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DECAY OF IMAGE CURRENTS AND SOME EFFECTS ON BEAM STABILITY†

STEPHEN BODNER, V. K. NEIL, AND LLOYD SMITH‡

Lawrence Radiation Laboratory, University of California, Livermore, California, U.S.A.

An unneutralized beam of particles is suddenly established off-axis in a conducting tube of circular cross section, and the time-dependent magnetic field of the image currents is investigated theoretically. The system is infinite in the axial direction. The tube wall has conductivity σ, inner radius b, and thickness τ. The calculation is valid to first order in the displacement of the beam axis from the axis of the tube. For $\tau/b \lesssim 0.1$ it is shown that 94% of the image field decays with a time constant $2\pi\sigma b\tau/c^2$, where c is the speed of light. The remaining portion of the field decays more rapidly. These results are then applied to a beam in an accelerator, and the effects of the time-dependent field on the dynamics of the beam are treated. As this field decays, the beam can cross an integral resonance, with the result that the displacement of the beam grows exponentially. An expression for the growth rate is presented.

1. INTRODUCTION

We consider the effect of image currents on the transverse motion of a beam of relativistic particles in a conducting vacuum tank of circular cross section. The magnetic field arising from the image currents has been investigated by Laslett,(1) who employed the thin wall approximation. (This approximation is discussed in Sec. 3 below.) In our treatment the vacuum tank wall has finite thickness τ, and for purposes of calculating the magnetic field of the beam we consider the tank to be straight and infinitely long (i.e., we " unwrap " the accelerator). Furthermore, we consider the displacement of the beam to be uniform along its entire length. In Secs. 2 and 4 we calculate the magnetic field from a beam that is suddenly established at a position off-axis in the pipe. From this result one can obtain the field from a beam in arbitrary transverse motion. This is done in Sec. 5, where the effect of image currents on beam stability is discussed.

2. LAPLACE TRANSFORM OF THE IMAGE FIELD

We consider that at time $t = 0$ the electron beam " appears " off-axis in an infinitely long pipe with cylindrical cross section. The geometry is shown in Fig. 1. The pipe has inner radius b, outer radius d, and conductivity σ. The density of electrons in the beam is taken as being uniform out to a radius " a " and zero beyond. We will calculate all quantities to first order in ξ, the displacement of the beam axis from the axis of the pipe. The displaced beam may be thought of as the undisplaced beam plus a surface current. The magnetic field of the undisplaced beam is unaffected by the decay of image currents, and will be discussed below. We consider here only the fields of the surface current. To first order in ξ the configuration of image currents is independent of the beam radius or the radial distribution of current in the beam. The

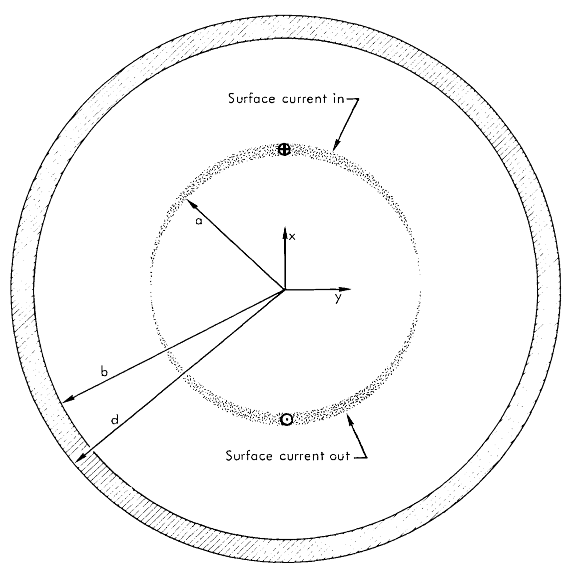

FIG. 1. Displaced Beam and Equivalent Surface Current.

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‡ Permanent address: Lawrence Radiation Laboratory, Berkeley, California.

source of the first order fields is then a current density $\bar{j}$ given by

$$\bar{j} = \frac{I\xi}{\pi a^2}\delta(r-a)\cos\theta\hat{z} \tag{2.1}$$

with I the beam current. The angle θ is measured from the line between the axes of the beam and pipe (x-axis).

At time $t = 0$ surface currents are established on the inner surface of the pipe so that no field penetrates into the metal. Thus initially the magnetic field is given by

$$\bar{B}_i(r,\theta) = -\frac{2I\xi}{ca^2}\begin{cases}\left[1-\left(\frac{a}{b}\right)^2\right]\hat{y}, & r<a\\ \left(\frac{a}{r}\right)^2(\sin\theta\,\hat{r}-\cos\theta\,\hat{\theta}) & \\ -\left(\frac{a}{b}\right)^2\hat{y}, & a<r<b\\ 0, & r>b,\end{cases} \tag{2.2}$$

in which $\hat{r}$, $\hat{\theta}$, and $\hat{y}$ are unit vectors. We have $\hat{y} = \hat{r}\sin\theta + \hat{\theta}\cos\theta$. Gaussian units are used throughout this calculation.

