Impact of cosmic neutrinos on the gravitational-wave background

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We obtain the equation governing the evolution of the cosmological gravitational-wave background, accounting for the presence of cosmic neutrinos, up to second order in perturbation theory. In particular, we focus on the epoch during radiation dominance, after neutrino decoupling, when neutrinos yield a relevant contribution to the total energy density and behave as collisionless ultrarelativistic particles. Besides recovering the standard damping effect due to neutrinos, a new source term for gravitational waves is shown to arise from the neutrino anisotropic stress tensor. The importance of such a source term, so far completely disregarded in the literature, is related to the high velocity dispersion of neutrinos in the considered epoch; its computation requires solving the full second-order Boltzmann equation for collisionless neutrinos.

I. INTRODUCTION

An important discriminator among different models for the generation of the primordial density perturbations is the level of the gravitational-wave background predicted by these models. For example, within the inflationary scenario the tensor (gravitational-wave) amplitude generated by tiny initial quantum fluctuations during the accelerated inflationary expansion of the Universe depends on the energy scale at which this inflationary period took place, and it can widely vary among different inflationary models [1,2]. On the other hand, some alternative scenarios, such as the curvaton model, typically predict an amplitude of primordial tensor modes that is far too small to be ever detectable by future satellite experiments aimed at observing the *B* mode of the cosmic microwave background (CMB) polarization.

There is however another background of stochastic gravitational waves of cosmological origin. Gravitational waves (as well as vector modes) are inevitably generated at second order in perturbation theory by scalar density perturbations [3–7]. This is due to the fact that the nonlinear evolution always involves quadratic source terms for tensor (and vector) perturbation modes made of linear scalar (density) perturbations.

Since the level of density perturbations is well determined by CMB anisotropy measurements and large-scale structure observation [8,9], we know that these secondary vector and tensor modes (produced after the primordial curvature perturbations have been generated) must exist and their amplitude must have a one-to-one relation with the level of density perturbations. In this sense, the scalar-induced contribution can be computed directly from the observed density perturbations and general relativity, and is independent of the specific cosmological model for generating the perturbations.[1]

Such a background of gravitational waves could be interesting in relation to future high-sensitivity CMB polarization experiments or for small-scale direct detectors, such as the space-based laser interferometer Big Bang Observer and the Deci-hertz Interferometer Gravitational Wave Observatory operating in the frequency range $0.1 - 1$ Hz [11] with an improved sensitivity (in terms of the closure energy density of gravitational waves (GW), $\Omega_{GW} \sim 10^{-17} - 10^{-15}$). In particular, in Ref. [12] the effects of secondary tensor and vector modes on the

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[1]See, however, Ref. [10], where in the context of the curvaton mechanism, second-order gravitational waves can be produced when the perturbations are still of isocurvature nature, thus resulting to be strongly model dependent.

large-scale CMB polarization have been computed, showing that they dominate over the primordial gravity-wave background if the tensor-to-scalar perturbation ratio on large scales is $r < 10^{-6}$. More recently, Ref. [13] computed the power spectrum of the secondary tensors accounting for their evolution during the radiation-dominated epoch, to see their effects on the scales relevant for small-scale direct detectors, and Ref. [14] extended this analysis by accounting for a more detailed study of the transfer function for the secondary tensor modes.

In this paper we consistently account for the presence of cosmic neutrinos to analyze their impact on the evolution of the second-order gravitational-wave background. At linear order it has been shown that there is a damping effect due to the anisotropic stress of free-streaming neutrinos that strongly affects the primordial gravitational-wave background on those wavelengths which enter the horizon during the radiation-dominated epoch (at the level of 30%) [15–21] (see also Ref. [22]). At second order, along with the analogous damping effect, we find that free-streaming neutrinos are an important source for the second-order gravitational-wave background during the radiation-dominated epoch. We find completely new source terms, arising because of the fact that neutrinos give a relevant contribution to the total energy density during this epoch and they behave as ultrarelativistic collisionless particles after their decoupling: their high velocity dispersion acts as an extra source for the second-order gravitational waves. To compute such a contribution we evaluate the second-order tensor part of the neutrinos' anisotropic stress tensor, that has been neglected so far. This is achieved by computing and solving the Boltzmann equation for neutrinos. Approximating the neutrino contribution as a perfect fluid of relativistic particles during the radiation era leads to a serious underestimate of their role. Let us stress that the new contribution is at least of the same order of magnitude as that computed by adopting a fluid treatment in the source of the scalar-induced gravitational waves. Moreover it has a clear physical interpretation. It arises in the Boltzmann equation from a "lensing" effect of the neutrinos as they travel through the inhomogeneities of the gravitational potential.

The paper is organized as follows. In Secs. II A and 3 we derive the evolution equation for the tensor (gravitational-wave modes) at second order, accounting for photons and neutrinos. In Sec. 4 we present the Boltzmann equation for neutrinos approximated as being collisionless massless particles (see Appendix A, B, and C for details about the Boltzmann equation for massive neutrinos) and give an integral solution for it. Section 6 contains the computation needed to determine the tensor part of the second-order anisotropic stress tensor of neutrinos, which leads to one of our main results, Eq. (52). Finally, in Sec. IV we derive the photon contribution, consistently accounting for the presence of neutrinos, which leads also to a new source of

gravitational waves. In Sec. V we present the summary and our main conclusions.

II. SECOND-ORDER GRAVITATIONAL WAVES

A. Metric perturbations in the Poisson gauge

The second-order metric perturbations around a flat Friedmann-Robertson-Walker background can be described by the line element in the Poisson gauge

$$ds^2 = a^2(\tau)[-e^{2\Phi}d\tau^2 + 2\omega_i dx^i d\tau$$
$$+ (e^{-2\Psi}\delta_{ij} + \chi_{ij})dx^i dx^j]. \quad (1)$$

In this gauge one scalar degree of freedom is eliminated from g_{0i} and one scalar and two vector degrees of freedom are removed from g_{ij}. As usual $a(\tau)$ is the scale factor and τ is the conformal time. The functions Φ and Ψ are scalar functions which correspond to the Newtonian potential and to the spatial curvature perturbations, respectively. Within the second-order perturbation theory, they consist in the sum of a linear and a second-order term, such that Φ and Ψ can be written as

$$\Phi = \Phi^{(1)} + \Phi^{(2)}/2 \quad \text{and} \quad \Psi = \Psi^{(1)} + \Psi^{(2)}/2. \quad (2)$$

Since the choice of the exponentials greatly helps in simplifying the computation of many expressions, they will be kept where it is convenient. It is worth remarking here that all the equations in the following where the exponential show up are meant to be second-order equations, therefore the exponentials are to be thought as implicitly truncated up to second order in all these expressions, e.g., $e^{2\Phi} \simeq 1 + 2\Phi^{(1)} + \Phi^{(2)} + 2(\Phi^{(1)})^2$.

The remaining functions that appear in Eq. (1) account for second-order vector (ω_i) and tensor (χ_{ij}) modes. Tensor perturbations are traceless and transverse: $\chi_i^i = 0$, $\partial_i \chi^{ij} = 0$ and vectors have vanishing spatial divergence: $\partial^i \omega_i = 0$. Linear vector modes have been neglected as they are not produced by standard mechanisms, such as inflation, that generate cosmological perturbations [23,24]. As discussed, for example, in Refs. [23,24], indeed linear vector modes have decreasing amplitudes and they are not generated in the presence of scalar fields, while the first-order tensor part gives a negligible contribution to second-order perturbations. Second-order vector and tensor modes however must be taken into account, even if they were initially zero. This is because scalar, vector, and tensor modes are dynamically coupled at this stage and second-order vectors and tensors are generated by first-order scalar mode-mode coupling. First-order perturbations behave as a source for the intrinsically second-order fluctuations [5].

