Kac-Moody spectrum of (half-)maximal supergravities

Eric A. Bergshoeff,[a] **Joaquim Gomis,**[bc] **Teake A. Nutma**[a] **and Diederik Roest**[b]

[a]*Centre for Theoretical Physics, University of Groningen,*
Nijenborgh 4, 9747 AG Groningen, The Netherlands
[b]*Departament Estructura i Constituents de la Materia,*
Facultat de Física, Universitat de Barcelona,
Diagonal 647, 08028 Barcelona, Spain
[c]*PH-TH Division, CERN,*
1211 Geneva 23, Switzerland
E-mail: `E.A.Bergshoeff@rug.nl`, `gomis@ecm.ub.es`, `T.A.Nutma@rug.nl`,
`droest@ecm.ub.es`

ABSTRACT: We establish the correspondence between, on one side, the possible gaugings and massive deformations of half-maximal supergravity coupled to vector multiplets and, on the other side, certain generators of the associated very extended Kac-Moody algebras. The difference between generators associated to gaugings and to massive deformations is pointed out. Furthermore, we argue that another set of generators are related to the so-called quadratic constraints of the embedding tensor. Special emphasis is placed on a truncation of the Kac-Moody algebra that is related to the bosonic gauge transformations of supergravity. We give a separate discussion of this truncation when non-zero deformations are present. The new insights are also illustrated in the context of maximal supergravity.

Contents

1. Introduction

In describing the field theoretic representation of a supersymmetry algebra, one usually specifies those fields that represent physical states only. It is known that other fields can be added to the supermultiplet that do not describe physical states but on which nevertheless the full supersymmetry algebra can be realized (for an early discussion of such potentials, see [1]). In this paper we will focus on the following two classes of such fields.

The first class consists of $(D-1)$-form potentials in D dimensions, which we will call "deformation potentials" for the following reason. The equations of motion of these deformation potentials can be solved in terms of integration constants that describe deformations of the supersymmetric theory. The foremost example of a deformation potential is

the nine-form potential of type IIA string theory that couples to the D8-brane [2–4]. The integration constant corresponding to this nine-form potential is the masslike parameter m of massive IIA supergravity [5]. The relation between the two is given by

$$d\,{}^{\star}F_{(10)}(A_{(9)}) = 0 \quad \Rightarrow \quad {}^{\star}F_{(10)}(A_{(9)}) \propto m\,. \tag{1.1}$$

The second class of fields that do not describe physical states consists of D-form potentials in D dimensions, which we will call "top-forms potentials", or top-forms for short. The prime example of a top-form is the Ramond-Ramond ten-form that couples to the D9-brane of type IIB string theory [2]. It turns out that this ten-form is part of a quadruplet of ten-forms transforming according to the **4** representation of the $\mathrm{SL}(2,\mathbb{R})$ duality group, while also a doublet **2** of ten-forms can be added in IIB supergravity [6].

It has been known for a number of years that one can reproduce the physical degrees of freedom of maximal supergravity from the very extended Kac-Moody algebra E_{11} [7–9]. Furthermore, this Kac-Moody algebra contains generators corresponding to the deformation potential of IIA [9, 10] and the top-form potentials of IIB [9, 11, 12]. Recently, the representations under the duality group of the deformation and top-form potentials of all maximal supergravities have been calculated [13, 14]. Remarkably, the E_{11} results on deformation potentials are in agreement with those of [15–24] where maximal gauged supergravities are classified within a supergravity approach.[1] In particular, this agreement shows that the components of the embedding tensor [15, 16, 18] can be identified with the masslike deformation parameters of the supergravity theory. Therefore, the field strength $F_{(D)}$ of the deformation potential $A_{(D-1)}$ is proportional to the embedding tensor Θ:

$$\,{}^{\star}F_{(D)}\left(A_{(D-1)}\right) \propto \Theta\,. \tag{1.2}$$

This relation can be viewed as a duality relation, like the ones between potentials and dual potentials.

It is natural to extend the analysis of [13, 14] to other cases. In this paper we will do this for the class of half-maximal supergravity theories. The Kac-Moody analysis for this case shows a number of new features. First of all, one can add matter vector multiplets and consider matter-coupled supergravity [9, 27]. Our results on the deformation and top-form potentials will depend on the number of vector multiplets. Another new feature is that one encounters duality groups that are *not* maximal non-compact. Only a limited number of vector multiplets lead to a maximal non-compact duality group. Finally, the duality groups are not necessarily simply laced, and hence we will have to address the issue of non-symmetric Cartan matrices and roots of different lengths. For more details on the latter, see appendix C.

An additional motivation to study the case of half-maximal supergravities is that for $D < 10$, e.g. $D = 4$ or $D = 6$, the corresponding matter-coupled supergravities are related to compactifications of string theory and M-theory with background fluxes. The nonzero

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[1]An exception to this correspondence are the gauging of the 'trombone' or scale symmetry of the field equations and Bianchi identities [25], as discussed in e.g. [26, 23, 24], for which no corresponding deformation potentials have been identified in E_{11}.

fluxes lead to the additional mass parameters. Especially the $D = 6$ case is interesting due to the existence of a chiral and a non-chiral theory. These two theories are related via S- and T-dualities between Type I string theory on T^4 and Type II string theory on K3. The mass parameters of these theories have been investigated [28, 29] and the massive dualities between them have been studied [30, 31].

In this paper we will pay particular attention to the bosonic algebra that the different p-form Kac-Moody generators with $p > 0$ satisfy amongst each other. We will call this algebra the "p-form algebra". This algebra, without the deformation and top-form generators, also occurs in [32, 33] as the bosonic gauge algebra of supergravity. The p-form potentials corresponding to these generators, together with gravity and the scalar fields, constitute the part of the very extended Kac-Moody spectrum that does not require the introduction of the dual graviton. We will show how the possible deformation and top-form potentials, with which the p-form algebra can be extended, follow from the Kac-Moody algebra. In particular, we will show that for the case of half-maximal supergravity the deformation potentials of the p-form algebra, and hence also the embedding tensor in generic dimensions, can be written in terms of the fundamental and three-form representation of the duality group.

One encounters the following subtlety in establishing the connection between the p-form algebra and supergravity: whereas for each physical state the Kac-Moody algebra gives rise to both the potential and the dual potential this is not the case for the deformation potentials. The Kac-Moody algebra does give rise to the deformation potentials but not to the dual embedding tensor. Indeed the duality relation (1.2) does not follow from the Kac-Moody approach. We know from supergravity that the inclusion of a mass parameter or an embedding tensor leads to deformations of the transformation rules. We will show that in specific cases these deformations cannot be captured by the p-form algebra alone but that, instead, one is forced to introduce further mixed symmetry generators whose interpretation has yet to be clarified.

This paper is organized as follows. In section 2 we briefly summarize the Kac-Moody approach to supergravity. In section 3 we introduce the p-form algebra and uncover interesting properties of the deformation and top-form potentials in the context of this algebra. We will use the case of maximal supergravity to elucidate a few of these general properties. In the next section we apply the Kac-Moody approach to the case of half-maximal supergravity. In section 5 we will show that the addition of the embedding tensor leads to the introduction of additional symmetry generators to obtain closure. Finally, in the conclusions we comment on our results. We have included four appendices. Appendix A shortly summarizes the terminology we introduce in this paper. Appendix B contains a brief summary of the physical degrees of freedom and duality groups of matter-coupled half-maximal supergravity. Appendix C covers some group-theoretical details concerning the Kac-Moody algebras that are non-simply laced. Finally, appendix D contains lists of tables with the relevant low level results of the spectrum of the relevant Kac-Moody algebra.

2. The Kac-Moody approach to supergravity

The spectrum of physical states of the different maximal supergravity theories can be

obtained from the very extended Kac-Moody algebra E_{11} [7–9]. This has been extended to the set of all possible deformation and top-form potentials in [13, 14]. A similar analysis could be done for E_{10} [34–36] except for the top-form potentials. In addition, non-maximal supergravity and the associated very extended Kac-Moody algebras have been discussed in [9, 27, 37]. In the present paper we will apply the "Kac-Moody approach" to extract the deformation and top-form potentials of half-maximal supergravity. In general this approach breaks down into four steps:

1. Reduce to $D = 3$ over a torus and determine the $G/K(G)$ scalar coset sigma model.

2. Take the very extension $G^{+++}/K(G^{+++})$.

3. Oxidize back to $3 \leq D \leq D_{\max}$.

4. Read off the spectrum by means of a level decomposition.

As steps 2, 3, and 4 can be automatically carried out on the computer [38], this approach is very simple to carry out in practice. We will now take a close look at each of these steps.

The first step is to determine the $G/K(G)$ scalar coset sigma model in three dimensions for the toroidally reduced supergravity in question, where $K(G)$ is the maximal compact subgroup of G. If there is no such a sigma model, which often is the case for theories with less than 16 supercharges, the Kac-Moody approach comes to a standstill. But when the coset does exist, as is the case for maximal and half-maximal supergravity, we can go on and take the very extension $G^{+++}/K(G^{+++})$. The first extension corresponds to the (untwisted) affine version of G, which has been shown to be the symmetry group of various supergravities in $D = 2$ [39]. Also the second (over) extension and the third (very) extension are conjectured to be symmetry groups of maximal supergravity: the former has been employed for a $D = 1$ coset [34–36] while the latter has been used for non-linear realisations of the higher-dimensional theory [7–9].

Once $G^{+++}/K(G^{+++})$ has been constructed, we are in the position to oxidize back to $3 \leq D \leq D_{\max}$ dimensions using group disintegrations. The valid disintegrations for G^{+++} are always of the type $G_D \otimes \mathrm{SL}(D, \mathbb{R})$, where G_D is the duality group in D dimensions and $\mathrm{SL}(D, \mathbb{R})$ refers to the space-time symmetries. Extended Dynkin diagrams are a useful tool to visualize these group disintegrations: the disintegrations then correspond to 'disabling' certain nodes of the diagram in order to obtain two disjoint parts, of which one is the $\mathrm{SL}(D, \mathbb{R})$ gravity line and the other is the G_D duality group. As an example we give the cases of maximal supergravity in $D = 11, 10$ in figure 1. Note that the duality group G_D contains an extra $\mathbb{R}^+$ factor whenever there is a second disabled node. This explains why the duality group of IIA supergravity is $\mathbb{R}^+$ and why those of IIB and $D = 11$ supergravity do not have such a factor.

