

# Space-time vector supersymmetry and massive spinning particle

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**ABSTRACT:** We construct the action of a relativistic spinning particle from a non-linear realization of a space-time odd vector extension of the Poincaré group. For particular values of the parameters appearing in the lagrangian the model has a gauge world-line supersymmetry. As a consequence of this local symmetry there are BPS solutions in the model preserving 1/5 of the supersymmetries. A supersymmetric invariant quantization produces two decoupled 4d Dirac equations.

**KEYWORDS:** Space-Time Symmetries, D-branes, Extended Supersymmetry.

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## 1. Introduction

Supersymmetry plays a crucial role in field theories, supergravities and String/M theory. In flat space-time supersymmetry is characterized by the presence of odd spinor charges that together with the generators of the Poincaré group form the target space Super Poincaré group.

Strings with the space-time supersymmetry are described by the Green-Schwarz action [1]. The fermionic components of the string are realized by spinorial fields in target space. The associated particle model (superparticle), that was introduced before [2, 3] has, for a particular value of the two coefficients of the lagrangian, i.e., for the parameters of Nambu-Goto and Wess-Zumino pieces, a fermionic gauge symmetry called kappa symmetry [4, 5]. The covariant quantization of this model is an unsolved issue.

There is an alternative to the GS string known as spinning string, or Neveu-Schwarz-Ramond string [6],<sup>1</sup> that has world "line" supersymmetry. The fermionic components of the string are described by odd vector fields in target space. A truncation of this theory known as GSO projection [7] produces a spectrum that is space-time supersymmetric invariant.

The corresponding particle model (spinning particle) was introduced in refs. [8–10]. The quantization of this model reproduces the four dimensional Dirac equation.

In this paper we will consider an odd vector extension of the Poincaré group, to be called the Vector Super Poincaré group  $G$ , first formulated in [8]. This symmetry was

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<sup>1</sup>formulated prior to the GS string

introduced with the aim of obtaining a pseudo-classical description of the Dirac equation. However this result was obtained by using a constraint breaking the symmetry itself. In this paper we want to take full advantage of this new symmetry. We will show that the massive spinning particle action of reference [8] can be obtained by applying the method of non-linear realizations [11] to this group. However, trying to preserve the target space supersymmetry under quantization we will obtain two copies of the 4d Dirac equation. On the other hand, breaking the rigid supersymmetry by a suitable constraint on the Grassmann variables we will recover the 4d Dirac equation of [8–10].

The lagrangian will contain a Dirac-Nambu-Goto piece and two Wess-Zumino terms. By construction the action is invariant under rigid vector supersymmetry. For particular values of the coefficients of the lagrangian, the model has world line gauge supersymmetry which is analogous to the fermionic kappa symmetry of the superparticle case. When we require world-line supersymmetry the model has bosonic BPS configurations that preserve 1/5 of the vector supersymmetry. The BPS configurations imply second order equations of motion.

The organization of the paper is as follows: In section 2 we will introduce the space-time vector supersymmetry. In section 3 we will construct the massive spinning particle action using the method of non-linear realizations. Section 4 is devoted to the canonical formalism. In section 5 we will quantize the model in terms of the unconstrained variables. In section 6 we will quantize the model in the Clifford representation. Section 7 will be devoted to classical BPS configurations and finally in section 8 we will give some conclusions.

## 2. Space-time vector supersymmetry

Let us consider the Poincaré algebra<sup>2</sup> with generators  $P_\mu, M_{\mu\nu}$  extended with odd graded generators, supertranslations, belonging to a pseudoscalar ( $G_5$ ) and a pseudovector ( $G_\mu$ ) representation of the Lorentz group and two central charges  $Z$  and  $\tilde{Z}$ . They satisfy

$$[M_{\mu\nu}, M_{\rho\sigma}] = -i\eta_{\nu\rho}M_{\mu\sigma} - i\eta_{\mu\sigma}M_{\nu\rho} + i\eta_{\nu\sigma}M_{\mu\rho} + i\eta_{\mu\rho}M_{\nu\sigma}, \quad (2.1)$$

$$[M_{\mu\nu}, P_\rho] = i\eta_{\mu\rho}P_\nu - i\eta_{\nu\rho}P_\mu, \quad [M_{\mu\nu}, G_\rho] = i\eta_{\mu\rho}G_\nu - i\eta_{\nu\rho}G_\mu, \quad (2.2)$$

$$[G_\mu, G_\nu]_+ = \eta_{\mu\nu}Z, \quad [G_5, G_5]_+ = \tilde{Z}, \quad (2.3)$$

$$[G_\mu, G_5]_+ = -P_\mu. \quad (2.4)$$

We first consider a coset of the vector Super Poincaré group,  $G/O(3, 1)^3$ , and parameterize the group element as

$$g = e^{iP_\mu x^\mu} e^{iG_5 \xi^5} e^{iG_\mu \xi^\mu} e^{iZc} e^{i\tilde{Z}\tilde{c}}. \quad (2.5)$$

$x^\mu$ ,  $\xi^\mu$  and  $\xi^5$  are the superspace coordinates and  $c$  and  $\tilde{c}$  are coordinates associated to the projective representations of G. This is analogous to the ordinary case of the N=2 Super Poincaré group with central charges. Notice that we assume the odd generators  $G_\mu$

<sup>2</sup>We consider a four dimensional space-time with metric  $\text{diag } \eta_{\mu\nu} = (-; + + +)$ .