Since our main task is to provide the second-order Einstein's equations that describe the evolution of tensor modes, we are interested in the spatial components of both the Einstein and the energy-momentum tensors. Here we find that, using the Christoffel symbols obtained in

Appendix A and accounting only for the terms up to second order, the spatial Einstein tensor reads

$$
\begin{aligned}
G^i_j = \frac{1}{a^2}\bigg[& e^{-2\Phi}\bigg(\mathcal{H}^2 - 2\frac{a''}{a} - 2\Psi'\Phi' - 3(\Psi')^2 \\
& + 2\mathcal{H}(\Phi' + 2\Psi') + 2\Psi''\bigg) + e^{2\Psi}(\partial_k\Phi\partial^k\Phi \\
& + \nabla^2\Phi - \nabla^2\Psi)\bigg]\delta^i_j + \frac{e^{2\Psi}}{a^2}(-\partial^i\Phi\partial_j\Phi - \partial^i\partial_j\Phi \\
& + \partial^i\partial_j\Psi - \partial^i\Phi\partial_j\Psi + \partial^i\Psi\partial_j\Psi - \partial^i\Psi\partial_j\Phi) \\
& - \frac{\mathcal{H}}{a^2}(\partial^i\omega_j + \partial_j\omega^i) - \frac{1}{2a^2}(\partial^i\omega'_j + \partial_j\omega^{i'}) \\
& + \frac{1}{a^2}\bigg(\mathcal{H}\chi^{i'}_j + \frac{1}{2}\chi^{i''}_j - \frac{1}{2}\nabla^2\chi^i_j\bigg),
\end{aligned} \tag{3}
$$

where $\mathcal{H} = a'/a$, and a prime denotes differentiation with respect to conformal time.

B. Second-order gravitational-wave evolution equation during the radiation-dominated era

In Fourier space the equation which describes the evolution of second-order GW can be put in the form

$$
\chi''_{k,\lambda} + 2\mathcal{H}\chi'_{k,\lambda} + k^2\chi_{k,\lambda} = 16\pi G a^2 S_{k,\lambda}, \tag{4}
$$

where the subscript λ refers to the two possible polarization states of a gravitational wave. Each mode χ_k is in fact transverse with respect to the direction along which it propagates and, for a mode traveling in the z direction, χ_{ij} can be written as:

$$
\chi_{ij} = \begin{pmatrix} \chi_+ & \chi_\times & 0 \\ \chi_\times & -\chi_+ & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{5}
$$

The 2 degrees of freedom account for the two polarization states χ_+ and $\chi_\times$.

The source term $S_{k,\lambda}$ for GW in the radiation era consists in the sum of three different parts: $E_{k,\lambda}$ that comes from the Einstein tensor, $\Pi^{(\nu)}_{k,\lambda}$ that comes from the neutrino anisotropic stress tensor term, and $\Pi^{(\gamma)}_{k,\lambda}$ that accounts for the photon contribution. We then have

$$
S_{k,\lambda} = E_{k,\lambda} + \Pi^{(\nu)}_{k,\lambda} + \Pi^{(\gamma)}_{k,\lambda}. \tag{6}
$$

In making the source term $S_{k,\lambda}$ explicit, the first step is to extract the tensor part of the Einstein and energy-momentum tensors to get the corresponding transverse traceless component.

This can be done by making use of the projection operator $\mathcal{P}^{is}_{rj}$, so that [2]

$$
(\Pi^i_j)_{TT} = \mathcal{P}^{is}_{rj}T^r_s. \tag{7}
$$

Here T^r_s contains both the neutrino and photon contributions

$$
T^r_s = T^{r(\nu)}_s + T^{r(\gamma)}_s. \tag{8}
$$

The definition of such an operator is given in Ref. [6]

$$
\mathcal{P}^{is}_{rj} = \mathcal{P}^i_r\mathcal{P}^s_j - \frac{1}{2}\mathcal{P}^i_j\mathcal{P}^s_r, \tag{9}
$$

where

$$
\mathcal{P}^i_j = \delta^i_j - (\nabla^2)^{-1}\partial^i\partial_j. \tag{10}
$$

Moving to Fourier space the two-indices operator $\mathcal{P}^i_j$ reads

$$
\tilde{\mathcal{P}}^i_j = (\delta^i_j - \hat{k}^i\hat{k}_j), \tag{11}
$$

and then the operator to apply is

$$
\begin{aligned}
\tilde{\mathcal{P}}^{is}_{rj} = & \delta^i_r\delta^s_j - \delta^i_r\hat{k}^s\hat{k}_j - \delta^s_j\hat{k}^i\hat{k}_r + \frac{1}{2}\hat{k}^i\hat{k}_r\hat{k}^s\hat{k}_j - \frac{1}{2}\delta^i_j\delta^s_r \\
& + \frac{1}{2}\delta^i_j\hat{k}^s\hat{k}_r + \frac{1}{2}\delta^s_r\hat{k}^i\hat{k}_j.
\end{aligned} \tag{12}
$$

The contribution to the source term coming from the Einstein tensor can then be written as

$$
\begin{aligned}
E^i_j = -\frac{1}{a^2}\mathcal{P}^{is}_{rj}(& -\partial^r\Phi\partial_s\Phi - (\partial^r\partial_s\Phi)2\Psi + 2(\partial^r\partial_s\Psi)\Psi \\
& - \partial^r\Phi\partial_s\Psi + \partial^r\Psi\partial_s\Psi - \partial^r\Psi\partial_s\Phi).
\end{aligned} \tag{13}
$$

To get $E_{k,\lambda}$ we now have to move to Fourier space and project along one of the two polarization states $\lambda = +, \times$.

If we take an orthonormal basis made up by the three unit vectors $\mathbf{e}$, $\bar{\mathbf{e}}$, and $\hat{\mathbf{k}}$, the two polarization tensors are defined as follows:

$$
\varepsilon^+_{ij}(\mathbf{k}) = \frac{1}{\sqrt{2}}[e_i(\mathbf{k})e_j(\mathbf{k}) - \bar{e}_i(\mathbf{k})\bar{e}_j(\mathbf{k})], \tag{14}
$$

$$
\varepsilon^\times_{ij}(\mathbf{k}) = \frac{1}{\sqrt{2}}[e_i(\mathbf{k})\bar{e}_j(\mathbf{k}) + \bar{e}_i(\mathbf{k})e_j(\mathbf{k})].
$$

Remembering that a gravitational wave is transverse with respect to the direction $\hat{\mathbf{k}}$ along which it propagates we find that [3] :

$$
\begin{aligned}
E^\lambda_k = & \varepsilon^{j\lambda}_i E^i_j \\
= & a^{-2}\int\frac{d^3k_1 d^3k_2}{(2\pi)^3}\delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k})\varepsilon^{j\lambda}_i k_{1j}k^i_2 \\
& \times [k_1k_2\Phi_{k_1}\Phi_{k_2} - 2k^2_1\Phi_{k_1}\Psi_{k_2} + 2k^2_1\Psi_{k_1}\Psi_{k_2} \\
& + k_1k_2\Phi_{k_1}\Psi_{k_2} - k_1k_2\Psi_{k_1}\Psi_{k_2} + k_1k_2\Psi_{k_1}\Phi_{k_2}]. \tag{15}
\end{aligned}
$$

The explicit form for the product $\varepsilon^{j+}_i k_{1j}k^i_2$ is given by the analogous of Eq. (36).

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[2] In the following, where not necessary, we will omit the *TT* subscript.

[3] The indices of the 3D unit vectors are lowered and raised by using δ_{ij} so that $\delta^i_j e^i e_j = 1 = \delta^i_j\bar{e}^i\bar{e}_j$.

The next sections are devoted to the computation of the neutrino source term $\Pi_{k,\lambda}^{(\nu)}$, since the term due to the photons is already known and we will recall it later. Notice however that Sec. IV presents a nontrivial case accounting for both neutrinos and photons: here in fact the contribution of the first-order collisionless neutrinos is fully accounted for the first time when explicitly calculating the second-order photon quadrupole in the tight coupled limit [Eq. (59)].

We will proceed by steps. First we solve the second-order Boltzmann equation for the neutrinos in order to give the expression for the neutrino energy-momentum tensor. Then we will give the expression for the spatial components of the energy-momentum tensor of neutrinos in terms of the their distribution function perturbed up to second order, and finally we can extract the transverse and traceless part of it.