The maximum oxidization dimension is determined by the largest $\mathrm{SL}(D_{\max}, \mathbb{R})$ chain possible starting from the very extended node in the (extended) Dynkin diagram of G^{+++} [40, 41]. In our conventions these will always start at the right hand side of the extended Dynkin diagram. The lower limit on the oxidization dimension stems from the
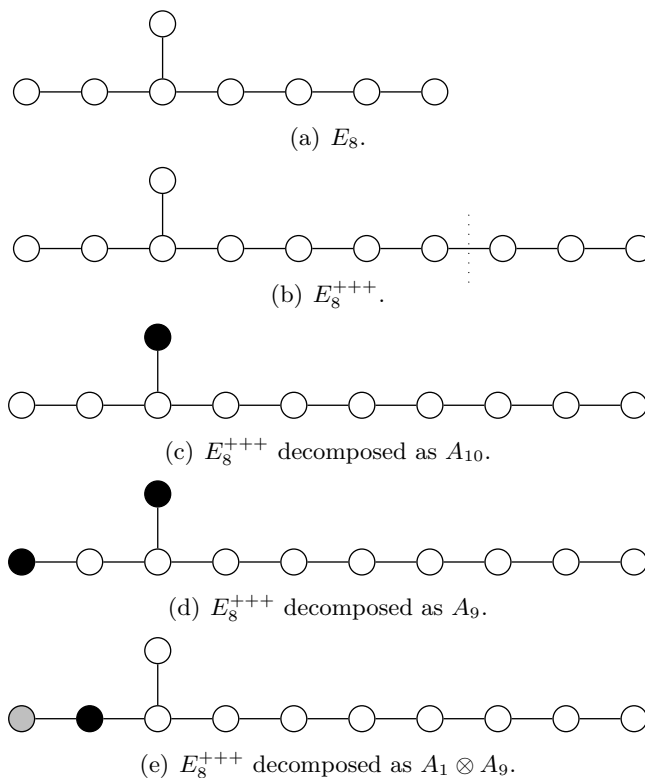
(a) E_8.

(b) E_8^{+++}.

(c) E_8^{+++} decomposed as A_{10}.

(d) E_8^{+++} decomposed as A_9.

(e) E_8^{+++} decomposed as $A_1 \otimes A_9$.

Figure 1: The Dynkin diagrams of E_8 1(a), the very extended E_8^{+++} 1(b), and its decompositions corresponding to 11D 1(c), IIA 1(d) and IIB 1(e) supergravity. In these decompositions the black nodes are disabled, the white nodes correspond to the gravity line $\mathrm{SL}(D, \mathbb{R})$ and the gray node in the last diagram corresponds to the duality group A_1.

fact that below $D = 3$ the duality group G_D becomes infinite-dimensional, and there are currently no computer-based tools available to analyze these cases.

After the group disintegration has been fixed, the generators of $G^{+++}/K(G^{+++})$ can be analyzed by means of a level decomposition [34, 35]. A level decomposition comes down to a branching of G^{+++} with respect to the $G_D \otimes \mathrm{SL}(D, \mathbb{R})$ disintegration. The disabled nodes then induce a grading on G^{+++} which will be indicated by the so-called levels. When G^{+++} is of real split form, i.e. maximally non-compact, modding out by the subgroup $K(G^{+++})$ implies truncating all the negative levels in the representation and generically also modding out the scalars at level 0 by the compact part of the duality group G_D. For clarity we will restrict our discussion to the split forms, although with some slight modifications everything also holds for the non-split cases, as follows from [42]. Indeed, we have verified for various non-split cases that the computer calculations give rise to the general results discussed in this paper.

The spectrum is obtained by associating to each generator a supergravity field in the same representation. This leads to the following fields at each level: At the lowest levels the physical states of the supergravity we started out with appear together with their duals.[2] The duality relations themselves are not reproduced by the level decomposition:

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[2]More precisely: corresponding to any p-form generator we also find a $(D-p-2)$-form. In addition there

in the absence of dynamics these relations have to be imposed by hand (for a discussion of dynamics in the context of E_{10} and E_{11}, see e.g. [34, 36, 11] and [7, 8] respectively). At higher levels there are the so-called "dual" generators, which can be interpreted as infinitely many exotic dual copies of the previously mentioned fields [43, 37]. The remaining "non-dual" generators do not correspond to any physical degrees of freedom. Amongst these are the $(D-1)$– and D-form potentials we are interested in.

In short, once the relevant $G/K(G)$ coset in three dimensions is known, all we have to do is consider the different decompositions of its very extended Dynkin diagram. The deformation and top-form potentials can then be read off from the spectrum the computer has calculated.

3. The p-form algebra

In this section we will consider the bosonic algebra that the different p-form Kac-Moody generators with $p > 0$ satisfy amongst each other. Subsequently it will be shown how the same algebra arises in supergravity. In the following two subsections we will discuss two classes of special generators. Frequently, we will clarify general features of the algebra by the example of maximal supergravities. In the next section we will discuss the case of matter-coupled half-maximal supergravities.

3.1 Truncation to p-forms

It is convenient to introduce a special algebra, which we call the "p-form algebra". It can be obtained as a truncation from the very extended Kac-Moody algebra in a particular $G_D \otimes \mathrm{SL}(D, \mathbb{R})$ decomposition by deleting all generators except those at positive levels in a purely antisymmetric $\mathrm{SL}(D, \mathbb{R})$ tensor representation of rank[3] $1 \leq p \leq D$. Embedded within the Kac-Moody algebra this is generically not a proper subalgebra (it is not closed), but on its own it nonetheless is a Lie algebra. What one ends up with after the truncation is an algebra of generators represented by components of p-forms that satisfy commutation relations of the form

$$\left[A_{\mu_1 \cdots \mu_p}, B_{\nu_1 \cdots \nu_q}\right] = C_{\mu_1 \cdots \mu_p \nu_1 \cdots \nu_q} \, . \tag{3.1}$$

Suppressing the $\mathrm{SL}(D, \mathbb{R})$ indices, we will write this more concisely as

$$[\boldsymbol{p}, \boldsymbol{q}] = \boldsymbol{r} \, . \tag{3.2}$$

Here we have introduced the shorthand notation $\boldsymbol{p}$, which will be used throughout this paper.[4] In the above commutator the ranks of the p-forms add up: $r = p + q$. In other words, the rank of the third form is equal to the sum of the ranks of the first and second forms.

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is a $(D-3, 1)$-form with mixed symmetries and possibly $(D-2)$-form generators, which are interpreted as the dual graviton and dual scalars, respectively.

[3]One could also include the $p = 0$ or scalar generators, which are the generators of the duality group G_D.

[4]Note that $\boldsymbol{p}$ indicates the components of p-forms and not p-forms themselves. In this way we avoid the anti-commutators which are used in [33].

An important property of the p-form algebra is the existence of "fundamental" p-forms whose multiple commutators give rise to the whole algebra by using the Jacobi identity. These fundamental p-forms correspond to the positive simple roots of the disabled nodes in the Kac-Moody algebra. From the decomposed Dynkin diagram one can thus deduce the number and type of these fundamental p-forms: any disabled node connected to the nth node of the gravity line (counting from the very extended node) gives rise to a $(D-n)$-form. Furthermore, if the disabled node in question is also connected to a node of the duality group the $(D - n)$-form carries a non-trivial representation of the duality group.

In the simplest case when there is only one disabled node, and thus only one fundamental p-form, we can schematically write for each p-form

$$\underbrace{[\boldsymbol{q}, \ldots, [\boldsymbol{q}[\boldsymbol{q}, \boldsymbol{q}]] \ldots]}_{\ell \text{ times}} = \boldsymbol{p} \,, \tag{3.3}$$

where q is the rank of the fundamental form and $p = \ell q$. The number of times the commutators are applied corresponds to the level ℓ at which the p-form occurs in the level decomposition of the Kac-Moody algebra. By definition the fundamental generators occur at level one. This is the structure of e.g. 11D supergravity, which only has a fundamental three-form $\boldsymbol{3}$, see figure 1(c). In addition there is a six-form $\boldsymbol{6}$, which can be obtained from the $\boldsymbol{3}$ by the commutation relation

$$[\boldsymbol{3}, \boldsymbol{3}] = \boldsymbol{6} \,. \tag{3.4}$$

According to the definition above the 6-form generator occurs at level $\ell = 2$. Note that this p-form algebra is defined in any dimension D. It is only after we impose dynamics, i.e. duality relations, that we should restrict to $D = 11$ in order to make contact with $D = 11$ supergravity.

Another example might further clarify the above. Consider again the Dynkin diagram of E_{11} and the embedding of an $\mathrm{SL}(10, \mathbb{R})$ gravity line that corresponds to IIB supergravity, see figure 1(e). We now associate to each generator a supergravity field. There are two nodes outside of the white gravity line. One is the grey node not connected to the gravity line. This node corresponds to the $\mathrm{SL}(2, \mathbb{R})$ duality symmetry. The other is the black node attached to the gravity line at the 8th position counted from the right, and hence corresponds to a fundamental two-form. Since this node is also connected to the internal symmetry node the two-form is in a non-trivial representation of $\mathrm{SL}(2, \mathbb{R})$: the IIB theory contains a doublet of NS-NS and R-R two-forms. We denote these two-forms by $\boldsymbol{2}^\alpha$. Using the same notation for the higher-rank forms we have the following p-form algebra

$$\begin{aligned}
\left[\boldsymbol{2}^\alpha, \boldsymbol{2}^\beta\right] &= \boldsymbol{4}\,\epsilon^{\alpha\beta} \,, \\
\left[\boldsymbol{2}^\alpha, \boldsymbol{4}\right] &= \boldsymbol{6}^\alpha \,, \\
\left[\boldsymbol{2}^\alpha, \boldsymbol{6}^\beta\right] &= \boldsymbol{8}^{\alpha\beta} \,, \\
\left[\boldsymbol{2}^\alpha, \boldsymbol{8}^{\beta\gamma}\right] &= \boldsymbol{10}^{\alpha\beta\gamma} + \epsilon^{\alpha(\beta}\boldsymbol{10}^{\gamma)} \,,
\end{aligned} \tag{3.5}$$

where all $SL(2, \mathbb{R})$ representations are symmetric and $\epsilon^{\alpha\beta}$ is the Levi-Civita tensor. There are other non-zero commutators but, due to the Jacobi identity, they follow from these basic ones involving the fundamental $\boldsymbol{2}^\alpha$-form generators. The commutators (3.5) specify the level ℓ at which each generator occurs. This level can be read off from these commutators by counting the number of times the fundamental generators $\boldsymbol{2}^\alpha$ occur in the multiple commutator that expresses the generator in terms of the fundamental ones. In this way we obtain that the generators $\boldsymbol{4}, \boldsymbol{6}^\alpha, \boldsymbol{8}^{\alpha\beta}, \boldsymbol{10}^{\alpha\beta\gamma}$ and $\boldsymbol{10}^\alpha$ occur at level $\ell = 2, 3, 4, 5$ and 5, respectively.

The p-form algebra contains generators corresponding to the following IIB supergravity fields: a doublet of two-forms, a singlet four-form potential, the doublet of six-form potentials that are dual to the two-forms, a triplet of eight-form potentials that are dual to the scalars [5] [44] and, finally, a doublet and quadruplet of ten-form potentials [6].