As time progresses, the image currents decay, and as $t\to\infty$ the magnetic field is given by

$$\bar{B}_f(r,\theta) = -\frac{2I\xi}{ca^2}\begin{cases}\hat{y}, & r<a.\\ \left(\frac{a}{r}\right)^2(\sin\theta\,\hat{r}-\cos\theta\,\hat{\theta}), & r>a.\end{cases} \tag{2.3}$$

The purpose of this calculation is to describe the time behavior of the magnetic field after time $t = 0$. It is convenient to introduce the vector potential, A_z, which for convenience in typesetting will carry no subscript in the following. Initially we have

$$A_i = \frac{2I\xi}{ca^2}\begin{cases}\left[1-\left(\frac{a}{b}\right)^2\right]r\cos\theta, & r<a\\ \frac{a^2}{r}\cos\theta-\left(\frac{a}{b}\right)^2 r\cos\theta, & a<r<b,\\ 0, & r>b.\end{cases} \tag{2.4}$$

The final form of A is given by

$$A_f = \frac{2I\xi}{ca^2}\begin{cases}r\cos\theta, & r<a,\\ \frac{a^2}{r}\cos\theta, & r>a.\end{cases} \tag{2.5}$$

We introduce a transient solution $A_t(r,\theta,t)$ such that the total vector potential for $t>0$ is given by

$$A(r,\theta,t) = A_f(r,\theta) + A_t(r,\theta,t). \tag{2.6}$$

Our transient solution gives the fields that result from the image currents alone, and must vanish as $t\to\infty$. At $t = 0$ we have

$$A_t(r,\theta,0) = A_i(r,\theta) - A_f(r,\theta)$$

so that

$$A_t(r,\theta,0) = -\frac{2I\xi}{c}\cos\theta\begin{cases}r/b^2, & r<b,\\ 1/r, & r>b.\end{cases} \tag{2.7}$$

The object is to obtain an expression for $A_t(r,\theta,t)$ with initial conditions given by Eq. (2.7). The equations satisfied by A_t are

$$\nabla^2 A_t = 0 \qquad r<b,\quad r>d,$$

$$\nabla^2 A_t = \frac{4\pi\sigma}{c^2}\frac{\partial A_t}{\partial t}, \qquad b<r<d. \tag{2.8}$$

We introduce Laplace transforms to solve Eqs. (2.8) and define

$$\tilde{A}(p) = \int_0^\infty A_t(t)e^{-pt}\,dt. \tag{2.9}$$

The inversion formula is

$$A_t(t) = \frac{1}{2\pi i}\int_{\epsilon-i\infty}^{\epsilon+i\infty}\tilde{A}(p)\,e^{pt}\,dp. \tag{2.10}$$

In terms of the transformed quantities, Eqs. (2.8) become

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\tilde{A}}{\partial r}\right)-\frac{\tilde{A}}{r^2} = \begin{cases}0 & r<b,\quad r>d,\\ \frac{4\pi\sigma}{c^2}(p\tilde{A}-A_t(0)), & r>b.\end{cases} \tag{2.11}$$

With the use of Eq. (2.7) for $A_t(0)$, we have

$$\tilde{A}(p) = \cos\theta\begin{cases}C_1 r, & r<b,\\ C_2 I_1(kr)+C_3 K_1(kr)-\frac{2I\xi}{pcr}, & b<r<d,\\ C_4/r, & r>d,\end{cases} \tag{2.12}$$

in which I_1 and K_1 are the modified Bessel functions of the first and second kind respectively, the C's are constants determined by boundary conditions, and

$$k^2 \equiv 4\pi\sigma p/c^2. \tag{2.13}$$

The coefficients in Eq. (2.12) are to be evaluated using the continuity conditions at $r = b$ and $r = d$. At $r = d$ we have both $\tilde{A}$ and $\partial\tilde{A}/\partial r$ continuous.

These conditions yield

$$C_3/C_2 = I_0(kd)/K_0(kd). \tag{2.14}$$

In obtaining Eq. (2.14) use has been made of the recursion relations;

$$K_1'(z) = -K_0(z) - \frac{K_1(z)}{z}, \quad I_1'(z) = I_0(z) - \frac{I_1(z)}{z}. \tag{2.15}$$

At $r = b$, we also employ the continuity of $\tilde{A}$ and $\partial\tilde{A}/\partial r$. ($A$ and $\partial A/\partial r$ are continuous for all $t > 0$; thus $\tilde{A}$ and $\partial\tilde{A}/\partial r$ are continuous for all p.)

