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COHERENT INSTABILITIES IN HIGH CURRENT LINEAR INDUCTION ACCELERATORS†

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Unstable coherent motion of the beam in a high current ($I \simeq 1000$ A) linear induction accelerator has been observed. The instability has been attributed to resonances with various structures in the beam's path. The effect of resonances with the accelerating units on both transverse and longitudinal coherent oscillations of the beam is investigated theoretically. The theory of Panofsky and Bander is extended and generalized to include both types of motion. The stabilizing mechanism of a spread in particle velocity and/or transverse oscillation frequency are included in the treatment of the transverse motion. Stability criteria are derived that place lower limits on these spreads. These criteria depend upon the quality factor, Q, of the structures, and it is shown that both a spread and a finite Q are necessary for stability.

1. INTRODUCTION

The so-called non-regenerative beam instability has been observed on the SLAC two-mile electron accelerator, and considerable experimental and theoretical work has been done toward understanding and alleviating the instability in that machine.[1–3] The instability is manifested by a transverse oscillation of the beam that results from the interaction of the beam with a non-axially symmetric mode of the accelerating structure. The basic theory of the instability is set forth in the analytical work of Panofsky and Bander,[4] which treats particles with extreme relativistic energy.

Our interest has been stimulated by the behavior of the beam in the Astron injector.[5] This injector accelerates electrons from an injection energy of 500 keV to a final energy of 4 MeV. The beam is of 0.3 μsec duration and carries a current of several hundred amperes. One purpose of our investigation is to extend the theory of Ref. (4) to the non-relativistic and intermediate energy ranges. In addition, we treat another instability that consists of longitudinal bunching of an axially uniform beam. This longitudinal bunching results from the interaction of the beam with an azimuthally-symmetric mode of the accelerator structure. Our analytical methods are quite different from those employed in Ref. (4), but our results agree in the proper limit.

In addition, we investigate possible stabilizing mechanisms that are present in the Astron injector. These are a spread in axial velocity of the beam particles, and a spread in transverse oscillation frequency (i.e., betatron frequency) that arises from non-linearities in the focusing magnets. Neither of these stabilizing mechanisms is present in SLAC. The stability criteria are derived through the use of the Vlasov equation, and unfortunately are rigorous only for a beam that is *not* being accelerated. The criteria place a lower limit on the spread in velocity and betatron frequency necessary to stabilize the beam against transverse oscillations. Numerical examples illustrate that these criteria are quite stringent for the operating parameters of the Astron injector. Indeed, stabilization by velocity spread cannot be envisioned. We make no attempt to calculate the effect of spreads that are *insufficient* for stability, although any spread in either velocity or betatron frequency is certain to reduce the growth rate.

Our mathematical model of the accelerator is a semi-infinite series of cylindrical cavities of radius b as shown in Fig. 1. All cavities are identical. (The effect of slight geometric variations from one cavity to the next is another stabilizing mechanism. It is not treated in this work.) Excitation of a particu-

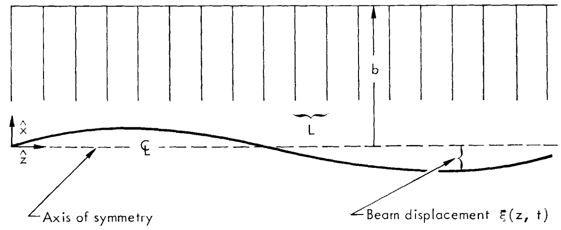

FIG. 1. Configuration of the idealized accelerator structure, showing beam displaced in transverse direction.

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lar mode of each cavity by the beam, and the forces exerted on the beam by the electromagnetic fields of the mode, are the sources of the two instabilities. The term 'non-regenerative' means that there is no propagation of electromagnetic fields from one cavity to the next. Information is carried only by perturbations on the beam.

2. TRANSVERSE INSTABILITY

The particular mode of a pill box cavity that gives rise to the transverse instability is the TM_{110} mode. The field configuration does not vary with axial distance z over the length of one cavity. The beam excites the mode if it passes through the cavity off axis, and the magnetic field on axis exerts a transverse force on the beam. In the equilibrium state, the beam is axially uniform with all particles having the same axial velocity $v(z)$ at any point z and energy $\gamma(z)mc^2$, with m the rest mass, c the speed of light and

$$\gamma(z) = [1-(v/c)^2]^{-1/2}.$$

We calculate the response of the system to a pulse disturbance at $z=0$ at $t=0$. The disturbance may be a displacement or a change in direction of an infinitesimal portion of the beam. Subsequently the displacement of the beam is $\xi(z,t)$ in a transverse (x) direction, where ξ satisfies the equation

$$\left(\frac{\partial}{\partial t}+v\frac{\partial}{\partial z}\right)\left[\gamma\left(\frac{\partial \xi}{\partial t}+v\frac{\partial \xi}{\partial z}\right)\right] = \frac{F_x(z,t)}{m}, \quad (1)$$

in which F_x is the force arising from the magnetic field (B_y) at position z.

The cavity mode of interest can be characterized by the vector potential $\mathbf{A}_l(\mathbf{r})$, which (neglecting the hole) is given by

$$\mathbf{A}_l(\mathbf{r}) = N_l J_1(\rho r)\cos\theta\,\hat{z}, \quad (2)$$

where J_1 is the first order Bessel function, $\rho b=3.83$ (the first zero of J_1), and the angle θ is measured from the x-axis. The normalization chosen is

$$\int \mathbf{A}_l^2\,dV = 4\pi c^2, \quad (3)$$

so that

$$N_l^2 = 8c^2/Lb^2J_0^2(\rho b), \quad (4)$$

where L is the length of a single cavity.

The equation obeyed by $\mathbf{A}_l$ is $\nabla^2\mathbf{A}_l+(\omega_l/c)^2\mathbf{A}_l=0$, with ω_l the eigenfrequency of the mode.