It was shown in [45] that the algebra of bosonic gauge transformations of IIB supergravity can be brought to precisely the form (3.5). This was achieved after making a number of redefinitions of the fields and gauge parameters, as was also done in the "doubled" formalism of [32]. The correspondence goes as follows. The p-form gauge transformations of IIB supergravity can be written as [45]

$$
\begin{aligned}
\delta A_{(2)}^\alpha &= \Lambda_{(2)}^\alpha \,, \\
\delta A_{(4)} &= \Lambda_{(4)} + \epsilon_{\gamma\delta} \Lambda_{(2)}^\gamma A_{(2)}^\delta \,, \\
\delta A_{(6)}^\alpha &= \Lambda_{(6)}^\alpha + \Lambda_{(4)} A_{(2)}^\alpha - 2\Lambda_{(2)}^\alpha A_{(4)} \,, \\
\delta A_{(8)}^{\alpha\beta} &= \Lambda_{(8)}^{\alpha\beta} + \Lambda_{(6)}^{(\alpha} A_{(2)}^{\beta)} - 3\Lambda_{(2)}^{(\alpha} A_{(6)}^{\beta)} \,, \\
\delta A_{(10)}^{\alpha\beta\gamma} &= \Lambda_{(10)}^{\alpha\beta\gamma} + \Lambda_{(8)}^{(\alpha\beta} A_{(2)}^{\gamma)} - 4\Lambda_{(2)}^{(\alpha} A_{(8)}^{\beta\gamma)} \,, \\
\delta A_{(10)}^\alpha &= \Lambda_{(10)}^\alpha + \tfrac{5}{27}\epsilon_{\beta\gamma}\Lambda_{(8)}^{\alpha\beta} A_{(2)}^\gamma - \tfrac{20}{27}\epsilon_{\beta\gamma}\Lambda_{(2)}^\beta A_{(8)}^{\gamma\alpha} + \Lambda_{(4)} A_{(6)}^\alpha - \tfrac{2}{3}\Lambda_{(6)}^\alpha A_{(4)} \,.
\end{aligned}
\tag{3.6}
$$

Here we use the notation $\Lambda_{(2n)} \equiv \partial\lambda_{(2n-1)}$, following [32]. By definition each parameter Λ is closed. In contrast to [45] we have redefined the gauge parameter of the doublet ten-form potential such that the gauge transformations of this potential precisely agree with the Kac-Moody algebra or its truncation to p-forms. This can always be done for top-form transformations. Note that the same structure also follows from a superspace calculation [46].

In order to compare with the p-form algebra we now truncate the bosonic gauge algebra to a finite-dimensional subalgebra as follows:

$$
\Lambda_{(2n)} \text{ is constant} \qquad \text{or} \qquad \lambda_{(2n-1)} = x \cdot \Lambda_{(2n)} \,, \tag{3.7}
$$

where it is understood that the spacetime coordinate x^μ is contracted with the first index of $\Lambda_{(2n)}$. Note that this indeed is a consistent truncation due to the fact that the local gauge parameters $\lambda_{(2n-1)} = \lambda_{(2n-1)}(x)$ always occurs in the transformation rules (3.6) with a derivative acting on it. Furthermore, we could have included a constant part in $\lambda_{(2n-1)}$, but this drops out of the gauge transformations for the same reason.

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[5] Supersymmetry will imply a single constraint on the nine-form field-strengths in order to produce the correct counting of physical degrees of freedom dual to the scalars [44].
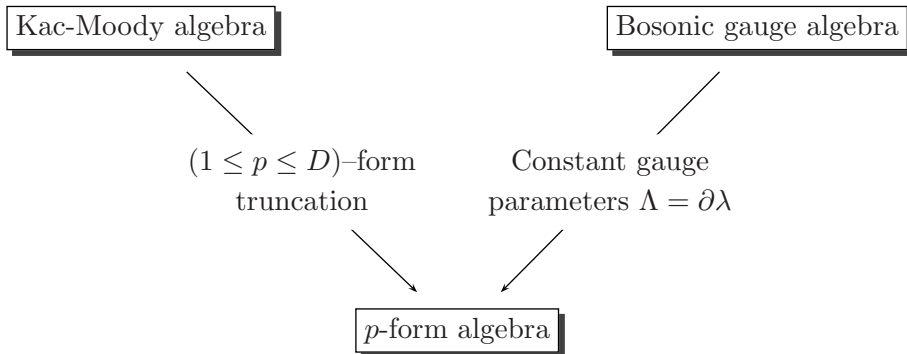
$(1 \leq p \leq D)$–form   Constant gauge
truncation   parameters $\Lambda = \partial\lambda$

p-form algebra

Figure 2: The p-form algebra as the respective limits of the Kac-Moody and the bosonic gauge algebra.

The commutator algebra corresponding to (3.6), for constant Λ, is now precisely of the form (3.5) provided we associate to each p-form in (3.5) the gauge transformation, with parameter $\Lambda_{(p)}$, of a p-form potential in (3.6). The p-form algebra arises as a Lie algebra truncation, defined by (3.7), of the bosonic gauge algebra,[6] see figure 2. The p-form gauge field transformations (3.6) can now be viewed as a nonlinear realisation of the p-form algebra (3.5).[7] Note that the p-form gauge fields not only transform under their own gauge transformations but also under those of the other gauge fields. Consequently, the curvatures of these gauge fields will contain Chern-Simons like terms.

We would like to stress that the truncation of the bosonic gauge algebra to a p-form Lie algebra is only possible in a particular basis of the supergravity theory. In particular, the gauge transformations need to be expressed only in terms of the gauge potentials and not their derivatives. It will not always be possible to bring the gauge transformations of any supergravity theory to such a form. An example of this will be discussed in section 5 in the context of gauged and massive supergravities.

The above reasoning can be applied to any very extended Kac-Moody algebra. It provides a useful truncation to the part of the spectrum that contains all p-form potentials, including the deformation and top-form potentials, which will be discussed next.

3.2 Deformation potentials

We now wish to discuss some properties of the deformation and top-form potentials in the light of the p-form algebra introduced above. We first consider the deformation potentials. It has been argued in, e.g., [2, 4, 13, 14] that deformation potentials are in one-to-one correspondence with deformations of the supergravity theory, such as gaugings or massive deformations. Indeed, the D-form curvatures of the $(D-1)$-form potentials can be seen as the duals of the deformation parameters: in the presence of a deformation, one can only realize the

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[6]The relation between the bosonic gauge algebra and the p-form algebra, in the sense that the latter is a Lie algebra truncation of the former, can also be introduced for diffeomorphisms. Restricting the general coordinate transformations to $x \rightarrow \Lambda x$, where Λ is a constant $GL(D)$ matrix, these span a Lie algebra.

[7]Note that the gauge transformations (3.6) contain no terms that are higher order in the potentials, as would occur in a generic non-linear representation. It turns out that all higher order terms (but not the 0th order one) can be eliminated by making appropriate field redefinitions.

D	Fundamental p-forms
11	**3**
IIA	**1**, **2**
IIB	**2**
3–9	**1**

Table 1: Fundamental p-forms for maximal supergravity [7, 8, 13, 14], where we have suppressed the duality group indices and multiplicities.

supersymmetry algebra on a $(D-1)$-form potential provided its field strength is the Hodge dual of the deformation parameter. As far as the deformation parameters are concerned one can distinguish between gauged and massive deformations, as we will discuss below.

The most familiar class of deformed supergravities are the so-called "gauged" supergravities. They are special in the sense that the deformations can be seen as the result of gauging a subgroup H of the duality group G. Not all deformed supergravities can be viewed as gauged supergravities. In the case of maximal supergravity there is one exception: massive IIA supergravity cannot be obtained by gauging the $\mathbb{R}^+$ duality group [5]. The gauged supergravities can be seen as the first in a series of "type p deformations". There is a a simple criterion that defines to which type of deformation parameter each deformation potential gives rise to. The central observation is that to each $(D-1)$-form one can associate a unique commutator

$$[\boldsymbol{p}, (\boldsymbol{D-p-1})] = (\boldsymbol{D-1}),\tag{3.8}$$

where $\boldsymbol{p}$ corresponds to a fundamental p-form and where we have suppressed the representation of the duality group. The deformation potential corresponding to such a deformation generator gives rise to a type p deformation parameter.

We observe that each type p deformation is characterized by the fact that a fundamental p-form gauge field becomes massive. For $p=1$ this leads to gauged supergravities, in which a vector can become massive by absorbing a scalar degree of freedom.[8] Note that other non-fundamental gauge fields may become massive as well. The case $p=2$ entails a fundamental two-form that becomes massive by "eating" a vector. The prime example of this is massive IIA supergravity in ten dimensions [5]. Another example is the non-chiral half-maximal supergravity in six dimensions [47]. An example of a $p=3$ deformation is the half-maximal supergravity theory of [48] where a fundamental three-form potential acquires a topological mass term. Due to the restricted number of dimensions it can be seen that there are no $p \geq 4$ deformations of supergravity theories.

It is interesting to apply these general observations to the case of maximal supergravity. In that case all deformations are gauge deformations except massive IIA supergravity. This can be easily understood from the Kac-Moody approach. In $D \leq 9$ all fundamental p-forms are vectors (see table 1) and thus one can never realize the

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[8]In the $p=0$ case there is only a massive vector when an isometry of the scalar manifold is being gauged.

commutation relation (3.8) for $p \neq 1$. Only in $D = 10$ we have a fundamental 2-form making massive supergravity possible. In $D = 10$ a deformation potential is a 9-form and eq. (3.8) becomes $[\boldsymbol{7}, \boldsymbol{2}] = \boldsymbol{9}$. Instead we have that $[\boldsymbol{8}, \boldsymbol{1}]$ vanishes. Note that IIA supergravity allows a massive deformation but IIB supergravity does not. The reason for this, from the Kac-Moody point of view, is that in writing the commutator $[\boldsymbol{7}, \boldsymbol{2}] = \boldsymbol{9}$ it is understood that there is either a fundamental 7-form, which is not the case, or this representation can be written as a multiple commutator of fundamental p-forms. The latter is only possible if there is at least one fundamental p-form with an odd number of indices. This condition is only satisfied in the case of IIA supergravity.

Note that in $D = 11$ we have a single fundamental 3-form, but in $D = 11$ a deformation potential is a 10-form which cannot be written as a multiple commutator of 3-forms. Therefore, there is no massive deformation in $D = 11$.

3.3 Top-form potentials

Finally, we consider the top-form potentials and point out an intriguing relation with the deformation potentials and parameters. Given a supergravity theory with deformation parameters in different representations of the duality group, it is not obvious that these deformation parameters can be turned on all at the same time. In fact, in the case of gauged supergravities it is known that the deformation parameters satisfy certain quadratic constraints [16, 18].