The algebraic details involved in applying the continuity conditions at $r = b$ will not be given. In simplifying the result, use is made of the relations:

$$\begin{aligned} K_1(z) - zK_1'(z) &= zK_2(z), \\ I_1(z) - zI_1'(z) &= -zI_2(z), \\ 2K_1(z) &= z[K_2(z) - K_0(z)], \\ 2I_1(z) &= z[I_0(z) - I_2(z)], \end{aligned} \tag{2.16}$$

The resulting expression for C_1 yields for the region $r < b$

$$\tilde{A}(p) = \frac{-2I\xi r\cos\theta}{cpb^2}\left[\frac{I_0(kd)\,K_0(kb) - K_0(kd)\,I_0(kb)}{I_0(kd)\,K_2(kb) - K_0(kd)\,I_2(kb)}\right]. \tag{2.17}$$

The Laplace transform $\tilde{B}(p)$ of the magnetic field in the region $r < b$ that results from currents in the wall is simply (this field is uniform and has only a y component)

$$\tilde{B}(p) = -\tilde{A}(p)/r\cos\theta. \tag{2.18}$$

3. RELATION TO " THIN WALL " APPROXIMATION

The so-called " thin wall " approximation to the problem under investigation assumes that the discontinuity in A_t across the metal can be neglected. For all time the vector potential A_t is given by

$$A_t = f(t)\begin{cases} \dfrac{r}{b^2}\cos\theta, & r < b, \\ \dfrac{\cos\theta}{r}, & r > b, \end{cases} \tag{3.1}$$

At the radius of the wall ($r = b$), A is continuous, and the discontinuity of $\partial A/\partial r$ is related to the surface current K which is the total current in the metal. The surface current is given by

$$K = -\frac{\sigma\tau}{c}\frac{\partial A}{\partial t}\bigg|_{r=b}, \tag{3.2}$$

in which τ is the thickness of the sheet. Applying the condition

$$\frac{\partial A^-}{\partial r} - \frac{\partial A^+}{\partial r} = \frac{4\pi K}{c},$$

we obtain the differential equation for $f(t)$ in Eq. (3.1), namely,

$$\frac{2f(t)}{b} = -\frac{4\pi\sigma\tau}{c^2}\frac{\mathrm{d}f}{\mathrm{d}t}. \tag{3.3}$$

Thus we have

$$A_t = A_t(t = 0)\,e^{-c^2t/2\pi\sigma b\tau}. \tag{3.4}$$

From Eq. (2.7) of the previous section, we have for $r < b$,

$$A_t = \frac{-2I\xi r}{cb^2}\cos\theta\,e^{-c^2t/2\pi\sigma b\tau}. \tag{3.5}$$

It is straightforward, although somewhat tedious, to obtain this result from Eqs. (2.17) and (2.10). To do so one uses the approximations

$$\begin{aligned} I_0(kd) &= I_0(kb) + k\tau I_0'(kb), \\ K_0(kd) &= K_0(kb) + k\tau K_0'(kb), \end{aligned} \tag{3.6}$$

in which $\tau = d - b$. Noting that $I_0' = I_1$ and $K_0' = -K_1$, and using relations in Eqs. (2.16) as well as the Wronskian relation, we reduce Eq. (2.17) to the form

$$\tilde{A}(p) = \frac{-2I\xi}{cb^2}\frac{r\cos\theta}{p + (c^2/2\pi\sigma b\tau)}. \tag{3.7}$$

Inverting the Laplace transform yields a result identical to Eq. (3.5).

As we shall show below, the expression for $\tilde{A}(p)$ in Eq. (2.17) has an infinite number of simple poles along the negative real p axis. For $\tau/b \simeq 0.1$ the thin wall approximation gives a rather good result for the contribution from the pole nearest to the origin. It completely neglects all other poles.

4. INVERSION OF THE LAPLACE TRANSFORM

In order to locate the poles of the expression in Eq. (2.17) we introduce the variable y by the definition

$$y \equiv ikb \tag{4.1a}$$

$$y^2 = -4\pi\sigma pb^2/c^2. \tag{4.1b}$$

Using the definitions

$$I_0(z) = J_0(iz), \qquad I_2(z) = -J_2(iz),$$
$$K_0(z) = \frac{\pi i}{2} H_0^{(1)}(iz), \quad K_2(z) = -\frac{\pi i}{2} H_2^{(1)}(iz), \tag{4.2}$$

we write Eq. (2.17) in the form;

$$\tilde{A}(p) = \frac{2I\xi r\cos\theta}{cpb^2}\left[\frac{J_0(y\mathrm{d}/b)\,N_0(y) - J_0(y)\,N_0(y\mathrm{d}/b)}{J_0(y\mathrm{d}/b)\,N_2(y) - J_2(y)\,N_0(y\mathrm{d}/b)}\right]. \tag{4.3}$$

The expression in square brackets in Eq. (4.3) has an infinite number of simple poles along the real y-axis (which, by virtue of Eq. (4.1b) corresponds to the negative p-axis). In spite of the factor p^{-1} in the expression for $A(p)$, an expansion of the expression in square brackets for small y shows that there is no pole at $p = 0$.

