\Delta\phi$ itself, we actually use the parameter $\varepsilon$ given by:

$$\varepsilon = \sin^2(\Delta\phi/4). \tag{21}$$

In the first four cases below, the formulas give all of the remaining parameters directly in terms of $\varepsilon$.

### 3.1. *Harmonics $n = 1$ and 2*

$$\cos\phi_1 = 1 - \varepsilon = \beta; \tag{22a}$$

$$\delta = 2\varepsilon^2/(1+2\beta^2); \tag{22b}$$

$$v_1 = 4\beta/(1+2\beta^2); \quad -v_2 = v_1/4\beta. \tag{22c}$$

### 3.2. *Harmonics $n = 1$, 2, and 3*

$$\cos\phi_{\frac{1}{2}} = \beta \pm \varepsilon/2; \quad \beta = 1 - \varepsilon; \tag{23a}$$

$$\delta = 2\varepsilon^3/(1+9\beta^2); \tag{23b}$$

$$v_3 = 1/(1+9\beta^2); \quad -v_2 = 6\beta v_3;$$

$$v_1 = 3\beta(2+3\beta)v_3. \tag{23c}$$

### 3.3. *Harmonics $n = 1$, 2, 3, and 4*

$$\cos\phi_2 = 1 - \varepsilon = \beta; \quad \cos\phi_{\frac{1}{3}} = \beta \pm \varepsilon/\sqrt{2}; \tag{24a}$$

$$\delta = 2\varepsilon^4/(1+24\beta^2+8\beta^3+2\beta^4); \tag{24b}$$

$$v_4 = -1/(1+24\beta^2+8\beta^3+2\beta^4);$$

$$v_3 = -8\beta v_4; \quad v_2 = 4\beta(2+5\beta)v_4;$$

$$v_1 = -8\beta(1+4\beta+2\beta^2)v_4. \tag{24c}$$

### 3.4. *Harmonics $n = 1$ and 3*

Although this case has been treated before,[1] we present the results again for the sake of completeness.

$$\cos\phi_1 = \alpha = (1 - 2\varepsilon + 4\varepsilon^2/3)^{\frac{1}{2}} \tag{25a}$$

$$\delta = \tfrac{1}{2}(1-\alpha)^2(1+2\alpha)/\alpha^3; \tag{25b}$$

$$v_3 = -1/(2\alpha)^3; \quad v_1 = 3(4\alpha^2 - 1)/(2\alpha)^3. \tag{25c}$$

### 3.5. *Harmonics* $n = 1$, 3, *and* 5

$$\cos\phi_1 = \xi; \quad \cos\phi_2 = \eta; \quad \zeta = 2\varepsilon-(1-\xi); \tag{26a}$$

$$5(\xi^2-\eta^2)(2\eta+1) = (1-\eta)(3+9\eta+8\eta^2); \tag{26b}$$

$$15\xi(\xi^2-\eta^2)-5(5\xi^2-\eta^2)\zeta+15\xi\zeta^2 = 3\zeta^3; \tag{26c}$$

$$\delta = [(\xi-\eta)/\eta]^3(\xi^2+\eta^2+3\xi\eta)/(5\xi^2-\eta^2); \tag{26d}$$

$$v_3/v_5 = -5(4\xi^2+4\eta^2-3)/3;$$

$$v_1/v_5 = 10(8\xi^2\eta^2-2\xi^2-2\eta^2+1);$$

$$v_1+v_3+v_5 = 1. \tag{26e}$$

Since these formulas are far less transparent than those found in the preceding cases, we present in Table I the values of $\delta$, $v_1$, $v_3$, and $v_5$ as a function of $\Delta\phi$.