To illustrate the need for quadratic constraints on the deformation parameters and observe the relation with the top-form potentials it is instructive to consider $D = 9$ maximal gauged supergravity with duality group $\mathbb{R}^+ \times \mathrm{SL}(2, \mathbb{R})$ [23]. There is a triplet $m_{\alpha\beta}$ of deformations, corresponding to the gauging of a one-dimensional subgroup of $\mathrm{SL}(2, \mathbb{R})$, and a doublet m_α of deformations, corresponding to an $\mathbb{R}^+$ gauging:

$$\boldsymbol{3} \quad \Leftrightarrow \quad m_{\alpha\beta} : \qquad \left.\begin{array}{c} \mathrm{SO}(2) \\ \mathrm{SO}(1,1) \\ \mathbb{R}^+ \end{array}\right\} \in \mathrm{SL}(2, \mathbb{R}) \qquad \text{- IIB (and mIIA) origin.} \tag{3.9a}$$

$$\boldsymbol{2} \quad \Leftrightarrow \quad m_\alpha : \qquad\qquad \mathbb{R}^+ \qquad\qquad \text{- IIA origin.} \tag{3.9b}$$

All components of the triplet and of the doublet can be obtained via generalized Scherk-Schwarz reductions of IIB and of massless IIA supergravity, respectively. In addition, one component of the triplet, corresponding to the $\mathbb{R}^+ \in \mathrm{SL}(2, \mathbb{R})$ gauging, can be obtained via a Kaluza-Klein reduction of the massive IIA theory. Note that it is impossible to perform a generalized Scherk-Schwarz reduction in the massive case, since the mass parameter breaks the relevant scale symmetry.

Due to the different origins of the triplet and the doublet it is impossible to obtain them simultaneously from ten dimensions. However, one might wonder whether they can be turned on at the same time, independent of any higher-dimensional origin. This question has been answered in the negative [23], which can be summarized by imposing the following

quadratic constraint:

$\qquad m_{\alpha\beta}\, m_\gamma = 0 \quad \Leftrightarrow \quad \mathbf{3} \, \times \, \mathbf{2} \, = \, \mathbf{4} \, + \, \mathbf{2}\,.$ $\qquad\qquad$ (3.10)

These constraints occur in the $\mathbf{4}$ and $\mathbf{2}$ representation which are in one-to-one correspondence with two of the three representations of top-forms, as can be seen in [13, 14]. So for each constraint there is a corresponding top-form potential.

Also in lower dimensions, both for maximal supergravity in $3 \leq D \leq 7$ [15, 16, 18] and half-maximal supergravity in $D = 3, 4, 5$ [17, 49], the quadratic constraints have been calculated with the embedding tensor approach. In each case we observe that there is an exact one-to-one correspondence between the quadratic constraints and the top-form potentials in terms of representations of the duality group.

In the embedding tensor approach this correspondence can be explained as follows. Starting with a gauged supergravity Lagrangian $\mathcal{L}_{\text{sugra}}$ one can always replace the constant embedding tensor by a scalar field that is constant only on-shell due to the field equation of a deformation potential $A_{(D-1)}$, see also e.g. [4, 33]. Furthermore, for each quadratic constraint one introduces a top-form Lagrange multiplier $A_{(D)}$ in the same representation to enforce the constraint.[9] The total Lagrangian thus becomes (suppressing duality group indices)

$$\mathcal{L} = \mathcal{L}_{\text{sugra}} + A_{(D-1)}\partial\Theta + A_{(D)}\Theta\Theta\,, \qquad\qquad (3.11)$$

where Θ is the embedding tensor. One might wonder how a gauge field can act as a Lagrange multiplier. It turns out that the gauge transformation of the top-form in (3.11) is cancelled by adding an extra term to the gauge transformation of the deformation potential in the following way

$$\begin{aligned} \delta A_{(D-1)} &= \partial\Lambda_{(D-2)} + \Lambda_{(D-1)}\Theta\,, \\ \delta A_{(D)} &= \partial\Lambda_{(D-1)}\,. \end{aligned} \qquad\qquad (3.12)$$

The field equation for the embedding tensor field leads to a duality relation of the form (1.2). This provides a concrete way of explaining why the deformation potentials and the embedding tensor must be in the same representation of the duality group, and similarly for the top-form potentials and the quadratic constraints.

We conclude that one can divide the top-form potentials into two classes: the first class consists of all top-forms that are Lagrange multipliers enforcing quadratic constraints on the deformation parameters, while the second class contains all the other independent top-forms whose role is unclear from the present point of view. Examples of supergravity theories with independent top-form potentials are the half-maximal chiral supergravity theory in six dimensions and IIB and $D = 9$ maximal supergravity. The first theory does not contain deformation potentials and hence no quadratic constraints. The same applies to IIB which contains an independent quadruplet of potentials that is related to the D9-brane and an independent doublet of top-forms that sofar has no brane interpretation. Finally, in $D = 9$ maximal supergravity there is another top-form representation, in addition to (3.10), that does not correspond to a quadratic constraint.

---

[9]We thank Henning Samtleben for pointing this out to us. See also the recent paper [50].

4. Matter coupled half-maximal supergravity

We now proceed with the case of matter coupled half-maximal supergravity. In subsection 4.1 we first investigate the structure of the Kac-Moody and p-form algebras. In the next two subsections we discuss the deformation and top-form potentials that are contained in it.

4.1 Kac-Moody and p-form algebras

Half-maximal supergravities, coupled to $D - 10 + n$ vector multiplets, reduce to the scalar coset $\mathrm{SO}(8, 8 + n) / \mathrm{SO}(8) \times \mathrm{SO}(8 + n)$ when reduced to three dimensions. In other words, the relevant groups for supergravity theories with 16 supercharges are the B and D series in the above real form. Of these, only three are of split real form, i.e. maximally non-compact, which are given by $n = -1, 0, +1$. These correspond to the split forms of B_7, D_8 and B_8, respectively.

We are interested in the decomposition of the very extensions of these algebras with respect to the possible gravity lines. An exhaustive list of the possibilities for the algebras of real split form is given in table 2. As can be seen from this table, these correspond to the unique D-dimensional supergravity theory with 16 supercharges coupled to $m + n$ vector multiplets with $m = 10 - D$. The corresponding duality groups G_D in D dimensions are also given in table 2. Note that there is no second disabled node and therefore no $\mathbb{R}^+$ factor in the duality group for the 6b case and in $D = 3, 4$.

In appendix D the result of the decomposition of the D_8^{+++} algebras with respect to the different $\mathrm{SL}(D, \mathbb{R})$ subalgebras is given. In addition, the decompositions of the other two split forms $B_{7,8}^{+++}$ can be found on the website of SimpLie [38]. It can be seen that these decompositions give rise to exactly the physical degrees of freedom [9]. In addition there are the deformation and top-form potentials in the Kac-Moody spectrum. In particular, table 3 summarises our results for the deformation and top-form potentials for half-maximal supergravity in D dimensions.

To discuss the p-form algebra it is easiest to start with $8 \leq D \leq 10$ dimensions where there is a unified result valid in any dimension $D = 8, 9, 10$. We will refer to this as the "generic" situation. In lower dimensions this generic pattern remains but there are extra generators specific for each dimension $D < 8$, see table 3. We will not discuss all the details of the lower dimensions but its should be clear that they follow the same pattern as the higher dimensions except that the expressions involved are a bit messier.

In $8 \leq D \leq 10$ dimensions the p-form algebra is given by the following generators and commutation relations. As can be seen from table 2, except in a few special (lower-dimensional) cases to be discussed below, in each dimension the fundamental p-forms of the algebra are a $\mathbf{1}^M$ and a $(\mathbf{D} - \mathbf{4})$. The former is in the fundamental representation of the duality group G_D while the latter is a singlet. The other generators follow from the

following basic commutators describing the p-form algebra:

$$\left[\mathbf{1}^M, \mathbf{1}^N \right] = \eta^{MN} \mathbf{2} \, ,$$
$$\left[\mathbf{1}^M, (\mathbf{D} - \mathbf{4}) \right] = (\mathbf{D} - \mathbf{3})^M \, ,$$
$$\left[\mathbf{1}^M, (\mathbf{D} - \mathbf{3})^N \right] = (\mathbf{D} - \mathbf{2})^{[MN]} + \eta^{MN}(\mathbf{D} - \mathbf{2}) \, ,$$
$$\left[\mathbf{1}^M, (\mathbf{D} - \mathbf{2})^{[NP]} \right] = \eta^{M[N}(\mathbf{D} - \mathbf{1})^{P]} + (\mathbf{D} - \mathbf{1})^{[MNP]} \, , \qquad (4.1)$$
$$\left[\mathbf{1}^M, (\mathbf{D} - \mathbf{2}) \right] = (\mathbf{D} - \mathbf{1})^M \, ,$$
$$\left[\mathbf{1}^M, (\mathbf{D} - \mathbf{1})^N \right] = \eta^{MN} \mathbf{D} + \mathbf{D}^{[MN]} \, ,$$
$$\left[\mathbf{1}^M, (\mathbf{D} - \mathbf{1})^{[NPQ]} \right] = \eta^{M[N} \mathbf{D}^{PQ]} + \mathbf{D}^{[MNPQ]} \, ,$$

where η^{MN} is the $\mathrm{SO}(m, m + n)$ invariant metric and the straight brackets indicate anti-symmetrization. In addition $[\mathbf{1}^M, \mathbf{2}]$ vanishes. From these commutators we read off the levels (ℓ_1, ℓ_2) of the different generators. Here ℓ_1, ℓ_2 is the number of times the fundamental $(\mathbf{D} - \mathbf{4}), \mathbf{1}^M$ generators occur in the multiple commutators expressing the generator in terms of the fundamental ones. Suppressing the duality indices we obtain that the generators $\mathbf{2}, (\mathbf{D} - \mathbf{3}), (\mathbf{D} - \mathbf{2}), (\mathbf{D} - \mathbf{1})$ and $\mathbf{D}$ occur at the levels $(0, 2), (1, 1), (1, 2), (1, 3)$ and $(1, 4)$, respectively. These results are in agreement with the tables in appendix C.[10]

We will now turn to the potentials associated to the different generators. We will first discuss the potentials corresponding to the physical degrees of freedom of half-maximal supergravity. A summary of these can be found in appendix B and in particular table 4. Afterwards we will discuss the non-propagating deformation and top-form potentials.

The $\mathbf{2}$ corresponds to the two-form potential, which is indeed present in any half-maximal supergravity. Note that the first commutator in (4.1) tells us that the two-form potential transforms in a Chern-Simons way under the vector gauge transformations:

$$\delta A_{(2)} = \partial \Lambda_{(1)} + \partial \Lambda_{(0)}^M \, A_{(1)}^N \, \eta_{MN} \, . \qquad (4.2)$$

Hence, the Kac-Moody approach automatically leads to the Chern-Simons gauge transformations that are crucial for anomaly cancellations in string theory [51]. The $(\mathbf{D} - \mathbf{3})^M$, $(\mathbf{D} - \mathbf{2})$ and $(\mathbf{D} - \mathbf{2})^{[MN]}$ correspond to the duals of the vectors, the dilaton and the scalar coset, respectively. Note that the number of $(D - 2)^{[MN]}$-forms exceeds that of the scalars, since the latter take values in the scalar coset G/K and hence are modded out by the compact subgroup K of G. Therefore, we expect that there will be a number of linear relations between the field strengths of the $(D - 2)$-forms, similar to what has been found for the 8-forms of IIB supergravity [44].