We wish to calculate the residues at the poles in the p-plane. For a function of the form $N(p)/D(p)$ such that at a point p_i we have $D(p_i) = 0$, $N(p_i) \neq 0$, and $D'(p_i) \neq 0$, the residues Res_i, at p_i is given by

$$\mathrm{Res}_i = N(p_i)/D'(p_i). \tag{4.4}$$

Thus the residues of the right-hand side of Eq. (4.13) are given by

$$\mathrm{Res}_i = \frac{2I\xi r\cos\theta}{cb^2 p_i}\left[J_0(y_i\mathrm{d}/b)\,N_0(y_i) - J_0(y_i)\,N_0(y_i\mathrm{d}/b)\right] \times\left\{\frac{\mathrm{d}y}{\mathrm{d}p}\frac{\mathrm{d}}{\mathrm{d}y}\left[J_0(y\mathrm{d}/b)\,N_2(y) - J_2(y)\,N_0(y\mathrm{d}/b)\right]\right\}^{-1}_{y=y_i} \tag{4.5}$$

From Eq. (4.1b) we have

$$p_i\left.\frac{\mathrm{d}y}{\mathrm{d}p}\right|_{y_i} = \frac{y_i}{2}.$$

Equation (4.5) can be simplified by using the relations:

$$J_0(y_i\mathrm{d}/b)\,N_2(y_i) = J_2(y_i)\,N_0(y_i\mathrm{d}/b), \tag{4.6}$$

$$J_0(y)\,N_0'(y) - J_0'(y)\,N_0(y) \equiv J_1(y)\,N_0(y) - N_1(y)J_0(y) = 2/\pi y, \tag{4.7}$$

and

$$J_2(y)N_0(y) - N_2(y)J_0(y) = 4/\pi y^2. \tag{4.8}$$

We then obtain

$$\mathrm{Res}_i = \frac{8I\xi r\cos\theta}{cb^2}\, y_i^{-2}\left[1 - \frac{J_2^2(y_i)}{J_0^2(y_i\mathrm{d}/b)}\right]^{-1}. \tag{4.9}$$

With this expression for the residues, as well as Eq. (4.1b), we may easily perform the inverse transform to obtain the final result:

$$A_t(r,\theta,t) = -\frac{8I\xi r\cos\theta}{cb^2}\sum_{i=1}^{\infty}\frac{\exp(-c^2y_i^2t/4\pi\sigma b^2)}{y_i^2\{[J_2(y_i)/J_0(y_i\mathrm{d}/b]^2 - 1\}} \tag{4.10}$$

A computer code has been used to evaluate the expression. However, a bit more qualitative information can be gleaned if the thickness of the wall $\tau \equiv d - b$ is very much less than b. We may find the values of y_i to a good approximation by employing the asymptotic forms

$$J_0(z) \simeq (2/\pi z)^{1/2}\cos\left(z - \frac{\pi}{4}\right), \qquad N_0(z) \simeq (2/\pi z)^{1/2}\sin\left(z - \frac{\pi}{4}\right),$$
$$J_2(z) \simeq (2/\pi z)^{1/2}\cos\left(z - \frac{5\pi}{4}\right), \qquad N_2(z) \simeq (2/\pi z)^{1/2}\sin\left(z - \frac{5\pi}{4}\right). \tag{4.11}$$

With these expressions we have

$$J_0(yd/b)\,N_2(y) - J_2(y)\,N_0(y\mathrm{d}/b) = -\frac{2}{\pi y}\left(\frac{b}{\mathrm{d}}\right)^{1/2}\sin\left[(\tau y/b) - \pi\right], \tag{4.12}$$

so that the approximate roots y_i are simply

$$y_i \simeq m\pi b/\tau, \tag{4.13}$$

with m an integer. *This approximation misses the first root.* That is, $m = 1$ in Eq. (4.13) gives $y_2 = \pi b/t$. The first root is the one corresponding to the thin wall approximation, namely, $y_1 \simeq (2b/\tau)^{1/2}$. (Numerical work gives $y_1 = 4.4335$ for $b = 10\tau$.)

With all y_i for $i > 1$ given by Eq. (4.13), we use Eq. (4.11) to show that

$$\frac{J_2^2(y_i)}{J_0^2(y_i\,\mathrm{d}/b)} - 1 = \frac{\tau}{b}. \tag{4.14}$$

Equation (4.10) now takes the form

$$A_t(r,\theta,t) = -\frac{8I\xi r\cos\theta}{cb^2}\left\{\frac{\exp(-c^2y_1^2t/4\pi\sigma b^2)}{y_1^2[J_2^2(y_1)/J_0^2(y_1\,\mathrm{d}/b) - 1]} + \frac{\tau}{\pi^2 b}\sum_{m=1}^{\infty}\frac{\exp(-c^2\pi m^2t/4\sigma\tau^2)}{m^2}\right\} \tag{4.15}$$

The first term in curly brackets dominates the sum at all times. For example, for $b = 10\tau$, the sum contributes 6% at $t = 0$. Thus the time behavior of 94% of the transient magnetic field is governed by the first term.

