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MATRIX METHOD FOR THREE-ELEMENT EXTRACTION SYSTEMS†

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This paper discusses the matrix-method calculation of the properties of a three-element regenerative extraction system for a synchrocyclotron. It is shown that the three-element-system stability calculation can be conveniently parameterized in matrix formalism separating the perturbation strengths from the field geometry.

1. INTRODUCTION

During the extraction-improvement studies for the Berkeley 184-inch synchrocyclotron we have considered three-element regenerative extraction systems. The motivation behind the three-element system has been to effect efficient radial extraction while preserving vertical stability for all turns through the regenerator. Two calculational methods have been used: exact integration of the equations of motion in the magnetic field[(1)] and the matrix method of LeCouteur.[(2)] This paper describes some of our work using the matrix method. In order to study the regenerative effect of these three-element systems, we have extended the standard two-element equations for the stability criteria in matrix formalism and show that a separation of the perturbation strength from the geometry (location of the perturbations) is possible.

The geometry of the extraction system considered is shown in Fig. 1. The perturbations Q_1, Q_2, and Q_3 begin at radii r_1, r_2 and r_3 and are separated by angles α, β, and γ respectively. These perturbations are considered superimposed on the normal weak-focusing field of the cyclotron.

The effect of field perturbations on regeneration is studied by calculating the matrices for the several field regions comprising a turn. The single-turn product matrix can be used to track a given particle vector one or more turns, or provide information on stability from the value of the trace of the matrix.[(3)] The matrix transforms a particle vector $\binom{x}{x'}$ where $x' = \mathrm{d}x/\mathrm{d}\theta$, θ being the azimuth formed by the right-handed polar coordinate system (x, z, θ). We define the field index describing the field fall-off as

$$n = -\frac{\mathrm{d}B}{\mathrm{d}r}\frac{\rho}{B_0},$$

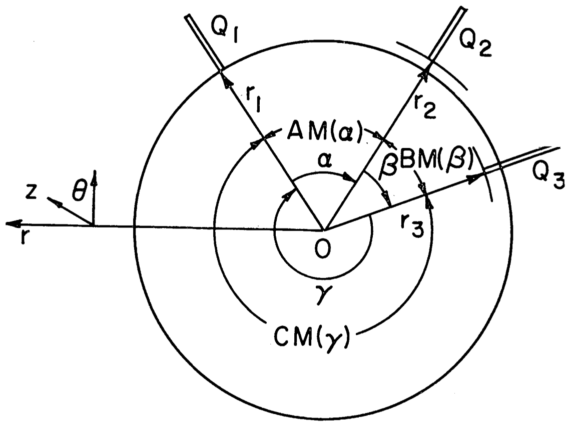

FIG. 1. Geometry of the three-region extraction system. The center of the cyclotron is at 0. The perturbations Q_1, Q_2, and Q_3 begin at radii r_1, r_2, and r_3 and are separated by angle α, β, and γ respectively. The matrices AM( ), BM( ), and CM( ) transform the particle vectors through the angles α, β, and γ.

where ρ is the radius of curvature, PC/eB_z, for a particle of momentum P in field B_z on the median plane. The perturbations produced by a peeler, regenerator, or additional extraction elements is represented by a matrix which produces a deflection of the particle without displacing the trajectory:

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -Q_j & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$

$$\begin{pmatrix} z \\ z' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ Q_j & 1 \end{pmatrix} \begin{pmatrix} z_0 \\ z'_0 \end{pmatrix}.$$

Here Q_j is the strength of the jth perturbation of width $\Delta\theta_j$, $Q_j = -n_j\Delta\theta_j$. Note that Q has the sign of the field gradient:

$$P(Q_j) = \begin{pmatrix} 1 & 0 \\ -Q_j & 1 \end{pmatrix}.$$

The effect of the weak-focusing cyclotron field of

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azimuthal extent α_j is given by the following matrix:

$$A(\alpha_j) = \begin{pmatrix} \cos \nu\alpha_j & 1/\nu \sin \nu\alpha_j \\ -\nu \sin \nu\alpha_j & \cos \nu\alpha_j \end{pmatrix},$$

where $\nu = \sqrt{n}$ for vertical motion and $\nu = \sqrt{1-n}$ for radial motion. A complete revolution about the cyclotron is simulated by the product of the matrices in the order encountered:

$$A(2\pi) = A(\gamma)P(Q_3)A(\beta)P(Q_2)A(\alpha)P(Q_1).$$

This single-turn matrix represents the effect of the three perturbations Q_1, Q_2, and Q_3 separated by angles α, β, and γ in the cyclotron (Fig. 1). Radial or vertical stability requires that the trace of the matrix be less than 2. Calculating the trace, we obtain:

$$\begin{aligned}
\text{Trace } A(2\pi) &= 2\cos\nu\alpha\cos\nu\beta\cos\nu\gamma - 2\sin\nu\alpha\sin\nu\beta\cos\nu\gamma - 2\cos\nu\alpha\sin\nu\beta\sin\nu\gamma - 2\sin\nu\alpha\cos\nu\beta\sin\nu\gamma \\
&\quad + Q_1(1/\nu\sin\nu\alpha\cos\nu\beta\cos\nu\gamma + 1/\nu\cos\nu\alpha\sin\nu\beta\cos\nu\gamma - 1/\nu\sin\nu\alpha\sin\nu\beta\sin\nu\gamma + 1/\nu\cos\nu\alpha\cos\nu\beta\sin\nu\gamma) \\
&\quad + Q_2(1/\nu\cos\nu\alpha\sin\nu\beta\cos\nu\gamma + 1/\nu\cos\nu\alpha\cos\nu\beta\sin\nu\gamma - 1/\nu\sin\nu\alpha\sin\nu\beta\sin\nu\gamma + 1/\nu\sin\nu\alpha\cos\nu\beta\cos\nu\gamma) \\
&\quad + Q_3(1/\nu\cos\nu\alpha\cos\nu\beta\sin\nu\gamma - 1/\nu\sin\nu\alpha\sin\nu\beta\sin\nu\gamma + 1/\nu\sin\nu\alpha\cos\nu\beta\cos\nu\gamma + 1/\nu\cos\nu\alpha\sin\nu\beta\cos\nu\gamma) \\
&\quad + Q_1 Q_2(1/\nu^2\sin\nu\alpha\sin\nu\beta\cos\nu\gamma + 1/\nu^2\sin\nu\alpha\cos\nu\beta\sin\nu\gamma) \\
&\quad + Q_1 Q_3(1/\nu^2\sin\nu\alpha\cos\nu\beta\sin\nu\gamma + 1/\nu^2\cos\nu\alpha\sin\nu\beta\sin\nu\gamma) \\
&\quad + Q_2 Q_3(1/\nu^2\cos\nu\alpha\sin\nu\beta\sin\nu\gamma + 1/\nu^2\sin\nu\alpha\sin\nu\beta\cos\nu\gamma) \\
&\quad + Q_1 Q_2 Q_3(1/\nu^3\sin\nu\alpha\sin\nu\beta\sin\nu\gamma).
\end{aligned}$$

Considerable insight into this horrendous expression can be obtained by decomposing the trace into the sum of vector elements W_j obtained by the matrix product of a perturbation matrix, P, and a geometry vector, G.

$$\text{Trace } A(2\pi) = \sum_j W_j,$$

$$W_j = \sum_k P_{jk} G_k,$$

$$P = \begin{bmatrix}
2 & 0 & 0 & 0 & 0 & -2\nu^2 & -2\nu^2 & -2\nu^2 \\
0 & -Q_1 & -Q_1 & -Q_1 & Q_1 & 0 & 0 & 0 \\
0 & -Q_2 & -Q_2 & -Q_2 & Q_2 & 0 & 0 & 0 \\
0 & -Q_3 & -Q_3 & -Q_3 & Q_3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & Q_1 Q_2 & Q_1 Q_2 \\
0 & 0 & 0 & 0 & 0 & Q_1 Q_3 & 0 & Q_1 Q_3 \\
0 & 0 & 0 & 0 & 0 & Q_2 Q_3 & Q_2 Q_3 & 0 \\
0 & 0 & 0 & 0 & \dfrac{-Q_1 Q_2 Q_3}{\nu^2} & 0 & 0 & 0
\end{bmatrix},$$

$$G \equiv \begin{bmatrix}
\cos\nu\alpha\cos\nu\beta\cos\nu\gamma \\
\cos\nu\alpha\sin\nu\beta\cos\nu\gamma/\nu \\
\cos\nu\alpha\cos\nu\beta\sin\nu\gamma/\nu \\
\sin\nu\alpha\cos\nu\beta\cos\nu\gamma/\nu \\
\sin\nu\alpha\sin\nu\beta\sin\nu\gamma/\nu \\
\cos\nu\alpha\sin\nu\beta\cos\nu\gamma/\nu^2 \\
\sin\nu\alpha\sin\nu\beta\cos\nu\gamma/\nu^2 \\
\sin\nu\alpha\cos\nu\beta\sin\nu\gamma/\nu^2
\end{bmatrix}$$

The geometry vector G depends on the azimuthal location of the perturbations and the weak-focusing strength of the cyclotron field while the perturbation strengths enter in an 8×8 matrix of simple structure. The advantage of this representation is that for a given geometry, the effect of change in perturbation strengths can be readily evaluated.