Provided that only one mode is appreciably excited, the vector potential $\mathbf{A}(\mathbf{r},t)$ in the cavity can be expressed as

$$\mathbf{A}(\mathbf{r},t) = q(t)\mathbf{A}_l(\mathbf{r}), \quad (5)$$

with the time dependent coefficient q obeying the equation

$$\frac{d^2q}{dt^2}+\frac{\omega_l}{Q}\frac{dq}{dt}+\omega_l^2 q = \frac{1}{c}\int \mathbf{j}\cdot\mathbf{A}_l\,dV. \quad (6)$$

The quantity Q in Eq. (6) is the usual quality factor. The integral extends over the volume of one cavity. The current density $\mathbf{j}$ is given by

$$\mathbf{j} = \hat{z}I\delta\,[r-\xi(z,t)]\,\delta(\theta)/r, \quad (7)$$

with I the total beam current. From Eqs. (2) and (7) we obtain (to the first order in ξ)

$$\frac{1}{c}\int \mathbf{j}\cdot\mathbf{A}\,dV = \frac{IN_lL\rho\xi}{2c}. \quad (8)$$

In the derivation of Eq. (8), it is assumed that the variation of ξ over the length of one cavity is negligible.

The transverse force F_x arises from the magnetic field $B_y = -\partial A_z/\partial x$. For this first order calculation we evaluate B_y on the axis, and from Eqs. (2) and (5) we obtain

$$F_x = -e\frac{v}{c}B_y = \frac{ev}{2c}N_l\rho q. \quad (9)$$

With the use of Eqs. (8) and (9), Eqs. (1) and (6) become

$$\left(\frac{\partial}{\partial t}+v\frac{\partial}{\partial z}\right)\left[\gamma\left(\frac{\partial \xi}{\partial t}+v\frac{\partial \xi}{\partial z}\right)\right] = \frac{evN_l\rho q}{2mc}, \quad (10)$$

$$\frac{d^2q}{dt^2}+\frac{\omega_l}{Q}\frac{dq}{dt}+\omega_l^2 q = \frac{IN_l}{2c}L\rho\xi. \quad (11)$$

The form of Eq. (10) assures a zero solution for $t<\int_0^z dz/v$. We solve Eq. (11) with the initial conditions $q=0$ and $dq/dt=0$ at $t=\int_0^z dz/v$. At this point we introduce the Fourier transforms $\tilde{q}$ and $\tilde{\xi}$ by the definitions

$$\tilde{q}(\omega) = \frac{1}{2\pi}\int_0^\infty q(t)\,e^{i\omega t}\,dt, \quad (12)$$

$$\tilde{\xi}(z,\omega) = \frac{1}{2\pi}\int_0^\infty \xi(z,t)\,e^{i\omega t}\,dt. \quad (13)$$

The inversion integrals are

$$q(t) = \int_{-\infty+i\epsilon}^{\infty+i\epsilon} \tilde{q}(\omega)\,e^{-i\omega t}\,d\omega, \quad (14)$$

$$\xi(z,t) = \int_{-\infty+i\epsilon}^{\infty+i\epsilon} \tilde{\xi}(z,\omega)\,e^{-i\omega t}\,d\omega. \quad (15)$$

Equation (11) now yields

$$\tilde{q} = (IN_lL\rho\tilde{\xi}/2c)\,[\omega_l^2-\omega^2-i(\omega\omega_l/Q)]^{-1}. \quad (16)$$

We choose

$$\tilde{\xi}(z,\omega) = g(z,\omega)\exp\left[i\omega\int_0^z dz/v\right] \quad (17)$$

so that Eq. (10) becomes

$$\frac{d}{dz}\left[\frac{v}{c}\gamma\frac{dg}{dz}\right] + K^2 g = 0, \quad (18)$$

with

$$K^2 = \frac{2Ie}{mc}\left[\frac{\rho}{bJ_0(\rho b)}\right]^2\left(\omega^2 - \omega_l^2 + i\frac{\omega\omega_l}{Q}\right)^{-1}. \quad (19)$$

Equation (18) can be solved exactly in the extreme relativistic limit, in the non-relativistic limit, and in the limit of zero acceleration. For our purposes it will suffice to consider the WKB solution to Equation (18) which is

$$(\gamma^2-1)^{1/8} g = \exp\left[\pm iK\int_0^z(\gamma^2-1)^{-1/4}dz\right]. \quad (20)$$

The desired form of the solution depends upon the initial conditions. We introduce the functions $f(\omega)$ and $h(\omega)$ by the definitions

$$f(\omega) = \tilde{\xi}(0,\omega) = g(0,\omega), \quad (21a)$$

$$h(\omega) = \frac{\partial\tilde{\xi}}{\partial z}(0,\omega) = \frac{i\omega}{v_i}g(0,\omega) + \frac{dg}{dz}(0,\omega). \quad (21b)$$

We neglect the dependence of $(\gamma^2-1)^{1/8}$ in Eq. (20) and obtain

$$g = \left(\frac{\gamma_i^2-1}{\gamma^2-1}\right)^{1/8}\left\{f(\omega)\cos Ku + (\gamma_i^2-1)^{1/4} \cdot \left[h(\omega) - \frac{i\omega}{v_i}f(\omega)\right]\frac{\sin Ku}{K}\right\}, \quad (22)$$

where we introduce the variable u defined by

$$u = \int_0^z(\gamma^2-1)^{-1/4}dz. \quad (23)$$

For an impulse displacement of the beam $f(\omega)$ is a constant and $h(\omega)$ is zero. We shall use the opposite extreme in the following, and let $f(\omega)$ be zero and $h(\omega)$ be constant. Since $\cos Ku = d(\sin Ku/K)/du$, we need perform the Fourier inversion of the $\sin Ku$ term only. In general, the functions are arbitrary, thus allowing for any desired initial conditions. From Eqs. (15), (17), and (22) we obtain

$$\tilde{\xi}(z,t) = \frac{(\gamma_i^2-1)^{3/8}}{(\gamma^2-1)^{1/8}}h\int_{-\infty+i\epsilon}^{\infty+i\epsilon}\exp\left[-i\omega(t-\int_0^z dz/v)\right] \cdot \frac{\sin Ku}{K}d\omega. \quad (24)$$

We introduce the following notation:

$$\tau = t - \int_0^z dz/v, \qquad \Omega_l = \omega_l[1-(1/2Q)^2]^{1/2},$$
$$\Omega = \omega + i(\omega_l/2Q), \qquad \lambda^2 = (\Omega^2-\Omega_l^2)K^2. \quad (25)$$

The singularities of the integrand in Eq. (24) occur at $\Omega = \pm\Omega_l$. These are essential singularities lying below the real axis in the Ω plane. We employ the expansion

$$\sin Ku = \sum_{s=0}^{\infty}(-1)^s\frac{(Ku)^{2s+1}}{(2s+1)!}, \quad (26)$$

and obtain

$$\frac{\sin Ku}{K} = u\sum_{s=0}^{\infty}\frac{(-1)^s(\lambda u)^{2s}}{(2s+1)!(\Omega^2-\Omega_l^2)^s}. \quad (27)$$