The behavior of the expression in Eq. (4.15) near $t = 0$ may be found by noting that

$$\frac{d}{dt}\sum_{m=1}^{\infty}\frac{\exp(-c^2\pi m^2 t/4\sigma\tau^2)}{m^2} = -\frac{c^2\pi}{4\sigma\tau^2}\sum_{m=1}^{\infty}\exp(-c^2\pi m^2 t/4\sigma\tau^2) \quad (4.16)$$

We use the relation[2]

$$\sum_{m=-\infty}^{\infty}\exp(-\pi m^2 x) = \frac{1}{x^{1/2}}\sum_{m=-\infty}^{\infty}\exp(-\pi m^2/x), \quad (4.17)$$

with $x \equiv c^2 t/4\sigma\tau^2$. Thus we have

$$\sum_{m=1}^{\infty}\exp(-\pi m^2 x) = \frac{1}{2x^{1/2}}\sum_{m=-\infty}^{\infty}\exp(-\pi m^2/x) - \tfrac{1}{2}. \quad (4.18)$$

For $t \ll 4\sigma\tau^2/c^2$, the right hand side of Eq. (4.18) is dominated by the $m = 0$ term in the sum, so that

$$\sum_{m=1}^{\infty}\exp(-\pi c^2 m^2 t/4\sigma\tau^2) \approx \frac{\tau}{c}\left(\frac{\sigma}{t}\right)^{1/2}. \quad (4.19)$$

From Eq. (4.16) we obtain

$$\sum_{m=1}^{\infty}\frac{\exp(-\pi c^2 m^2 t/4\sigma\tau^2)}{m^2} = \sum_{m=1}^{\infty}\frac{1}{m^2} - \frac{c\pi}{2\tau}\left(\frac{t}{\sigma}\right)^{1/2}. \quad (4.20)$$

We set the exponential in the first term on the right in Eq. (4.15) equal to unity. This first term then combines with the Σm^{-2} ($= \pi^2/6$) contribution from the second term to yield 0.025. We thus obtain

$$A_t(r,\theta,t) = -\frac{2I\xi r\cos\theta}{cb^2}\left[1 - \frac{2c}{\pi b}\left(\frac{t}{\sigma}\right)^{1/2}\right]. \quad (4.21)$$

Equation (4.21) exhibits the proper early time behavior, namely $(t/\sigma)^{1/2}$ and independent of τ. This approximate form of A_t will be useful in Sec. 5.

5. BEAM DYNAMICS

If the beam is not displaced, each particle would undergo incoherent transverse motion along a trajectory described by $x_0(t)$. We have assumed throughout the above analysis that the beam is displaced rigidly a distance ξ in the x-direction. Therefore at time t each particle in the displaced beam is instantaneously at a position

$$x = x_0(t) + \xi[z(t), t],$$

with $z(t)$ the axial position of the particle. All particles are assumed to have the same velocity v in the z-direction. Although a definite 'model' of the beam has been used, the results are valid (to first order in ξ) even if the beam radius 'a' is less than ξ. The equation governing the particles' motion in the x-direction is

$$\frac{d^2x}{dt^2} = \frac{d^2x_0}{dt^2} + \frac{d^2\xi}{dt^2} = \frac{e[E_x^{(t)} - (v/c)B_y^{(t)}]}{\gamma m_0}, \quad (5.1)$$

in which m_0 is the particle's rest mass and γ the particle's energy in units of m_0c^2. We designate E_{x0} and B_{y0} as the field components present if the beam is not displaced. We use the symbols E_x and B_y to denote the field components that arise from the additional sources present when the beam is displaced. These include the effects of image charges and currents. One additional source, namely the surface current, is given by Eq. (2.1). The other source is the associated charge density, ρ, is given (in terms of the total charge per unit length $\lambda \equiv I/v$) by

$$\rho = \frac{2\lambda\xi}{\pi a^2}\delta(r-a)\cos\theta. \quad (5.2)$$

The total fields to be used in Eq. (5.1) are given by

$$E_x^{(t)} = E_{x0} + \xi\frac{\partial E_{x0}}{\partial x} + E_x, \quad (5.2a)$$

$$B_y^{(t)} = B_{y0} + \xi\frac{\partial B_{y0}}{\partial x} + B_y + B_{ye} + (x_0+\xi)\left.\frac{\partial B_{ye}}{\partial x}\right|_{x=0}, \quad (5.2b)$$

in which B_{ye} is the external magnetic guide field.

