Control Problem for Nonlinear Systems Given by Klein-Gordon-Maxwell Equations with Electromagnetic Field*

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Abstract— **This paper is aim at realizing control for systems described by Klein-Gordon-Maxwell (K-G-M) equation. Theoretic approach will be formulated in the framework of variational theory. On the other hand, computational insight using semi-discrete numerical algorithm is consist of finite element method. Lastly, numerical experiments are evident the completely combination of theoretic and computation aspects.**

I. INTRODUCTION

A. Physics background

Using the Klein-Gordon-Maxwell electrodynamics, Schrödinger demonstrated that charged particles may be described by real fields. The rationale are considered for the Klein-Gordon-Maxwell electrodynamics, where the sets of solutions with real-valued electron-positron fields. Schrödinger considered interacting scalar charged field and electromagnetic field, and the Klein-Gordon-Maxwell equations of motion (cf. [3], [7], [16]).

B. Problem description

As is well known that the Klein-Gordon-Maxwell equations is extensively studied using numerous methodologies. Such as assigned electromagnetic fields in [1], [2] and [6], the interacting with an electromagnetic fields see references [4], [5], [8], [9], [10], [11] and [13]. Literatures survey show some formulations of K-G-M equation for physics consideration. For example, here is the most common description: $D_\mu D^\mu \phi = c^2\phi, \partial^\nu F_{\mu\nu} = \frac{1}{c}\mathcal{I}(\phi\overline{D_\mu\phi})$, where c is light speed, $\phi \in \mathbf{C}$ represents a particle field and $F_{\mu\nu}$ is the electromagnetic field tensor, and $\mathcal{I}$ is the imaginary part. For control problem, a new description will be adopted.

Let Ω be an open bounded set of $\mathbf{R}^3$. Set $Q = (0,T)\times\Omega$. The Klein-Gordon-Maxwell systems is described by

$$\begin{cases} -\psi_{xx} + e^2\phi^2\psi = -e\omega\phi^2 + u, \\ -\frac{\hbar^2}{m}\phi_{xx} + [m^2 - (\omega + e\psi)^2]\phi = |\psi|^{p-2}\phi + v, \\ \psi(0) = \psi_0, \phi(0) = \phi_0, \end{cases} \tag{1}$$

where $\psi, \phi : R^3 \to R$. Here $m > 0$ and $e > 0$ are the mass and the charge of the particle respectively, while $\omega > 0$ denotes the phase, and $\hbar$ is the Planck's constant. The variables of the system are the field ψ, ϕ associated to the particle and the electric potential. Here in (1), u and v are control inputs, and it's meaningful to make the assumption $m > \omega > 0$ and $2 \le p < 6$ for infinitely many radially symmetric solutions having bounded energy. The presence of the nonlinear term simulates the interaction between particles or external nonlinear perturbations. In [8] the regularity are $\psi \in H^1(\mathbf{R}^3), \phi \in D^{1,2}(\mathbf{R}^3)$, where $D^{1,2}(\mathbf{R}^3)$ is the completion of $C_0^\infty(\mathbf{R}^3, \mathbf{R})$ with respect to the norm of $\|\phi\|_{D^{1,2}} \equiv \left(\int_{\mathbf{R}^3}|\nabla\phi|^2dx\right)^{1/2}$. However the more common regularity is considered in our case.

This work is to explore the control problem for Klein-Gordon-Maxwell equations using control theory based on variational approach (cf. [15]). Furthermore, the computational issue is involved for one-dimensional case. The new theoretical contribution of the paper will concrete on attempting of two-particles control and its numerical realization.

The contents of this paper is consist of several sections. Section II is to establish the theoretic control theory for Klein-Gordon-Maxwell system. Section III will explore the numerical study of K-G-M equations using finite element approximate. In section IV, the laboratory simulation is carried out for interpreting the established control theory. Section V give concluding remark and future work.

II. CONTROL THEORY FOR K-G-M SYSTEMS

A. Weak solution

Define two Hilbert spaces $H = L^2(\Omega)$ and $V = H_0^1(\Omega)$ with usual norm and inner products. Then the pairing (V, H) is a Gelfand triple space, $V \hookrightarrow H \hookrightarrow V'$, their embedding are continuous, dense and compact.

Definition 2.1: The Hilbert space $W(0,T)$ is solution space defined by

$$W(0,T) = \Big\{(\psi,\phi) \mid \psi \in L^2(0,T;V), \psi' \in L^2(0,T;V'), \phi \in L^2(0,T;V), \phi' \in L^2(0,T;V')\Big\}.$$

Definition 2.2: Let $T > 0$, the pairing (ψ,ϕ) are weak solutions of (1) when ψ, $\phi \in W(0,T)$ satisfy

$$\begin{cases} \int_0^T\int_\Omega [\psi_x\eta_x + e^2\phi^2\psi\eta]\ dxdt \\ = -\int_0^T\int_\Omega e\omega\phi^2\eta\ dxdt + \int_0^T\int_\Omega u\eta\ dxdt \\ \frac{\hbar}{m}\int_0^T\int_\Omega \phi_x\rho_x + [m^2 - (\omega + e\psi)^2]\phi\rho\ dxdt \\ = \int_0^T\int_\Omega |\psi|^{p-2}\rho\ dxdt + \int_0^T\int_\Omega v\rho\ dxdt \end{cases} \tag{2}$$

for all $\eta, \rho \in C^1(0,T;V)$ and such that $\eta(T) = \rho(T) = 0$ a.e. $t \in [0,T]$.