III. NEUTRINOS AND SECOND-ORDER TENSOR MODES

A. Solution of the second-order Boltzmann equation for neutrinos

The Boltzmann equation up to second-order for decoupled neutrinos can be written as

$$\frac{\partial F_\nu}{\partial \tau} + \frac{\partial F_\nu}{\partial x^i} \cdot \frac{dx^i}{d\tau} + \frac{\partial F_\nu}{\partial E} \cdot \frac{dE}{d\tau} + \frac{\partial F_\nu}{\partial n^i} \cdot \frac{dn^i}{d\tau} = 0, \quad (16)$$

where τ is the conformal time, $E = \sqrt{m_\nu^2 + p^2}$, $p^2 = g_{ij}P^iP^j$ is the squared neutrinosthree momentum, and P^μ is the neutrino four momentum defined as $P^\mu = dx^\mu/d\lambda$. Here λ parametrizes the particle's path and $x^\mu = (t, \mathbf{x})$ represents a space-time point. The unit vector $\mathbf{n}$, with components n^i, represents the neutrino momentum direction and it is therefore such that: $\mathbf{p} = p\mathbf{n}$ and $n^i n^j \delta_{ij} = 1$. Equation (16) refers to the general case of massive neutrinos ($g_{\mu\alpha}P^\mu P^\alpha = -m_\nu^2$).

The neutrino distribution function F_ν at this stage will consist in an unperturbed, a first-order and a second-order term, such that we can put it in the form

$$F_\nu = \overline{F}_\nu(E, \tau) + F_\nu^{(1)}(E, \tau, x^i, n^i) + \frac{F_\nu^{(2)}}{2}(E, \tau, x^i, n^i). \tag{17}$$

The explicit forms for the different contributions in Eq. (16) are listed in Appendix C.

Since we are interested in evaluating the neutrino contribution during the radiation-dominated epoch, we can specialize the second-order Boltzmann equation for ultra-relativistic (massless) neutrinos. Replacing E with the comoving three momentum q as one of the independent variables in Eq. (16) such that $p = E$ and $q = ap$, the relevant equation takes the form

$$\frac{1}{2}\frac{\partial F_\nu^{(2)}}{\partial \tau} + \frac{1}{2}\frac{\partial F_\nu^{(2)}}{\partial x^i}n^i$$
$$= -\frac{\partial F_\nu^{(1)}}{\partial x^i}n^i(\Phi^{(1)} + \Psi^{(1)}) - q\frac{\partial F_\nu^{(1)}}{\partial q}(-n^j\Phi_{,j}^{(1)})$$
$$+ \Psi^{(1)\prime}) - \frac{\partial F_\nu^{(1)}}{\partial n^i}[n^i n^r(\Phi_{,r}^{(1)} + \Psi_{,r}^{(1)}) - \Psi_{,}^{(1)i} - \Phi_{,}^{(1)i}]$$
$$- q\frac{\partial \overline{F}}{\partial q}\left[-n^j\left(\frac{1}{2}\Phi_{,j}^{(2)} + \Phi_{,j}^{(1)}(\Psi^{(1)} + \Phi^{(1)})\right)\right.$$
$$\left. + \frac{1}{2}\Psi^{(2)\prime} + V_{II} - \frac{1}{2}\chi_{ij}'n^i n^j\right]. \tag{18}$$

Going to Fourier space it can be put in the form

$$F_{\mathbf{k}}^{(2)\prime} + ik\mu F_{\mathbf{k}}^{(2)} = G_{\mathbf{k}}(\tau) - T_{\mathbf{k}}(\tau), \tag{19}$$

where

$$\mu = \hat{\mathbf{k}} \cdot \hat{\mathbf{n}}, \tag{20}$$

and

$$G_{\mathbf{k}}(\tau) = -q\frac{\partial \overline{F}}{\partial q}(-ik\mu\Phi_{\mathbf{k}}^{(2)} + \Psi_k^{(2)\prime} + 2V_{II})$$
$$- 2\int \frac{d^3k_1 d^3k_2}{(2\pi)^3}\delta^D(k_1 + k_2 - k)[(\Phi_{k_2}^{(1)} + \Psi_{k_2}^{(1)})$$
$$\times (ik_1\mu F_{\mathbf{k}_1}^{(1)} - i\mathbf{k}_2 \cdot \mathbf{n}\frac{\partial F_{\mathbf{k}_1}^{(1)}}{\partial n^i}n^i + q\frac{\partial \overline{F}}{\partial q}ik_1\mu\Phi_{k_1}^{(1)})$$
$$+ q\frac{\partial F_{\mathbf{k}_1}^{(1)}}{\partial q}(-ik_2\mu\Phi_{k_2}^{(1)} + \Psi_{k_2}^{(1)})]. \tag{21}$$

Here V_{II} accounts for the second-order vector perturbations modes, see Eq. (C17). We will keep it in this implicit form, since we already know that second-order vectors do not take part to the tensor contribution we are interested in.

The term $T_{\mathbf{k}}$ represents the "pure" tensor contribution and is defined by

$$T_{\mathbf{k}} = q\frac{\partial \overline{F}}{\partial q}\sum_{\lambda=\times,+}\chi_{\mathbf{k},\lambda}'(\tau)\varepsilon_{sr}^\lambda n^s n^r. \tag{22}$$

The Fourier expansion for the tensor modes reads

$$\chi_{ij}(\mathbf{x}, \tau) = \sum_{\lambda=\times,+}\int \frac{d^3k}{(2\pi)^3}\chi_{\mathbf{k},\lambda}(\tau)e^{i\mathbf{k}\mathbf{x}}\varepsilon_{ij}^\lambda \tag{23}$$

where ε_{ij}^λ are the polarization tensors defined in Eq. (14) with $\lambda = \times, +$ accounting for the two possible polarization states of a gravitational wave.

At this point, following the standard procedure [25,26], we can write down an integral solution of the second-order Boltzmann equation as

$$F_{\mathbf{k}}^{(2)}(\tau) = \int_{\tau_{\text{dec}}}^{\tau} d\tau' e^{ik\mu(\tau'-\tau)}[G_k(\tau') - T_k(\tau')]. \tag{24}$$

As usual, $\tau_{\rm dec}$ refers to the neutrino decoupling conformal time [4].

For $F_{\mathbf{k}_1}^{(1)}$ we take the formal solution of the first-order neutrino Boltzmann equation integrated by parts,

$$F_{\mathbf{k}_1}^{(1)}(\tau) = q \frac{\partial \overline{F}}{\partial q}\Big[\Phi_{k_1}^{(1)}(\tau) - \int_{\tau_{\rm dec}}^{\tau} d\tau' e^{ik_1\mu_1(\tau'-\tau)}(\Phi_{k_1}^{(1)\prime}(\tau') + \Psi_{k_1}^{(1)\prime}(\tau'))\Big], \tag{25}$$

where $\mu_1 = \hat{\mathbf{k}}_1 \cdot \mathbf{n}$. In order not to weigh further the notation, $F_{\mathbf{k}_1}^{(1)}$ will be explicitly inserted only later, when strictly necessary.

B. The second-order neutrino energy-momentum tensor

The contribution to the energy-momentum tensor of a given species "i" is

$$T_{\nu}^{\mu(i)} = g_i \frac{1}{\sqrt{-g}} \int \frac{d^3P}{(2\pi)^3} \frac{P^\mu P_\nu}{P^0} F_i, \tag{26}$$

where $P^\mu = dx^\mu/d\lambda$ four momentum of the particle and F_i is the distribution function of the given species.

If we now want an expression for the second-order spatial component of the neutrino energy-momentum tensor, we find that it will consist of the sum of four parts. There are in fact four terms which contain the perturbations: the product of the three momentum $P^i P_j$, the component P^0, the distribution function F_ν, and the determinant of the metric (1) g. As far as the first and the second term are concerned we must remember that

$$P^i = \frac{\overline{q}^i}{a} e^\Psi \Big(1 - \frac{1}{2}\chi_{ml}n^m n^l\Big), \tag{27}$$

$$P_j \equiv g_{\mu j} P^\mu = q[2\omega_j + (e^{-\Psi}n_j - \tfrac{1}{2}\chi_{mn}n^m n^n n_j + \chi_{ij}n^i)], \tag{28}$$

$$P^0 = \frac{q}{a^2} e^{-\Phi}(1 + \omega_i n^i), \tag{29}$$

where we define $q^2 = a^2 g_{ij}P^i P^j$ and we introduce the momentum $\mathbf{q} = q\mathbf{n}$ of magnitude q and direction n^i, see the notations of Refs. [27,28]. The overline refers to unperturbed quantities and we have $\overline{q}^i = a^{-1}qn^i$, $\overline{q}_j = aqn_j$, and $\overline{q}^0 = q/a$.