<sup>3</sup>Note that the algebra of vector supersymmetry is a subalgebra of the N=2 topological supersymmetry algebra in four dimensions. The topological algebra contains also an odd self dual tensor generator, see for example [12] and the references there.

and  $G_5$  anticommute with the Grassmann coset parameters  $\xi^\mu$  and  $\xi^5$ . It follows that the representation of the group is unitary if these generators are anti-hermitian.

The MC 1-form associated to this coset is

$$\Omega = -ig^{-1}dg = P_\mu(dx^\mu - i\xi^\mu d\xi^5) + G_5 d\xi^5 + G_\mu d\xi^\mu + Z\left(dc - \frac{i}{2}d\xi_\mu \xi^\mu\right) + \tilde{Z}\left(d\tilde{c} - \frac{i}{2}d\xi^5 \xi^5\right). \quad (2.6)$$

From this we get the even differential 1-forms

$$L_x^\mu = dx^\mu - i\xi^\mu d\xi^5, \quad L_Z = dc + \frac{i}{2}\xi^\mu d\xi_\mu, \quad L_{\tilde{Z}} = d\tilde{c} + \frac{i}{2}\xi^5 d\xi^5 \quad (2.7)$$

and the odd ones by

$$L_\xi^\mu = d\xi^\mu, \quad L_{\xi^5} = d\xi^5. \quad (2.8)$$

The supersymmetry transformations leaving the MC form invariant are

$$\delta x^\mu = i\epsilon^\mu \xi^5, \quad \delta \xi^\mu = \epsilon^\mu, \quad \delta c = \frac{i}{2}\xi_\mu \epsilon^\mu \quad (2.9)$$

and

$$\delta \xi^5 = \epsilon^5, \quad \delta \tilde{c} = \frac{i}{2}\xi^5 \epsilon^5. \quad (2.10)$$

These vector supersymmetry transformations are generated by  $G^\mu$  and  $G^5$  respectively and were first discussed in [8]. The transformations generated by  $P_\mu, Z, \tilde{Z}$  are the infinitesimal translations of the coordinates  $x^\mu, c$  and  $\tilde{c}$ ,

$$\delta x^\mu = a^\mu, \quad \delta c = \epsilon_Z, \quad \delta \tilde{c} = \epsilon_{\tilde{Z}}. \quad (2.11)$$

The vector fields generating the previous transformations are

$$\begin{aligned} X_\mu^G &= -i\left(\frac{\partial}{\partial \xi^\mu} + i\xi^5 \frac{\partial}{\partial x^\mu} - \frac{i}{2}\xi_\mu \frac{\partial}{\partial c}\right), & X_5^G &= -i\left(\frac{\partial}{\partial \xi^5} - \frac{i}{2}\xi^5 \frac{\partial}{\partial \tilde{c}}\right), \\ X_\mu^P &= -i\frac{\partial}{\partial x^\mu}, & X_Z &= -i\frac{\partial}{\partial c}, & X_{\tilde{Z}} &= -i\frac{\partial}{\partial \tilde{c}}. \end{aligned} \quad (2.12)$$

The algebra of these vector fields<sup>4</sup> is

$$[X_\mu^G, X_\nu^G]_+ = -\eta_{\mu\nu} X_Z, \quad [X_5^G, X_5^G]_+ = -X_{\tilde{Z}}, \quad [X_\mu^G, X_5^G]_+ = X_\mu^P. \quad (2.13)$$

With the Lorentz generators  $X_{\mu\nu}^M$ , these vector fields give a realization of the Vector Super Poincaré algebra.<sup>5</sup> The representations of this algebra will be studied in their general aspects in.<sup>6</sup>

<sup>4</sup>Remember  $\frac{\partial}{\partial \xi^\mu}$  and  $\frac{\partial}{\partial \xi^5}$  are hermitian and  $X_\mu^G$  and  $X_5^G$  are anti-hermitian.

<sup>5</sup>The reason for the overall sign difference from the starting algebra is that now the generators are *active* operators.

<sup>6</sup>The representations of this extension of the Poincaré algebra will be studied in a future paper.

### 3. Massive spinning particle lagrangian

In this section we will study how to construct the massive spinning particle action from a non-linear realization [11] of the Vector Super Poincaré group  $G$ . The relevant coset is  $G/O(3)$  defined by the little group of a massive particle  $O(3)$  as the unbroken (stability) group of the coset. We write the elements of the coset as

$$g = g_L U, \quad U = e^{iM_{0i}v^i}, \quad (3.1)$$

where  $g_L$  is the same as in (2.5)

$$g_L = e^{iP_\mu x^\mu} e^{iG_5 \xi^5} e^{iG_\mu \xi^\mu} e^{iZc} e^{i\tilde{Z}\tilde{c}} \quad (3.2)$$

and  $U$  represents a finite Lorentz boost with parameters  $v^i$ .