TABLE I
Optimum flat-topping parameters for $n = 1$, 3, and 5 harmonic combination

| $\Delta\phi$ (deg) | $\log\delta$ | $v_1$ | $v_3$ | $v_5$ |
|---|---|---|---|---|
| 35 | −4.779 | 1.1773 | −0.2036 | 0.0264 |
| 40 | −4.425 | 1.1789 | −0.2063 | 0.0273 |
| 45 | −4.111 | 1.1808 | −0.2092 | 0.0285 |
| 50 | −3.828 | 1.1828 | −0.2126 | 0.0298 |
| 55 | −3.571 | 1.1850 | −0.2164 | 0.0314 |
| 60 | −3.334 | 1.1874 | −0.2206 | 0.0332 |
| 65 | −3.115 | 1.1899 | −0.2252 | 0.0353 |
| 70 | −2.910 | 1.1924 | −0.2302 | 0.0377 |
| 75 | −2.717 | 1.1951 | −0.2356 | 0.0406 |
| 80 | −2.535 | 1.1977 | −0.2415 | 0.0438 |
| 85 | −2.362 | 1.2003 | −0.2479 | 0.0476 |
| 90 | −2.197 | 1.2026 | −0.2547 | 0.0521 |
| 95 | −2.039 | 1.2047 | −0.2619 | 0.0572 |
| 100 | −1.887 | 1.2062 | −0.2694 | 0.0632 |

## 4. DISCUSSION

Figure 1 shows curves of $\Delta\phi$ versus $\log\delta$ over the range $\delta = 10^{-4}$–$10^{-2}$ for the five different types of harmonic combinations considered above. For comparison, a pure sinusoidal voltage would yield $\Delta\phi = 1.6°$–$16°$ over the same $\delta$ range. The improvement in duty factor with each added harmonic, as manifested by the curves in Fig. 1, is therefore quite impressive. For a given $k$ value (number of added harmonics) these curves confirm that the largest duty factor is obtained for the set of harmonics $n = 1, 2, \ldots, k+1$, that is, for the set of lowest possible frequencies. Each of the curves in Fig. 1 represents the optimum result which can be achieved only under ideal conditions.

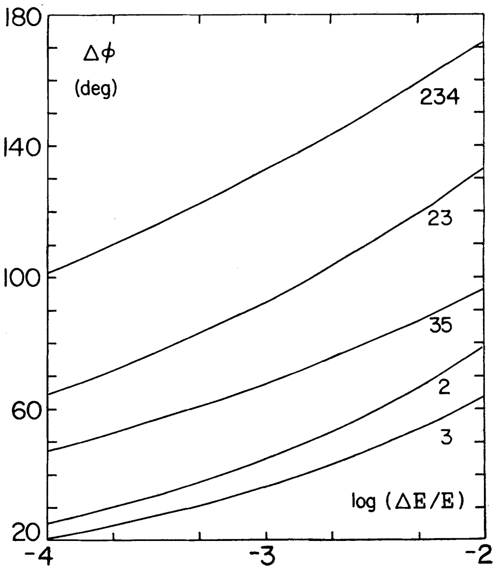


FIG. 1. Curves showing the maximum phase width $\Delta\phi$ (in degrees) of a beam which can be achieved for a given energy resolution and for a given combination of harmonics. The range of energy resolution covered is: $(\Delta E)/E = 10^{-4}$ to $10^{-2}$. Reading from bottom to top, the curves correspond to the harmonic combinations: $n = 1$ and 3; $n = 1$ and 2; $n = 1$, 3, and 5; $n = 1$, 2, and 3; $n = 1$, 2, 3, and 4.

Figures 2 and 3 display curves of the effective voltage wave form $v(\phi)$, defined in (13), which are obtained when $\delta = 10^{-3}$ by using the optimum flat-topping formulas given in Secs. 3.1–3.5. The beam is confined within the flat-top region shown on these curves for $|\phi| < (\Delta\phi)/2$. The plots in Fig. 3 extend only to $\phi = \pi/2$, since $v(\pi-\phi) = -v(\phi)$ in these cases where only odd harmonics are involved. The large negative values near $\phi = \pi$ in Fig. 2 constitute no difficulty in themselves since they lie far outside the flat-top region within which the beam is confined. These values do, however, indicate that when even and odd harmonics are mixed, the available dee voltages are utilized rather inefficiently. The value of $v_1 = V_1/V_0$ provides a good measure of this inefficiency since $v_1 = 1$ for a