The determinant of the metric, g, up to second order is such that

$$(-g)^{-1/2} = a^{-4} e^{3\Psi - \Phi} \tag{30}$$

while for the distribution function we use the decomposi-

tion in Eq. (17). By performing the variable change $P_j \to q_j$ in order to make all the perturbations explicit in Eq. (26) and by combining all the terms, we find that the neutrino energy-momentum tensor at second order in perturbation theory reads:

$$(\delta T_j^i)_\nu^{(2)} = a^{-4} g_i \int \frac{d^3q}{(2\pi)^3} qn^i n_j F_\nu^{(2)}. \tag{31}$$

Similar to the linear case, vectors and tensors do not show up because of the angular integration.

With this results we are able to express the spatial component of the second-order energy-momentum tensor Eq. (31) in Fourier space

$$(\delta T_j^{i(\nu)})_{\mathbf{k}}^{(2)} = 2a^{-3} \int \frac{d^3q}{(2\pi)^3} qn^i n_j\Big[\int_{\tau_{\rm dec}}^{\tau} d\tau' e^{ik\mu(\tau'-\tau)} \times [G_k(\tau') - T_k](\tau')\Big] \tag{32}$$

C. The neutrino contribution to the source term

In order to get the transverse traceless part of the neutrino energy-momentum tensor we now have to make use of the operator defined in Eq. (12). Before proceeding, notice that we are interested only in the tensor contribution to the energy-momentum tensor. Therefore it is very useful to keep in mind the decomposition of the distribution function into its scalar, vector and tensor parts, according to the splitting of Ref. [22]. For example the tensor part is given by

$$\delta F = \sum_\lambda f_\lambda(\mathbf{k}, \tau, q, \hat{\mathbf{n}})\epsilon_{ij}^\lambda n^i n^j. \tag{33}$$

This greatly helps in simplifying all the expressions: it is telling us that the only tensor contributions to the energy-momentum tensor are those that can be built out of the product of two momentum direction n^i. Looking back to Eq. (31) and (18) we see that, apart from the straightforward term depending on the gravitational waves χ_{ij}, there is just one term of this type, namely,

$$-\frac{\partial F_\nu^{(1)}}{\partial n^i}[n^i n^k(\Phi_{,k}^{(1)} + \Psi_{,k}^{(1)})]. \tag{34}$$

Therefore, from now on we will focus just on this term in Eq. (32), and we will drop all the others since they correspond to scalar and vector modes.[5] It is interesting to notice that the term (34) has a clear and simple physical interpretation. It arises in the Boltzmann equation from a lensing effect of the neutrinos as they travel through the inhomogeneities of the gravitational potential. In the

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[4]Notice that we are assuming that at a time right before $\tau_{\rm dec}$ the distribution of the neutrinos is the homogeneous one since they are still in thermal equilibrium.

[5]In fact we have explicitly verified that all the remaining terms vanish once the energy-momentum tensor is projected along the two polarization tensors (14) and the angular integration in Eq. (32) is performed.

Boltzmann equation (16) it derives from Eq. (C11), which describes how the neutrino's momentum direction changes in time due to the potential wells they pass trough.

Let us now continue our computation and apply the projection operator (12) to Eq. (32), keeping only the term (34) in $G_{\mathbf{k}}(\tau')$ and the pure tensor contribution $T_{\mathbf{k}}(\tau')$. We see that this operator will act on the product of the two direction unit vectors $n^r n_s$ contained in $T_s^{r(\nu)}$. Since $\delta_r^s n^r n_s = 1$ and the product $\hat{\mathbf{k}} \cdot \hat{\mathbf{p}}$ defines the cosine μ between the direction along which the perturbation propagates and the neutrino momentum, this means that $(\Pi_j^i)_{\mathbf{k}}^{(\nu)}$ will contain the term

$$n^i n_j - \mu(\hat{k}_j n^i + \hat{k}^i n_j) + \tfrac{1}{2}\hat{k}^i \hat{k}_j (1 + \mu^2) - \tfrac{1}{2}\delta_j^i (1 - \mu^2).$$

$$(35)$$

As already done in Sec. 3, we can now choose one of the two polarization states $\lambda = +, \times$ and project the transverse traceless part we are interested in into the corresponding polarization tensors ε_{ij}^λ. Since a gravitational wave is transverse with respect to the direction $\hat{\mathbf{k}}$ along which it propagates, we then have

$$\varepsilon_i^{\lambda j} n^i n_j = \frac{1}{\sqrt{2}}(1 - \mu^2)(\delta^{+\lambda}\cos 2\varphi_{\mathbf{n}} + \delta^{\times\lambda}\sin 2\varphi_{\mathbf{n}}), \quad (36)$$

where the angle $\varphi_{\mathbf{n}}$ is the azimuthal angle of the neutrino momentum direction $\mathbf{n}$ in the orthonormal basis $\mathbf{e}$, $\bar{\mathbf{e}}$, and $\hat{\mathbf{k}}$. In Fourier space the term (34) evaluated at τ' becomes (for simplicity we omit the convolution integral)

$$- (\mathbf{k}_1 \cdot \mathbf{n})(\mathbf{k}_2 \cdot \mathbf{n})(\Phi_{\mathbf{k}_2}(\tau') + \Psi_{\mathbf{k}_2}(\tau'))q\frac{\partial \overline{F}}{\partial q}$$

$$\times \int_{\tau_{\mathrm{dec}}}^{\tau'} d\tau'' e^{ik_1\mu_1(\tau''-\tau')}(\tau'' - \tau')(\Phi_{\mathbf{k}_1}'(\tau'') + \Psi_{\mathbf{k}_1}'(\tau'')),$$

$$(37)$$

where we have written explicitly the term $\partial F_{\mathbf{k}_1}/\partial n^i$ as

$$\frac{\partial F_{\mathbf{k}_1}}{\partial n^i} = q\frac{\partial \overline{F}}{\partial q}\int_{\tau_{\mathrm{dec}}}^{\tau} d\tau' ik_1^i(\tau - \tau')[\Phi_{k_1}'(\tau') + \Psi_{k_1}'(\tau')]$$

$$\times e^{ik_1\mu_1(\tau'-\tau)}, \quad (38)$$

using the solution of Eq. (25). It proves convenient to take the tensor part of $(\mathbf{k}_1 \cdot \mathbf{n})(\mathbf{k}_2 \cdot \mathbf{n})$ which reads

$$(k_{2l}k_{1m}\varepsilon^{+lm})\varepsilon_{rs}^+ n^r n^s + (k_{2l}k_{1m}\varepsilon^{\times lm})\varepsilon_{rs}^\times n^r n^s. \quad (39)$$

Notice that this step is equivalent to follow the decomposition (33) of Ref. [22] which allows one to isolate the tensor contributions to the distribution function.[6]

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[6]Also in this case we have verified that the scalar and vector components of $(\mathbf{k}_1 \cdot \mathbf{n})(\mathbf{k}_2 \cdot \mathbf{n})$ give a vanishing contribution once the energy-momentum tensor is projected along the two polarization tensors (14) and the angular integration in Eq. (32) is performed.

With these results we are now able to write the expression for $\Pi_{\mathbf{k},\lambda}^{(\nu)} = \varepsilon_i^{\lambda j}\Pi_{j\mathbf{k}}^{i(\nu)}$ at second order in the perturbations

$$\Pi_{\mathbf{k},\lambda}^{(\nu)} = -2a^{-4}g_i \iint \frac{d^3k_1 d^3k_2}{(2\pi)^3}\delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k})$$

$$\times \left[\int \frac{d^3q}{(2\pi)^3}q^2\frac{\partial \overline{F}}{\partial q}\int_{\tau_{\mathrm{dec}}}^{\tau} d\tau' e^{ik\mu(\tau'-\tau)}\varepsilon_i^{\lambda j}n^i n_j\right.$$

$$\times [(k_{2l}k_{1m}\varepsilon^{+lm})\varepsilon_{rs}^+ n^r n^s + (k_{2l}k_{1m}\varepsilon^{\times lm})\varepsilon_{rs}^\times n^r n^s]$$

$$\times A_{\mathbf{k}_2}(\tau')\int_{\tau_{dec}}^{\tau'} d\tau'' e^{ik_1\mu_1(\tau''-\tau')}(\tau'' - \tau')A_{\mathbf{k}_1}'(\tau'')]$$

$$+ a^{-4}g_i \int \frac{d^3q}{(2\pi)^3}\frac{q^2}{4}\frac{\partial \overline{F}}{\partial q}(1 - \mu^2)^2$$

$$\times \int_{\tau_{\mathrm{dec}}}^{\tau} d\tau' \chi_{\mathbf{k},\lambda}'(\tau')e^{ik\mu(\tau'-\tau)}, \quad (40)$$

where $A_{\mathbf{k}}(\tau) \equiv \Phi_{\mathbf{k}}(\tau) + \Psi_{\mathbf{k}}(\tau)$.