The Maurer-Cartan 1-form is now

$$\Omega = -ig^{-1}dg = U^{-1}\Omega_L U - iU^{-1}dU, \quad \Omega_L \equiv -ig_L^{-1}dg_L. \quad (3.3)$$

Using the commutation relations of the Lorentz generators with the four-vectors  $P_\mu$  and  $G_\mu$  (2.2) we find

$$U^{-1}P_\mu U = P_\nu \Lambda^\nu_\mu(v), \quad U^{-1}G_\mu U = G_\nu \Lambda^\nu_\mu(v), \quad (3.4)$$

where  $\Lambda^\nu_\mu(v)$  is a finite Lorentz boost

$$\Lambda^\nu_\mu = \begin{pmatrix} \cosh v & -v^i \frac{\sinh v}{v} \\ -\frac{\sinh v}{v} v_j & \delta_j^i + \frac{v^i v_j}{v^2} (\cosh v - 1) \end{pmatrix}, \quad v \equiv \sqrt{(v^i)^2}. \quad (3.5)$$

It follows

$$\Omega = P_\nu \tilde{L}_x^\nu + G_\nu \tilde{L}_\xi^\nu + G_5 L_\xi^5 + ZL_Z + \tilde{Z}L_{\tilde{Z}} + M_{0i}L_v^i + \frac{1}{2}M_{ij}L_v^{ij}, \quad (3.6)$$

where

$$\tilde{L}_x^\nu = \Lambda^\nu_\mu(v)L_x^\mu, \quad \tilde{L}_\xi^\nu = \Lambda^\nu_\mu(v)L_\xi^\mu \quad (3.7)$$

and the 1-forms  $L$ 's are given in (2.7) and (2.8).  $L_v^i$  and  $L_v^{ij}$  are given by

$$L_v^i = dv^i + dv^j \left( \delta_j^i - \frac{v^j v^i}{v^2} \right) \left( \frac{\sinh v}{v} - 1 \right),$$

$$L^{ij} = \frac{dv^i v^j - dv^j v^i}{v^2} (\cosh v - 1). \quad (3.8)$$

The invariant action for the particle is a sum of the manifest  $O(3)$  invariant one forms. By taking the pull-back, the (Goldstone) super-coordinates become functions of the parameter  $\tau$  that parameterizes the worldline of the particle, see for example [13]. The action is

$$S[x(\tau), \xi(\tau), v(\tau)] = \int \left( -\mu \tilde{L}_x^0 - \beta L_Z - \gamma L_{\tilde{Z}} \right)^* = \int L d\tau, \quad (3.9)$$

where  $*$  means pull-back on the world line and  $\mu, \beta$  and  $\gamma$  are real constants to be identified with the mass and the central charges,  $m, Z$  and  $\tilde{Z}$  respectively. The action is invariant under global Vector Super Poincaré transformations. In addition it is invariant under a local supersymmetry transformation if the parameters satisfy

$$-\beta\gamma = \mu^2. \quad (3.10)$$

In this paper we will study the case in which this condition is satisfied. If we use the explicit form of the finite Lorentz boost (3.5) the action is

$$S[x(\tau), \xi(\tau), v(\tau)] = \int \left[ -\mu \left( L_x^0 \cosh v - L_x^i v^i \frac{\sinh v}{v} \right) - \beta L_Z - \gamma L_{\tilde{Z}} \right]^*. \quad (3.11)$$

The explicit form of the local supersymmetry transformation is

$$\delta\xi^0 = -\frac{\mu}{\beta} \cosh v \kappa^5(\tau), \quad \delta\xi^i = -\frac{\mu}{\beta} \frac{v^i}{v} \sinh v \kappa^5(\tau), \quad \delta\xi^5 = \kappa^5(\tau) \quad (3.12)$$

and

$$\delta x^\mu = i\xi^\mu \kappa^5(\tau), \quad \delta c = -\frac{i}{2} \xi_\mu \delta\xi^\mu, \quad \delta\tilde{c} = -\frac{i}{2} \xi^5 \kappa^5(\tau), \quad \delta v^a = 0. \quad (3.13)$$

Now we will see how one eliminates the boost parameters  $v^i$  from the theory. This can be done by using their equations of motion

$$\frac{\delta S}{\delta v^i} = 0 = L_\tau^0 \frac{v^i}{v} \sinh v - L_\tau^j \left[ \delta^{ij} \frac{\sinh v}{v} + \frac{v^i v^j}{v^2} \left( \cosh v - \frac{\sinh v}{v} \right) \right]. \quad (3.14)$$

By solving this equation we get

$$\frac{v^i}{v} = \frac{L_\tau^i}{\sqrt{(L_\tau^j)^2}}, \quad \sinh v = \frac{\sqrt{(L_\tau^j)^2}}{\sqrt{(L_\tau^0)^2 - (L_\tau^j)^2}}, \quad \cosh v = \frac{L_\tau^0}{\sqrt{(L_\tau^0)^2 - (L_\tau^j)^2}}, \quad (3.15)$$

where  $L_\tau^\mu$  is the pull-back of the  $L_x^\mu$

$$L_\tau^\mu = \dot{x}^\mu - i\xi^\mu \dot{\xi}_5. \quad (3.16)$$

Using them in (3.11) we get

$$S[x(\tau), \xi(\tau)] = \int d\tau \left( -\mu \sqrt{-\left( \dot{x}^\mu - i\xi^\mu \dot{\xi}_5 \right)^2} - \beta \left( \dot{c} + \frac{i}{2} \xi^\mu \dot{\xi}_\mu \right) - \gamma \left( \dot{\tilde{c}} + \frac{i}{2} \xi^5 \dot{\xi}_5 \right) \right). \quad (3.17)$$

The local super transformation for the action (3.17) is

$$\delta\xi^\mu = -\frac{p^\mu}{\beta} \kappa^5(\tau), \quad \delta\xi^5 = \kappa^5(\tau), \quad (3.18)$$

where  $p_\mu$  is the momentum conjugate to  $x^\mu$  whereas the transformations for the other variables remain the same as in (3.13). Note that this transformation is analogous to the

kappa symmetry transformation of the massive superparticle action [4]. In the literature it is known as gauge world-line supersymmetry.