---

[10]Note that we refer to the generic situation. There are special cases. For instance, pure $D = 10$ half-maximal supergravity has no 1-form generators and the fundamental generators are a 2-form and a 6-form. Another exception is pure $D = 9$ half-maximal supergravity which has an $\mathbb{R}^+ \times \mathrm{SO}(1, 1)$ duality group. We now need three level numbers (ℓ_1, ℓ_2, ℓ_3) in order to distinguish between the different Poincare dualities under $\mathrm{SO}(1, 1)$.

D	G_D	Multiplets	$B_7^{+++}(n=-1)$	$D_8^{+++}(n=0)$	$B_8^{+++}(n=1)$
10	$\mathbb{R}^+ \times SO(n)$	GV^n	–		
9	$\mathbb{R}^+ \times SO(1,1+n)$	GV^{n+1}			
8	$\mathbb{R}^+ \times SO(2,2+n)$	GV^{n+2}			
7	$\mathbb{R}^+ \times SO(3,3+n)$	GV^{n+3}			
6a	$\mathbb{R}^+ \times SO(4,4+n)$	GV^{n+4}			
6b	$SO(5,5+n)$	GT^{n+4}			
5	$\mathbb{R}^+ \times SO(5,5+n)$	GV^{n+5}			
4	$SO(6,6+n) \times$ $\times SL(2,\mathbb{R})$	GV^{n+6}			
3	$SO(8,8+n)$	GV^{n+7}			

Table 2: The decompositions of B_7^{+++}, D_8^{+++} and B_8^{+++} with respect to the possible gravity lines. The duality groups G_D and the multiplet structures (where G is the graviton, V the vector and T the self-dual tensor multiplet) are also given.

Extreme cases occur when the symmetry group is compact, i.e. $m = 0$ or $m + n = 0$. These correspond to ten dimensions or pure supergravity, without vector multiplets, respectively. These theories do not have any other scalars than the dilaton and hence one expects supersymmetry to require all of the field strengths of the $(D-2)^{[MN]}$-forms to vanish. Although we are not aware of a discussion of this phenomenon in the context of half-maximal supergravity, it has recently been encountered in pure $D = 5$, $N = 2$ supergravity [52].

We should also mention two exceptions that differ from the above pattern. The $D = 10$ theory without vector multiplets and the $D = 6b$ theories do not contain any vectors. Rather, the simple roots correspond to a $\mathbf{2}$ and a $\mathbf{6}$ in the $D = 10$ case and to a $\mathbf{2}^M$ in the $D = 6b$ case. These generate the following gauge transformations:

$$D = 10,\ n = 0:\quad \begin{cases} [\mathbf{2}, \mathbf{2}] = 0\,, \\ [\mathbf{2}, \mathbf{6}] = \mathbf{8}\,, \\ [\mathbf{2}, \mathbf{8}] = \mathbf{10}\,, \end{cases} \tag{4.3}$$

for the ten-dimensional theory and

$$D = 6b:\quad \begin{cases} \left[\mathbf{2}^M, \mathbf{2}^N\right] = \mathbf{4}^{[MN]}\,, \\ \left[\mathbf{2}^M, \mathbf{4}^{[NP]}\right] = \eta^{M[N}\mathbf{6}^{P]} + \mathbf{6}^{(M[N)P]}\,, \end{cases} \tag{4.4}$$

for the six-dimensional case.

In yet lower dimensions a similar pattern occurs. The main difference, however, is that for half-maximal supergravity in $D \leq 5$ dimensions, all fundamental p-forms are vectors. This is in contrast to higher dimensions where there are also fundamental p-forms of higher rank. The explicit formulae are a bit messier in the lower dimensions, as can also be seen from table 4, and hence we will refrain from giving them. It should be stressed that they follow exactly the same pattern as above. The same applies to the deformation and top-form potentials discussed in the next two subsections where there is a plethora of representations in the lower dimensions, see table 3.

This finishes the discussion of the physical degrees of freedom and their duals [9, 27]. The generators corresponding to these potentials are already present in the affine Kac-Moody extension [41]. Potentials of yet higher rank do not correspond to propagating degrees of freedom and only occur in the over– and very-extended Kac-Moody algebras. We now discuss the deformation and top-form potentials of half-maximal supergravity.

4.2 Deformation potentials

Turning first to the $(D-1)$-forms, it follows from (4.1) that in generic dimensions these occur in a fundamental and an anti-symmetric three-form representation. In dimensions $8 \leq D \leq 10$ this is the complete story as well.

In lower dimensions, however, there are more possibilities. For example, in $D = 7$ one can generate an additional deformation potential 6 from a multiple commutator of the dual two-form $\boldsymbol{D-4}$, which is a $\mathbf{3}$ in this case. This corresponds to the singlet in table 3. In

D	$(D-1)$-forms			D-forms	
	$p=1$	$p=2$	$p=3$	constraints on $p=1$	other
$10,9,8$	□ ⊟			1 ⊟ ⊟	
7	□ ⊟		1	1 ⊟ ⊟	□
$6a$	□ ⊟	□		1 ⊟ ⊟	1 1 ⟐ ⊟
$6b$					□ ⬚
5	□ ⊟ ⊟			1 □ □ ⊟ ⬚ ⊟ ⊟	
4	$(\mathbf{2},□)\ \left(\mathbf{2},\ \substack{⊟}\right)$			$(\mathbf{3},1)\ (\mathbf{3},⊟)\ \left(\mathbf{3},\substack{⊟}\right)$ ⊟ ⊟ ⬚	
3	1 ⟐ ⊟			⟐ ⊟ ⊟ ⬚ ⬚	

Table 3: The representations of deformation — and top-forms in all half-maximal supergravities. The representations refer to the duality group G_D given in table 2. We also indicate which type p of deformations they correspond to, and to which top-forms one can associate a quadratic constraint on type 1 deformation parameters.

addition there is another top-form representation $\mathbf{7}^M$. The additional commutators are:

$$D = 7 : \quad \begin{cases} [\mathbf{3}, \mathbf{3}] = \mathbf{6} \,, \\ [\mathbf{3}, \mathbf{4}^M] = \mathbf{7}^M \,. \end{cases} \tag{4.5}$$

In fact, also the commutator $[\mathbf{1}^M, \mathbf{6}]$ is non-vanishing and leads to a $\mathbf{7'}^M$. However, this commutator is related to the one above by the Jacobi identity and hence $\mathbf{7}^M$ and $\mathbf{7'}^M$ are linearly dependent.

In $D = 6$ one has $D - 4 = 2$, i.e. the dual of the two-form is itself again a two-form. To avoid confusion, we will denote the fundamental 2-form by $\mathbf{2}$ and the one coming from the commutator of the vectors by $\mathbf{2'}$. In this theory there are again a number of extra commutators that contribute to the deformation and top-form potentials:

$$D = 6a : \quad \begin{cases} [\mathbf{2}, \mathbf{3}^M] = \mathbf{5'}^M \,, \\ [\mathbf{2}, \mathbf{4}^{[MN]}] = \mathbf{6'}^{[MN]} \,, \\ [\mathbf{2}, \mathbf{4}] = \mathbf{6'} \,, \\ [\mathbf{1}^M, \mathbf{5'}^N] = \eta^{MN} \mathbf{6'} + \mathbf{6}^{(MN)} \,. \end{cases} \tag{4.6}$$

There are also other non-vanishing commutators but these are related by the Jacobi identity.

We now turn to the question to which types of deformations the deformation potentials correspond. Given that there are only a few of these, we will start with the massive deformations. As can be seen from the previous discussion, type 3 deformations of half-maximal supergravity are only possible in $D = 7$, for the simple reason that only here there is a fundamental 3-form. The deformation is a singlet of the symmetry group $\mathbb{R}^+ \times SO(3, 3 + n)$ and has been explicitly constructed for $n = -3$ [48]. Similarly, type 2 deformations of half-maximal supergravity are only possible in $D = 6a$ and occur in the fundamental representation of the symmetry group $\mathbb{R}^+ \times SO(4, 4 + n)$. The deformed theory has been explicitly constructed for the special cases of $n = -4$ [47] and $n = 16$ [29].

All remaining deformation potentials correspond to type 1 deformations, i.e. to gaugings. Note that in every dimension $D \geq 4$ there is a fundamental and three-form representation of such deformation potentials. To be able to do more general gaugings one needs more space-time vectors than only the fundamental representation, which is present in all these dimensions. For example, in $D = 5$ an additional vector is provided by the dual of the two-form, giving rise to an extra two-form representation of possible gaugings. In $D = 4$ the extra vectors are the Hodge duals of the original ones, leading to an $SL(2, \mathbb{R})$ doublet of possible gaugings. Finally, in $D = 3$ scalars are dual to vectors. This is the underlying reason for the symmetry enhancement in three dimensions, and also gives rise to the more general possibilities of gaugings in this case.

Many of these gaugings have been obtained in the literature. Explicit calculations of the possible gaugings using the embedding tensor formalism in $D = 3, 4, 5$ have uncovered exactly the same representations [15, 49]. In addition, one can obtain components of the three-form representation of gauging in any dimension D by a Scherk-Schwarz reduction from $D + 1$ dimensions using the $SO(m, m + n)$ symmetry, see e.g. [28, 31].

A prediction that follows from the above analysis is that in the dimensions where the possible gaugings have not yet been fully analyzed, i.e. in $D \geq 6$, it will be possible to introduce a fundamental and a three-form representation of gaugings. In terms of the embedding tensor, which describes the embedding of the gauge group in the duality group G [16, 18], this would read

$$
\begin{aligned}
\Theta_M{}^{NP} &= f_M{}^{NP} + \delta_M^{[N} \xi^{P]} \,, \\
\Theta_N{}^0 &= \xi_N \,,
\end{aligned}
\tag{4.7}
$$

where $\xi_M = \eta_{MN} \xi^N$ and f_{MNP} are the fundamental and three-form representations of gaugings, respectively. The notation here is as follows: the subscript index $M = 1, \ldots, 2m + n$ refers to the generators of the gauge group and the superscript indices $\{0, MN\}$ label the generators of the duality group $\mathbb{R}^+ \times SO(m, m + n)$. The embedding tensor thus encodes which subgroup is gauged by the vectors $\mathbf{1}^M$. Note that the $\mathbb{R}^+$ factor is crucial for the introduction of ξ_N, as can be seen from the $\Theta_N{}^0$ component. The different components of $\Theta_M{}^{NP}$ and $\Theta_N{}^0$ specify which linear combinations of the gauge fields are used to gauge $\mathbb{R}^+$ and a subgroup $H \subset SO(m, m + n)$, respectively:

$$
\begin{aligned}
\Theta_M{}^{NP} \mathbf{1}^M : & \qquad H \subset SO(m, m + n) \,, \\
\Theta_N{}^0 \mathbf{1}^N : & \qquad \mathbb{R}^+ \,.
\end{aligned}
\tag{4.8}
$$

4.3 Top-form potentials

Subsequently, we consider the top-form potentials and their relation to the quadratic constraints.