If we consider the usual two-element system made of perturbations Q_1 and Q_2 separated by angles α and β, we obtain, after elimination of zero elements:

$$W = \begin{bmatrix} 2 & 0 & 0 & -2\nu^2 \\ 0 & -Q_1 & -Q_1 & 0 \\ 0 & -Q_2 & -Q_2 & 0 \\ 0 & 0 & 0 & Q_1 Q_2 \end{bmatrix} \begin{bmatrix} \cos\nu\alpha\cos\nu\beta \\ \cos\nu\alpha\sin\nu\beta/\nu \\ \sin\nu\alpha\cos\nu\beta/\nu \\ \sin\nu\alpha\sin\nu\beta/\nu^2 \end{bmatrix}$$

This gives the usual expression for the trace of a regenerator-peeler system:[4]

$$\text{Trace} = 2\cos\nu(\alpha+\beta) - \frac{1}{\nu}(Q_1+Q_2)\sin\nu(\alpha+\beta) + \frac{Q_1 Q_2}{\nu}\sin\nu\alpha\sin\nu\beta.$$

Here Q_1 and Q_2 are positive for a rising field and negative for a falling field. From the field of the 184-inch cyclotron, Fig. 2, we have $Q_1 = -0.4$, $Q_2 = 0.6$, $\nu_r = 0.98$, $\nu_z = 0.2$, $\alpha = 90\,\text{deg}$ and $\beta = 270\,\text{deg}$, giving:

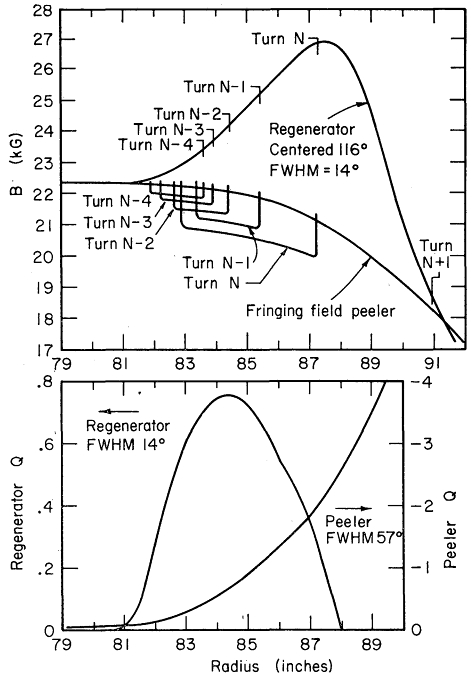

FIG. 2. Field and peeler-regenerator strengths for 184-inch cyclotron.

$$W_r = \begin{pmatrix} 1.984 \\ -.05116 \\ .07673 \\ .2486 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 & -1.9208 \\ 0 & +.4 & .4 & 0 \\ 0 & -.6 & -.6 & 0 \\ 0 & 0 & 0 & -\ .24 \end{pmatrix} \begin{pmatrix} -.00296 \\ -.09598 \\ -.03191 \\ -1.0361 \end{pmatrix},$$

$$W_z = \begin{pmatrix} .6180 \\ -1.902 \\ 2.853 \\ -1.500 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 & -0.080 \\ 0 & -.4 & -.4 & 0 \\ 0 & .6 & .6 & 0 \\ 0 & 0 & 0 & -\ .24 \end{pmatrix} \begin{pmatrix} .5590 \\ .90818 \\ 3.8471 \\ 6.250 \end{pmatrix}.$$

The radial trace is $\sum_j W_{rj} = 2.258$ and the vertical trace is $\sum_j W_{zj} = 0.0691$; this then gives regeneration without vertical instability.