The action (3.17) coincides with the action originally proposed in [8] after we make a suitable rescaling of the coordinates

$$\xi'^{\mu} = \sqrt{|\beta|} \xi^{\mu}, \quad \xi'^5 = \sqrt{\gamma} \xi^5, \quad c' = |\beta|c, \quad \tilde{c}' = \gamma\tilde{c}. \quad (3.19)$$

Correspondingly the local super transformation becomes

$$\delta x'^{\mu} = \frac{i}{\mu} \xi'^{\mu} \kappa'^5, \quad \delta \xi'^{\mu} = \frac{p^{\mu}}{\mu} \kappa'^5(\tau), \quad \delta \xi'^5 = \kappa'^5(\tau), \quad (3.20)$$

where the parameter has also been rescaled as  $\kappa'^5 = \sqrt{\gamma} \kappa^5$ .

#### 4. Canonical formalism

The canonical momentum conjugate to  $x^{\mu}$  defined by the lagrangian (3.9) is

$$p_{\mu} = \frac{\partial L}{\partial \dot{x}^{\mu}} = -\mu \Lambda_{\mu}^0(v). \quad (4.1)$$

Since  $\Lambda_{\mu}^0(v)$  is a time-like vector, the momentum verifies the mass-shell constraint  $p^2 + \mu^2 = 0$ . Therefore the parameter  $\mu$  is identified with the mass of the particle and the lagrangian (3.9) can be written in the first order form as

$$L^C = p_{\mu}(\dot{x}^{\mu} - i\xi^{\mu}\xi^5) - \beta \frac{i}{2} \xi_{\mu} \dot{\xi}^{\mu} - \gamma \frac{i}{2} \xi^5 \dot{\xi}^5 - \frac{e}{2}(p^2 + \mu^2), \quad (4.2)$$

where  $e$  is a lagrange multiplier (ein-bein). In the lagrangian we have omitted  $\dot{c}$  and  $\dot{\tilde{c}}$  terms since they are total derivatives. By eliminating  $p_{\mu}$ 's and  $e$  using their equations of motion, the lagrangian goes back to the covariant one in eq. (3.17).

Let us study the constraints and symmetries of  $L^C$ . The canonical momenta are defined by using *left derivatives* as the Grassmann variables are concerned,

$$\begin{aligned} p_{\mu}^x &= \frac{\partial L^C}{\partial \dot{x}^{\mu}} = p_{\mu}, & p_{\mu}^p &= \frac{\partial L^C}{\partial \dot{p}^{\mu}} = 0, & p^e &= \frac{\partial L^C}{\partial \dot{e}} = 0, \\ \pi_{\mu} &= \frac{\partial L^C}{\partial \dot{\xi}^{\mu}} = \beta \frac{i}{2} \xi_{\mu}, & \pi_5 &= \frac{\partial L^C}{\partial \dot{\xi}^5} = \gamma \frac{i}{2} \xi^5 + i p_{\mu} \xi^{\mu}. \end{aligned} \quad (4.3)$$

All of them give rise to primary constraints

$$\begin{aligned} \phi_{\mu}^x &= p_{\mu}^x - p_{\mu} = 0, & \phi_{\mu}^p &= p_{\mu}^p = 0, & \phi^e &= p^e = 0, \\ \chi_{\mu} &= \pi_{\mu} - \beta \frac{i}{2} \xi_{\mu} = 0, & \chi_5 &= \pi_5 - \gamma \frac{i}{2} \xi^5 - i p_{\mu} \xi^{\mu} = 0. \end{aligned} \quad (4.4)$$

The Hamiltonian,  $H \equiv \dot{q}p - L$ , is

$$H = \lambda_x^{\mu} \phi_{\mu}^x + \lambda_p^{\mu} \phi_{\mu}^p + \lambda_e \phi^e + \lambda_{\xi}^{\mu} \chi_{\mu} + \lambda_{\xi}^5 \chi_5 + \frac{e}{2}(p^2 + \mu^2), \quad (4.5)$$

where the  $\lambda$ 's are Dirac multipliers. Using the graded Poisson brackets  $\{p, q\} = -1$  we study the stability of the constraints. We get the following secondary constraint

$$\phi \equiv \frac{1}{2}(p^2 + \mu^2) = 0, \quad (4.6)$$

and

$$\lambda_\mu^p = 0, \quad \lambda_\mu^x = e p_\mu + i \xi_\mu \lambda_\xi^5, \quad \lambda_\mu^\xi = -\frac{1}{\beta} p_\mu \lambda_\xi^5, \quad (4.7)$$

where we have used the condition (3.10),  $\beta\gamma = -\mu^2$ . The secondary constraint (4.6) is preserved in time

$$\dot{\phi} = 0, \quad (4.8)$$

and it does not generate further constraints.  $\phi_\mu^x = \phi_\mu^p = 0$  are second class constraints and are used to eliminate  $p_\mu^x$  and  $p_\mu^p$ . The second class constraints  $\chi_\mu = 0$  are used to eliminate  $\pi_\mu$ . The Dirac bracket for the remaining variables are

$$\{p_\mu, x^\nu\}^* = -\delta_\mu^\nu, \quad \{\xi^\mu, \xi^\nu\}^* = \frac{i}{\beta} \eta^{\mu\nu}, \quad \{\pi_5, \xi^5\}^* = -1 \quad (4.9)$$

and the Hamiltonian becomes

$$H = \lambda_e \phi^e + e \phi + \lambda_\xi^5 \chi_5 = \lambda_e p^e + \frac{e}{2}(p^2 + \mu^2) + \lambda_\xi^5 \left( \pi_5 - \gamma \frac{i}{2} \xi^5 - i p_\mu \xi^\mu \right). \quad (4.10)$$