We can write

$$\frac{1}{(\Omega^2-\Omega_l^2)^s} = \frac{1}{(s-1)!}\left(\frac{\partial}{\partial\Omega_l^2}\right)^{s-1}\frac{1}{(\Omega^2-\Omega_l^2)}. \quad (28)$$

The $s = 0$ term in the sum will make no contribution to the integral, so we replace $s-1$ by p and write Eq. (24) in the form

$$\xi(z,t) = -uh\frac{(\gamma_i^2-1)^{3/8}}{(\gamma^2-1)^{1/8}}\sum_{p=0}^{\infty}\frac{(-1)^p(\lambda u)^{2p+2}}{p!(2p+3)!} \cdot \left(\frac{\partial}{\partial\Omega_l^2}\right)^p\int_{-\infty}^{\infty}\frac{e^{-i\omega\tau}d\omega}{(\Omega^2-\Omega_l^2)}. \quad (29)$$

The integral in Eq. (29) is easily performed. The result is

$$\int_{-\infty}^{\infty}\frac{e^{-i\omega\tau}d\omega}{(\Omega^2-\Omega_l^2)} = \begin{cases}0 & \tau<0,\\ 2\pi e^{-\omega_l\tau/2Q}(\sin\Omega_l\tau)/\Omega_l, & \tau>0.\end{cases} \quad (30)$$

The derivatives in Eq. (29) may be expressed in terms of Bessel functions of order $p+\frac{1}{2}$, but we won't do this. If we consider time τ such that $\Omega_l\tau \gg 1$ (which is satisfied for τ longer than a few periods of cavity oscillation), we can use the approximation

$$\left(\frac{\partial}{\partial\Omega_l^2}\right)^p\frac{\sin\Omega_l\tau}{\Omega_l} \simeq \frac{1}{\Omega_l}\left(\frac{w}{2\Omega_l^2}\right)^p\left(\frac{d}{dw}\right)^p\sin w, \quad (31)$$

with $w \equiv \Omega_l\tau$. This approximation retains only the term proportional to τ each time a derivative is taken. We note that

$$\left(\frac{d}{dw}\right)^p\sin w = \begin{cases}(-1)^{(p-1)/2}\cos w, & p \text{ odd},\\ (-1)^{p/2}\sin w, & p \text{ even},\end{cases}$$

and insert Eqs. (30) and (31) into Eq. (29) to obtain

$$\xi(z,t) = 2\pi h \frac{(\gamma_i^2-1)^{3/8}}{(\gamma^2-1)^{1/8}} \frac{\lambda^2 u^3}{\Omega_l} \Big\{ \cos\Omega_l\tau \sum_{p\,\mathrm{odd}} \cdot \frac{(-1)^{(p-1)/2}}{p!(2p+3)!}\left(\frac{\lambda u}{\Omega_l}\right)^{2p}\left(\frac{\Omega_l\tau}{2}\right)^p - \sin\Omega_l\tau \sum_{p\,\mathrm{even}} \frac{(-1)^{p/2}}{p!(2p+3)!}\left(\frac{\lambda u}{\Omega_l}\right)^{2p}\left(\frac{\Omega_l\tau}{2}\right)^p\Big\} e^{-\omega_l\tau/2Q}. \qquad (32)$$

The sums are functions of a single variable, namely W, defined to be

$$W = \left[\frac{\lambda u}{\Omega_l}\right]^2 \frac{\Omega_l\tau}{2}. \qquad (33)$$

With the use of Eqs. (19) and (25) we have

$$W = \frac{Ie}{mc}\left[\frac{\rho u}{bJ_0(\rho b)}\right]^2 \frac{\tau}{\Omega_l}. \qquad (34)$$

Because of the double factorial, these sums are quickly evaluated. The results are shown in Fig. 2,

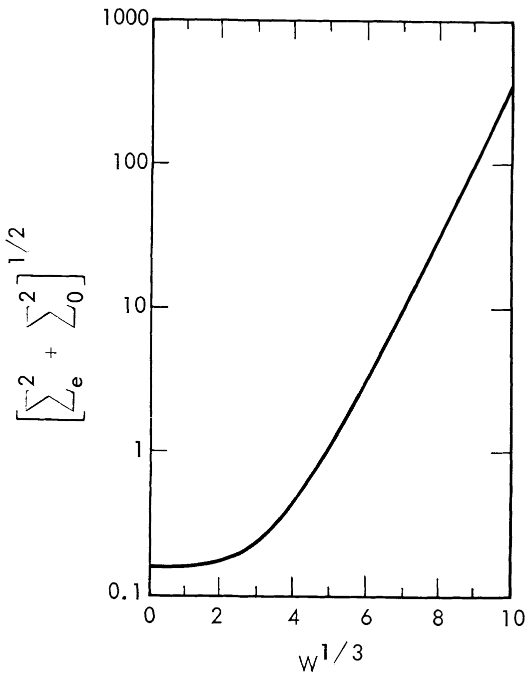

FIG. 2. Plot of $[\Sigma_e^2 + \Sigma_0^2]^{1/2}$ *vs.* $W^{1/3}$, where Σ_e and Σ_0 are respectively the even and odd sums in Eq. (32), and W is defined by Eq. (34).

and closely resemble the results in Ref. (4). The curve is valid for all values of W, but the condition $\Omega_l\tau \gg 1$ must be satisfied. The amplitude grows approximately[(6)] as $\exp[1.15(W^{1/3}-5)]$. Regardless of the energy or acceleration, at any point z the instability grows as $\tau^{1/3}$. Since λ^2 is proportional to the beam current I, the growth at any position is determined by the cube root of the total charge that has passed that point.

The growth along the beam depends upon the injection energy and the acceleration rate. For non-relativistic energies we have from Eq. (23) with $\gamma^2 - 1 = v^2/2c^2$,

$$u = 2^{1/4}c^{1/2}\int_0^z dz/v^{1/2},$$

and for constant acceleration $\dot{v}$ such that $v = \sqrt{v_i^2 + 2\dot{v}z}$,

$$u = 2^{5/4}c^{1/2}(v^{3/2} - v_i^{3/2})/3\dot{v}.$$

Thus when $2\dot{v}z \gg v_i^2$, $W^{1/3}$ varies as $z^{1/2}/\dot{v}^{1/6}$. For a coasting beam γ is constant and $W^{1/3}$ varies as $z^{2/3}/(\gamma^2-1)^{1/6}$. In the extreme relativistic range $u = (\gamma^{1/2} - \gamma_i^{1/2})/\gamma'$, and when $\gamma' z \gg \gamma_i$, $W^{1/3}$ varies as $(z/\gamma')^{1/3}$.