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Theorem 2.3: Given $\psi_0, \phi_0 \in V$, then there exists a unique weak solution of system (1).

The proof of Theorem 2.3 can be completed referring to Faedo-Galerkin method in [12].

B. Control problem

Let $\mathbf{u} = (u, v)$, and $\mathcal{U}$ is control space of variables $u, v \in \mathcal{U}$. The optimal criteria associated with (1) is given by

$$J(\mathbf{u}) = \|\psi(\mathbf{u}, T) - z_d^1\|_V^2 + \|\phi(\mathbf{u}, T) - z_d^2\|_V^2 + (\mathbf{u}, \mathbf{u})_{\mathcal{U}}, \quad (3)$$

for all $\mathbf{u} \in \mathcal{U}_{ad} \times \mathcal{U}_{ad}$, where $z_d^1, z_d^2 \in V$ are desired values of $\psi(\mathbf{u})$ and $\phi(\mathbf{u})$, respectively. Let $\mathcal{U}_{ad}$ be a closed and convex subset of $\mathcal{U}$, which called the admissible set.

Suppose $\mathcal{U} = L^2(0, T)$, then Theorem 2.3 deduce that there exists a unique weak solution $\psi(\mathbf{u}), \phi(\mathbf{u}) \in W(0, T)$ for any $\mathbf{u} \in \mathcal{U}$. Furthermore, by the analogs manipulation as in [15] and [12] to prove Theorem 2.4 and 2.5.

Theorem 2.4: Let $\psi_0, \phi_0 \in V$. If $\mathcal{U}_{ad}$ is bounded, then there exists at least one optimal control $\mathbf{u}^* = (u^*, v^*)$ for cost (3) subject to systems (1).

Theorem 2.5: The optimal control $\mathbf{u}^*$ for cost (3) is characterized by optimality system consisting of state equation (1), adjoint equation (4) and necessary condition (5):

$$\begin{cases} -p_{xx} + e^2\phi^2(\mathbf{u}^*)p \\ = -2[\omega + e\psi(\mathbf{u}^*)]\phi(\mathbf{u}^*)q + u_0 \quad \text{in } Q, \\ -\frac{\hbar}{m}q_{xx} + [(m^2 - (\omega + e\psi(\mathbf{u}^*))^2]q \\ = 2[\omega + e\psi(\mathbf{u}^*)]\phi(\mathbf{u}^*)p + |\phi(\mathbf{u}^*)|^{p-2}q + v_0 \quad \text{in } Q, \\ p(T) = \psi(\mathbf{u}^*, T) - z_d^1 \text{ in } (0, l). \\ q(T) = \phi(\mathbf{u}^*, T) - z_d^2 \quad \text{in } (0, l). \\ p, q \in W(0, T). \end{cases} \quad (4)$$

$$(u^*, u - u^*)_{\mathcal{U}} + (v^*, v - v^*)_{\mathcal{U}} + \int_Q p(\mathbf{u}^*)(u - u^*)\, dxdt$$
$$+ \int_Q q(\mathbf{u}^*)(v - v^*) dxdt \geq 0 \quad \forall \mathbf{u} = (u, v) \in \mathcal{U}_{ad}^2. \quad (5)$$

The highlighted point in this section is attempting firstly to seek theoretic control conclusions of quantum optimal control for two particles system described by KGM equations.

III. NUMERICAL STUDY

A. Numerical solution

Let $0 = x_0 < x_1 < \cdots < x_N < x_{N+1} = l$ be a partition of the interval $[0, l]$ into subintervals $I_e = [x_{e-1}, x_e)$ of length $h^e = x_e - x_{e-1}, e = 1, 2, \cdots, N+1$. Let V_h be a set of functions b_i^e for $i = 1, 2, 3$. $e = 1, 2, \cdots, N+1)$ such that b_i^e is quadratic function on each interval I_e, and continuous on $[0, l]$. Then it's clear that $V_h \subset H_0^1(0, l)$(cf. [18]). The $b_i^e \in V_h$ is given by

$$\begin{cases} b_1^e(x) = \Big(1 - \dfrac{x - x_e}{h^e}\Big)\Big(1 - \dfrac{2(x - x_e)}{h^e}\Big), \\ b_2^e(x) = \dfrac{4(x - x_e)}{h^e}\Big(1 - \dfrac{x - x_e}{h^e}\Big), \\ b_3^e(x) = -\dfrac{(x - x_e)}{h^e}\Big(1 - \dfrac{2(x - x_e)}{h^e}\Big). \end{cases}$$

Its interpolation properties see [17]. The total approximate solution can be represented as

$$\begin{cases} \psi_h(t, x) = \displaystyle\sum_{e=1}^N \psi_h^e(t, x) = \sum_{e=1}^N \sum_{i=1}^3 \xi_i^e(t) b_i^e(x) \in V_h, \\ \phi_h(t, x) = \displaystyle\sum_{e=1}^N \phi_h^e(t, x) = \sum_{e=1}^N \sum_{i=1}^3 \zeta_i^e(t) b_i^e(x) \in V_h. \end{cases}$$