The constraints  $\phi^e, \phi$  and  $\chi_5$  appearing here are the first class constraints. In particular we have

$$\{\chi_5, \chi_5\}^* = i\gamma - \frac{i}{\beta} p_\mu p_\nu \eta^{\mu\nu} = -\frac{2i}{\beta} \phi. \quad (4.11)$$

$\chi_5$  generates the local kappa variation corresponding to (3.12),

$$\delta x^\mu = i \xi^\mu \kappa^5(\tau), \quad \delta \xi^\mu = -\frac{p^\mu}{\beta} \kappa^5(\tau), \quad \delta \xi^5 = \kappa^5(\tau), \quad \delta e = -\frac{2i}{\beta} \xi^5 \kappa^5(\tau), \quad (4.12)$$

under which the lagrangian transforms as

$$\delta L^C = \frac{d}{d\tau} \left( \frac{i}{2} (p_\mu \xi^\mu - \gamma \xi^5) \kappa^5(\tau) \right). \quad (4.13)$$

The global vector supersymmetry transformations are

$$\delta x^\mu = i \epsilon^\mu \xi^5, \quad \delta \xi^\mu = \epsilon^\mu, \quad \delta \xi^5 = \epsilon^5, \quad (4.14)$$

and again the lagrangian changes by a total derivative

$$\delta L^C = \frac{d}{d\tau} \left( -\beta \frac{i}{2} \epsilon^\mu \xi_\mu - \gamma \frac{i}{2} \epsilon^5 \xi^5 \right). \quad (4.15)$$

The generators of the global supersymmetries are

$$\begin{aligned} G_\mu &= \pi_\mu + \beta \frac{i}{2} \xi_\mu + i p_\mu \xi^5 = i \beta \xi_\mu + i p_\mu \xi^5, \\ G_5 &= \pi_5 + \gamma \frac{i}{2} \xi^5. \end{aligned} \quad (4.16)$$



They satisfy

$$\{G_\mu, G_\nu\}^* = -i\beta\eta_{\mu\nu}, \quad \{G_\mu, G_5\}^* = -ip_\mu, \quad \{G_5, G_5\}^* = -i\gamma. \quad (4.17)$$

At the quantum level we have

$$[G_\mu, G_\nu]_+ = +\beta\eta_{\mu\nu}, \quad [G_\mu, G_5]_+ = +p_\mu, \quad [G_5, G_5]_+^* = +\gamma. \quad (4.18)$$

This is a canonical realization of the starting algebra (2.3) and (2.4) with the central charges<sup>7</sup>

$$Z = -\beta, \quad \tilde{Z} = -\gamma, \quad Z\tilde{Z} = -\mu^2. \quad (4.19)$$

## 5. Quantization in reduced space

In this section we discuss the quantization of this system in terms of the unconstrained variables. Let us see the classical form of the canonical action in the reduced space. The starting point is the canonical lagrangian  $L^C$  defined in eq. (4.2). It is locally supersymmetric and it is invariant under reparametrization in  $\tau$ . These two local symmetries are generated by the first class constraints  $\chi_5 = 0$  in (4.4) and  $\phi = 0$  in (4.6) respectively. We fix these gauge freedom by imposing the conditions

$$x^0 = \tau, \quad \xi^5 = 0. \quad (5.1)$$

The first class constraints  $\phi = \chi_5 = 0$  become second class and are solved for  $p_0$  and  $\pi_5$  as

$$p_0 = \pm\sqrt{\vec{p}^2 + \mu^2}, \quad \pi_5 = ip_\mu\xi^\mu. \quad (5.2)$$

Other second class constraints are also used to reduce the variables leaving  $x^i, p_i$  and  $\xi^\mu$  as the independent variables. The non-trivial Dirac brackets with respect to all these new second class constraints are

$$\{p_i, x^j\}^{**} = -\delta_i^j, \quad \{\xi^\mu, \xi^\nu\}^{**} = \frac{i}{\beta}\eta^{\mu\nu}. \quad (5.3)$$

The canonical form of the lagrangian (4.2) in the reduced space<sup>8</sup> becomes

$$L^{C*} = \pm\sqrt{\vec{p}^2 + \mu^2} + \vec{p}\cdot\dot{\vec{x}} - \beta\frac{i}{2}\xi_\mu\dot{\xi}^\mu. \quad (5.4)$$

Now we quantize the model. The basic canonical (anti-)commutators are

$$[x^i, p_j] = i\delta^i_j, \quad [\xi^\mu, \xi^\nu]_+ = -\frac{1}{\beta}\eta^{\mu\nu}. \quad (5.5)$$

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<sup>7</sup>It is possible to show (see footnote 6) that the representations of the Vector Super Poincaré algebra are characterized, besides the momentum square and the Pauli-Lubanski invariant, by the quantity  $\sqrt{|Z\tilde{Z}|}$  and by the signs of  $Z$  and  $\tilde{Z}$ .

<sup>8</sup>An analogous discussion for the lagrangian of spinning particle of [9] was done in reference [14].