In generic dimensions these top-forms occur in a singlet and anti-symmetric two- and four-form representations. Using the embedding tensor approach, an analysis of the quadratic constraints on the possible deformations has been explicitly carried out in $D = 3, 4, 5$ [15, 49]. It turns out that the representations of the quadratic constraints exactly coincide with the representations of the possible top-forms in these dimensions.

For $D \geq 6$ the embedding approach has not yet been applied and the Kac-Moody approach leads to a prediction. The generic top-forms occur in the singlet, two- and four-form representations. In addition, from the lower-dimensional analysis [15, 49] one would expect them to correspond to a quadratic constraint. In terms of the embedding tensor Θ these would take the following form:

$$
\begin{aligned}
f_{MNP}\xi^P &= 0\,, \\
3f_{R[MN}f_{PQ]}{}^R &= 2f_{[MNP}\xi_{Q]}\,, \\
\eta^{MN}\xi_M\xi_N &= 0\,.
\end{aligned}
\tag{4.9}
$$

The prediction is that the most general gauging of half-maximal supergravity in $D \geq 6$ is described by the embedding tensor (4.7) subject to the quadratic constraints (4.9). It would be interesting to explicitly construct these gauged theories. Note that the fundamental representation ξ_M cannot be non-zero in $D = 10$, since the quadratic constraint requiring it to be a null vector can not be satisfied for an $\mathrm{SO}(n)$ representations. The gauging with deformation parameters f_{MNP} can be viewed as a gauging of a subgroup $H \subset \mathrm{SO}(n)$ with structure constants f_{MNP}. In this sense $D = 10$ half-maximal matter-coupled supergravity with gauge groups $\mathrm{SO}(32)$ or $E_8 \times E_8$, i.e. the low-energy limit of type I or heterotic string theory, can be viewed as the gauged deformation of $D = 10$ half-maximal supergravity coupled to 496 Maxwell multiplet.[11]

In addition, it would be interesting to investigate the possibilities of including the type 2 and 3 massive deformations in six and seven dimensions, respectively, in the gauged theories; in other words, to see which types of deformations can be turned on simultaneously.

5. Deformations

Up to this point we have only considered the role of the deformation potentials in the Kac-Moody or p-form algebra but not the deformation parameters themselves. These parameters can be seen as the duals of (the field strengths of) the deformation potentials, see eqs. (1.1) and (1.2). This is in contradistinction with the lower-rank potentials in which case the p-form algebra gives rise to both the potentials and their duals. In this section we will briefly consider how the inclusion of the deformation parameters in supergravity effects the p-form algebra. In particular, we will discuss how the bosonic gauge transformations

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[11]We thank Axel Kleinschmidt for a discussion on this point.

could be truncated to a Lie algebra in the deformed case, first for massive IIA supergravity and in the next subsection for gauged half-maximal supergravity.

5.1 Massive IIA supergravity

For massless IIA supergravity the fundamental generators are a 1-form generator $\mathbf{1}$ and a 2-form generator $\mathbf{2}$, see figure 1(d). The other generators are the R-R generators $\mathbf{3}, \mathbf{5}, \mathbf{7}, \mathbf{9}$ and the NS-NS generators $\mathbf{6}, \mathbf{8}, \mathbf{10}, \mathbf{10}'$. The basic commutators are given by

$$
\begin{aligned}
[\mathbf{2}, \mathbf{1}] &= \mathbf{3}, & [\mathbf{1}, \mathbf{7}] &= \mathbf{8}, \\
[\mathbf{2}, \mathbf{3}] &= \mathbf{5}, & [\mathbf{2}, \mathbf{7}] &= \mathbf{9}, \\
[\mathbf{1}, \mathbf{5}] &= \mathbf{6}, & [\mathbf{2}, \mathbf{8}] &= \mathbf{10}, \\
[\mathbf{2}, \mathbf{5}] &= \mathbf{7}, & [\mathbf{1}, \mathbf{9}] &= \mathbf{10}'.
\end{aligned}
\tag{5.1}
$$

We have not included the 0-form generator corresponding to the duality group $\mathbb{R}^+$. In fact, also the commutator $[\mathbf{2}, \mathbf{6}]$ is non-vanishing and leads to an $\mathbf{8}'$. However, this commutator is related to $[\mathbf{1}, \mathbf{7}]$ by the Jacobi identity and hence $\mathbf{8}$ and $\mathbf{8}'$ are linearly dependent. As can be seen from (5.1), the IIA theory has a type 2 deformation potential and two top-form potentials. There is no quadratic constraint associated to either of the top-forms.

Let us now turn to the realisation of this symmetry on the IIA potentials. In the following we will only consider the truncation to the low-level potentials corresponding to $\mathbf{1}, \mathbf{2}, \mathbf{3}$ as this will be sufficient for our purpose. The gauge transformations of massless IIA supergravity are given by

$$
\begin{aligned}
\delta A_{(1)} &= \Lambda_{(1)}, \\
\delta A_{(2)} &= \Lambda_{(2)}, \\
\delta A_{(3)} &= \Lambda_{(3)} + 3\Lambda_{(1)}A_{(2)},
\end{aligned}
\tag{5.2}
$$

where the gauge parameters are all closed and hence $\Lambda_{(p)} = \partial\lambda_{(p-1)}$. The restriction to the p-form algebra corresponds to constant $\Lambda_{(p)}$'s, or equivalently $\lambda_{(p-1)}$'s with linear coordinate dependence, as discussed around (3.7). It can easily be verified that this truncation to a Lie algebra satisfies the first commutator of (5.1). Similarly, it is possible to include all potentials of IIA and truncate to the p-form Lie algebra, that satisfies the full (5.1) [45].

There are several formulations of the massive IIA theory. The original formulation by Romans [5] contains a constant mass parameter m and no deformation potential. Later, it was shown that there is an alternative description with a scalar mass function $m(x)$ and a 9-form deformation potential [4]. There is even a third formulation [4] with only a deformation potential and no parameter but the bosonic gauge transformations of this formulation are highly non-linear and have not been explicitly worked out yet. We will consider the original Romans formulation here.

It turns out that the IIA bosonic gauge transformations can be written as in the massless case:

$$
\begin{aligned}
\delta A_{(1)} &= \tilde{\Lambda}_{(1)}, \\
\delta A_{(2)} &= \Lambda_{(2)}, \\
\delta A_{(3)} &= \Lambda_{(3)} + 3\tilde{\Lambda}_{(1)}A_{(2)},
\end{aligned}
\tag{5.3}
$$

but with a different parameter $\tilde{\Lambda}_{(1)}$ that is not closed: $\partial \tilde{\Lambda}_{(1)} = -m\Lambda_{(2)}$. Note that, up to this level, the massive modification of the gauge transformations only occurs via $\tilde{\Lambda}_{(1)}$. The $\Lambda, \tilde{\Lambda}$ parameters can be expressed in terms of the local gauge parameters $\lambda_{(p-1)}$ as follows:

$$\tilde{\Lambda}_{(1)} = \partial \lambda_{(0)} - m\lambda_{(1)} \,,$$
$$\Lambda_{(2)} = \partial \lambda_{(1)} \,,$$
$$\Lambda_{(3)} = \partial \lambda_{(2)} \,. \tag{5.4}$$

If we perform the same truncation (3.7) as in the massless case to $\lambda_{(p)}$'s with linear coordinate dependence,[12] the massive transformation parameters reduce to

$$\tilde{\Lambda}_{(1)} = \Lambda_{(1)} - mx \cdot \Lambda_{(2)} \,, \tag{5.5}$$

and $\Lambda_{(2)}$ and $\Lambda_{(3)}$, with constant $\Lambda_{(i)}$'s.

A new feature is the appearance of an explicit coordinate dependence in the massive case. This has been interpreted from the point of view of the p-form algebra in the following way. The coordinate x^μ can be seen as a new potential with its associated symmetry being the translations [53]. Using the terminology of [33][13] we will call x^μ a "(-1)–form potential". Following [53] the massive deformation parameter m can be introduced to the p-form algebra by including an additional generator, which will be denoted by $-\mathbf{1}$. Subsequently one must define the commutators between the translation generator and the fundamental generators of the p-form algebra. In the case of massive IIA, the non-zero commutators are [53]

$$[\mathbf{2}_{\mu\nu}, -\mathbf{1}^\rho] = m \, \mathbf{1}_{[\mu}\delta^\rho_{\nu]}. \tag{5.6}$$

This commutator is realized by the truncated massive IIA gauge transformations (5.5) due to the term with explicit coordinate dependence.

The commutator (5.6), or equivalently the gauge transformation (5.4), tells us that the 1-form is transforming with a shift, proportional to m, under the gauge transformations of the 2-form. Therefore, the 1-form is "eaten up" by the 2-form and the two potentials $(2,1)$ together form a so-called Stückelberg pair describing a *massive* 2-form. The commutator (5.6) defines a deformation of the direct sum of the p-form algebra and the translation generator [53].

It is not guaranteed that the truncation (3.7) is consistent in the massive case, since $\lambda_{(1)}$ also appears without an accompanying derivative. Therefore, closing the algebra might force us to introduce more symmetries. Indeed, we find that the following commutator does not close:

$$[\delta_2, \delta_{2'}]A_{\mu\nu\rho} = 3m \left(x^\sigma \Lambda_{\sigma[\mu}\Lambda'_{\nu\rho]} - x^\sigma \Lambda'_{\sigma[\mu}\Lambda_{\nu\rho]} \right) \,. \tag{5.7}$$

Although the three-form potential transforms with a shift by a closed three-form, this is not covered by the present Ansatz for the gauge parameter $\lambda_{(2)}$, as it leads to a constant,

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[12]Note that in the massive case the constant part of $\lambda_{(1)}$ does not drop out, but it can be absorbed by a redefinition of $\Lambda_{(1)}$. Hence we will not consider this constant part.

[13]Actually, reference [33] proposes a different way of introducing the deformation parameters which will be discussed later.

x-independent, shift only. To obtain closure one must introduce an additional term of the form

$$\lambda_{\mu\nu} = x^\sigma \Lambda_{\sigma\mu\nu} + x^\sigma x^\tau \Lambda_{\sigma,\tau\mu\nu}\,, \quad \Rightarrow \quad \tilde{\Lambda}_{\mu\nu\rho} = \Lambda_{\mu\nu\rho} + \tfrac{4}{3} x^\sigma \Lambda_{\sigma,\mu\nu\rho}\,. \tag{5.8}$$

The algebra then closes provided

$$\Lambda_{\sigma,\mu\nu\rho} = \tfrac{9}{4} m \left(\Lambda_{\sigma[\mu} \Lambda'_{\nu\rho]} - \Lambda'_{\sigma[\mu} \Lambda_{\nu\rho]} \right)\,. \tag{5.9}$$

The additional parameter is anti-symmetric in the last three-indices and satisfies $\Lambda_{[\sigma,\tau\mu\nu]} = 0$. In terms of Lorentz representations, this corresponds to a $(\boldsymbol{3}, \boldsymbol{1})$ representation with mixed symmetry and its trace, which is a $\boldsymbol{2}$. Since the trace properties play no role here, we will denote both together by $\Lambda_{(3,1)}$.