For a beam of uniform density in the region $r < a$ we have

$$E_{x0} = \frac{2\lambda r}{a^2}\cos\theta,\ \frac{\partial E_{x0}}{\partial x} = \frac{2\lambda}{a^2}, \quad (5.3a)$$

$$B_{y0} = \frac{2Ir}{ca^2}\cos\theta,\ \frac{\partial B_{y0}}{\partial x} = \frac{2I}{ca^2}. \quad (5.3b)$$

The electric field arising from the surface charge in Eq. (5.2) is, in the region $r < a$,

$$E_x = -\frac{2\lambda\xi}{a^2}\left[1 - \left(\frac{a}{b}\right)^2\right]. \quad (5.4)$$

Thus we have

$$\xi\frac{\partial E_{x0}}{\partial x} + E_x = \frac{2\lambda\xi}{b^2}. \quad (5.5)$$

which is simply the electric field from the image charges. Similarly, we have (in the notation of Sec. 2)

$$B_y = B_{yf} + B_{yt},$$

with $B_{yf} = -2I\xi/ca^2$ as given in Eq. (2.3). Thus

$$\xi \frac{\partial B_{y0}}{\partial x} + B_y = B_{yt}, \tag{5.6}$$

which is simply the magnetic field from the image currents as calculated above. We have

$$B_{yt} = -A_t/r \cos\theta,$$

with A_t given by Eq. (4.10). The equation of motion for a particle in the undisplaced beam is

$$\frac{d^2x_0}{dt^2} = \frac{e}{\gamma m_0}\left[E_{x0} - \frac{v}{c}\left(B_{y0} + B_{ye} + x_0 \frac{\partial B_{ye}}{\partial x}\right)\right], \tag{5.7}$$

so that Eq. (5.1) reduces to

$$\frac{\mathrm{d}^2\xi}{\mathrm{d}t^2} = \frac{e}{\gamma m_0}\left[\frac{2\gamma\xi}{b^2} - \frac{v}{c}\left(B_{yt} + \xi\frac{\partial B_e}{\partial x}\right)\right]. \tag{5.8}$$

Having exhausted the utility of the 'unwrapped' vacuum tank, we revert to cylindrical coordinates such that $z \to \bar{R}\phi$. The circumference of the machine is $2\pi\bar{R}$, and the particles' angular circulation frequency is $\omega_0 \equiv v/\bar{R}$. We introduce the quantities ν_0, ν_e, ν_m by the following definitions:

$$\omega_0^2\nu_0^2 = \frac{ev}{\gamma m_0 c}\frac{\partial B_e}{\partial x}, \qquad \omega_0^2\nu_e^2 = \frac{2e\lambda}{\gamma m_0 b^2},$$

$$\omega_0^2\nu_m^2\xi = evB_{yt}/\gamma_0 mc, \tag{5.9}$$

The quantity ν_0 is the betatron wave number of a particle in the external guide field, and the quantity ν_e represents the change in time arising from the electric images in the wall of the vacuum tank. The symbol ν_m^2 represents an operator on ξ (as will be discussed below) and gives the tune shift arising from the magnetic images. The definition of ν_0^2 in Eq. (5.9) is accurate only for motion 'up and down'. For radial motion the centrifugal force term must be included. In the following, however, ν_0 may be interpreted as either without error. With these definitions, Eq. (5.8) takes the form

$$\frac{\mathrm{d}^2\xi}{\mathrm{d}t^2} + \omega_0^2(\nu_0^2 - \nu_e^2 + \nu_m^2)\xi = 0. \tag{5.10}$$

In any real machine there will be errors in the guide field, so that the right-hand side of Eq. (5.10) will contain a driving term of the form $g(\phi)$. This driving term may be Fourier decomposed

$$g(\phi) = \sum_{n=-\infty}^{\infty} \epsilon_n e^{in\phi}.$$

Likewise, we may expand

$$\xi(\phi, t) = \sum_{n=-\infty}^{\infty} \xi_n(t) e^{in\phi}$$

The procedure followed below will give the exponential growth or decay time of a particular ξ_n. In particular, the decay of the image currents can cause the beam to pass through an integral resonance, and we wish to obtain an expression for the growth rate of the transverse motion if this occurs.

In the previous sections we have obtained an expression for B_{yt} as a function of time after an instantaneous displacement of the beam at $t = 0$. The expression is of the form $B_{yt} = \xi F(t)$. From this we have, for a beam in arbitrary motion,[1]

$$B_{yt}(t) = \int_{-\infty}^{t} \frac{\partial\xi(t')}{\partial t'} F(t - t')\,\mathrm{d}t'. \tag{5.11}$$

From Eq. (4.10) we thus have

$$B_{yt} = \frac{8I}{cb^2}\sum_{i=1}^{\infty}\left[\int_{-\infty}^{t}\frac{\partial\xi}{\partial t'}\exp\left(-(t-t')/T_i\right)\right]y_i^{-2} \cdot \left[\frac{J_2^2(y_i)}{J_0^2(y_i\,\mathrm{d}/b)} - 1\right]^{-1}\mathrm{d}t', \tag{5.12}$$

in which

$$T_i = 4\pi\sigma b^2/c^2y_i^2. \tag{5.13}$$

From Eqs. (5.9) and (5.12) we have

$$\nu_m^2\xi = \sum_{i=1}^{\infty}\nu_i^2\int_{-\infty}^{t}\frac{\partial\xi}{\partial t'}\exp\left(-(t-t')/T_i\right)\,\mathrm{d}t', \tag{5.14}$$

with ν_i^2 given by ($\beta \equiv v/c$, $I \equiv \lambda v$),

$$\nu_i^2 = \frac{8e\lambda\bar{R}^2}{\gamma m_0 c^2 b^2}y_i^{-2}\left[\frac{J_2^2(y_i)}{J_0^2(y_i\,\mathrm{d}/b)} - 1\right]^{-1}. \tag{5.15}$$