Note that the sign degree of freedom of  $p_0$  must be taken into account in the quantum theory. The hamiltonian for the lagrangian (5.4) is an operator

$$P_0 = \begin{pmatrix} \omega & 0 \\ 0 & -\omega \end{pmatrix}, \quad \omega \equiv \sqrt{\vec{p}^2 + \mu^2}, \quad (5.6)$$

taking eigenvalues  $\pm\omega$  on the positive and negative energy eigenstates. The Schrödinger equation becomes

$$i\partial_\tau \Psi(\vec{x}, \tau) = P_0 \Psi(\vec{x}, \tau), \quad \Psi = \begin{pmatrix} \Psi_+ \\ \Psi_- \end{pmatrix}, \quad (5.7)$$

where  $\Psi_+$  and  $\Psi_-$  are positive and negative energy states. Now we look for a realization of  $\xi^\mu$ . Since they satisfy the anti-commutators in (5.5) and must commute with all the bosonic variables, in particular with the energy  $P_0$  in (5.6). We can realize them in terms of 8-dimensional gamma matrices

$$\xi^\mu = \sqrt{\frac{-1}{2\beta}} \Gamma^\mu \Gamma^5 = \sqrt{\frac{-1}{2\beta}} \begin{pmatrix} \gamma^\mu & 0 \\ 0 & -\gamma^\mu \end{pmatrix} \begin{pmatrix} \gamma^5 & 0 \\ 0 & -\gamma^5 \end{pmatrix}, \quad (5.8)$$

where the  $\gamma^\mu$  and  $\gamma^5$  are the ordinary 4-component gamma matrices in 4-dimensions.  $\Psi$  is an 8 component wave function and  $\Psi_+$  and  $\Psi_-$  are four dimensional spinors associated to positive and negative energy states. We rewrite the Schrödinger equation in a more familiar form using a unitary transformation, an inverse Foldy-Whouthuysen (FW) transformation,

$$i\partial_\tau \tilde{\Psi} = \begin{pmatrix} \boldsymbol{\alpha}^i p_i + \boldsymbol{\beta} \mu & \\ & \boldsymbol{\alpha}^i p_i - \boldsymbol{\beta} \mu \end{pmatrix} \tilde{\Psi}, \quad \Psi \equiv S \begin{pmatrix} U_4 U_3^+ U_4 U_2 U_1 & \\ & U_4 U_3^- U_4 U_2 U_1 \end{pmatrix} \tilde{\Psi}, \quad (5.9)$$

where  $\boldsymbol{\beta}$  and  $\boldsymbol{\alpha}^i$  are usual Dirac matrices and

$$S = \begin{pmatrix} 1_2 & & \\ & 1_2 & \\ & & 1_2 \end{pmatrix}, \quad U_3^+ = \begin{pmatrix} e^{i\frac{\theta_3}{2}\sigma_2} & \\ & e^{-i\frac{\theta_3}{2}\sigma_2} \end{pmatrix}, \quad U_3^- = \begin{pmatrix} e^{i\frac{\pi-\theta_3}{2}\sigma_2} & \\ & e^{-i\frac{\pi-\theta_3}{2}\sigma_2} \end{pmatrix},$$

$$U_4 = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}, \quad U_1 = \begin{pmatrix} e^{i\frac{\theta_1}{2}\sigma_3} & \\ & e^{i\frac{\theta_1}{2}\sigma_3} \end{pmatrix}, \quad U_2 = \begin{pmatrix} e^{i\frac{\theta_2}{2}\sigma_2} & \\ & e^{i\frac{\theta_2}{2}\sigma_2} \end{pmatrix},$$

$$\tan \theta_1 = \frac{p_2}{p_1}, \quad \tan \theta_2 = \frac{\sqrt{p_2^2 + p_1^2}}{p_3}, \quad \tan \theta_3 = \frac{\sqrt{\vec{p}^2}}{\mu}. \quad (5.10)$$

Multiplying by  $\Gamma^0 = \begin{pmatrix} \boldsymbol{\beta} & \\ & -\boldsymbol{\beta} \end{pmatrix}$  on the equation (5.9), we get

$$\Gamma^0 i\partial_\tau \tilde{\Psi} = (\Gamma^i (-i\partial_{x^i}) + \mu) \tilde{\Psi}, \quad (5.11)$$

where

$$\gamma^i = \boldsymbol{\beta} \boldsymbol{\alpha}^i, \quad \boldsymbol{\beta}^2 = 1, \quad \Gamma^\mu = \begin{pmatrix} \gamma^\mu & \\ & -\gamma^\mu \end{pmatrix}. \quad (5.12)$$

(5.11) is the 8-components Dirac equation reducible into two 4-components Dirac equations with mass  $\mu$ ,

$$(\Gamma^\mu(-i\partial_{x^\mu}) + \mu)\tilde{\Psi}(x^\mu) = \begin{pmatrix} -i\partial_{x^\mu}\gamma^\mu + \mu & \\ & i\partial_{x^\mu}\gamma^\mu + \mu \end{pmatrix}\tilde{\Psi}(x^\mu) = 0. \quad (5.13)$$

## 6. Quantization in Clifford representation

In this section the system is quantized in a covariant manner by requiring the first class constraints to hold on the physical states. The Dirac brackets are replaced by the following graded-commutators,

$$[p_\mu, x^\nu] = -i\delta_\mu^\nu, \quad [\xi^\mu, \xi^\nu]_+ = -\frac{1}{\beta}\eta^{\mu\nu}, \quad [\pi_5, \xi^5]_+ = -i. \quad (6.1)$$

The odd variables define a Clifford algebra. This is better seen by introducing a new set of variables with appropriate normalization. We define

$$\lambda^\mu = \sqrt{-2\beta}\xi^\mu, \quad \lambda^5 = -i\sqrt{\frac{2}{\gamma}}\left(\pi_5 - \frac{i}{2}\gamma\xi^5\right), \quad \lambda^6 = -i\sqrt{\frac{2}{\gamma}}\left(\pi_5 + \frac{i}{2}\gamma\xi^5\right). \quad (6.2)$$

The reason to introduce the  $i$ 's is that the momentum  $\pi_5$  is anti-hermitian at the pseudo-classical level and in this way all the dynamical variables are real,  $(\lambda^A)^* = \lambda^A$ .