One can see the need to include such a symmetry also from the p-form algebra point of view. Given the commutator (5.6) between the translation generator and the fundamental $\boldsymbol{2}$, the Jacobi identity between the $\{\boldsymbol{2}, \boldsymbol{2}, -\boldsymbol{1}\}$ generators implies

$$[[\boldsymbol{2}_{\mu\nu}, \boldsymbol{2}_{\rho\sigma}], -\boldsymbol{1}^\tau] = m\boldsymbol{3}_{\mu\nu[\rho}\delta^\tau_{\sigma]} + m\boldsymbol{3}_{\rho\sigma[\mu}\delta^\tau_{\nu]}\,. \tag{5.10}$$

Hence, in the massive case, $[\boldsymbol{2}_{\mu\nu}, \boldsymbol{2}_{\rho\sigma}]$ must be non-vanishing. It is anti-symmetric in a pair of two anti-symmetric indices and hence has $\tfrac{1}{8}(D+1)D(D-1)(D-2)$ components in D dimensions. This is equal to the number of components of a (traceful) $(\boldsymbol{3}, \boldsymbol{1})$ representation. Therefore we write

$$[\boldsymbol{2}_{\mu\nu}, \boldsymbol{2}_{\rho\sigma}] = m(\boldsymbol{3}, \boldsymbol{1})_{[\mu,\nu]\rho\sigma} - m(\boldsymbol{3}, \boldsymbol{1})_{[\rho,\sigma]\mu\nu}\,. \tag{5.11}$$

The above Jacobi identity is then satisfied provided

$$[(\boldsymbol{3}, \boldsymbol{1})_{\mu,\nu\rho\sigma}, -\boldsymbol{1}^\tau] = \delta^\tau_\mu \boldsymbol{3}_{\nu\rho\sigma} - \delta^\tau_{[\mu} \boldsymbol{3}_{\nu\rho\sigma]}\,. \tag{5.12}$$

We have checked that the first commutator of (5.1) together with (5.6), (5.11) and (5.12) lead to a closed Lie algebra. Schematically we have

$$\begin{aligned} [\boldsymbol{2}, \boldsymbol{1}] &= \boldsymbol{3}\,, & [\boldsymbol{2}, -\boldsymbol{1}] &= m\boldsymbol{1}\,, \\ [\boldsymbol{2}, \boldsymbol{2}] &= m(\boldsymbol{3}, \boldsymbol{1})\,, & [(\boldsymbol{3}, \boldsymbol{1}), -\boldsymbol{1}] &= \boldsymbol{3}\,. \end{aligned} \tag{5.13}$$

Note that $-\boldsymbol{1}$ does not appear on the right-hand side of any commutator, i.e. the complementary generators form an ideal, and the former can therefore be quotiented out. However, the same cannot be said for the $(\boldsymbol{3}, \boldsymbol{1})$ due to the commutator (5.11).

We conclude that a truncation of the massive IIA gauge transformations forces us to consider extensions of the p-form algebra with additional mixed symmetry generators. It is expected that more such generators are needed when also the higher rank potentials are included. It remains to be seen whether a consistent truncation exists when all p-form generators are included.

It is interesting to compare the present result with the approach of [33] which takes the same massive IIA gauge transformation rules as their starting point. Before doing any

truncation one first rewrites the massive transformation rules such that every parameter occurs with a derivative, like in the massless case. This makes it possible to perform the same truncation as in the massless case. For this to work it is crucial that one first formulates the transformation rules in terms of forms and next formally write the 0-form m as the exterior derivative of a "(-1)–form potential" $A_{(-1)}$:

$$m = dA_{(-1)} \,. \tag{5.14}$$

Once every parameter occurs under a derivative one can write the transformation rules as the non-linear realization of an algebra that includes a formal "(-1)–form generator". We understand that in this procedure one should not convert to component notation in the presence of the (-1)–form potential. Only after all (-1)–form potentials have been converted into deformation parameters a transition to component notation can be made. In particular, one should not consider a component formulation of (5.14) since this would lead us back to our earlier discussion with the need to introduce extra mixed symmetry generators. It would be interesting to see whether the "(-1)–forms" needed in this procedure can be given a rigorous mathematical basis.

Sofar, we have discussed two ways to proceed in the massive case. Either one starts extending the direct sum of the p-form algebra and the translation generators with new mixed symmetry generators or one extends the p-form algebra with the formal concept of a new "(-1)–form generator". There is even a third way to proceed in the massive case which uses E_{10} instead of E_{11} [54]. The spectrum of E_{10} leads to precisely the same representations as E_{11} except for the top-forms which only follow from E_{11}. By using E_{10} one is able to not only consider kinematics but also dynamics consistent with E_{10}. By using the dynamics the authors of [54] seem to be able to derive the equations of motion of massive IIA supergravity without the need to introduce new symmetry generators. It would be interesting to more carefully compare the different approaches and to obtain a better understanding of what the role of the dynamics is.

5.2 Half-maximal supergravity

We now discuss the case of half-maximal supergravity. For simplicity we will consider only the three-form representation f_{MNP} and not the most general deformation. This will be sufficient for the present purpose.

The starting point will be the original ungauged p-form algebra of half-maximal supergravity, which we truncate to the vectors $\mathbf{1}^M$. In addition we include the scalars $\mathbf{0}^{MN}$, which are the generators of the special orthogonal part of the duality group $G_D = \mathbb{R}^+ \times SO(m, m+n)$. These generators satisfy

$$\left[\mathbf{1}^M, \mathbf{0}^{NP}\right] = \mathbf{1}^{[N}\eta^{P]M} \,, \tag{5.15}$$

while other commutators vanish (including $\left[\mathbf{1}^M, \mathbf{1}^N\right]$ in this truncation).

Subsequently we introduce the three-form deformation f_{MNP}, which is defined by the following non-zero commutators between the translation generator $-\mathbf{1}$ and the fundamental generators, see also [13]:

$$\left[\mathbf{1}^M, -\mathbf{1}\right] = f^M{}_{NP} \, \mathbf{0}^{NP} \,. \tag{5.16}$$

Based on our experience with the massive IIA case we do not expect the above deformation to lead to a closed algebra. Indeed, from the $\{\mathbf{1}, \mathbf{1}, -\mathbf{1}\}$ Jacobi identity it follows that one is led to extend the algebra with a new generator that transforms in the symmetric $(1, 1)$ representation. The additional commutators take the form

$$\left[\mathbf{1}_\mu^M, \mathbf{1}_\nu^N\right] = 2 f^{MN}{}_P (\mathbf{1}, \mathbf{1})_{\mu\nu}^P ,$$
$$\left[(\mathbf{1}, \mathbf{1})_{\mu\nu}^M, -\mathbf{1}^\tau\right] = \delta^\tau_{(\mu} \mathbf{1}_{\nu)}^M . \tag{5.17}$$

The above Jacobi identity then vanishes.

Unlike in the massive IIA case, there are additional non-trivial Jacobi identities, for example of the form $\{\mathbf{1}, -\mathbf{1}, (\mathbf{1}, \mathbf{1})\}$. To satisfy these one needs to introduce additional symmetric three-index tensor generators with commutator relations similar to (5.17). Subsequently one finds that there are Jacobi identities involving the symmetric three-index tensors, that require the introduction of symmetric four-index tensors. This iterative procedure does not terminate. In terms of the local gauge parameters λ^M of the vector transformations, the new symmetries can be understood as the expansion

$$\lambda^M = \Lambda_\mu^M x^\mu + \Lambda_{\mu,\nu}^M x^\mu x^\nu + \cdots , \tag{5.18}$$

where Λ_μ^M and $\Lambda_{\mu,\nu}$ are the parameters corresponding to the $\mathbf{1}^M$ and $(\mathbf{1}, \mathbf{1})^M$ generators, respectively. Hence it appears that the gauge transformations of gauged half-maximal supergravities can only be truncated to an infinite number of generators.

It would be interesting to see if there exist an interpretation (or modification) of the approaches [53, 13, 33, 54] that can reproduce all the results that follow from the embedding tensor method [15, 16, 18].

6. Conclusions

In the first part of this paper we have refined the correspondence between the Kac-Moody spectrum of deformation and top-form potentials and the gaugings and massive deformations of the associated supergravity. It was shown that there is a truncation of the Kac-Moody algebra to a Lie algebra of p-forms, which encodes all the relevant information for the physical states (apart from gravity and scalars) plus the non-propagating deformation and top-form potentials. A special role is played by the fundamental p-forms, from which all other potentials can be constructed via commutators. In particular, one has commutators of the form (3.8) giving rise to the $(D-1)$-forms, from which the corresponding type of supergravity deformation can be deduced. In addition, the p-form algebra contains commutators leading to D-forms, and these may be associated to quadratic constraints on the deformation parameters. We should stress that the properties derived from (3.8) and the relation to the quadratic constraints are empirical observations. It would be interesting to understand how these follow from the bosonic gauge transformations of supergravity.

In the second part we have established that the correspondence also holds for half-maximal supergravity. In particular, in table 3 the spectrum of deformation and top-form potentials of the associated Kac-Moody algebras is summarized. These possibilities agree

perfectly with the known gaugings and massive deformations of half-maximal supergravity and the ensuing quadratic constraints, respectively. In addition it gives a prediction for the most general gaugings in $6 \leq D \leq 10$: these are encoded in a fundamental and three-form representation of the duality groups subject to the quadratic constraints (4.9).

Note that we have only realized a finite-dimensional part of the Kac-Moody algebra as a symmetry. However, in different dimensions, this p-form algebra constitutes a different truncation of the Kac-Moody algebra. The latter contains all symmetry groups of half-maximal supergravity in D dimensions. This shows how the very extended Kac-Moody algebra $\mathrm{SO}(8, 8+n)^{+++}$ plays a unifying role in describing the symmetries of half-maximal supergravity coupled to $10 - D + n$ vector multiplets.

Finally, we considered the effect of the deformation itself on the p-form algebra. It was found that in the deformed case, the bosonic gauge algebra can not be truncated to a p-form algebra. Instead, to obtain a closed algebra, one needs to include additional generators with mixed symmetries whose role from the Kac-Moody point of view remains to be clarified.

In addition to the open issues mentioned above, we see a number of interesting venues to extend the present results. First of all, a relevant question is whether the above correspondence, which holds for maximal and half-maximal supergravity, can also be extended to theories with less supersymmetry. A number of such supergravities are given by a scalar coset $G/K(G)$ after reduction to three dimensions. Restricting to groups G with a real split form these cosets have been classified [40]. It is natural to investigate whether the over and very extensions of G contain deformation and top-form potentials corresponding to all deformations of the associated supergravities as well. Furthermore, these coset models $G/K(G)$ are special points in a landscape of more general geometries. It would be interesting to learn more about the deformation and top-form potentials associated to the general non-coset geometries.