At this point we want to solve the angular and the momentum dependence of the transverse, traceless neutrino energy-momentum tensor $\Pi_{\mathbf{k}}^{(\nu)}$. This implies solving the d^3q integral in the previous expression.

First of all notice that the pure tensor part in the last line of Eq. (40) represents the second-order analogue of the damping effect found and discussed in Refs. [15–22]. We are then able to deal with it (see, for example, Refs. [18,21,22]) and we find

$$a^{-4}\int \frac{dq q^4}{(\pi)^2}\frac{\partial \overline{F}}{\partial q}\int_{u_{\mathrm{dec}}}^{u} dU \chi_k'(U)\frac{1}{15}\left[j_0(s) + \frac{10}{7}j_2(s)\right.$$

$$\left. + \frac{3}{7}j_4(s)\right], \quad (41)$$

which becomes

$$- 8\bar{\rho}_\nu(\tau)\int_{u_{\mathrm{dec}}}^{u} dU \chi_k'(U)\frac{1}{15}\left[j_0(s) + \frac{10}{7}j_2(s) + \frac{3}{7}j_4(s)\right],$$

$$(42)$$

where we have used Eqs. (50) and (51), the variable change $\tau \rightarrow u = k\tau$, $s = u - U$, with $U = k\tau'$, and the derivative is with respect to U. This means that also at second order gravitational waves are damped by neutrino free-streaming, as expected on general grounds. Here $\bar{\rho}_\nu$ is the unperturbed neutrino energy density given in Eq. (50).

We now have to deal with the reminder of Eq. (40), giving an additional source term that represents a completely new result.

1. Angular integration

We notice that there is a μ dependence hidden in the exponential

$$e^{i\mu_1 s'} = \sum_{l=0}^{+\infty} i^l (2l+1) j_l(s') P_l(\mu_1)$$

$$= \sqrt{4\pi} \sum_{l=0}^{+\infty} i^l j_l(s') \sum_{m=-l}^{+l} a_{lm} e^{im\varphi_\mathbf{n}} P_{lm}(\mu) Y_{lm}^*(\hat{\mathbf{k}}_1),$$

$$(43)$$

where $a_{lm} = \sqrt{\frac{(2l+1)(l-m)!}{(l+m)!}}$ and $s' = k_1(\tau' - \tau) \equiv (U_1 - u_1)$. For the integration over the angle $\varphi_\mathbf{n}$ we need to take into account the product $(\varepsilon_i^{\lambda j} n^i n_j)(\varepsilon_{rs}^{\lambda'} n^r n^s)$ in Eq. (40) and, using Eq. (36), we find

$$\int d\varphi_\mathbf{n} e^{im\varphi_\mathbf{n}} (\varepsilon_i^{\lambda j} n^i n_j)(\varepsilon_{rs}^{\lambda'} n^r n^s) = \frac{\pi}{2}(1-\mu^2)^2 \delta^{\lambda\lambda'} \delta_{m0},$$

$$(44)$$

which means that the cross terms vanish, while the squared ones select $m = 0$. When $m = 0$ the P_{lm} correspond to the Legendre polynomials P_l and we have

$$\sqrt{4\pi} \sum_{l=0}^{+\infty} j_l(s') i^l \sqrt{2l+1} Y_{l0}^*(\hat{\mathbf{k}}_1) P_l(\mu). \qquad (45)$$

Therefore by expanding $e^{ik\mu(\tau'-\tau)}$ as in Eq. (43) we are led to an integration over μ of the following quantity:

$$\int_{-1}^{+1} d\mu \sum_{ll'} A_l P_l(\mu) B_{l'} P_{l'}(\mu)(1-\mu^2)^2, \qquad (46)$$

where

$$A_l = i^l \sqrt{4\pi(2l+1)} Y_{l0}^*(\hat{\mathbf{k}}_1) j_l[k_1(\tau'' - \tau')],$$
$$B_{l'} = i^{l'}(2l'+1) j_{l'}[k(\tau' - \tau)]. \qquad (47)$$

Such an integral is easily performed using the formulas of Appendix D, and Eq. (46) becomes

$$\sum_l \frac{1}{2l+1} [A_l B_l - A_l B_l^{(1)} - A_l^{(1)} B_l + A_l^{(1)} B_l^{(1)}], \qquad (48)$$

where

$$A_l^{(1)} = \frac{l(l-1)}{(2l-3)(2l-1)} A_{l-2} + \left[\frac{(l+1)^2}{(2l+1)(2l+3)}\right.$$
$$\left. + \frac{l^2}{(2l+1)(2l-1)}\right] A_l + \frac{(l+2)(l+1)}{(2l+3)(2l+5)} A_{l+2},$$

$$(49)$$

and similar for $B_l^{(1)}$.

2. Integration over the comoving momentum q.

We can treat the momentum integration over q in Eq. (40) independently from the angular part. There is just one type of q dependence, namely $q^4 \partial \overline{F} / \partial q$. Remember that $\overline{F}$ is the unperturbed neutrino distribution function given by the Fermi-Dirac distribution and

$$\overline{\rho}_\nu = a^{-4} \frac{g_i}{2\pi^2} \int \overline{F} q^3 dq = \frac{7}{8} \frac{\pi^2}{30} T_\nu^4 \qquad (50)$$

is the unperturbed neutrino energy density. Hence, integrating by parts when necessary, we find [the $\pi/2$ factor comes from Eq. (44)]

$$a^{-4} g_i \frac{\pi}{2} \int \frac{dq}{(2\pi)^3} \frac{\partial \overline{F}}{\partial q} q^4 = -\frac{1}{2} \overline{\rho}_\nu. \qquad (51)$$

3. Final expression for the neutrino contribution

Collecting the previous results we arrive at the contribution of streaming neutrinos to the evolution of gravitational waves

$$\Pi_{\mathbf{k},\lambda}^{(\nu)} = \overline{\rho}_\nu \iint \frac{d^3 k_1 d^3 k_2}{(2\pi)^3} \delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k})$$

$$\times (k_{2r} k_{1s} \varepsilon_\lambda^{rs}) \sum_l \frac{1}{2l+1} \int_{\tau_{\text{dec}}}^\tau d\tau' (\Phi_{\mathbf{k}_2}(\tau')$$

$$+ \Psi_{\mathbf{k}_2}(\tau'))[B_l - B_l^{(1)}] \int_{\tau_{\text{dec}}}^{\tau'} d\tau'' (\tau'' - \tau')(\Phi_{\mathbf{k}_1}'(\tau'')$$

$$+ \Psi_{\mathbf{k}_1}'(\tau''))[A_l - A_l^{(1)}] - 8\overline{\rho}_\nu(\tau) \int_{\tau_{\text{dec}}}^\tau d\tau' \chi_k'(\tau')$$

$$\times \frac{1}{15}\left[j_0(s) + \frac{10}{7} j_2(s) + \frac{3}{7} j_4(s)\right] \qquad (52)$$

where $s = k(\tau - \tau')$, and the functions A_l, $A_l^{(1)}$, B_l, and $B_l^{(1)}$ are defined in Eqs. (47) and (49).

This equation is clearly telling us that collisionless free-streaming neutrinos contribute with new terms to the source of second-order gravitational waves, with respect to the fluid treatment adopted in the literature so far, where the tensor part of $\Pi_j^{i(\nu)}$ at second order has never been taken into account.

Even with a qualitative approach, we can state that if low l contributions (up to $l = 2$) can have a correspondence with respect to a source where neutrinos and photons are treated as a single-fluid radiation, higher multipoles surely do not. These in fact come from the high neutrino velocity dispersion and can be found only if neutrinos are treated as collisionless particles.