Thus by (2) to find ψ_h^e and ϕ_h^e satisfy

$$\begin{cases} \displaystyle\sum_{i=1}^3 \xi_i^e(b_{ix}^e, b_{jx}^e) + e^2\Big(\sum_{i=1}^3 \zeta_i^e b_i^e\Big)^2 \sum_{i=1}^3 \xi_i^e(b_i^e, b_j^e) \\ = -\omega e\Big((\displaystyle\sum_{i=1}^3 \zeta_i^e b_i^e)^2, b_j^e\Big) + \sum_{i=1}^3 (u, b_j^e), \\ \dfrac{\hbar}{m}\displaystyle\sum_{i=1}^3 \zeta_i^e(b_{ix}^e, b_{jx}^e) + [m^2 - (\omega + e\sum_{i=1}^3 \xi_i^e b_i^e)^2]\sum_{i=1}^3 \zeta_i^e(b_i^e, b_j^e) \\ = \Big(|\displaystyle\sum_{i=1}^3 \xi_i^e b_i^e|^{p-2}, b_j^e\Big) + \sum_{i=1}^3 (v, b_j^e). \end{cases} \quad (6)$$

with $\sum_{i=1}^3 \xi_i^e(b_i^e, b_j^e) = \psi_0, \sum_{i=1}^3 \zeta_i^e(b_j^e, b_j^e) = \phi_0$ and $\sum_{i=1}^3 \zeta_i^e(t)(b_j^e, b_j^e) = \phi_1$. Set

$$B^e = ((b_{ij})) = (b_i^e, b_j^e)_{i=1,2,3}^{j=1,2,3} \in M_{3\times3}(\mathbf{R}),$$
$$D^e = ((d_{ij})) = (\nabla b_i^e, \nabla b_j^e)_{i=1,2,3}^{j=1,2,3} \in M_{3\times3}(\mathbf{R}),$$
$$\Xi^e(t) = [\xi_1^e(t), \xi_2^e(t), \xi_3^e(t)]^t \in M_{3\times1}(\mathbf{R}),$$
$$\Sigma^e(t) = [\zeta_1^e(t), \zeta_2^e(t), \zeta_3^e(t)]^t \in M_{3\times1}(\mathbf{R}),$$
$$L_1^e(t) = (l_{11}^e, l_{12}^e, l_{13}^e) \in M_{3\times1}(\mathbf{R}), \text{ where}$$
$$l_{1j}^e = e^2\Big(\sum_{i=1}^3 \zeta_i^e b_i^e\Big)^2 \xi_j^e(b_i^e, b_j^e), \ j = 1, 2, 3.$$
$$N_1^e(t) = (n_{11}^e, n_{12}^e, n_{13}^e) \in M_{3\times1}(\mathbf{R}), \text{ where}$$
$$n_{1j}^e = \Big(-\omega e(\sum_{i=1}^3 \zeta_i^e b_i^e)^2, b_j^e\Big), \ j = 1, 2, 3.$$
$$U^e(t) = [(u(t), b_1^e), (u(t), b_2^e), (u(t), b_3^e)]^t \in M_{3\times1}(\mathbf{R}),$$
$$L_2^e(t) = (l_{21}^e, l_{22}^e, l_{23}^e) \in M_{3\times1}(\mathbf{R}), \text{where}$$
$$l_{2j}^e = \Big(m^2 - (\omega + e\sum_{i=1}^3 \xi_i^e b_i^e)^2\Big)\zeta_j^e(b_i^e, b_j^e), \ j = 1, 2, 3.$$
$$N_2^e(t) = (n_{21}^e, n_{22}^e, n_{23}^e) \in M_{3\times1}(\mathbf{R}), \text{ where}$$
$$n_{2j}^e = \Big(|\sum_{i=1}^3 \xi_i^e b_i^e|^{p-2}, b_j^e\Big), \ j = 1, 2, 3.$$
$$X_0^e = [(\psi_0, b_1^e), (\psi_0, b_2^e), (\psi_0, b_3^e)]^t \in M_{3\times1}(\mathbf{R}).$$

$V^e(t)$ has the same structure with $U^e(t)$ just instead of $u(t)$ with $v(t)$. Y_0^e has the same structure with X_0^e just replace the ψ_0 with ϕ_0. The continuity of $\xi_i^e(t), \zeta_i^e(t)$ on $[0, T]$ implies that $\xi_3^e(t) = \xi_1^{e+1}(t), \zeta_3^e(t) = \zeta_1^{e+1}(t)$ for $e = 1, 2, ..., N$.

Let's introduce the following matrixes and vectors.

$$D=\begin{bmatrix} d^1_{11} & d^1_{12} & d^1_{13} & & & & \\ d^1_{21} & d^1_{22} & d^1_{23} & & & & \\ d^1_{31} & d^1_{32} & d^1_{33}+d^2_{11} & & 0 & & \\ & & d^2_{21} & & & & \\ & & d^2_{31} & & & & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\ & & 0 & & d^{N-1}_{33}+d^N_{11} & d^N_{12} & d^N_{13} \\ & & & & d^N_{21} & d^N_{22} & d^N_{23} \\ & & & & d^N_{31} & d^N_{32} & d^N_{33} \end{bmatrix}.$$

Hence B has same structure as D with b_{ij} instead of d_{ij}.