It is not clear that the accelerating units in the Astron injector play the role of simple cavities as characterized in this theory. We shall, however, use the dimensions of these structures for a numerical example. We take the parameters given in Table I. The value for τ is the total time for the

TABLE I

Parameters of the Astron injector

Cavity radius	$b = 30$ cm
Radial wave number	$\rho = 0.127\ \mathrm{cm}^{-1}$
Beam pulse duration	$\tau = 2 \times 10^{-7}$ sec
Initial value of γ	$\gamma_i = 2$
Final value of γ	$\gamma_f = 11$
Eigenfrequency	$\Omega_l \simeq \omega_l = c\rho$
	$= 3.81 \times 10^9\ \mathrm{sec}^{-1}$
Beam current	$I = 850$ A
Cavity quality factor	$Q = 50$

beam pulse to pass, and the value of Q is somewhat arbitrary. The accelerator is built of 23 sections each 0.51 m long. There are straight pipes and various gadgets between the sections. For this example we assume the accelerator consists of nothing but accelerating units, and has a total length of 11.7 m. This gives $\gamma' = (1/130)\ \mathrm{cm}^{-1}$. At the end of the machine after a time τ we find $W = 6.75 \times 10^4$, or $W^{1/3} \simeq 40$. Taking into account the damping factor $\exp(-\omega_l\tau/2Q)$ in Eq. (32), but ignoring the factor u^3, we find that the amplitude of the transverse oscillation has undergone thirty-three e-folds at the end of the machine after a time $\tau = 2 \times 10^{-7}$ sec. Another calculation shows five e-folds in the first section alone.

3. LONGITUDINAL INSTABILITY

The TM_{010} mode is a characteristic mode of the cavities in the chain illustrated in Fig. 1. Such an azimuthally symmetric mode will be excited by density fluctuations in the beam. The modulated beam density drives the mode by delivering energy to the axial electric field, which then reacts back on the beam in such a way as to increase the density modulation. The equation of motion is quite complicated if the beam is being accelerated, so we shall consider here only a coasting beam with $\gamma \equiv \gamma_0$.

Again neglecting the beam hole, the mode in question can be characterized by the vector potential $\mathbf{A}_l(\mathbf{r})$ given by

$$\mathbf{A}_l(\mathbf{r}) = N_l J_0(\sigma r)\hat{z}, \tag{35}$$

with $J_0(\sigma b) = 0$. With the normalization of Eq. (3), we have

$$N_l^2 = 4c^2/Lb^2 J_l^2(\sigma b). \tag{36}$$

We consider an element of the beam to be displaced a distance η in the z direction, where η is uniform over the beam cross-section. This displacement gives rise to a first order charge density ρ_1 and current density j_{1z} given by

$$\rho_1 = -\rho_0\, \partial\eta/\partial z, \tag{37a}$$

$$j_{1z} = \rho_0\, \partial\eta/\partial t. \tag{37b}$$

Assuming the beam radius is such that $J_0(\sigma r) \simeq 1$ over the beam cross-section, we have

$$\frac{1}{c}\int \mathbf{j}\cdot\mathbf{A}_l\, dV = \frac{\Lambda}{c} N_l L \frac{\partial\eta}{\partial t}, \tag{38}$$

where Λ is the charge per unit length in the beam. The total velocity v_t, of a particle is $v_t = v + d\eta/dt$.

If the particle energy does not vary with z, the first order equation of motion is simply

$$\frac{d}{dt}(\gamma v_t) \equiv \gamma_0^3\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial z}\right)^2 \eta = \frac{eE_z}{m}, \tag{39}$$

with E_z given by

$$E_z = -\frac{1}{c}\frac{dq}{dt}A_{lz} = -\frac{N_l}{c}\frac{dq}{dt}. \tag{40}$$

In Eq. (40) we have again used the approximation $J_0(\sigma r) = 1$. There is a complication with regard to initial conditions in this calculation that was not present in the previous section. The complication comes about because the front of the beam will excite this azimuthally symmetric mode. The calculation of the subsequent behavior of the system is therefore not a linear problem. We will circumvent this state of affairs by pretending that there isn't any head to the beam. The equilibrium state is one where the beam is uniform in z and on axis until time $t = 0$, at which time an impulse variation in charge density occurs at $z = 0$, and proceed as in Section 2.

We insert Eq. (38) into Eq. (6) and Eq. (40) into Eq. (39), and take the Fourier time transform to obtain

$$\tilde{q} = -i\omega\tilde{\eta}(\Lambda N_l L/c)\,[\omega_l^2 - \omega^2 - i(\omega\omega_l/Q)]^{-1}. \tag{41}$$

$$\left(\frac{\partial}{\partial z} - i\frac{\omega}{v}\right)^2 \tilde{\eta} + K^2\tilde{\eta} = 0, \tag{42}$$

with K^2 defined by

$$K^2 = \frac{4\omega^2 e\Lambda}{\gamma_0^3 m[vbJ_1(\sigma b)]^2}\left(\omega^2 - \omega_l^2 + i\frac{\omega\omega_l}{Q}\right)^{-1}. \tag{43}$$

In this section the symbol ω_l is used for the eigenfrequency of the mode with vector potential given by Eq. (35) and is simply $\omega_l = c\sigma$. The symbol Q also refers to the quality factor of that mode, and the symbol Ω_l is the frequency analogous to that given in Eq. (25). An impulse perturbation in charge density corresponds to a $\partial\eta/dz$ at $z = 0$ that is a delta function in time. ($\partial\tilde{\eta}(0,\omega)/\partial z = P$, a constant.) With this initial condition the solution to Eq. (42) is

$$\tilde{\eta} = P\, e^{i\omega z/v}(\sin Kz)/K. \tag{44}$$

The fact that K^2 has an ω^2 in the numerator makes the Fourier inversion a bit more tedious than that in Section 2. However, if we adopt the same approximation that $\Omega_l\tau \gg 1$, we may set $\omega^2 = \Omega_l^2$ in the numerator of K^2. The mathematical details are then just the same as those in Section 2, and the results are identical, but with the quantity W for this mode (which we designate W_η) given by

$$W_\eta = \frac{2e\Lambda z^2\Omega_l\tau}{\gamma_0^3 m[vbJ_1(\sigma b)]^2}. \tag{45}$$