Moreover, during the radiation-dominated epoch, the neutrino fraction $f_\nu(\tau) = \Omega_\nu / \Omega_R$ is not negligible. Since all the terms in $\Pi_{\mathbf{k}}^{(\nu)}$ are multiplied by $\overline{\rho}_\nu$, they are therefore non-negligible at that time.

IV. THE PHOTON TENSOR QUADRUPOLE

In order to complete the expression of the source term for gravitational waves during the radiation era in Eq. (4), we now have to add the photon contribution. Following [29], the second-order photon quadrupole in the tight coupling limit is given by

$$\Pi_\gamma^{(2)ij} \simeq \frac{8}{3}(v_\gamma^{(1)i} v_\gamma^{(1)j} - \frac{1}{3}\delta^{ij} v_\gamma^{(1)2}). \tag{53}$$

To get the photon contribution to the second-order gravitational waves' source we then have to extract the transverse and traceless component.

Since the operator to apply is such that $\mathcal{P}_{rj}^{is}\delta_s^r = 0$, we will find that the tensor quadrupole takes the form

$$(\Pi_{\gamma j}^{(2)i})_{TT} \simeq \frac{8}{3}\mathcal{P}_{rj}^{is} v_\gamma^{(1)r} v_{\gamma s}^{(1)}. \tag{54}$$

By making use of the first-order space-time component of the Einstein equations (see Appendix B)

$$\frac{1}{a^2}\partial^i(\Psi' + \mathcal{H}\Phi) = -\frac{16\pi G}{3}(\overline{\rho}_\gamma v_\gamma^{(1)i} + \overline{\rho}_\nu v_\nu^{(1)i}), \tag{55}$$

we can define the first-order photon velocity as

$$v_\gamma^{(1)i} = -\left[\frac{3}{16\pi Ga^2\overline{\rho}_\gamma}\partial^i(\Psi' + \mathcal{H}\Phi) + \frac{\overline{\rho}_\nu}{\overline{\rho}_\gamma}v_\nu^{(1)i}\right], \tag{56}$$

where $v_\nu^{(1)i}$ represents the first-order neutrino velocity. During the radiation-dominated epoch neutrinos are still relativistic so that, in Fourier space, the velocity can be written as

$$v_\nu^{(1)i} = \frac{1}{\rho_\nu + P_\nu}\int\frac{d^3p}{(2\pi)^3}F_\nu^{(1)}p^i = -12\pi i\mathcal{N}_1\hat{k}^i. \tag{57}$$

The term $\mathcal{N}_1$ refers to the neutrino dipole and P_ν is the neutrino pressure. By making use of the formal solution (25) of the first-order neutrino Boltzmann equation, the dipole can be expressed explicitly in terms of the perturbations to give

$$\mathcal{N}_1(k) = \frac{i}{2}\frac{\partial \ln\overline{F}_\nu}{\partial \ln q}\int_{\tau_{dec}}^{\tau}d\tau'\left[\frac{ik_1}{3}\Phi'_{k_1}(j_0(s) - 2j_2(s))\right.$$
$$\left. + i\Psi'_{k_1}j_1(s)\right]. \tag{58}$$

Projecting along one of the polarization states (+), we have that, in Fourier space, the photon contribution we are searching for takes the form

$$\Pi_{\mathbf{k}}^{(\gamma)\lambda} = \int\frac{d^3k_1 d^3k_2}{(2\pi)^3}\delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k})\varepsilon_i^{j\lambda}k_{1j}k_2^i$$
$$\times\left[\frac{3k_1 k_2}{32(\pi Ga^2\overline{\rho}_\gamma)^2}(\Psi'_{k_1} + \mathcal{H}\Phi_{k_1})(\Psi'_{k_2} + \mathcal{H}\Phi_{k_2})\right.$$
$$+ 32\cdot12\left(\frac{\overline{\rho}_\nu\pi}{\overline{\rho}_\gamma}\right)^2\mathcal{N}_1(k_1)\mathcal{N}_1(k_2)$$
$$+ 6\pi\frac{\overline{\rho}_\nu}{\overline{\rho}_\gamma}k_2\mathcal{N}_1(k_1)(\Psi'_{k_2} + \mathcal{H}\Phi_{k_2})$$
$$\left. + \frac{6\overline{\rho}_\nu}{Ga^2\overline{\rho}_\gamma^2}k_1\mathcal{N}_1(k_2)(\Psi'_{k_1} + \mathcal{H}\Phi_{k_1})\right]. \tag{59}$$

It is worth stressing that this expression contains at most multipoles with $l = 2$. Therefore, if we had used a fluid

treatment for the neutrinos as well (e.g., $\Pi_\nu^{ij(2)} \propto v_\nu^{(1)i} v_\nu^{(1)j}$), we would have found in the second-order neutrino contributions terms with l not higher then $l = 2$. This is clearly in contrast with the neutrino source term Eq. (52).

V. CONCLUSIONS

This paper represents the first step towards the quantitative evaluation of the impact of cosmic neutrinos on the evolution of the gravitational-wave background; it provides, first of all, a complete study of the Boltzmann equations for neutrinos at second order and the expression for the second-order anisotropic stress tensor.

Free-streaming neutrinos are an important source of second-order gravitational waves during the radiation-dominated epoch. Along with the fact that neutrinos yield a relevant contribution to the total energy during this epoch, this is due to the large neutrino velocity dispersion and it emerges from the calculations performed in this paper when assuming that neutrinos are collisionless particles, as it is the case after their decoupling.

The fluid treatment adopted so far for describing neutrinos turns out therefore to be a poor approximation that leads, in particular, to underestimating the role of neutrino free-streaming as a source of gravitational waves. In particular, in this paper we have made the first full consistent computation of the second-order tensor part of the neutrino anisotropic stress tensor, Eq. (52). This has been achieved by computing and solving the second-order Boltzmann equation for the neutrino distribution function. Besides recovering the second-order counterpart of the damping effect studied in Refs. [15–22], Eq. (52) represents a completely new source term for the evolution of gravitational waves.

ACKNOWLEDGMENTS

APPENDIX A: THE SECOND-ORDER EINSTEIN TENSOR

In this appendix we provides the definitions for the connection coefficients and the expression of the second-order Einstein tensor for the metric (1)

$$ds^2 = a^2(\tau)[-e^{2\Phi}d\tau^2 + 2\omega_i dx^i d\tau + (e^{-2\Psi}\delta_{ij} + \chi_{ij})dx^i dx^j].$$

The space-time metric $g_{\mu\nu}$ has signature $(-, +, +, +)$. The connection coefficients are defined as

$$\Gamma^{\alpha}_{\beta\gamma} = \frac{1}{2} g^{\alpha\rho}\left(\frac{\partial g_{\rho\gamma}}{\partial x^{\beta}} + \frac{\partial g_{\beta\rho}}{\partial x^{\gamma}} - \frac{\partial g_{\beta\gamma}}{\partial x^{\rho}}\right). \qquad \text{(A1)}$$

Greek indices $(\alpha, \beta, ..., \mu, \nu, ...)$ run from 0 to 3, while Latin indices $(a, b, \ldots, i, j, k, \ldots, m, n, \ldots)$ run from 1 to 3. In particular, their explicit expression with our metric reads

$$\Gamma^{0}_{00} = \mathcal{H} + \Phi', \qquad \Gamma^{0}_{0i} = \frac{\partial \Phi}{\partial x^{i}} + \mathcal{H}\omega_{i},$$

$$\Gamma^{i}_{00} = \omega^{i\prime} + \mathcal{H}\omega^{i} + e^{2\Psi+2\Phi}\frac{\partial \Phi}{\partial x_{i}},$$

$$\Gamma^{0}_{ij} = -\frac{1}{2}\left(\frac{\partial \omega_{j}}{\partial x^{i}} + \frac{\partial \omega_{i}}{\partial x^{j}}\right) + e^{-2\Psi-2\Phi}(\mathcal{H} - \Psi')\delta_{ij}$$

$$+ \frac{1}{2}\chi'_{ij} + \mathcal{H}\chi_{ij}, \qquad \text{(A2)}$$

$$\Gamma^{i}_{0j} = (\mathcal{H} - \Psi')\delta_{ij} + \frac{1}{2}\chi'_{ij} + \frac{1}{2}\left(\frac{\partial \omega_{i}}{\partial x^{j}} - \frac{\partial \omega_{j}}{\partial x^{i}}\right),$$

$$\Gamma^{i}_{jk} = -\mathcal{H}\omega^{i}\delta_{jk} - \frac{\partial \Psi}{\partial x^{k}}\delta^{i}_{j} - \frac{\partial \Psi}{\partial x^{j}}\delta^{i}_{k} + \frac{\partial \Psi}{\partial x_{i}}\delta_{jk}$$

$$+ \frac{1}{2}\left(\frac{\partial \chi^{i}_{j}}{\partial x^{k}} + \frac{\partial \chi^{i}_{k}}{\partial x^{j}} - \frac{\partial \chi_{jk}}{\partial x_{i}}\right).$$

The Einstein equations are written as $G_{\mu\nu} = 8\pi G_{\mathrm{N}}T_{\mu\nu}$, where G_{N} is the usual Newtonian gravitational constant, $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$ is the Einstein tensor, and $T_{\mu\nu}$ is the energy-momentum tensor.