$$\Xi=\begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \\ \xi_5 \\ \vdots \\ \xi_{2N-1} \\ \xi_{2N} \\ \xi_{2N+1} \end{bmatrix}=\begin{bmatrix} \xi^1_1 \\ \xi^1_2 \\ \xi^1_3(=\xi^2_1) \\ \xi^2_2 \\ \xi^2_3(=\xi^3_1) \\ \vdots \\ \xi^{N-1}_3(=\xi^N_1) \\ \xi^N_2 \\ \xi^N_3 \end{bmatrix},\Sigma=\begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \\ \zeta_4 \\ \zeta_5 \\ \vdots \\ \zeta_{2N-1} \\ \zeta_{2N} \\ \zeta_{2N+1} \end{bmatrix}=\begin{bmatrix} \zeta^1_1 \\ \zeta^1_2 \\ \zeta^1_3(=\zeta^2_1) \\ \zeta^2_2 \\ \zeta^2_3(=\zeta^3_1) \\ \vdots \\ \zeta^{N-1}_3(=\zeta^N_1) \\ \zeta^N_2 \\ \zeta^N_3 \end{bmatrix}.$$

$$L_1=\begin{bmatrix} L^1_1 \\ L^2_1 \\ L^3_1 \\ L^4_1 \\ L^5_1 \\ \vdots \\ L^{2N-1}_1 \\ L^{2N}_1 \\ L^{2N+1}_1 \end{bmatrix}=\begin{bmatrix} l^1_{11} \\ l^1_{12} \\ l^1_{13}+l^2_{11} \\ l^2_{12} \\ l^2_{13}+l^3_{11} \\ \vdots \\ l^{N-1}_{13}+l^N_{11} \\ l^N_{12} \\ l^N_{13} \end{bmatrix},U=\begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \\ \vdots \\ U_{2N-1} \\ U_{2N} \\ U_{2N+1} \end{bmatrix}=\begin{bmatrix} u^1_1 \\ u^1_2 \\ u^1_3+u^2_1 \\ u^2_2 \\ u^2_3+u^3_1 \\ \vdots \\ u^{N-1}_3+u^N_1 \\ u^N_2 \\ u^N_3 \end{bmatrix}.$$

Hence L_2 has same structure as L_1 with L^j_2 instead of L^j_1 and l^e_{2j} instead of l^e_{1j}. V has same structure as U with V_j instead of U_j and v^e_j instead of u^e_j. The matrix N_1, N_2 have the same configures with L_1, L_2, just their components is composed of N^e_1, N^e_2, respectively.

$$X_0=\begin{bmatrix} X^1_{01} \\ X^1_{02} \\ X^1_{03}(=X^2_{01}) \\ X^2_{02} \\ X^2_{03}(=X^3_{01}) \\ \vdots \\ X^{N-1}_{03}(=Y^N_{01}) \\ X^N_{02} \\ X^N_{03} \end{bmatrix},\quad Y_0=\begin{bmatrix} Y^1_{01} \\ Y^1_{02} \\ Y^1_{03}(=Y^2_{01}) \\ Y^2_{02} \\ X^2_{03}(=Y^3_{01}) \\ \vdots \\ Y^{N-1}_{03}(=Y^N_{01}) \\ Y^N_{02} \\ Y^N_{03} \end{bmatrix}.$$

Hence X_0 has same structure as Y_0 with X^e_{0j} instead of Y^e_{0j}. Then by (6), the overall equation in the vector form can be expressed as

$$\begin{cases} D\Xi + L_1(\Xi,\Sigma) = N_1 + U, \\ \frac{\hbar}{m} D\Sigma + L_2(\Xi,\Sigma) = N_2 + V, \\ \Xi(0) = X_0,\ \Sigma(0) = Y_0. \end{cases} \tag{7}$$

As in [17], applying Gauss-Legendre integrate method to the components of g_i, divide the element interval $[x_e, x_{e+1}]$ into $m = 6$ points to obtain the abscissas $p^e_1, p^e_2, \cdots, p^e_m$ on $[x_e, x_{e+1}]$ and weights $r_1, r_2, \cdots, r_m$. Thus S_1, S_2 are approximated by the new function $\hat{S}_1, \hat{S}_2$, respectively. By introducing

$$\mathbf{M}=\begin{pmatrix} D & 0 \\ 0 & \frac{\hbar}{m}D \end{pmatrix},\ \mathbf{L}=\begin{pmatrix} L_1(\Xi,\Sigma) \\ L_2(\Xi,\Sigma) \end{pmatrix},\ \mathbf{N}=\begin{pmatrix} N_1 \\ N_2 \end{pmatrix}.$$

$$\mathbf{\Sigma}=\begin{pmatrix} \Xi \\ \Sigma \end{pmatrix},\mathbf{U}=\begin{pmatrix} U \\ V \end{pmatrix},\ \mathbf{\Sigma}(0)=\begin{pmatrix} \Xi(0) \\ \Sigma(0) \end{pmatrix}$$

with initial $\mathbf{\Sigma}(0) = [X_0, Y_0]^t$. Since the inverse of $\mathbf{M}$ is exists, then (7) implies that

$$\mathbf{\Sigma} = \mathbf{M}^{-1}(-\mathbf{L} + \mathbf{N} + \mathbf{U}). \tag{8}$$

with initial guess $\mathbf{\Sigma}(0)$. Therefore, the first order ODE (8) can be solved using the 4th order Runge-Kutta method (cf.[18]). Using ξ^e_i and $\zeta^e_i (i = 1, 2, 3)$ to obtain the numerical solution on domain $[0, T] \times (0, l)$. The convergency of the approach refer to [17].

B. Numerical control solution

Let $\mathbf{u}_h = (u_h, v_h)$ be the approximate control of $\mathbf{u} = (u, v)$. The formulation of finite element approximation is by minimizing the approximate cost functions,

$$J_h = J(\mathbf{u}_h) = \int_0^l (\psi_h(T) - z^1_d)^2 dx + \int_0^l (\phi_h(T) - z^2_d)^2 dx + \int_0^T (u_h - u^d)^2 dt + \int_0^T (v_h - v^d)^2 dt. \tag{9}$$

Theorem 3.1: The existence theorem of optimal control in [14] implies that there exists at least a minimizer to the finite element problems (9).

