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PARTONS OF A SPHERICAL BOX

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A B S T R A C T

Calculation of parton distributions in the "cavity approximation" to the MIT bag model gives a divergent sum of positive terms. This suggests that Bjorken scaling does not hold for the deep inelastic scattering in this version of the model.

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We report an attempt to find parton distributions for the "cavity approximation" to the MIT bag model ¹⁾. In this version of the model the flexible movable bag is replaced by an inflexible immovable box ¹⁾. The calculation was initiated by Jaffe ²⁾ and Hughes ³⁾. For two of the three Feynman diagrams (omitting the "bubble diagram" of his figure 3c) Jaffe verified Bjorken scaling for deep inelastic scattering, and interpreted the scaling limit in terms of parton and antiparton distributions. Hughes showed how these same distributions can be obtained directly by analysis of the null plane quantum state. In the case of a one-dimensional version of the model it has been verified ⁴⁾ that the correspondence between Bjorken-scaling structure functions and null-plane parton distributions holds good even when the contributions omitted by Jaffe and Hughes are retained. Those which they kept were identified as "valence" contributions and those which they omitted as "sea" contributions. It remains to calculate the sea in the three-dimensional case. That is what we attempt here. We do not examine deep inelastic scattering or Bjorken scaling, but look directly at null-plane parton distributions. Our result, as divergent series of positive terms, suggests that Bjorken scaling does not hold in this version of the model ^{*}).

Consider a second quantized Dirac field with position dependent "mass". Energy levels E_m and corresponding wave functions u_m are defined by

$$E_m u_m(\vec{r}) = \left(\frac{\vec{\alpha}}{i} \cdot \frac{\partial}{\partial \vec{r}} + \beta m(r) \right) u_m(\vec{r}) \quad (1)$$

and normalization

$$\int d^3 \vec{r} |u_m|^2 = 1 .$$

The quark field operator is given by

$$q(\vec{r}, t) = \sum_m b_m u_m(\vec{r}) e^{-iE_m t} \quad (2)$$

^{*}) According to A.C. Davis, and to R.L. Jaffe, this is indeed so (private communications to J.S.B.).

where the b 's are quasiparticle destruction or creation operators according to whether E_m is positive or negative. When specializing to positive E_m we will replace the index m by p , and when specializing to negative E_m will replace m by n . Then the "empty box" $|\Omega\rangle$ is defined by

$$b_n^+ |\Omega\rangle = b_p |\Omega\rangle = 0 \quad (3)$$

for all n and p . Consider the particular state $|T\rangle$ obtained from $|\Omega\rangle$ by adding a quasiparticle in the lowest positive level, $p=0$:

$$|T\rangle = b_0^+ |\Omega\rangle \quad (4)$$

Then, generalizing to three-dimensions the formulae of Ref. 4), the corresponding parton distribution f and antiparton distribution \bar{f} are

$$\left. \begin{aligned} f(x, \vec{p}_\perp) &= \left(\frac{1}{2\pi}\right)^2 \frac{M}{\pi} \left| \int d^3\vec{r} \left(\frac{1+\vec{\alpha}\cdot\hat{n}}{2} \right) u_0(\vec{r}) e^{i(E_0-Mx)\vec{r}\cdot\hat{n} - i\vec{p}_\perp\cdot\vec{r}_\perp} \right|^2 + f_s(x, \vec{p}_\perp) \\ \bar{f}(x, \vec{p}_\perp) &= -\left(\frac{1}{2\pi}\right)^2 \frac{M}{\pi} \left| \int d^3\vec{r} \left(\frac{1+\vec{\alpha}\cdot\hat{n}}{2} \right) u_0(\vec{r}) e^{i(E_0+Mx)\vec{r}\cdot\hat{n} + i\vec{p}_\perp\cdot\vec{r}_\perp} \right|^2 + f_s(x, \vec{p}_\perp) \end{aligned} \right\} \quad (5)$$

$$f_s(x, \vec{p}_\perp) = \left(\frac{1}{2\pi}\right)^2 \frac{M}{\pi} \sum_p \left| \int d^3\vec{r} \left(\frac{1+\vec{\alpha}\cdot\hat{n}}{2} \right) u_p(\vec{r}) e^{i(E_p+Mx)\vec{r}\cdot\hat{n} + i\vec{p}_\perp\cdot\vec{r}_\perp} \right|^2 \quad (6)$$

where \hat{n} is a unit vector in the direction distinguished by choice of null plane, and \vec{p}_\perp and \vec{r}_\perp are components perpendicular to this. We will be interested also in the x distributions resulting from integration over \vec{p}_\perp ,

$$f_s(x) = \int d^2\vec{p}_\perp f_s(x, \vec{p}_\perp) \quad (7)$$

The first terms in f and \bar{f} , with appropriate allowance for colour and flavour, are the "valence" contributions [in the language of Ref. 4] considered by Jaffe and Hughes. Here we will concentrate on f_s , the "sea" or "empty box" contribution.