We shall examine solutions to Eq. (5.10) of the form

$$\xi = \xi_0 e^{in\phi - pt}. \tag{5.16}$$

With this simple form the integrals in Eq. (5.14) may be performed with the result

$$\int_{-\infty}^{t}\frac{\partial\xi}{\partial t'}\exp\left(-(t-t')/T_i\right)\mathrm{d}t = \frac{p\xi_0 e^{in\phi-pt}}{(p - T_i^{-1})}. \tag{5.17}$$

The operator $\mathrm{d}/\mathrm{d}t$ in Eq. (5.10) is to be interpreted as

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \omega_0\frac{\partial}{\partial\phi} = in\omega_0 - p, \tag{5.18}$$

but we shall look for solutions such that $|p| \ll n\omega_0$ and interpret d/dt as simply $in\omega_0$. With this approximation and the use of Eqs. (5.14) and (5.17), Eq. (5.10) gives the equation for the allowed values of p, which is

$$n^2 - (\nu_0^2 - \nu_e^2) = p \sum \frac{\nu_i^2}{(p - T_i^{-1})}. \qquad (5.19)$$

The right-hand side of Eq. (5.19) is sketched qualitatively in Fig. 2. A real negative root of Eq. (5.19) implies exponential growth of our solution. If we have $n^2 < \nu_0^2 - \nu_e^2$, no real negative roots occur. There is one real positive root corresponding to each term in the sum. However, if the conditions

$$0 < n^2 - (\nu_0^2 - \nu_e^2) < \sum \nu_i^2 \qquad (5.20)$$

hold, a negative real root does occur. We wish to estimate the growth rate for such a situation.

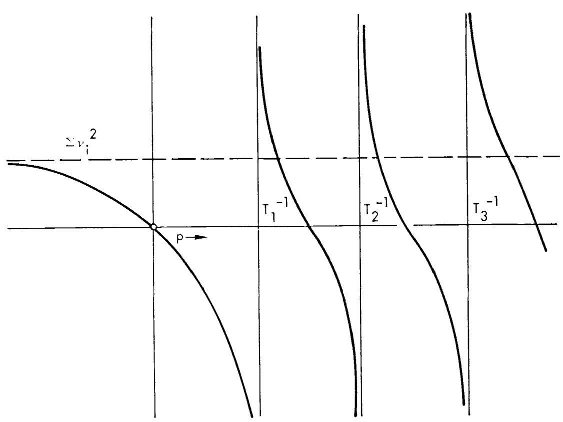

FIG. 2. Qualitative Sketch of the Function $p \sum \nu_i^2/(p - T_i^{-1})$.

As a reasonable numerical example, we take $b = 30$ cm, $\tau = 3$ cm, and $\sigma = 3 \times 10^{17}$ (aluminum). Computation yields $y_1 = 4.4335$ and $y_2 = 32.04$. From Eq. (5.13) we find $T_1 = 0.2$ sec and $T_2 = 3.8$ msec. Furthermore, we find $\nu_1^2/\nu_2^2 = 24$. Apparently the presence of all terms except the first will have a very small influence on the shape of the curve for $p < 0$ in Fig. 2. A good approximation to the negative root can thus be obtained by using only the first term. This procedure is equivalent to the thin wall approximation of Sec. 3. We have

$$p = \frac{1}{T_1} \frac{[n^2 - (\nu_0^2 - \nu_e^2)]}{[n^2 - (\nu_0^2 - \nu_e^2) - \nu_1^2]}. \qquad (5.21)$$

It is convenient, and probably more realistic, to replace ν_1^2 by $\sum \nu_i^2$, which is given by

$$\sum \nu_i^2 = \frac{2e\lambda \bar{R}^2}{\gamma m_0 c^2 b^2} \equiv \frac{2Nr_e \bar{R}^2}{\gamma b^2}, \qquad (5.22)$$

in which N is the number of particles per unit length in the beam and r_e is the electron radius. In terms of these quantities, we also have

$$\nu_e^2 = \frac{2Nr_e \bar{R}^2}{\gamma \beta^2 b^2} \equiv \frac{1}{\beta^2} \sum \nu_i^2. \qquad (5.23)$$

For 1000 A, $Nr_e = 1/17$. We choose $\gamma = 12$ and $\bar{R} = 4.8$ m (circumference $2\pi\bar{R} = 30$ m). We find $\nu_e^2 = 2.6$, and $\nu_e^2 - \nu_m^2 = \nu_e^2/\gamma^2 = 1.8 \times 10^{-2}$. For illustration we choose $\nu_0 = 1.75$. With these values, $n = 1$ is the only integer for which Eq. (5.21) gives a negative value of p, and that value is $p = -1.3$ sec^{-1}. If we choose $\nu_0 = 2.3$, then $n = 2$ yields a negative value of p, namely $p = -5$ sec^{-1}.