Denote the Gâteaux derivative of J_h at any point $\mathbf{u}_h$ and given direction $\bar{\varphi}_h \in V_h$ by $J'_h(\mathbf{u}_h)\bar{\varphi}_h$. The Gâteaux derivative for solution ψ_h at any direction $\bar{\varphi}_h \in V_h$ denotes as $\psi'_h\bar{\varphi}_h$ satisfying (2). The discrete adjoint system and necessary optimality condition (5) for $\mathbf{u}^* = (u^*_h, v^*_h) \in U_{ad} \times U_{ad}$ can be obtained easily.

Theorem 3.2: Let $\{\mathbf{u}^k\}$ be a sequence of minimizer to finite element problems (9). Then each subsequence of $\{\mathbf{u}^k\}$ has a sub-subsequence convergence in $L^2(0, l)$ minimizer of the continuous problems (9).

C. Computational procedure

Suppose $\mathbf{u}^k_h(t) = (u^k_h(t), v^k_h(t))$ is available at iteration k, $\mathbf{u_k} = \{u^k_h\}, k = 1, 2, \cdots$ are minimize sequence of $\{u^k_h\}$ such that the cost function (9) achieve minimization.

Step 1 For given $\mathbf{z}^d = (z^1_d, z^2_d) =$ and $\mathbf{u}(0) \in \mathcal{U}^2_{ad}$, using construct approximate solution $\psi_h(x, t), \phi_h(x, t)$ to solve the directly problems for state equation for ψ. Let $P^k(t) := -\bigtriangledown J(\mathbf{u}(0))$; $k := 0$.

Step 2 Compute the search step size β^k such that

$$J_h(\mathbf{u}^k_h + \beta^k \mathbf{u}^k_h) = \min\{J_h(\mathbf{u}^k_h + \beta\mathbf{u}^k_h); \beta \geq 0\}.$$

Given $0 < \xi < \dfrac{1}{2}$ and $0 < \tau < 1$. Let $\rho^0 = 1$, for $m = 0, 1, 2, \cdots$, do, If

$$J_h(\mathbf{u}_h^k - \rho^m P^k) \le J_h(\mathbf{u}_h^k) - \xi \rho^m J_h'(\mathbf{u}_h^k)^2,$$

then $\beta^n = \rho^m$; else $\rho^{m+1} = \tau \rho^m$.

Step 3 $\mathbf{u}_h^{k+1} := \mathbf{u}_h^k + \beta^k P^k$. The convergence of iteration procedure in minimizing J_h is guaranteed in [14].

Step 4 The stopping criterion ε is a small specified number. If $J_h(\mathbf{u}_h^{k+1}) < \varepsilon$, then stop ($\mathbf{u}_h^{k+1}$ is the solution).

Step 5 Compute the gradient of $J_h'(t)$ to obtain $J_h'(\mathbf{u}_h^k)\varphi_h$ for all $\varphi_h \in V_h$.

Step 6 Compute the updated conjugate coefficients

$$\gamma^k = \frac{\varepsilon_1 \int_0^T (J_h'^k)^2 dt}{\varepsilon_2 \int_0^T (J_h'^{(k-1)})^2 dt}, \quad \text{with} \quad \gamma^0 = 0.$$

Using $\varepsilon_1, \varepsilon_2$ to get proper γ^k.

Step 7 Compute the directions of descent $P^k(t) := J_h'^k(t) + \gamma^k P^{k-1}(t)$. $k := k + 1$; return to step 2, and so on.

D. *Convergence of nonlinear algorithm*

The convergency proof can be given as in [17], hence $J_h(\mathbf{u}_h)$ can be minimized by sequence $\{\mathbf{u}_h^k\}$. Let $\mathbf{u}_h$ denotes the solution of discrete problem such that $\mathbf{u}_h \to \mathbf{u}^*$. It's clearly that $\mathbf{u}_h$ towards to $\mathbf{u}^*$ in the order of $O(h)$ as $h \to 0$.

IV. LABORATORY EXPERIMENTS

Let $\Omega = (0,1)$, $t_0 = 0.0, T = 1.0, h = \frac{1}{23}$ and take $\varepsilon = 0.02$ in Step 4 of part C in Section III. The desired state $z_d^1 = e^{(-\frac{5(x-0.5)}{0.5})^{10}} \sin(\omega(x - 0.5))$ and $z_d^2 = 0.5\sin(3\pi x)$. Take initial functions $\psi(0) = e^{(-\frac{x-0.5}{0.5})^{10}} \sin(\omega(x-0.5)), \phi(0) = \sin(3\pi x)$. Set the physics constants $m = 9.10938188 \times 10^{-31}$, $\omega = 9.10938188 \times 10^{-32}$, $p = 5$, $e = 1.602176462 \times 10^{-19}$ and $\hbar = 1.0545715964207855 \times 10^{-34}$. Let $c_1 = 0.05, c_2 = 0.5 \times 10^{-4}$, and experiment control inputs as $c_1 u$ and $c_2 v$. The initial and desired states are shown in Figures 1-2. The start controls functions $u_0(t) = 1 + 2\sin(\frac{5\pi t}{T}), v_0(t) = 1 + 2\cos(\frac{7\pi t}{T})$, the desired controls $u_T(t) = 1 + 0.0001\sin(\frac{5\pi t}{T}), v_T(t) = 1 + 0.0002\cos(\frac{7\pi t}{T})$. See Figures 3-4.