Given T_r, the desired value of the trace of the radial matrix, and the strength of one regenerator and peeler, say T_r, Q_1 and Q_3 the value of Q_2 required to produce T_r can be calculated from the expression

$$Q_2 = \frac{T_r - 2g_1 + 2\nu_r^2(g_6+g_7+g_8) + (Q_1+Q_3)(g_2+g_3+g_4-g_5) - Q_1 Q_3(g_6+g_8)}{(g_2+g_3+g_4-g_5) + Q_1(g_7+g_8) - Q_3(g_6+g_7) + Q_1 Q_3 g_5/\nu_r}$$

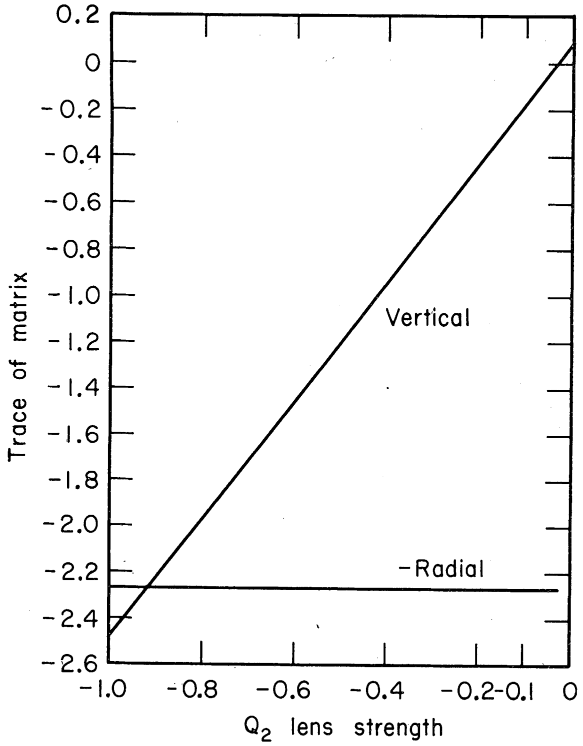

FIG. 3. Effect of lens placed at radial node. Here $\alpha = 40$ deg, $\beta = 230$ deg, $\gamma = 90$ deg, $Q_1 = 0.6$, $Q_3 = -0.4$, and $n = 0.04$.

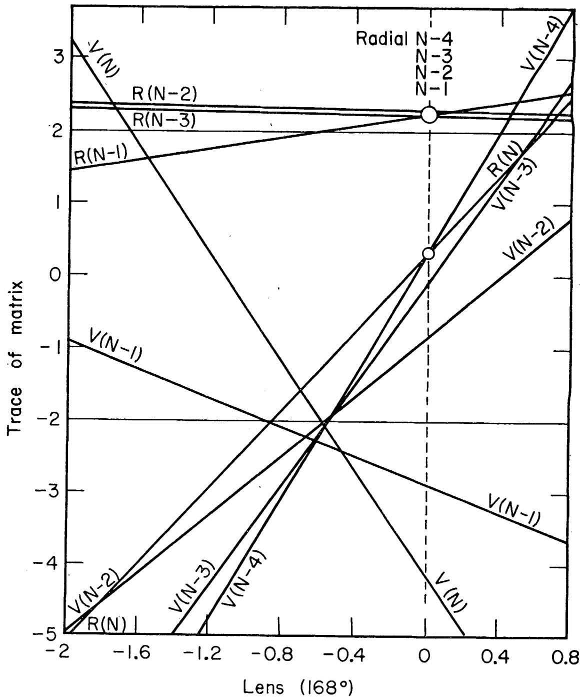

FIG. 4. Vertical and radial trace of transfer matrix as a function of lens strength at radial regeneration node for last five resolutions in 184-inch cyclotron. The matrices extend from 60, 116, 168 to 60 deg and were calculated numerically by orbit integration in the existing magnetic field of the 184-inch cyclotron for typical 730-MeV protons.

where g_j are the elements of the geometry vector G evaluated with $\nu = \nu_r = \sqrt{1-n}$. The trace in the vertical plane produced by this value of Q_2 is obtained from

$$T_z = 2h_1 - 2\nu_z(h_6+h_7+h_8) - (h_2+h_3+h_4-h_5) \cdot (Q_1+Q_2+Q_3) + Q_1 Q_2(h_7+h_8) + Q_1 Q_3(h_6+h_8) + Q_2 Q_3(h_6+h_7) - Q_1 Q_2 Q_3 h_5 \nu_z^{-2},$$

where h_j are the elements of the geometry vector G evaluated with $\nu = \nu_z = \sqrt{n}$. This equation simplifies to a straight line when plotting T_z vs Q_2 (Fig. 3):

$$T_z = a + bQ_2,$$
$$a = 2h_1 - 2\nu_z(h_6+h_7+h_8) - (h_2+h_3+h_4-h_5) \cdot (Q_1+Q_3) + Q_1 Q_3(h_6+h_8),$$
$$b = Q_3(h_6+h_7) + Q_1(h_7+h_8) - (h_2+h_3+h_4-h_5) \cdot (Q_1+Q_3) - Q_1 Q_3 Q_5 \nu_z^{-2}.$$