An E-folding time of 1/5 sec is considerably longer than the lifetime of the beam set by neutralization of the background, even at a pressure of 5×10^{-9} torr. It is therefore unlikely that the above effect of decaying image currents in the vacuum tank wall can be studied in the storage ring.(3) However, the insertion of appropriate liners in all or a portion of the circumference will allow the effect to be investigated.

We note that this work has been limited to a linear treatment of the beam displacement. However, one conclusion is that the thin wall approximation is really pretty good, at least for the treatment of low frequency oscillations. A complete treatment of the magnetic field from image currents, to all orders in beam displacement, has been carried out by Laslett using the thin wall approximation.(1) Our conclusion supports the validity of Laslett's work.

6. RELATION TO PREVIOUS RESULTS

It is, of course, anticipated that the transverse resistive wall instability will be troublesome.(4) The mechanism for this instability is contained in the above formalism. To show this, we assume a solution

$$\xi = \xi_0 e^{i(n\phi - \omega t)}. \qquad (6.1)$$

If the skin depth at the frequency ω is much smaller than τ, the approximate expression, Eq. (4.21) for A_t is appropriate, and the equation analogous to Eq. (5.12) is

$$B_{yt} = \frac{2I}{cb^2} \int_{-\infty}^{t} \frac{\partial \xi}{\partial t'} \left[1 - \frac{2c}{\pi b}\left(\frac{t - t'}{\sigma}\right)^{1/2}\right] \mathrm{d}t'. \qquad (6.2)$$

With Eq. (6.1) inserted for ξ, we obtain

$$B_{yt} = \frac{2I\xi}{cb^2}\left[1 - \frac{(1+i)}{b}\left(\frac{c^2}{2\pi\omega\sigma}\right)^{1/2}\right]. \qquad (6.3)$$

We note that $I = \pi a^2 \rho_0 v$ and see that the second term on the right in Eq. (6.3) is identical to Eq. (2.24) of Ref. 4. It is the imaginary part of this term that leads to the instability. The first term on the right in Eq. (6.3) is the usual ($\sigma = \infty$) contribution to B_y *from the image currents only*. Equation (2.21b) of Ref. 4 contains this term plus the field from the first order current density (see Eqs. (2.1) and (2.3) above).

It has been proven possible to suppress this instability by means of electronic feedback, and this method will be attempted in the storage ring. Further, it should be possible to suppress the relatively high frequency oscillations associated with the resistive wall instability and still study the slowly growing effect discussed in the last section.

The above formalism may also be used to obtain one of the results of Morton, Neil and Sessler.[5] In Ref. 5 the electromagnetic fields of a bunch of charged particles oscillating transversely while moving down a conducting pipe are evaluated at a large distance behind the charge. Specifically, the distance d behind the charge satisfies the condition

$$b \ll d \ll 4\pi v\sigma b^2/c^2. \qquad (6.4)$$

In Ref. 5 the charged bunch is assumed to occupy the entire cross section of the pipe ($a = b$), and it is further assumed that the thickness of the pipe τ is infinite. To reproduce the results of Ref. 5 we need to consider a current such that

$$I = eN\delta(t), \qquad (6.5)$$

with N the total number of particles in the bunch. In a word, we are considering here a *current* that is time dependent, rather than a displacement ξ. In Ref. 5 the displacement does in fact vary with time, but it is only its phase at a given point that determines the phase of the electromagnetic fields at that point after the punch has long since passed [i.e., condition (6.4) is satisfied]. Here we solve the problem for a charged bunch moving down the pipe off-axis but not oscillating.

Our formalism is not valid if $\tau = \infty$, but this will not affect the agreement with the results of Ref. 5. We make a somewhat more stringent limitation on d, namely

$$b \ll d \ll 4\pi\sigma v\tau^2/c^2.$$

Identifying d as vt we see that this condition allows the use of Eq. (4.21). The equation for B_{yt} analogous to Eq. (6.2) then becomes

$$B_{yt} = \frac{2\xi}{cb^2}\int_{-\infty}^{t} \frac{\partial I}{\partial t'}\left[1 - \frac{2c}{\pi b}\left(\frac{t-t'}{\sigma}\right)^{1/2}\right] \mathrm{d}t'. \qquad (6.7)$$

Inserting Eq. (6.5) into the integral, we obtain

$$B_{yt} = -\frac{2eN\xi}{\pi b^3(\sigma t)^{1/2}}. \qquad (6.8)$$

Equation (6.8) is in agreement with Eqs. (2.19e) and (2.19f) of Ref. 5 in the limit of zero oscillation frequency.

It is not surprising that our technique should yield results in agreement with Refs. 4 and 5. Displacement currents in the metal were neglected in both those papers. The axial (z) variation of ξ is ignored in calculating the magnetic field in Ref. 4. In Ref. 5 the charge distribution and EM fields are Fourier analyzed in z, but only the behavior near $k = 0$ was considered in calculating the fields at large distance.

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