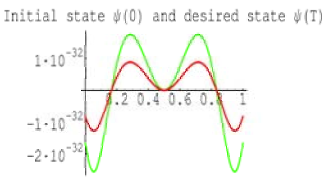

Fig. 1 $\psi(0), \psi(T)$

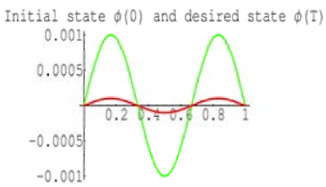

Fig. 2 $\phi(0), \phi(T)$

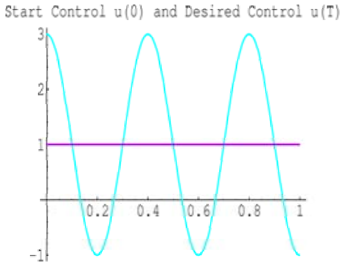

Fig.3 $u(0), u(T)$

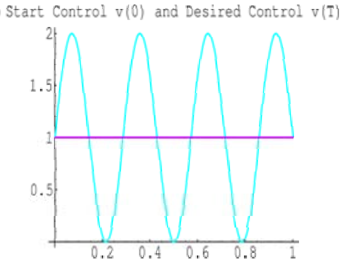

Fig.4 $v(0), v(T)$

Contour plots of solution at some iterations see Figures 5-16.