2. VERTICAL FOCUSING LENS AT RADIAL NODE

We have examined the effect of placing a vertically focusing lens Q_3, at the radial regeneration node.[5] The vertical and radial trace space is shown in Fig. 4 as a function of lens strength for values of Q_1, Q_2, and n appropriate for the turn under consideration. For a zero lens strength we have the values of the vertical trace for the various turns calculated in the preceding section. As we change the lens strength the vertical trace can be made to increase or decrease in value for any given turn while maintaining radial instability, but nowhere is there a lens value for which the vertical trace is less than 2 for all turns. It then must be concluded that a vertical lens placed at the radial node cannot correct the vertical over-focusing produced by the peeler on turns $N-1$ and N for the 184-in. cyclotron.

3. MAGNETIC BUMP BEAM STRETCHER

Consider the stability diagram and regeneration properties of a gradient coil used to stretch the beam. This scheme[6] calls for the acceleration of the beam into a field region that off-centers the

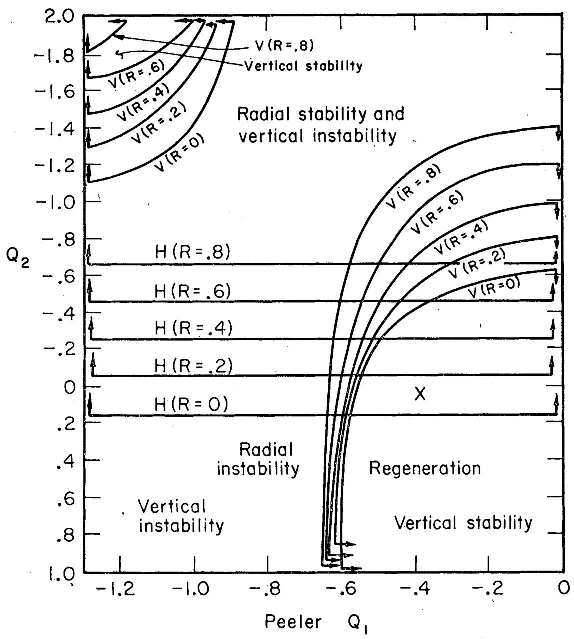

FIG. 5. Stability diagram for magnetic bump beam stretcher. The lines are drawn for trace of matrix equal to 2.0 with arrows pointing in direction of trace less than 2.0. The curves are marked H for horizontal and V for vertical for regenerator strengths R as indicated, the point marked X is the current operating point where radial trace is greater than 2 and the vertical trace is less than 2, $R=0.6$.

beam so as to prevent its entry into the regenerator. The acceleration is then turned off and extraction accomplished by a slow reduction of the magnetic bump, slowly bringing the beam into the regenerator, large radial amplitudes first. Figure 5 shows the stability diagram. The area marked 'vertical instability' is the area for which the absolute value of the trace of the vertical matrix is greater than 2; the area marked 'regeneration' is the area for which the radial trace is greater than 2 and the absolute value of the vertical trace is less than 2. The effect on the stable area for various values of the gradient coil Q_2 is indicated. This figure is obtained from the results of a three-dimensional map produced by varing the perturbations Q_1 (peeler), Q_2 (gradient coil), and Q_3 (regenerator) by a computer code REGEN.[7]

It will be noted that as the harmonic coil Q_2 pushes the particles off center, out into the fringing field, the value of the strength of the fringe field will increase (Fig. 2). When the orbits are out to about 85 in., $Q_1 = 0.6$, $Q_3 = 0$, and we have vertical instability (Fig. 5). The action of Q_2 under acceleration can be investigated by accelerating out to a radius of 81.15 in. with the gradient bump Q_2 on ($Q_2 = -0.2$). Regeneration does not occur since the trace of the radial matrix is less than 2 with the perturbation on. This is effected by the off-centering of the orbits so that they do not enter the regenerator, so $Q_3 = 0$. The perturbation is now turned off and the particles enter the regenerator and are extracted.

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