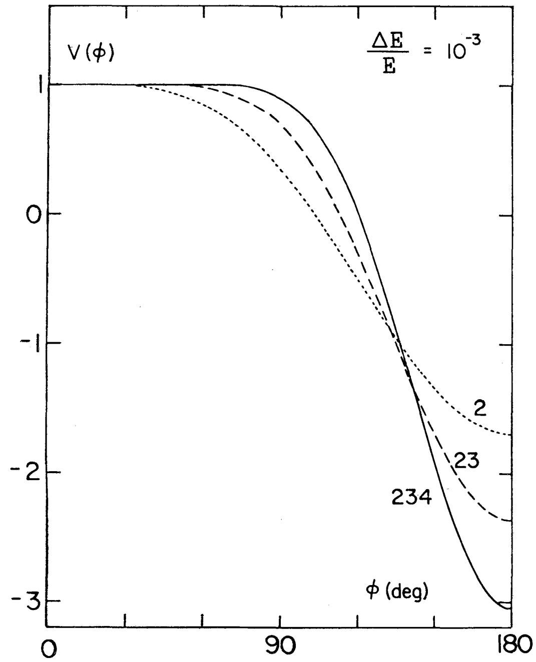


FIG. 2. Optimum flat-topping curves showing the effective voltage wave form $v(\phi)$ obtained for an energy resolution $\delta = (\Delta E)/E = 10^{-3}$, and for the following different harmonic combinations: $n = 1$ and 2; $n = 1, 2$, and 3; $n = 1, 2, 3$, and 4. These curves are symmetric about $\phi = 0$. On the scale of this figure, the ripple within the flat-top region is invisible.

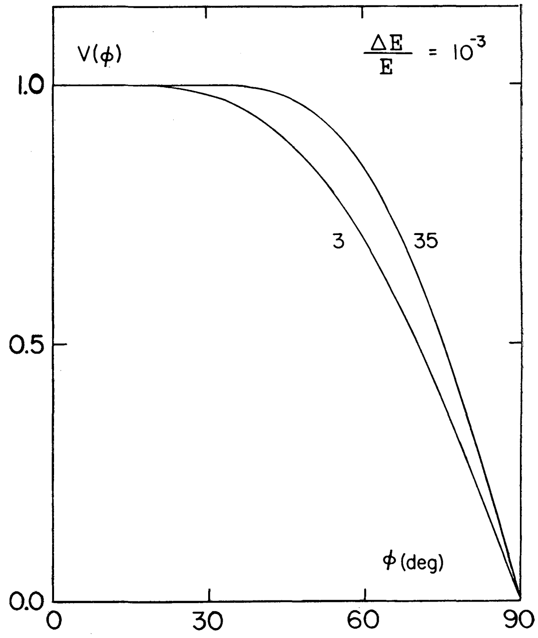


FIG. 3. Optimum flat-topping curves showing the effective voltage wave form $v(\phi)$ obtained for an energy resolution $\delta = (\Delta E)/E = 10^{-3}$, and for the following different harmonic combinations: $n = 1$ and 3; $n = 1$, 3, and 5. These curves are symmetric about $\phi = 0$, and anti-symmetric about $\phi = 90°$. On the scale of this figure, the ripple within the flat-top region is invisible. Compare with Fig. 2.

purely sinusoidal voltage. The curves shown in Fig. 2 require: $v_1(n = 1, 2) = 1.35$, $v_1(n = 1, 2, 3) = 1.55$, and $v_1(n = 1, 2, 3, 4) = 1.67$; by way of comparison, the odd harmonic curves shown in Fig. 3 require: $v_1(n = 1, 3) = 1.13$ and $v_1(n = 1, 3, 5) = 1.19$.