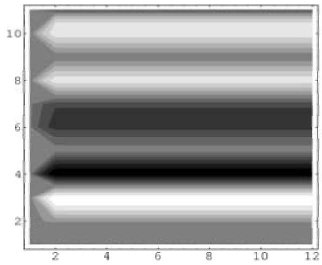

Fig. 5 Contour plot of ψ

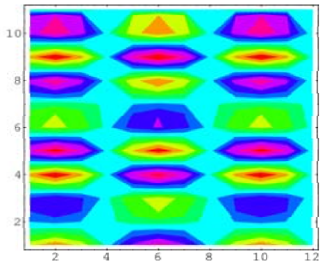

Fig. 6 Contour plot of ϕ

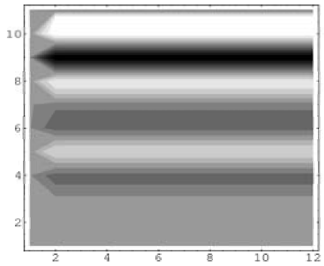

Fig. 7 Contour plot of ψ

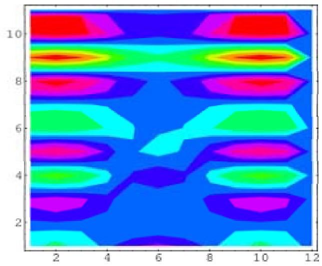

Fig. 8 Contour plot of ϕ

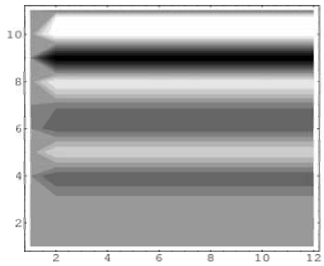

Fig. 9 Contour plot of ψ

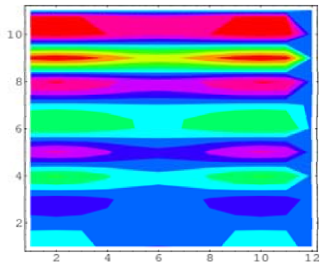

Fig. 10 Contour plot of ϕ

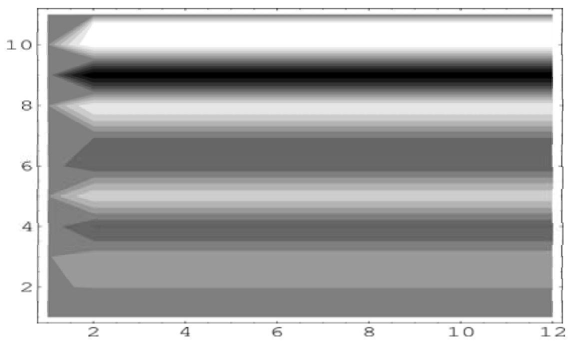

Fig. 11 Contour plot of ψ

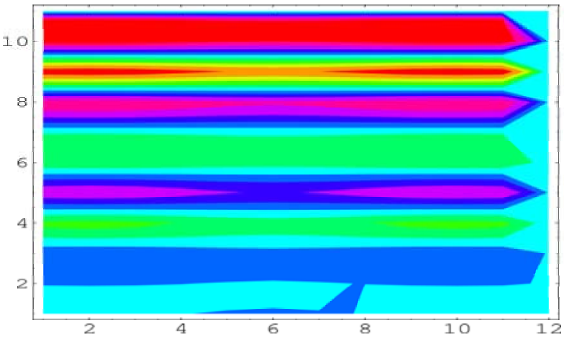

Fig. 12 Contour plot of ϕ

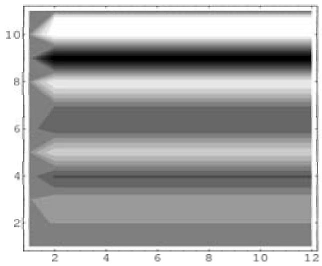

Fig. 13 Contour plot of ψ

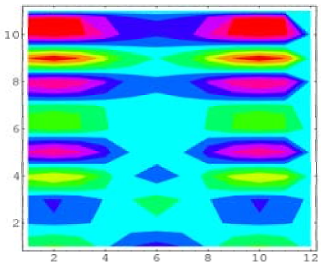

Fig. 14 Contour plot of ϕ

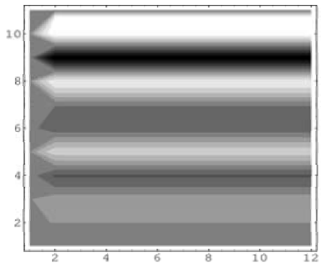

Fig. 15 Contour plot of ψ

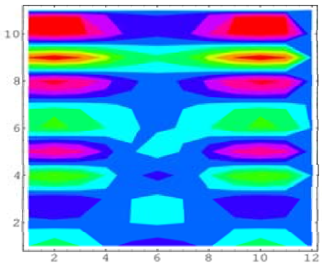

Fig. 16 Contour plot of ϕ

Remark 4.1: The Maxwell equation express the electromagnetic phenomena, and the KG equation changes its state from wave phenomena to electromagnetic phenomena in control iterations.

The vector plots of electromagnetic field for Maxwell equations in first and last iterations are shown in Figures 17-18.

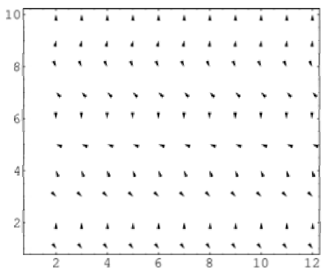

Fig. 17 ψ in first step

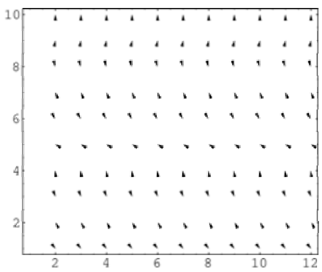

Fig. 18 ψ in last step

Optimal control functions obtained as

$$u^* = 0.999933 + 0.000199858\cos(5\pi t) + 0.00128123\sin(5\pi t);$$
$$v^* = 0.95173 - 0.822073\cos(7\pi t) + 0.00021738\sin(7\pi t).$$

Optimal control graphics of two systems in Figures 19-20.

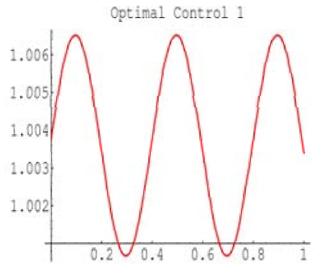

Fig. 19 Optimal control u^*

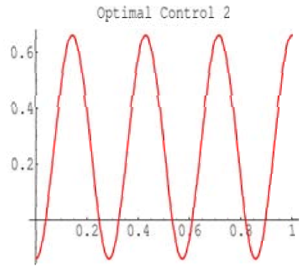

Fig. 20 Optimal control v^*

Controls functions iteration are listed in Figures 21-22.

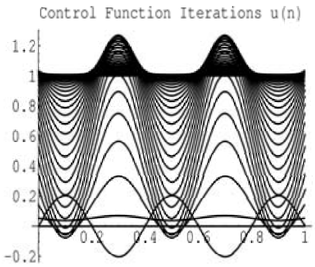

Fig. 21 Controls iteration u^k

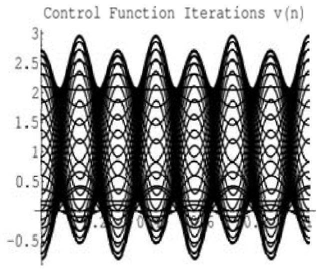

Fig. 22 Control iterations v^k

Optimal cost function value calculated as:

$$J(\mathbf{u}^*) = 0.461625.$$

The cost functions is shown in Figure 23.

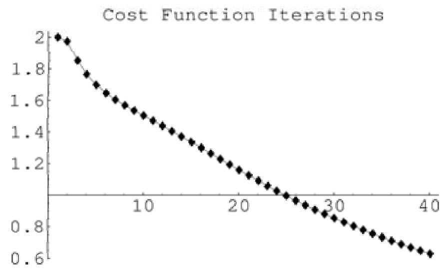

Fig. 23 Cost iterations $J(\mathbf{u}^k)$

For $\mathbf{u} = (u, v)$, the cost iterations are calculated in below.

$J(1) = 2.00002$; $J(2) = 1.97408$; $J(3) = 1.85106$;
$J(4) = 1.76527$; $J(5) = 1.69754$; $J(6) = 1.64626$;
$J(7) = 1.60478$; $J(8) = 1.5686$; $J(9) = 1.53517$;
$J(10) = 1.50289$; $J(11) = 1.47076$; $J(12) = 1.43814$;
$J(13) = 1.40472$; $J(14) = 1.37041$; $J(15) = 1.33534$;
$J(16) = 1.29977$; $J(17) = 1.26400$; $J(18) = 1.22832$;
$J(19) = 1.19299$; $J(20) = 1.15818$; $J(21) = 1.12405$;
$J(22) = 1.09067$; $J(23) = 1.05811$; $J(24) = 1.0264$;
$J(25) = 0.965562$; $J(26) = 0.965562$; $J(27) = 0.936434$;
$J(28) = 0.908152$; $J(29) = 0.880701$; $J(30) = 0.854062$;
$J(31) = 0.828218$; $J(32) = 0.803147$; $J(33) = 0.778828$;
$J(34) = 0.755241$; $J(35) = 0.732365$ $J(36) = 0.71018$;
$J(37) = 0.688665$; $J(38) = 0.66780$; $J(39) = 0.647567$;
$J(40) = 0.627946$; $J(41) = 0.608919$; $J(42) = 0.590469$;
$J(43) = 0.572577$; $J(44) = 0.555227$; $J(45) = 0.538402$;
$J(46) = 0.522088$; $J(47) = 0.506267$; $J(48) = 0.490926$;
$J(49) = 0.47605$; $J(50) = 0.461625$.