It is significant to compare W_η to W given by Eq. (34). In the latter we set $u = z/\gamma_0^{1/2}$ appropriate for a coasting beam. For relativistic particles we may set $v = c$ and $I = c\Lambda$. Using the appropriate values for the two Ω_l's, we find (hereinafter we use $\omega_l \simeq \Omega_l$)

$$\frac{W_\eta}{W} = \frac{2}{\gamma_0^2}\frac{\sigma}{\rho}\left[\frac{J_0(\rho b)}{J_1(\sigma b)}\right]^2. \tag{46}$$

The ratio of σ/ρ is $2.41/3.83 = 0.645$, and the ratio $[J_0(\rho b)/J_1(\sigma b)]^2$ is 0.618. Thus $W_\eta/W = 0.78/\gamma_0^2$. From this ratio one might conclude that the longitudinal instability is of less concern than the transverse. Certainly the γ_0^{-2} is significant. But we have assumed that the structure will support either mode equally well. In any practical structure this may not be so.

4. DAMPING MECHANISMS FOR TRANSVERSE MOTION

With the use of the Vlasov equation, we now determine stability criteria. These criteria will place lower limits on the spread in axial velocity and/or transverse oscillation frequency necessary to completely quench the instability treated in Section 2. We treat only a coasting beam.

It is convenient to introduce the four-velocity

$$u_\mu = (\gamma \mathbf{v}/c, i\gamma), \tag{47}$$

where γ may be expressed in terms of u as $\gamma^2 = 1 + \mathbf{u}^2$. Let $\Psi(x, y, z, u_x, u_y, u_z, t)$ be the distribution function for particles in the beam. The relativistic Vlasov equation then takes the form

$$\frac{\gamma}{c}\frac{\partial \Psi}{\partial t} + \mathbf{n}\cdot\nabla\Psi + \frac{e(\gamma\mathbf{E} + \mathbf{u}\times\mathbf{B})}{mc^2}\cdot\nabla_u\Psi = 0. \tag{48}$$

As we are considering only forces in the x and z directions, one may integrate Eq. (48) over y and u_y and define

$$\psi(x, z, u_x, u_z, t) = \int \Psi \, du_y \, dy. \tag{49}$$

We assume a solution of the form

$$\psi(x, z, u_x, u_z, t) = \psi_0(x, u_x, u_z) + \psi_1(x, u_x, u_z)e^{i(kz - \omega t)} \tag{50}$$

and solve Eq. (48) to first order in ψ_1. We will obtain a dispersion relation of the form $D(\omega, k) = 0$. If it is found that for all real k, $-\infty < k < \infty$, there are no roots in the upper half ω plane, then the system will not grow in space or time. Throughout this section the factor $\exp[i(kz - \omega t)]$ is understood to be appended to all first order quantities.

The linearized form of Eq. (48) is, if $\Psi \propto \delta(u_y)$,

$$-i\left(\frac{\gamma\omega}{c} - ku_z\right)\psi_1 + u_x\frac{\partial\psi_1}{\partial x} - \frac{e}{mc^2}u_z B_{0y}\frac{\partial\psi_1}{\partial u_x} = -\frac{e\gamma}{mc^2}E_{1z}\frac{\partial\psi_0}{\partial u_z} + \frac{eu_z}{mc^2}B_{1y}\frac{\partial\psi_0}{\partial u_x}. \tag{51}$$

The quantity B_{0y} is the externally applied magnetic focusing field, and is assumed to exist along the entire length of the beam. This is not true in any linear accelerator, but the model for transverse motion employed below will give some idea of the amount of focusing needed to provide stability. The quantities E_{1z} and B_{1y} must be found self-consistently from the first order distribution function ψ_1 and Eq. (6). The current density j_z is given by

$$j_z = ec\int \frac{\Psi}{\gamma} u_z \, d\mathbf{u}.$$

With the approximation $A_{lz} = N_l \rho x/2$ for the transverse mode we have

$$\frac{1}{c}\int \mathbf{j}\cdot\mathbf{A}_l \, dV = \frac{e}{2}N_l\rho\int x\,dx\,dy\,dz\int\frac{\Psi}{\gamma}u_z\,d\mathbf{u}. \tag{52}$$

We now assume that u_x and u_y are very much less than u_z for all particles in the beam and we may write $\gamma^2 \simeq 1 + u_z^2$, and we consider $kL \ll 1$. Thus we obtain, since ψ_0 is assumed even in x,

$$\frac{1}{c}\int \mathbf{j}\cdot\mathbf{A}_l \, dV = eN_l\rho LY/2, \tag{53}$$

in which the quantity Y is defined by

$$Y \equiv \int \frac{\psi_1}{\gamma} x u_z \, du_z \, du_x \, dx. \tag{54}$$

We have from Eqs. (6) and (53)

$$q = \frac{eN_l\rho LY}{2(\Omega_l^2 - \Omega^2)}, \tag{55}$$

with Ω and Ω_l defined by Eq. (25). From Eqs. (4) and (55) we obtain

$$B_{1y} = -\frac{2eYc^2[\rho/bJ_0(\rho b)]^2}{(\Omega_l^2 - \Omega^2)}. \tag{56}$$

The term containing E_{1z} in Eq. (51) is present, but merely complicates the algebra without altering the results significantly. We shall neglect it in this treatment of the transverse instability.

Our model for the transverse (x) motion in the unperturbed state will be the following: The externally applied focusing gives rise to motion in the x direction of the form

$$x = a\sin(\dot{\phi}t + \phi_0),$$
$$u_x = \frac{\gamma a}{c}\dot{\phi}\cos(\dot{\phi}t + \phi_0). \tag{57}$$