The Ricci tensor $R_{\mu\nu}$ is a contraction of the Riemann tensor, $R_{\mu\nu} = R^{\alpha}_{\mu\alpha\nu}$, and in terms of the connection coefficient it is given by

$$R_{\mu\nu} = \partial_{\alpha}\Gamma^{\alpha}_{\mu\nu} - \partial_{\mu}\Gamma^{\alpha}_{\nu\alpha} + \Gamma^{\alpha}_{\sigma\alpha}\Gamma^{\sigma}_{\mu\nu} - \Gamma^{\alpha}_{\sigma\nu}\Gamma^{\sigma}_{\mu\alpha}. \qquad \text{(A3)}$$

The Ricci scalar is the trace of the Ricci tensor, $R = R^{\mu}_{\mu}$. The components of Einstein's tensor up to second-order read

$$G^{0}_{0} = -\frac{e^{-2\Phi}}{a^{2}}[3\mathcal{H}^{2} - 6\mathcal{H}\Psi' + 3(\Psi')^{2}$$

$$- e^{2\Phi+2\Psi}(\partial_{i}\Psi\partial^{i}\Psi - 2\nabla^{2}\Psi)], \qquad \text{(A4)}$$

$$G^{i}_{0} = 2\frac{e^{2\Psi}}{a^{2}}[\partial^{i}\Psi' + (\mathcal{H} - \Psi')\partial^{i}\Phi] - \frac{1}{2a^{2}}\nabla^{2}\omega^{i}$$

$$+ \left(4\mathcal{H}^{2} - 2\frac{a''}{a}\right)\frac{\omega^{i}}{a^{2}}, \qquad \text{(A5)}$$

$$G^{i}_{j} = \frac{1}{a^{2}}\left[e^{-2\Phi}\left(\mathcal{H}^{2} - 2\frac{a''}{a} - 2\Psi'\Phi' - 3(\Psi')^{2}\right.\right.$$

$$+ 2\mathcal{H}(\Phi' + 2\Psi') + 2\Psi''\bigg) + e^{2\Psi}(\partial_{k}\Phi\partial^{k}\Phi$$

$$+ \nabla^{2}\Phi - \nabla^{2}\Psi)\bigg]\delta^{i}_{j} + \frac{e^{2\Psi}}{a^{2}}(-\partial^{i}\Phi\partial_{j}\Phi - \partial^{i}\partial_{j}\Phi$$

$$+ \partial^{i}\partial_{j}\Psi - \partial^{i}\Phi\partial_{j}\Psi + \partial^{i}\Psi\partial_{j}\Psi - \partial^{i}\Psi\partial_{j}\Phi)$$

$$- \frac{\mathcal{H}}{a^{2}}(\partial^{i}\omega_{j} + \partial_{j}\omega^{i}) - \frac{1}{2a^{2}}(\partial^{i}\omega'_{j} + \partial_{j}\omega^{i\prime})$$

$$+ \frac{1}{a^{2}}\left(\mathcal{H}\chi^{i\prime}_{j} + \frac{1}{2}\chi^{i\prime\prime}_{j} - \frac{1}{2}\nabla^{2}\chi^{i}_{j}\right). \qquad \text{(A6)}$$

The exponentials are maintained since they help in simplifying lots of calculations, however notice that in all these expressions they are implicitly truncated up to second order.

APPENDIX B: FIRST-ORDER PERTURBATIONS OF EINSTEIN EQUATIONS FOR PHOTONS AND NEUTRINOS

Carrying out the calculation, the first-order Einstein's equations, expressed in Fourier space in terms of the perturbations Φ and Ψ, take the form

$$-k^{2}\Psi - 3\frac{\dot{a}}{a}\left(\dot{\Psi} - \Phi\frac{\dot{a}}{a}\right) = 16\pi G a^{2}[\overline{\rho}_{\gamma}\Theta_{\gamma 0} + \overline{\rho}_{\nu}\Theta_{\nu 0}] \tag{B1}$$

$$k^{2}(\Phi - \Psi) = -32\pi G a^{2}\overline{\rho}_{\nu}\Theta_{\nu 2}. \tag{B2}$$

Here $\Theta_{\gamma 0}$ and $\Theta_{\nu 0}$ are, respectively, the photon and the neutrino monopole contribution, while $\Theta_{\nu 2}$ refers to the neutrino scalar quadrupole.

Remember that the first-order perturbation to the neutrino distribution function is defined by $F^{(1)}_{\nu} = \overline{F}_{\nu}\mathcal{N}$, where

$$\mathcal{N} = -\frac{\partial \ln\overline{F}_{\nu}}{\partial \ln p}\Theta_{\nu} \tag{B3}$$

and $\Theta_{\nu}(t, \vec{x}, n) = \frac{\delta T_{\nu}}{T_{\nu}}$. Similarly it happens to the photon contribution.

In general, the lth multipole of the temperature field Θ can be defined as

$$\Theta_{l} = \frac{1}{(-i)^{l}}\int_{-1}^{1}\frac{d\mu}{2}\mathcal{P}_{l}(\mu)\Theta, \tag{B4}$$

where $\mathcal{P}_{l}(\mu)$ are the Legendre polynomials.

The neutrino monopole is then defined by

$$\Theta_{\nu 0} = \int_{-1}^{1}\frac{d\mu}{2}\Theta_{\nu}, \tag{B5}$$

while the quadrupole, corresponding to $l = 2$, is

$$\Theta_{\nu 2} = -\int_{-1}^{1} \frac{d\mu}{2} \mathcal{P}_2(\mu)\Theta_{\nu}. \tag{B6}$$

APPENDIX C: THE SECOND-ORDER NEUTRINO BOLTZMANN EQUATION

In this appendix we provide the explicit form for the different contributions in the neutrino Boltzmann equation $\frac{\partial F_\nu}{\partial \tau} + \frac{\partial F_\nu}{\partial x^i} \cdot \frac{dx^i}{d\tau} + \frac{\partial F_\nu}{\partial E} \cdot \frac{dE}{d\tau} + \frac{\partial F_\nu}{\partial n^i} \cdot \frac{dn^i}{d\tau} = 0$:

(i) $\frac{dx^i}{d\tau} = \frac{dx^i}{d\lambda}\frac{d\lambda}{d\tau} \equiv \frac{P^i}{P^0}$:
For this term we have

$$\frac{dx^i}{d\tau} = \frac{p}{E} n^i e^{(\Psi+\Phi)} \left[1 - \omega_j n^j \frac{p}{E} - \frac{1}{2}\chi_{ij}n^i n^j \right]. \tag{C1}$$

(ii) $\frac{dE}{d\tau}$:
Deriving an expression for this term is more lengthy since it involves the use of the geodesic equation. We can start noting that, using Eq. (29), we obtain

$$\frac{dP^0}{d\tau} = e^{-\Phi}\frac{E}{a}\left[-\frac{d\Phi}{d\tau}B + \frac{dB}{d\tau} \right.$$
$$\left. + B\left(\frac{1}{E}\frac{dE}{d\tau} - \frac{1}{a}\frac{da}{d\tau}\right) \right], \tag{C2}$$

where $B = 1 + \frac{p}{E}\omega_i n^i$.
To simplify a bit the notation, from now on we will set

$$g' \equiv \frac{\partial g}{\partial \tau} \quad \text{and} \quad g_{,j} \equiv \frac{\partial g}{\partial x^j}, \quad \forall g \equiv g(\tau, x^i). \tag{C3}$$