For the box model we take the limiting case in which

$$\left. \begin{aligned} m(r) &= 0 & \text{for} & \quad r \leq 1 \\ m(r) &= +\infty & \text{for} & \quad r > 1 \end{aligned} \right\} \quad (8)$$

Then for finite energy the u vanish outside the box and satisfy

$$(-i\beta \vec{\alpha} \cdot \vec{r} / r) u = u \quad (9)$$

on the inside of the boundary. In the usual representation,

$$\beta = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \quad \vec{\alpha} = \begin{pmatrix} & \sigma^1 \\ \sigma^2 & \end{pmatrix} \quad (10)$$

the wave functions have the form ⁵⁾

$$u_{k\ell sm} = \frac{1}{\sqrt{\pi D}} \begin{pmatrix} j_{\ell}(kr) s \sqrt{\frac{\ell+m\bar{s}+1/2}{\ell+1/2}} Y_{\ell, m-1/2}(\hat{r}) \\ j_{\ell}(kr) \sqrt{\frac{\ell-m\bar{s}+1/2}{\ell+1/2}} Y_{\ell, m+1/2}(\hat{r}) \\ -j_{\bar{\ell}}(kr) \bar{s} \sqrt{\frac{\bar{\ell}+m\bar{s}+1/2}{\bar{\ell}+1/2}} Y_{\bar{\ell}, m-1/2}(\hat{r}) \\ -j_{\bar{\ell}}(kr) \sqrt{\frac{\bar{\ell}-m\bar{s}+1/2}{\bar{\ell}+1/2}} Y_{\bar{\ell}, m+1/2}(\hat{r}) \end{pmatrix} \quad (11)$$

where with zero internal mass, (8),

$$k = E$$

Here the j 's are spherical Bessel functions and the Y 's are spherical harmonics (with the conventions of Landau and Lifshitz ⁶⁾) and k is a solution of the eigenvalue equation [arising from Eq. (9)]

$$j_{\bar{\ell}}(k) = s j_{\ell}(k) \quad (12)$$

In (11) and (12) l (integral) and m (half-integral) are the usual orbital and magnetic quantum numbers. In terms of the usual total angular momentum number j ,

$$\left. \begin{aligned} s &= 2j - 2l \\ \bar{s} &= 2j - 2\bar{l} = -s \\ \bar{l} &= 2j - l = l + s \end{aligned} \right\} \quad (13)$$

The normalization factor D is given by

$$D = \frac{4}{\pi} \left(j_l(k) \right)^2 \left(1 - s \frac{j+1/2}{k} \right) \quad (14)$$

We have taken R , the radius of the box, as unit of length, and as usual $\hbar = c = 1$.

The f 's involve Fourier transforms of the u 's

$$\tilde{u}(\vec{k}) = \int d^3 \vec{r} u(\vec{r}) e^{i\vec{k} \cdot \vec{r}} \quad (15)$$

Using the Dirac equation (1), the boundary condition (9), and some partial integration, the \tilde{u} can be expressed as surface integrals

$$\tilde{u}(k) = (\epsilon + \vec{\alpha} \cdot \vec{k})^{-1} \beta \int ds e^{i\vec{k} \cdot \vec{r}} u(\vec{r}) \quad (16)$$

- integrated over the surface $r = |\vec{r}| = 1$. Take the z axis along \vec{k} , so that

$$e^{i\vec{k} \cdot \vec{r}} = e^{ikz} = 2\sqrt{\pi} \sum_l (-1)^l \sqrt{\frac{l+1}{2l+1}} j_l(kr) Y_{l0}^*(\hat{r}) \quad (17)$$

Then it is clear that \tilde{u} vanishes except for magnetic quantum number

$$m = \pm 1/2 \quad (18)$$

In that case, using the eigenvalue equation (13) to simplify the expression,

$$\tilde{u} = \frac{2}{\sqrt{D}} \frac{\sqrt{2j+1}}{E^2 - K^2} j_2(K) (-1)^l \begin{pmatrix} (1/2+m)(E s j_2(K) - K j_{\bar{2}}(K)) \\ (1/2-m)(E j_2(K) - K s j_{\bar{2}}(K)) \\ (1/2+m)(E j_{\bar{2}}(K) - K s j_2(K)) \\ (1/2-m)(-E s j_{\bar{2}}(K) + K j_2(K)) \end{pmatrix} \quad (19)$$

For the evaluation of (6), we need

$$K = ((E + Mx)^2 + p_{\perp}^2)^{1/2} \quad (20)$$

$$\vec{\alpha} \cdot \hat{n} = \alpha_z \cos \theta + \alpha_x \sin \theta \quad (21)$$