The quantity $\dot{\phi}$ will in general be a function of u_z, but we shall neglect this dependence and concentrate on the dependence of $\dot{\phi}$ on the amplitude, a, of transverse oscillations. The dependence on a arises from the non-linearities in the focusing magnets. We shall, in fact, consider $\dot{\phi}$ a constant everywhere except in a certain resonant denominator. With these approximations and assumptions, the amplitude, a, is a constant of the motion, namely

$$a^2 = x^2 + (cu_x/\gamma\dot{\phi})^2. \tag{58}$$

We choose ψ_0 to be of the form

$$\psi_0(x, u_x, u_z) = G(u_z)H(a), \tag{59}$$

with normalization such that

$$I = ec\int \frac{\psi_0}{\gamma} u_z\, du_z\, du_x\, dx.$$

Transforming to the variable a and $\phi \equiv \dot{\phi}t + \phi_0$ this expression becomes

$$I = -e\dot{\phi}\int \psi_0 u_z a\, da\, du_z\, d\phi. \tag{60}$$

With the use of Eqs. (58) and (59) we have

$$\frac{\partial\psi_0}{\partial u_x} = \frac{\cos\phi}{\mu} G \frac{dH}{da}, \tag{61}$$

with $\mu \equiv \gamma\dot{\phi}/c$. Using Eqs. (56), (57) and (61) together with the relation $\gamma\, du_x/c\, dt = -eu_z B_{0y}/mc^2$, we may write Eq. (51) in the form

$$-i\left(\frac{\gamma}{c}\omega - ku_z\right)\psi_1 + \mu\frac{\partial\psi_1}{\partial\phi} = -\frac{2e^2 Yu_z(\rho/bJ_0)^2}{m\mu(\Omega_l^2 - \Omega^2)} \cdot \cos\phi\, G\frac{dH}{da}. \tag{62}$$

The expression for ψ_1 that satisfies Eq. (62) has one term ψ_{1c}, proportional to $\cos\phi$ and another term, ψ_{1s}, proportional to $\sin\phi$. Only the latter contributes to the integral in Eq. (54). This term is given by

$$\psi_{1s} = \frac{2e^2 Yu_z[\rho/bJ_0]^2}{m(\Omega_l^2 - \Omega^2)} \frac{\sin\phi\, G(dH/da)}{\{[(\gamma\omega/c) - ku_z]^2 - \mu^2\}}. \tag{63}$$

We complete the self-consistent treatment by inserting this expression into Eq. (54), transforming to the variables a and ϕ, and performing the integration over ϕ. The resulting dispersion relation is

$$1 = \frac{\pi e K^2\dot{\phi}}{I}\int \frac{G(dH/da)u_z^2 a^2\, du_z\, da}{\{[(\gamma\omega/c) - ku_z]^2 - \mu^2\}}, \tag{64}$$

with K^2 defined by Eq. (19).

First we consider the effect of momentum spread in the absence of focusing, and choose

$$\dot{\phi}H(a) = 4\delta(a^2),$$

$$G(u_z) = -\frac{(I\epsilon/2\pi^2 eu_0)}{(u_z - u_0)^2 + \epsilon^2}. \tag{65}$$

These expressions are consistent with the normalization condition, Eq. (60), since

$$\int_0^\infty \delta(a^2)a\, da = \frac{1}{2}\int_0^\infty \delta(a^2)\, d(a^2) = \frac{1}{4}.$$

We note also that $\dot{\phi}\int(dH/da)a^2 da = -2$. Furthermore, we set $\mu = 0$ and replace u_z^2 by u_0^2 in the numerator of the integrand in Eq. (64). The presence of γ in the integrand creates branch points in the complex u_z plane, but we circumvent that difficulty by introducing the quantity $U = u_z - u_0$ and expanding γ to first order in U. Defining $\gamma_0^2 \equiv 1 + u_0^2$, we obtain

$$(\gamma\omega/c) - ku_z \simeq (\gamma_0\omega/c) - ku_0 + U[(u_0\omega/\gamma_0 c) - k]. \tag{66}$$

Since ω is assumed to have a positive imaginary part, the integrand has a double pole in the upper half U plane in addition to the simple poles at $U = \pm i\epsilon$. The integral is easily performed by contour integration, and the resulting dispersion relation is

$$1 = K^2 u_0\{(\gamma_0\omega/c) - ku_0 - i\epsilon[(u_0\omega/\gamma_0 c) - k]\}^{-2}. \tag{67}$$

After some rearrangement, this expression takes the form (to first order in ϵ)

$$1 = (\lambda^2 v_0 c/\gamma_0)(\Omega^2 - \Omega_l^2)^{-1}[\omega - kv_0 + i(kc\epsilon/\gamma_0^3)]^{-2}, \tag{68}$$

in which $v_0 \equiv cu_0/\gamma_0$, and λ^2 is defined by Eq. (25).

The detailed analysis of this 4th order equation is pretty tedious, and only the results will be given. In the limit $\epsilon \to 0$ and $Q \to \infty$ there are two real values of ω for all real values of k. The other two roots form a complex conjugate pair if $kv_0 \simeq \Omega_l$. With finite Q and ϵ the value of k which requires the most stringent conditions on Q and ϵ is found from the relation

$$(kc\epsilon/\gamma_0^3) - (\omega_l/2Q) = \sqrt{3}(kv_0 - \Omega_l).$$

Stability of a perturbation of this wavelength (and thus of all others) is attained if the following condition is satisfied:

$$\frac{8}{Q}\left(\frac{kc\epsilon}{\gamma_0^3}\right)^2 > 3^{3/2}\frac{\lambda^2 v_0 c}{\gamma_0\Omega_l^2}, \tag{69}$$

in which we shall use $k = \Omega_l/v_0$. The physical significance of $c\epsilon/\gamma_0^3$ is that it represents the spread in axial velocity Δv arising from a spread $\Delta p = c\epsilon/m$ in axial momentum. Only the spread in *velocity* is instrumental in suppressing the instability. Condition (69) may be re-written as

$$\frac{8}{Q}\left(\frac{\Delta v}{v_0}\right)^2 > 3^{3/2}\frac{\lambda^2 v_0 c}{\gamma_0\Omega_l^4}. \tag{70}$$

We note also that *both* a velocity spread and a finite Q are necessary. Neither alone is sufficient.

For the parameters of the Astron injector as given by Table I, we have $\lambda^2 v_0 c/\gamma_0 \Omega_l^4 = 4.25 \times 10^{-2} v_0/c\gamma_0$. Applying the condition (70) at injection ($\gamma_0 = 2$, $v_0 = \sqrt{3}c/2$) we find the condition to be

$$(\Delta v/v_0)^2 > 1.2 \times 10^{-2} Q.$$

Even for a Q as low as 50, this criterion requires that $\Delta v \simeq v_0$. What we have done is examine the stabilizing influence of a small velocity spread (indeed, our entire treatment is limited to $\Delta v/v_0 \ll 1$) and found that influence negligible for the parameters of the Astron injector. The condition (70) is a valid one only if it results in a spread of no more than a few percent.