Error estimates for cost functions at each listed as follows.

eJ[1]= −−;	eJ[2]= 0.0259364;	eJ[3]= 0.123026;
eJ[4]= 0.0857827;	eJ[5]= 0.0677341;	eJ[6]= 0.0512748;
eJ[7]= 0.0414829;	eJ[8]= 0.0361784;	eJ[9] = 0.0334364;
eJ[10]= 0.322785;	eJ[11]= 0.321319;	eJ[12]= 0.0326146;
eJ[13]= 0.0334246;	eJ[14]= 0.0343081;	eJ[15] = 0.0770371.
eJ[16]= 0.0355712;	eJ[17]= 0.0357717;	eJ[18]= 0.035678;
eJ[19]= 0.035336;	eJ[20]= 0.0348034;	eJ[21] = 0.034135;
eJ[22]= 0.0333758;	eJ[23]= 0.03256;	eJ[24]= 0.0317124;
eJ[25]= 0.0299864;	eJ[26]= 0.0299864;	eJ[27]= 0.0291284;
eJ[28]= 0.0282821;	eJ[29]= 0.0274512;	eJ[30]= 0.0266382;
eJ[31]= 0.0258446;	eJ[32]= 0.0250713;	eJ[33]= 0.0243187;
eJ[34]= 0.0235868;	eJ[35]= 0.0228758;	eJ[36]= 0.0221854;
eJ[37]= 0.021515;	eJ[38]= 0.0208645;	eJ[39]= 0.0202333;
eJ[40]= 0.0196209;	eJ[41]= 0.0190269;	eJ[42]= 0.0184507;
eJ[43]= 0.0178919;	eJ[44]= 0.0173499;	eJ[45]= 0.0168243;
eJ[46]= 0.0163146;	eJ[47]= 0.0158203;	eJ[48]= 0.015341;
eJ[49]= 0.0148762;	eJ[50]= 0.0144254.	

Let η_i represents the left hand of necessary condition (5) at ith iteration, then

$$\begin{aligned}
&\eta_1 = -3.1372, && \eta_2 = -3.15292, && \eta_3 = -2.8756,\\
&\eta_4 = -2.55266, && \eta_5 = -2.19919, && \eta_6 = -1.84938,\\
&\eta_7 = -1.5099, && \eta_8 = -1.18069, && \eta_9 = -0.862185,\\
&\eta_{10} = -0.556352, && \eta_{11} = -0.267066, && \eta_{12} = -0.252251,\\
&\eta_{13} = 0.237728, && \eta_{14} = 0.438674, && \eta_{15} = 0.595879,\\
&\eta_{16} = 0.703969, && \eta_{17} = 0.759743, && \eta_{18} = 0.76237,\\
&\eta_{19} = 0.713267, && \eta_{20} = 0.615796, && \eta_{21} = 0.47492,\\
&\eta_{22} = 0.296857, && \eta_{23} = 0.088758, && \eta_{24} = -0.14159,\\
&\eta_{25} = -0.386042 && \eta_{26} = -0.63636, && \eta_{27} = -0.884463,\\
&\eta_{28} = -1.12266, && \eta_{29} = -1.34385, && \eta_{30} = -1.54173,\\
&\eta_{31} = -1.71093, && \eta_{32} = -1.84715, && \eta_{33} = -1.94725,\\
&\eta_{34} = -2.00928, && \eta_{35} = -2.03254 && \eta_{36} = -2.01753,\\
&\eta_{37} = -1.96589, && \eta_{38} = -1.88033, && \eta_{39} = -1.7645,\\
&\eta_{40} = -1.62288, && \eta_{41} = -1.46056, && \eta_{42} = -1.28312,\\
&\eta_{43} = -1.09643, && \eta_{44} = -0.906454, && \eta_{45} = -0.719063,\\
&\eta_{46} = -0.53988, && \eta_{47} = -0.374105, && \eta_{48} = -0.226369,\\
&\eta_{49} = -0.100616, && \eta_{50} = 0.
\end{aligned}$$

Necessary optimality condition is checked numerically.

V. CONCLUSIONS AND FUTURE WORKS

This work investigated the control problem for Klein-Gordon-Maxwell equations. Both theoretic and numerical study are considered completely. Experiment demonstration interpret that the approach is efficiently and can be applied to widely nonlinear control systems. For example, the Klein-Gordon-Schrodinger system (cf. [19]) and diffusion neural network system (cf. [20],[21]).

Several key problems will be proposed in the future research. (i). The laboratory realization for the control problems. (ii). Large scale computations. (iii). Different fields interdisciplinary research. (iv). Research topics concerning with control processing. (v). Other controls (e.g. initial control) application. (vi). Two and three spatial dimensions control problems. All of these will be hopeful directions in the future.

VI. ACKNOWLEDGMENTS

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