Successful application of the voltage flat-topping technique demands very precise control over the amplitudes and phases of the different harmonic voltages. For a given phase width $\Delta\phi$, the energy resolution of the emergent beam can be adversely affected either through a change in the shape of the voltage wave form or through a fluctuation in its overall amplitude. Assuming the phases $\psi_n$ in (5b, 12) remain fixed, a small fluctuation in $V_n$ having an amplitude $dV_n$ will produce an increase in $\delta$ by an amount:

$$d\delta \simeq (dV_n)/V_0. \tag{27a}$$

Similarly, a phase error of magnitude $d\psi_n$ in one of the harmonics will, by itself, produce an increment:

$$d\delta \simeq (V_n/V_0)\, d\psi_n, \tag{27b}$$

assuming $\sin(n\phi) \simeq 1$ somewhere in the given $\phi$ range. Each of these errors, or the root-mean-square combination of all such errors, must satisfy the condition: $d\delta \ll \delta$ before the optimum flat-topping results can be realized in practice.

As discussed previously,[1] deviation of the magnetic field shape from that required for isochronism will produce phase excursions in the beam, thereby reducing the total phase width which can be maintained within the flat-top region of the effective voltage wave form. Similar effects are produced by a time dependent fluctuation in the magnetic field or in the main rf frequency. A change in the frequency of one of the harmonic voltages by an amount $d\omega_n = \omega_1(dn)$ will produce a phase shift $d\psi_n$ given by:

$$d\psi_n = 2\pi\tau h(dn), \tag{28}$$

and, as indicated above, the energy resolution will deteriorate accordingly.

Since the entire voltage flat-topping discussion is based on Eqs. (5a, b), the approximations involved in obtaining these equations should also be examined. Second and higher order corrections to these approximate equations will result from the radial and axial oscillations about the reference orbit, and from the nonadiabatic nature of the acceleration. Such corrections could be quite significant whenever a very precise energy resolution is being sought over a wide phase range. In particular, it should be kept in mind that the cyclotron is the least adiabatic of all cyclic accelerators.

The incoherent radial and axial oscillations can be limited by controlling the emittance of the beam from the ion source or injector. Coherent oscillations will be produced by magnetic field imperfections or by misalignments of electrical and magnetic components. Such oscillations can be readily detected with suitable beam probes and can therefore be limited if appropriate correction mechanisms have been provided. However, unavoidable coherent radial oscillations will be induced by the acceleration process, and the amplitude of these oscillations can vary significantly for different $\phi$ values. Moreover, these radial oscillations will produce corresponding oscillations in $\phi$. Furthermore, fluctuations in $E$ will result from these oscillations and from the acceleration process itself. The magnitude of these effects will depend on the injection energy and on the details of the rf system.

Under certain circumstances we can justify the use of Eqs. (5a, b) even when the acceleration is quite nonadiabatic. In these cases, the reference orbit should be chosen to coincide with the accelerated 'central ray' orbit which spirals outward. The isochronous magnetic field is then specified by requiring that the rotation period of this orbit be independent of energy. If, in addition, the potential of Eq. (3) changes insignificantly over a radial distance equal to the radius gain per turn of this orbit, then Eq. (5a, b) will be reasonably valid. It must, however, be verified that a unique central ray orbit exists independent of the $\phi$ value within the desired $\Delta\phi$ range. This is certainly the case when a square wave voltage (such as shown in Fig. 3) is supplied to each dee. Such may not be the case when the harmonics are supplied to separate dees which are widely spaced apart.

Ordinarily, 'isochronous' cyclotrons have sufficient flexibility regarding phase excursions to permit the acceleration of the beam into the non-isochronous edge region of the magnetic field prior to extraction. This procedure leads to greater turn separation at the entrance to the electrostatic channel, and also reduces the electric field required by this channel for successful beam deflection. However, this extraction procedure cannot be applied to cyclotrons which utilize the voltage flat-topping technique since the isochronism condition must be carefully maintained in these machines. Despite this drawback, and despite the difficulties outlined in the preceding paragraphs, the potential advantages of the voltage flat-topping technique are sufficiently attractive to justify the effort required to successfully implement this technique.

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