The  $\lambda^A$ 's define a Clifford algebra  $C_6$ ,

$$[\lambda^A, \lambda^B]_+ = 2\bar{\eta}^{AB}, \quad \bar{\eta}^{AB} = (-, +, +, +, +, -), \quad (A, B = 0, 1, 2, 3, 5, 6). \quad (6.3)$$

These variables can be identified as a particular combination of the elements of another  $C_6$  algebra having within its generators the  $\Gamma^\mu$ 's isomorphic to the Dirac matrices already used in eq. (5.8). This algebra is defined by the following elements

$$\Gamma^\mu = \begin{pmatrix} \gamma^\mu & 0 \\ 0 & -\gamma^\mu \end{pmatrix}, \quad \Gamma^5 = \begin{pmatrix} \gamma^5 & 0 \\ 0 & -\gamma^5 \end{pmatrix}, \quad \Gamma^6 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad (6.4)$$

satisfying

$$[\Gamma^A, \Gamma^B]_+ = 2\tilde{\eta}^{AB}, \quad \tilde{\eta}^{AB} = (+, -, -, -, +, -). \quad (6.5)$$

Of course, both Clifford algebras have the same automorphism group  $SO(4, 2)$ . They are related in the following way

$$\lambda^A = \begin{cases} \Gamma^A\Gamma^5, & A = 0, 1, 2, 3, \\ \Gamma^5, & A = 5, \\ i\Gamma^5\Gamma^6, & A = 6. \end{cases} \quad (6.6)$$

Let us start considering the  $C_6$  generated by the  $\lambda^A$ 's. The unitarity of the representation in terms of  $\Gamma^A$ 's requires an extra measure,  $\Gamma_*$ , in the inner product,

$$\langle \Phi | \Psi \rangle = \int d^4x \Phi^\dagger(x) \Gamma_* \Psi(x) \equiv \int d^4x \bar{\Phi}(x) \Psi(x). \quad (6.7)$$

In the quantization process we are going to require that the operators in the matrix basis satisfy, with respect to the metric  $\Gamma_*$ , the same reality property as in the classical case. We find that the following operator satisfy our requirement

$$\Gamma_* = -i\lambda^0\lambda^6, \quad \Gamma_*^\dagger = \Gamma_*, \quad \Gamma_*^2 = 1, \quad (6.8)$$

in fact

$$\bar{\lambda}^A \equiv \Gamma_*(\lambda^A)^\dagger\Gamma_* = \lambda^A. \quad (6.9)$$

In terms of  $\Gamma^A$  we have also

$$\Gamma_* = \Gamma^0\Gamma^6. \quad (6.10)$$

The expressions for the generators of vector SUSY transformations (4.16) are obtained by inverting the relations (6.2)

$$\pi_5 = \frac{i}{2}\sqrt{\frac{\gamma}{2}}(\lambda^5 + \lambda^6), \quad \xi^5 = -\frac{1}{\sqrt{2\gamma}}(\lambda^5 - \lambda^6). \quad (6.11)$$

We find

$$\begin{aligned} G^\mu &= i\beta\xi^\mu + ip^\mu\xi^5 = \frac{i}{\sqrt{2\gamma}}(\mu\lambda^\mu - p^\mu(\lambda^5 - \lambda^6)) \\ &= \frac{i}{\sqrt{2\gamma}}\Gamma^5(\mu\Gamma^\mu - p^\mu(1 - i\Gamma^6)) \end{aligned} \quad (6.12)$$

and

$$G_5 = \pi_5 + i\frac{\gamma}{2}\xi^5 = i\sqrt{\frac{\gamma}{2}}\lambda^6 = -\sqrt{\frac{\gamma}{2}}\Gamma^5\Gamma^6. \quad (6.13)$$

The generators have the following conjugation properties

$$\bar{G}_\mu = -G_\mu, \quad \bar{G}_5 = -G_5. \quad (6.14)$$

In analogous way we get the expression for the odd first class constraint

$$\chi_5 = \pi_5 - i\frac{\gamma}{2}\xi^5 - ip_\mu\xi^\mu = \frac{i}{\sqrt{-2\beta}}(\mu\lambda^5 - p_\mu\lambda^\mu) = \frac{i}{\sqrt{-2\beta}}\Gamma^5(p_\mu\Gamma^\mu + \mu). \quad (6.15)$$

The requirement that the first class constraint  $\chi_5$  holds on the physical states is equivalent to require the Dirac equation on an 8-dimensional spinor  $\Psi$

$$(p_\mu\Gamma^\mu + \mu)\Psi = 0. \quad (6.16)$$

The other first class constraint (4.6),  $\phi = \frac{1}{2}(p^2 + \mu^2) = 0$ , is then automatically satisfied, since

$$\phi\Psi = \frac{1}{2}\mathcal{D}^2\Psi = 0, \quad \text{with } \mathcal{D} \equiv \Gamma^5(p_\mu\Gamma^\mu + \mu). \quad (6.17)$$