(where we have taken the y axis at right angles to \hat{n} and \vec{p}_{\perp}) with

$$\cos \theta = (E + Mx) / K \quad (22)$$

$$\sin \theta = - p_{\perp} / K \quad (23)$$

The final result is

$$f_s(x, \vec{p}_{\perp}) = \frac{M}{16\pi^3} \sum_{\mathbf{p}} \left| (1 + \vec{\alpha} \cdot \hat{n}) \tilde{u}(K) \right|^2 = \left. \sum_{k, l, s} \frac{M(j+1/2)}{4\pi^2(1-(j+1/2)s/k)} \left(\frac{a^2(1+\cos\theta)}{(K+k)^2} + \frac{b^2(1-\cos\theta)}{(K-k)^2} \right) \right\} \quad (24)$$

where the summation is over $s = \pm 1$, $l = 0, 1, 2, \dots$, all positive roots k of (13), with K and $\cos\theta$ given by (20) and (22), and

$$a = s j_l(k) + j_{\bar{l}}(k) \quad (25)$$

$$b = s j_l(k) - j_{\bar{l}}(k) \quad (26)$$

where $\bar{l} = l + s$.

In (24) we integrate over p_1^2 to obtain

$$f_s(x) = \pi \int d p_1^2 f_s(x, \bar{p}_1) \quad (27)$$

It will be shown that the resulting series for $f_s(x)$ is divergent.

Since the terms in (24) are all positive, it will be sufficient to show divergence of a subseries integrated over a subregion. We will consider just the b terms, for $l \geq 1$, $s = +1$ ($j = l + \frac{1}{2}$, $\bar{l} = l + 1$), and with

$$k \gg N \epsilon \pi \quad (28)$$

$$p_1^2 \leq \delta k \quad (29)$$

where N is some large integer and ϵ some small constant. Then (20) and (22) imply

$$K - k = Mx + o(\delta) \quad (30)$$

$$(1 - \cos \theta) = \frac{1}{2} p_1^2 (k + Mx)^{-2} (1 + o(\delta/k)) \quad (31)$$

For the Bessel functions we use

$$k j_l(k) = \sin(k + \phi - \epsilon\pi/2) + \epsilon$$

$$\phi = \int_k^\infty dr (1 - \sqrt{1 - l(l+1)/r^2}) \quad (32)$$

where for $l(l+1)/k^2$ not too large *)

$$|\varepsilon| < C l(l+1) / k^2 \quad (33)$$

where C is some constant, independent of l and k . The eigenvalues k determined by (for $s=+1$)

$$b(k) = j_l(k) - j_{l+1}(k) = 0 \quad (34)$$

are

$$k = \pi(Nl + p + 3/4)(1 + o(N^{-2})) \quad (35)$$

where p is a positive integer or zero. Because of (34), in estimating $b(k)$ we need use (32) only to estimate the variation of the Bessel functions between arguments k and K . Then

$$|K b(k)| = \sin Mx + o(\delta) + o(N^{-2}) \quad (36)$$

Using (30), (31) and (36) in (24) we find a contribution, with k given by (35),

$$\approx \sum_{l \neq 0, p} \frac{M(l+3/2)}{4\pi^2} \frac{1}{2} \frac{\delta^2}{k^2} \left(\frac{\sin Mx}{kMx} \right)^2 \quad (37)$$

where the percentage error in each term can be reduced uniformly to any desired degree, for fixed x , by taking N big enough and δ small enough. The corresponding contribution to $f_s(x)$, by integration over p_1^2 subject to (29), is

$$\approx \sum_{l \neq 0, p} \frac{M(l+3/2)}{16\pi} \frac{\delta^2}{k^2} \left(\frac{\sin Mx}{Mx} \right)^2 \quad (38)$$

*) This can be seen for example by considering corrections to the LGJ-WKB approximation to solution of the differential equation for $rj_l(r)$ [by what amounts to a variation of Prüfer's method 7) (we are indebted to A. Martin for this reference)].

$$\approx \frac{M S^2}{16 \pi^3} \left(\frac{\sin Mx}{Mx} \right)^2 \sum_{\ell \neq 0, p} \frac{(\ell + 3/2)}{(N\ell + p)(N\ell + p + 1)} \quad (39)$$

(still with an arbitrarily small percentage error in each term)

$$\propto \sum_{\ell \neq 0} \frac{\ell + 3/2}{N\ell} \quad (40)$$

- which diverges.

It might be that convergence would be obtained with a less violent confining potential than an infinitely deep square well ^{*)}. And even with the latter, convergence would presumably result if we could project out a translationally invariant state, in view of the sum rule

$$\int dx \ x \ (f(x) + \bar{f}(x)) = 1$$

It would be good, of course, to be able to handle something which, like the bag model ¹⁾, and unlike the box model, is translationally invariant from the beginning.

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^{*)} Related ideas have been described by McCall and Squires ⁸⁾.

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