We now turn to the stabilizing effect of a spread in the transverse oscillation frequency $\dot{\phi}$. For purposes of this investigation, we neglect the spread in u_z and choose $G(u_z) = -I\delta(u_z - u_0)/2\pi e u_0$. After some rearrangement, Eq. (64) takes the form

$$1 = \frac{-K^2 v_0 c}{2\gamma_0} \dot{\phi} \int \frac{(dH/da) a^2\, da}{[(\omega - kv_0)^2 - \dot{\phi}^2]}. \qquad (71)$$

It is convenient to introduce $\zeta \equiv a^2$ and choose, as an illustration, a function H given by

$$\dot{\phi} H(\zeta) = 4(\zeta_m - \zeta)/\zeta_m^2 \quad \zeta < \zeta_m, \qquad (72)$$
$$= 0 \quad \zeta > \zeta_m.$$

We take the dependence of $\dot{\phi}$ on ζ to be

$$\dot{\phi} = \dot{\phi}_0 - s\zeta. \qquad (73)$$

Depending upon the configuration of the focusing magnets, s may be positive or negative, but we consider here only $s > 0$. Qualitatively the results do not depend on the sign of s, nor on the exact form of H.

The unstable root lies near $\omega = kv_0 - \dot{\phi}$, and in the absence of damping, the value of k that leads to the largest $\mathrm{Im}\,\omega$ is found from $kv_0 - \dot{\phi} = \Omega_l$. We make the approximation

$$(\omega - kv_0)^2 - \dot{\phi}^2 \simeq -2\dot{\phi}_0(\omega - kv_0 + \dot{\phi}), \qquad (74)$$

and define

$$\zeta_1 = (\omega - kv_0 + \dot{\phi}_0)/s. \qquad (75)$$

With the use of Eqs. (72)–(75), Eq. (71) takes the form

$$\frac{s\zeta_m \gamma_0 \dot{\phi}_0}{\lambda^2 v_0 c}(\Omega^2 - \Omega_l^2) = \frac{1}{\zeta_m}\int_0^{\zeta_m} \frac{\zeta\, d\zeta}{(\zeta - \zeta_1)}. \qquad (76)$$

We use one further approximation, namely

$$\Omega^2 - \Omega_l^2 \simeq 2\omega_l[\omega - \omega_l + i(\omega_l/2Q)], \qquad (77)$$

and seek the condition on $s\zeta_m$ such that ω and (thus ζ_1) lies on the real axis. In accordance with Eqs. (14) and (15), ζ_1 approaches the real axis from above, so that we may write

$$\frac{1}{\zeta_m}\int_0^{\zeta_m} \frac{\zeta\, d\zeta}{\zeta - \zeta_1} = 1 + \frac{\zeta_1}{\zeta_m} \ln\left(\frac{\zeta_m}{\zeta_1} - 1\right) + i\pi \frac{\zeta_1}{\zeta_m}. \qquad (78)$$

With Eqs. (77) and (78) inserted into Eq. (76), the imaginary part of that equation yields

$$\frac{s\zeta_m \dot{\phi}_0 \gamma_0 \omega_l^2}{\pi \lambda^2 v_0 c Q} = \frac{\zeta_1}{\zeta_m}. \qquad (79)$$

Equation (79) is in fact the stability criterion. It remains only to find the exact ratio ζ_1/ζ_m from the real part of Eq. (76). It is not necessarily true that $kv_0 = \omega_l + \dot{\phi}_0$ gives the most stringent condition, but it should be quite a good approximation. We therefore have $\omega - \omega_l = s\zeta_1$. We introduce the quantity C by the definition

$$C = 2(\pi Q \lambda)^2 v_0 c/\gamma_0 \omega_l^3 \dot{\phi}_0, \qquad (80)$$

and write the real part of Eq. (76) in the form

$$C\left(\frac{\zeta_1}{\zeta_m}\right)^3 = 1 + \frac{\zeta_1}{\zeta_m} \ln\left(\frac{\zeta_m}{\zeta_1} - 1\right), \qquad (81)$$

in which we have also used Eq. (79). The coefficient C on the left is known in terms of the parameters of the accelerator. We simply put in the appropriate value and solve Eq. (81) for the value of ζ_1/ζ_m to put into Eq. (79). A plot of ζ_m/ζ_1 vs. C is shown in Fig. 3. For $C < 10^{-1}$, ζ_m/ζ_1 has a value very

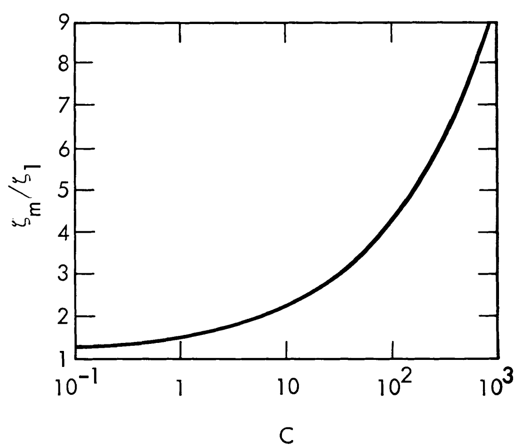

FIG. 3. Plot of the value of ζ_m/ζ_1 that satisfies Eq. (81) *vs.* the constant C.

near 1.3. For $C = 8$, $\zeta_m/\zeta_1 = 2$. For values of $C \gtrsim 8$, the approximation $\zeta_m/\zeta_1 = C^{1/3}$ should give results well within the accuracy of the theory, which is limited by the choice of the distribution function.

As a numerical example, we apply the criterion to the Astron injector at full energy where $v_0 \simeq c$ and $\gamma_0 = 10$. It is not clear what equivalent betatron frequency $\dot{\phi}_0$ is provided by the system of focusing magnets in the accelerator, but we will seek the spread $s\zeta_m$ necessary if the condition $\dot{\phi}_0 = \omega_l/2$ were to hold. This corresponds to an equivalent betatron wavelength ($\lambda_\beta = 2\pi c/\dot{\phi}_0$) of 100 cm. For the parameters listed in Table I, we find $C = 425$, so that the approximation $\zeta_m/\zeta_1 = C^{1/3}$ is quite valid. Inserting this expression into Eq. (79) yields the criterion

$$\frac{s\zeta_m}{\omega_l} = \left(\frac{\pi Q}{2}\right)^{1/3}\left(\frac{\omega_l}{\dot{\phi}_0}\,\frac{\lambda^2 v_0 c}{\gamma_0 \omega_l^4}\right)^{2/3}. \qquad (82)$$

Numerically we have $s\zeta_m/\omega_l = 0.182$, thus the spread ($s\zeta_m$) in $\dot{\phi}$ must be $0.36\,\dot{\phi}_0$. Qualitatively the condition (82) is correct. However, we should not take the form of Eq. (82) too literally, as it is dependent on the exact form of the distribution function.