If we now make the total derivatives explicit, according to the fact that

$$\frac{d\Phi}{d\tau} = \Phi' + \Phi_{,i}\frac{P^i}{P^0}, \tag{C4}$$

and

$$\frac{dB}{d\tau} \equiv \frac{p}{E}n^i\frac{d\omega_i}{d\tau} = \frac{p}{E}n^i\left(\omega_i' + \omega_{i,j}\frac{P^j}{P^0}\right), \tag{C5}$$

we find

$$\frac{1}{E}\frac{dP^0}{d\tau} = \frac{e^{-\Phi}}{a}\left(1 + \frac{p}{E}\omega_i n^i\right)\left[-\left(\Phi' + \Phi_{,i}\frac{P^i}{P^0}\right) \right.$$
$$\left. + \frac{dE}{d\tau} - \frac{1}{a}\frac{da}{d\tau} + \frac{p}{E}n^i\left(\omega_i' + \omega_{i,j}\frac{P^j}{P^0}\right) \right]. \tag{C6}$$

By making use of the geodesic equation, we can express the time component as a sum of three terms

$$-\Gamma_{\alpha\beta}^0\frac{P^\alpha P^\beta}{P^0} = -\Gamma_{00}^0 P^0 - 2\Gamma_{0j}^0 P^j - \Gamma_{ij}^0\frac{P^\alpha P^\beta}{P^0}. \tag{C7}$$

If we now insert the connection coefficients and the perturbed components of the four momentum (27) and (29), we come at last to the required term

$$\frac{1}{E}\frac{dE}{d\tau} = -\left(\frac{p}{E}\right)^2\mathcal{H} - \frac{p}{E}n^j\Phi_{,j}e^{\phi+\Psi} + \left(\frac{p}{E}\right)^2\Psi'$$
$$- \frac{p}{E}\left[n^i\omega_i' + 2n^j\omega_j\mathcal{H} - 2\mathcal{H}\omega_r n^r\left(\frac{p}{E}\right)^2 \right]$$
$$- \left(\frac{p}{E}\right)^2\frac{1}{2}\chi_{ij}'n^i n^j. \tag{C8}$$

We can note that the first term in the right-hand side of the latter equation is the zeroth-order time component of the geodesic equation.

(iii) $\frac{dn^i}{d\tau}$:
This term requires some lengthy algebra as well. To obtain it the spatial component of the geodetic equation must be used

$$\frac{dP^i}{d\tau} = -\Gamma_{\alpha\beta}^i\frac{P^\alpha P^\beta}{P^0}. \tag{C9}$$

Carrying on the calculation in the same way as we did for $dE/d\tau$, at the end we recover the expression

$$\frac{dP^i}{d\tau} = \frac{p}{a}\left[\frac{dn^i}{d\tau} + n^i\frac{d\Psi}{d\tau} + \frac{n^i}{p}e^{\Psi}\left(-2\frac{p}{E}E\mathcal{H}\right.\right.$$
$$\left.\left. - n^j\Phi_{,j}E + \frac{p}{E}E\Psi'\right) \right]. \tag{C10}$$

The term we are looking for has the form

$$\frac{dn^i}{d\tau} = -\frac{E}{p}\Phi^{,i} - \frac{E}{p}\Psi^{,i} + n^i n^j\frac{E}{p}\Phi_{,j} + \frac{p}{E}n^i n^k\Psi_{,k}. \tag{C11}$$

It is worth noticing that, since $\partial F_\nu/\partial n^i$ is already a first-order term, we must consider this equation just up to first order. The scalars that appear here are therefore the linear components of the perturbations.

The purpose of this appendix is to find the second-order Boltzmann equation for (decoupled) neutrinos. It is then useful to use the relations in Eqs. (17) and (2) for making the second-order terms explicit.

The second-order contributions of each part of Eq. (16) are listed below in the following expressions:

$$\frac{\partial F_\nu}{\partial x^i} \frac{dx^i}{d\tau}\bigg|_{2^{\mathrm{nd}}\,\mathrm{ord.}} = \frac{\partial F_\nu^{(1)}}{\partial x^i} \frac{p}{E} n^i (\Phi^{(1)} + \Psi^{(1)})$$
$$+ \frac{1}{2} \frac{\partial F_\nu^{(2)}}{\partial x^i} \frac{p}{E} n^i. \tag{C12}$$

The energy dependence term is a bit more complicated since it consists of three terms such that

$$\frac{\partial F_\nu}{\partial E} \frac{dE}{d\tau}\bigg|_{2^{\mathrm{nd}}} = \frac{\partial \overline{F_\nu}}{\partial E}\left(\frac{dE}{d\tau}\right)\bigg|_{2^{\mathrm{nd}}} + \frac{\partial F_\nu^{(1)}}{\partial E}\left(\frac{dE}{d\tau}\right)\bigg|_{1^{\mathrm{st}}}$$
$$+ \frac{1}{2}\frac{\partial F_\nu^{(2)}}{\partial E}\left(\frac{dE}{d\tau}\right)\bigg|_{0^{\mathrm{th}}}. \tag{C13}$$

(The overline refers to the zeroth-order neutrino distribution function.)

We then have

$$\frac{1}{E}\left(\frac{dE}{d\tau}\right)\bigg|_{2^{\mathrm{nd}}} = -\frac{p}{E} n^j \left[\frac{1}{2}\Phi^{(2)}_{,j} + \Phi^{(1)}_{,j}\Psi^{(1)} + \Phi^{(1)}_{,j}\Phi^{(1)}\right]$$
$$+ \left(\frac{p}{E}\right)^2 \frac{1}{2}\Psi^{(2)\prime} - V_{II} - \left(\frac{p}{E}\right)^2 \frac{1}{2}\chi'_{ij} n^i n^j, \tag{C14}$$

while

$$\frac{1}{E}\frac{dE}{d\tau}\bigg|_{1^{\mathrm{st}}} = -\frac{p}{E} n^j \Phi^{(1)}_{,j} + \left(\frac{p}{E}\right)^2 \Phi^{(1)\prime}, \tag{C15}$$

and

$$\frac{1}{E}\frac{dE}{d\tau}\bigg|_{0^{\mathrm{th}}} = -\left(\frac{p}{E}\right)^2 \mathcal{H}. \tag{C16}$$

The term V_{II} accounts for the second-order vector contribution. This already appears in Eq. (C8) and is defined as

$$V_{II} = \frac{p}{E}\left[n^i \omega'_i + 2n^j \omega_j \mathcal{H} - 2\mathcal{H}\omega_r n^r \left(\frac{p}{E}\right)^2\right]. \tag{C17}$$

Finally, we use Eq. (C11) to deal with the dependence on the momentum direction. Matching together all these terms in Eq. (16), we have now all the tools needed to obtain the second-order Boltzmann equation.

APPENDIX D: SOME FORMULAS USED FOR THE ANGULAR INTEGRATION

The coefficients $A_l^{(1)}$ and $B_l^{(1)}$ appearing in Eq. (48) are simply given in terms of Eq. (47) as

$$A_l^{(1)} = \frac{l(l-1)}{(2l-3)(2l-1)} A_{l-2} + \left[\frac{(l+1)^2}{(2l+1)(2l+3)}\right.$$
$$\left. + \frac{l^2}{(2l+1)(2l-1)}\right]A_l + \frac{(l+2)(l+1)}{(2l+3)(2l+5)} A_{l+2}, \tag{D1}$$

and similar for $B_l^{(1)}$. The result in Eq. (48) is obtained using the orthogonality of the Legendre polynomials and applying to Eq. (46) the formula (here a_l is a generic function of l)

$$\sum_l a_l j_l(x)\mu^2 P_l(\mu) = \sum_l \tilde{a}_l P_l(\mu), \tag{D2}$$

where

$$\sum_l \tilde{a}_l = \frac{l(l-1)}{(2l-3)(2l-1)} a_{l-2} j_{l-2}(x) + \left[\frac{(l+1)^2}{(2l+1)(2l+3)}\right.$$
$$\left. + \frac{l^2}{(2l+1)(2l-1)} a_l j_l(x)\right]$$
$$+ \frac{(l+2)(l+1)}{(2l+3)(2l+5)} a_{l+2} j_{l+2}(x), \tag{D3}$$

which derives from the recursion relation of the Legendre polynomials (see also [30]).

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