At the pseudo-classical level the first class constraints are invariant under the supersymmetry transformation. At the quantum level this is reflected by the following properties

$$[G_\mu, \mathcal{D}]_+ = [G_5, \mathcal{D}]_+ = 0. \quad (6.18)$$

From these relations we can define the corresponding symmetry transformations on the wave function if we can construct a matrix, call it  $E$ , anticommuting with all the dynamical variables,  $\lambda^A$ 's. In this case we have

$$[EG_\mu, \mathcal{D}] = [EG_5, \mathcal{D}] = 0 \quad (6.19)$$

and the transformations generated by  $EG_\mu$  and  $EG_5$  leave invariant the action of the theory

$$\int d^4x \bar{\psi} \mathcal{D} \psi. \quad (6.20)$$

The corresponding unitary transformations (with respect to the metric  $\Gamma_*$ ) are

$$e^{i\alpha_5 EG_5}, \quad e^{i\alpha^\mu EG_\mu}, \quad (6.21)$$

where  $\alpha_5$  and  $\alpha^\mu$  are even parameters defining the transformations.<sup>9</sup> The operator  $E$  can be easily constructed since our Clifford algebra is defined in a even dimensional space. Therefore,

$$E = \lambda^7 = i\lambda^0\lambda^1\lambda^2\lambda^3\lambda^5\lambda^6 = -\Gamma^0\Gamma^1\Gamma^2\Gamma^3\Gamma^6 = i\Gamma^5\Gamma^7, \quad \Gamma^7 \equiv i\Gamma^0\Gamma^1\Gamma^2\Gamma^3\Gamma^5\Gamma^6, \quad (6.22)$$

anti-commutes with all  $\lambda^A$ 's. One could ask if it is possible to recover the result of ref. [9], that is a Dirac equation in a 4-dimensional spinor space. This was indeed done in ref. [8] where it was imposed a further constraint

$$\pi_5 + i\frac{\gamma}{2}\xi^5 = 0. \quad (6.23)$$

In this way the quantization can be done by using only a  $C_5$  algebra which can be realized in a 4-dimensional space. Note that if we impose this condition the supersymmetry generator  $G_5$  vanishes identically. Therefore one loses the rigid supersymmetry although the local one remains.

## 7. BPS configurations

Here we will consider the BPS equations for the massive spinning particle. The corresponding bosonic supersymmetric configurations appear only when the lagrangians have a gauge world-line supersymmetry.

The lagrangian of the massive spinning particle (3.9) has a gauge symmetry when the parameters  $\mu, \beta, \gamma$  verify the condition (3.10),

$$-\beta\gamma = \mu^2. \quad (7.1)$$

Now we look for supersymmetric bosonic configurations. For consistency we look for transformations of the fermionic variables not changing their initial value that is supposed to

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<sup>9</sup>Note that  $\overline{EG_5} = EG_5$ ,  $\overline{EG_\mu} = EG_\mu$

vanish

$$\begin{aligned}
 0 &= \delta\xi^5|_{\text{fermions}=0} = \epsilon^5 + \kappa^5, \\
 0 &= \delta\xi^0|_{\text{fermions}=0} = \epsilon^0 - \frac{\mu}{\beta} \cosh v \kappa^5 \\
 0 &= \delta\xi^i|_{\text{fermions}=0} = \epsilon^i - \frac{\mu}{\beta} \frac{v^i}{v} \sinh v \kappa^5.
 \end{aligned}
 \tag{7.2}$$

The previous equations have a non-trivial solution if

$$v^i = \text{constant}. \tag{7.3}$$

Then, all the parameters can be expressed in terms of an independent global supersymmetry parameter,  $\epsilon^5$ , as

$$\kappa^5 = -\epsilon^5, \quad \epsilon^0 = -\frac{\mu}{\beta} \cosh v \epsilon^5, \quad \epsilon^i = -\frac{\mu}{\beta} \frac{v^i}{v} \sinh v \epsilon^5. \tag{7.4}$$

Equation (7.3) is the BPS equation of this model. If we write this expression in terms of space-time coordinates, using the solutions (3.15) of (3.14), we get

$$\frac{\dot{x}^\mu}{\sqrt{-\dot{x}^2}} = \text{constant}. \tag{7.5}$$

In Hamiltonian terms this implies that the momentum is constant. Note that this BPS equation implies the second order equations of motion of a free relativistic particle. Therefore the BPS configurations (7.5) preserve 1/5 of the supersymmetry. Notice that the fraction of preserved supersymmetry is different from the ordinary (spinor realization of) Super Poincaré group as, for example, in the case of the superparticle.

## 8. Discussions

In this paper we use the rigid space-time vector supersymmetry to construct the action of the massive spinning particle from the non-linear realization method.

For particular values of the coefficients of the lagrangian, the model has world line gauge supersymmetry which is the analogous of the fermionic kappa symmetry of the superparticle case. By quantizing the model in such a way to respect the rigid supersymmetry, we find two decoupled 4d Dirac equations with the same mass. The supersymmetry transformations at quantum level mix the two 4d Dirac equations.

At classical level we find BPS configurations that preserve 1/5 of the supersymmetry. The BPS equations imply second order equations of motion.

In a future work<sup>6</sup> we will study the representations of the Vector Super Poincaré algebra which, in the massive case, are characterized by the quantity  $\sqrt{|Z\tilde{Z}|}$  and by the sign of  $Z$  and  $\tilde{Z}$ . The massless spinning particle will be also considered.

Two interesting questions are: the possible physical role of the space-time vector supersymmetry in quantum field theories and the relation of this approach with the one based on space-time spinorial supersymmetry.

**Note added.** After this paper was put on the archive. M. Plyushchay has informed of a previous work [15] which has some overlap with our work. In particular about the constraints analysis of the model we have considered and the non-equivalence among the models of references [8] and [9].

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