Let us consider as a second example the distribution function

$$\dot{\phi} H(\zeta) = (2/\zeta_0)\exp(-\zeta/\zeta_0). \qquad (83)$$

This distribution function leads to two relations analogous to Eqs. (79) and (81). These relations are

$$\frac{2s\zeta_0\dot{\phi}_0\gamma_0\omega_l^2}{\pi\lambda^2 v_0 c Q} = \frac{\zeta_1}{\zeta_0}\, e^{-(\zeta_1/\zeta_0)}, \qquad (84)$$

$$C = 2(\zeta_0/\zeta_1)^3 e^{2\zeta_1/\zeta_0}[1 - (\zeta_1/\zeta_0)e^{-\zeta_1/\zeta_0}\, Ei(\zeta_1/\zeta_0)]. \qquad (85)$$

In Eq. (85) the function Ei is the exponential integral defined by

$$Ei(x) = -\fint_{-x}^{\infty} \frac{e^{-t}}{t}\, dt,$$

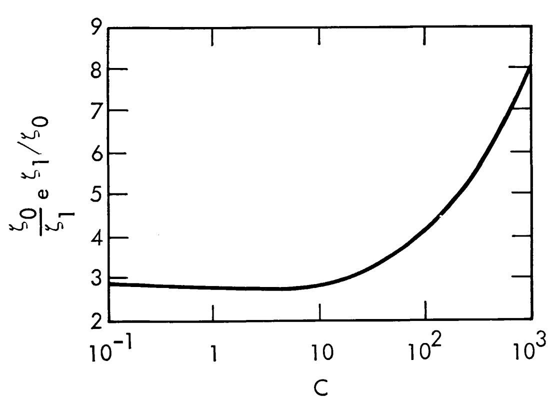

FIG. 4. Plot of the value of $(\zeta_0/\zeta_1)\exp(\zeta_1/\zeta_0)$ that satisfies Eq. (85) *vs.* the constant C.

in which the bar across the integral sign signifies the principal part. A graph of the solution to Eq. (85) as a function of C is given in Fig. 4. It is apparent that for all values of C the stability criteria derived from the two distribution functions are qualitatively the same.

Let us examine the dispersion relation in the absence of damping by setting $s = 0$ in Eq. (71). With the approximations of Eqs. (74) and (77), and recalling that $\dot{\phi}\int H' a^2 da = -2$, we obtain

$$\left(\omega - \omega_l + \frac{i\omega_l}{2Q}\right)(\omega - kv_0 + \dot{\phi}) + \frac{\lambda^2 v_0 c}{4\gamma_0\omega_l\dot{\phi}_0} = 0. \qquad (86)$$

The maximum value of $\mathrm{Im}\,\omega$ occurs when $kv_0 - \dot{\phi} = \omega_l$, and is given by

$$\mathrm{Im}\,\omega = \frac{\omega_l}{4Q}\left[1 + \frac{4Q^2\lambda^2 v_0 c}{\gamma_0\omega_l^3\dot{\phi}}\right]^{1/2} - \frac{\omega_l}{4Q}. \qquad (87)$$

The second term under the square root is just $2C/\pi^2$, and if that term is very much less than unity, the growth rate is simply

$$\mathrm{Im}\,\omega \simeq \frac{Q\lambda^2 v_0 c}{2\omega_l^2\dot{\phi}\gamma_0}.$$

In this limit the above treatment yields $\zeta_1/\zeta_m = 0.77$ and Eq. (79) requires that the spread in $\dot{\phi}$ must be of the order of $\mathrm{Im}\,\omega$. Physically this means that a typical particle in the beam must get out of phase with the wave in one growth time. In the limit $C \gg 1$, the growth rate becomes independent of Q, but the stability criterion does not. A finite Q, as well as a spread in $\dot{\phi}$, is necessary for stability.

Although the structures considered above are highly idealized, the results indicate that it is quite difficult to stabilize a beam with current of the order of 1000 A in the presence of resonant structures. Although we have used the parameters of the Astron injector in numerical examples throughout this work, it must be pointed out that the results of our theory are somewhat too pessimistic to agree with the observed phenomena in that accelerator. With careful 'tuning' of the focusing magnets, the Astron injector *will* accelerate a beam carrying a current of the order of 800 A, even though transverse oscillations of the beam are observed somewhere along the machine under all conditions.

It is possible that a spread in betatron frequency, of the order of the value required for stability, is present in the beam. Strictly speaking, we have considered only the spread resulting from nonlinearities in the focusing magnets, and it is doubtful that this alone would be sufficient. There is another source of spread in an intense beam, namely the effect of the coherent electromagnetic self-fields of the beam. The inclusion of the stabilizing effect of this spread is consistent with our theory. Just how much spread arises depends upon the radial density of particles in the beam (a radially uniform density gives no spread), and the degree of cancellation between electric and magnetic radial forces, neither of which are known.

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REFERENCES

1. R. Helm, 'Computer Study of Wave Propagation, Beam Loading and Beam Blowup in the SLAC Accelerator,' *Proc. Lin. Acc. Conference*, *Los Alamos*, *New Mexico*, p. 254 (1966).
2. O. H. Altenmueller, *et al.*, 'Beam Break-Up Experiments at SLAC,' *Proc. Lin. Acc. Conference*, *Los Alamos*, *New Mexico*, p. 267 (1966).
3. E. V. Farinholt, *et al.*, 'Progress Report on Beam Break-Up at SLAC,' *Sixth International Conference on High Energy Accelerators*, *Cambridge*, *Mass.*, p. 90 (1967).
4. W. K. H. Panofsky and M. Bander, *Rev. Sci. Instr.*, **39,** 206 (1968).
5. J. W. Beal, N. C. Christofilos and R. E. Hester, 'The Astron Linear Accelerator,' *IEEE Trans.*, **NS-16,** No. 3, 294 (1969).
6. Steepest descent calculations indicate a growth of $\exp 1.64W^{1/3}$ for very large values of W. Our numerical evaluation shows that at $W = 10^6$ the growth is approximately $\exp 1.60W^{1/3}$.

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