Recently, an example where the correspondence between supergravity and very extended algebras does not hold straightforwardly was found in the theory that reduces to the coset model $G_2/\mathrm{SO}(4)$ in three dimensions. While minimal $D = 5$ simple supergravity allows for a triplet of deformation potentials, related to the gauging of a $\mathrm{U}(1)$ subgroup of the $\mathrm{SU}(2)$ R-symmetry, there are no such potentials in the associated Kac-Moody algebra G_2^{+++} [52]. A possible explanation for this phenomenon may be that in this case the R-symmetry does not act on the original bosonic fields of the theory [37]. Another possibility may be that there is an extension of G_2^{+++} that does take the gauging into account. It would be worthwhile to find more examples of this phenomenon and to understand it in more detail.

It would also be interesting to study the brane interpretation of the deformation and top-form potentials. They naturally couple to domain walls and space-filling branes, respectively. It is known that in IIA supergravity the deformation potential $\boldsymbol{9}$ couples to the half-supersymmetric D8-brane and that the top-form potential $\boldsymbol{10'}$ couples to a half-supersymmetric space-filling brane whose string interpretation has yet to be clarified [45]. The other top-form potential $\boldsymbol{10}$ couples to a non-supersymmetric space-filling brane. Similarly, the quadruplet $\boldsymbol{4}$ of top-form potentials of IIB supergravity couples to a half-supersymmetric nonlinear doublet of 9-branes, including the D9-brane [55]. The

doublet **2** of top-form potentials couples to half-supersymmetric space-filling branes whose string interpretation is yet unclear. It would be interesting to perform a similar analysis for the other dimensions as well and see how all these branes fit into string theory.

Furthermore, while in this paper the possibilities of adding deformation and top-form potentials to matter coupled supergravity theories have been discussed, one may ask whether matter multiplets not coupled to supergravity can be extended with such potentials as well. It turns out that this is indeed the case. In fact, it has been suggested that a domain wall structure on a D-brane, interpolating between different values of the brane tension, should be described by a worldvolume deformation potential [56], similar to the way strings ending on such a brane are described by a worldvolume vector. In the case of the D9-brane this means that the $D = 10$ Maxwell multiplet can be extended with a nine-form potential, which is indeed possible [56]. This could correspond to the fundamental representation of deformation forms in table 3. This fundamental representation does not correspond to a deformation of supergravity due to the third quadratic constraint in (4.9). We expect that all fundamental representations in table 3 correspond to possible extensions of the $D < 10$ vector multiplets with deformation and top-form potentials as well. It might be worthwhile to consider the brane interpretation of these possibilities in further detail.

Acknowledgments

A. Terminology and notation

Below we shortly summarize the terminology we have introduced in this paper.

Deformation potential. A $(D - 1)$-form potential in D dimensions.

Top-form potential. A D-form potential in D dimensions.

p-form algebra. Truncation of the Kac-Moody algebra in a particular $G_D \otimes \mathrm{SL}(D, \mathbb{R})$ decomposition by restricting to only the generators at positive levels in a purely antisymmetric $\mathrm{SL}(D, \mathbb{R})$ tensor representation of rank $1 \leq p \leq D$. Also arises by considering the bosonic gauge algebra with constant gauge parameters of the associated supergravity.

Fundamental *p*-form. A *p*-form corresponding to a positive simple root of one of the disabled nodes in the decomposed Dynkin diagram of the Kac-Moody algebra. Generates the *p*-form algebra.

Type p deformation. A deformation of the *p*-form algebra in which a fundamental *p*-form becomes massive.

Furthermore we indicate components of *p*-forms by a boldface italic number equal to their rank, e.g. $\boldsymbol{5}$ stands for $\boldsymbol{5}_{\mu_1\cdots\mu_5}$. This is not to be confused with group representations, which are represented with a boldface number equal to their dimension. Our convention for the normalisation of products of *p*-forms is the same as in [45]; in particular, we (anti-)symmetrize with weight one.

B. Physical states of half-maximal supergravity

We consider half-maximal supergravity in any dimension and coupled to an arbitrary number of vector multiplets. Starting with the graviton multiplet of D-dimensional half-maximal supergravity, its bosonic part consists of a metric, m vector gauge fields with $m = 10 - D$, a two-form gauge field and a single scalar which is the dilaton. It has a global $\mathrm{SO}(m)$ symmetry, under which the vectors transform in the fundamental representation. The only exceptions are $D = 4$ and $D = 3$ where there are non-trivial hidden symmetries. In $D = 4$ there is one extra scalar due to the duality of the two-form potential to an axionic scalar. Together with the dilaton this leads to an enhanced $\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SO}(6)$ hidden symmetry. Similarly, in $D = 3$ there are 7 extra scalars due to the duality in $D = 3$ dimensions between the vectors and scalars. In this case all physical degrees of freedom are carried by a scalar coset $\mathrm{SO}(8, 1)/\mathrm{SO}(8)$.

The other possible multiplet in generic dimensions is the vector multiplet, which contains a vector and m scalars. The effect of adding $m + n$ vector multiplets is to enlarge the symmetry group from $\mathrm{SO}(m)$ to $\mathrm{SO}(m, m + n)$. The scalars parameterize the corresponding scalar coset while the vectors transform in the fundamental representation. In four dimensions the symmetry becomes $\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SO}(6, 6 + n)$ while in three dimensions it is given by $\mathrm{SO}(8, 8 + n)$. In the latter case there again is symmetry enhancement due to the equivalence between scalars and vectors. The entire theory can be described in terms of the corresponding scalar coset (coupled to gravity).

The above multiplets belong to non-chiral half-maximal supergravity and are the correct and complete story in generic dimensions. In six dimensions, however, the half-maximal theory can be chiral or non-chiral, similar to the maximal theory in ten dimensions. The non-chiral theory is denoted by $D = 6a$ and follows the above pattern. The chiral theory, $D = 6b$, instead has different multiplets. In particular, the graviton multiplet contains gravity, five scalars and five self-dual plus one anti-self-dual two-form gauge fields. The global symmetry is given by $\mathrm{SO}(5, 1)$. The other possible multiplet is that of the tensor, which contains an anti-self-dual two-form and five scalars. Adding $4 + n$ of such tensor multiplets to the graviton multiplet enhances the symmetry to $\mathrm{SO}(5, 5 + n)$.

D	Cont	$g_{\mu\nu}$	$p = 0$	$p = 1$	$p = 2$
10	GV^n	$\mathbf{35} \times \mathbf{1}$	$\mathbf{1} \times 1$	$\mathbf{8} \times \mathbf{n}$	$\mathbf{28} \times \mathbf{1}$
9	GV^{n+1}	$\mathbf{27} \times \mathbf{1}$	$\mathbf{1} \times \left(1 + (1n+1)\right)$	$\mathbf{7} \times (\mathbf{n+2})$	$\mathbf{21} \times \mathbf{1}$
8	GV^{n+2}	$\mathbf{20} \times \mathbf{1}$	$\mathbf{1} \times \left(1 + (2n+4)\right)$	$\mathbf{6} \times (\mathbf{n+4})$	$\mathbf{15} \times \mathbf{1}$
7	GV^{n+3}	$\mathbf{14} \times \mathbf{1}$	$\mathbf{1} \times \left(1 + (3n+9)\right)$	$\mathbf{5} \times (\mathbf{n+6})$	$\mathbf{10} \times \mathbf{1}$
6a	GV^{n+4}	$\mathbf{9} \times \mathbf{1}$	$\mathbf{1} \times \left(1 + (4n+16)\right)$	$\mathbf{4} \times (\mathbf{n+8})$	$\mathbf{6} \times \mathbf{1}$
6b	GT^{n+4}	$\mathbf{9} \times \mathbf{1}$	$\mathbf{1} \times (5n+25)$		$\mathbf{6} \times \frac{1}{2}\,(\mathbf{10+n})$
5	GV^{n+5}	$\mathbf{5} \times \mathbf{1}$	$\mathbf{1} \times \left(1 + (5n+25)\right)$	$\mathbf{3} \times (\mathbf{n+11})$	
4	GV^{n+6}	$\mathbf{2} \times \mathbf{1}$	$\mathbf{1} \times \left((1,2) + (6n+36,1)\right)$	$\mathbf{2} \times (\mathbf{n+12})$	
3	GV^{n+7}	$\mathbf{—} \times \mathbf{1}$	$\mathbf{1} \times (8n+64)$		

Table 4: The physical states of all $D = 10-m$ half-maximal supergravities coupled to $m+n$ vector multiplets. The multiplet structures (where G is the graviton, V the vector and T the self-dual tensor multiplet) are also given.

Upon dimensional reduction over a circle, the graviton multiplet splits up into a graviton multiplet plus a vector multiplet. A vector (or tensor) multiplet reduces to a vector multiplet in the lower dimensions. This was the reason for adding $m+n$ instead of n vector or tensor multiplets in any dimension; it can easily be seen that n remains invariant under dimensional reduction. That is, a theory with a certain value of n reduces to a theory with the same value of n in lower dimensions.

For the reader's convenience we have given the physical states corresponding to $D = 10 - m$ half-maximal supergravity coupled to $m + n$ vector (or tensor) multiplets in table 4, see, e.g., [57].

C. Group theory

In this appendix we will generalize the analysis of [14] to allow for non-simply laced Dynkin diagrams. The key difference between a simply laced and a non-simply laced diagram is that for the latter the associated Cartan matrix is not symmetric, and no longer fulfills the role of a metric on the root space. Moreover, the metric on the weight space is no longer given by the inverse of the Cartan matrix.

The root space metric is important in constructing the root system — one needs it to compute inner products between roots. The weight space metric plays a similar role for the highest weight representations, which are a necessary ingredient for the level decomposition. We will show how both metrics can be obtained from appropriate symmetrizations of the (inverse) Cartan matrix.

We start out from the defining equation for the Cartan matrix, which reads

$$A_{ij} = 2\frac{(\alpha_i \,|\, \alpha_j)}{(\alpha_j \,|\, \alpha_j)}\,. \tag{C.1}$$

Here α_i are the simple roots which span the whole root system Δ, and $(\cdot \,|\, \cdot)$ is the norm inferred from the Killing norm. The indices run over the rank of the associated Lie algebra.

Any root α of Δ can be expressed as a linear combination of simple roots,

$$\alpha = m^i \alpha_i \,, \tag{C.2}$$

where contracted indices are being summed over. The values of m^i are also known as the root labels.

Because the Killing norm is symmetric and bilinear, an inner product between two roots $\alpha = m^i \alpha_i$ and $\beta = n^i \alpha_i$ can be written as

$$(\alpha \,|\, \beta) = B_{ij} m^i n^j \,, \tag{C.3}$$

where the metric B on the root space is defined as

$$B_{ij} \equiv (\alpha_i \,|\, \alpha_j) = A_{ij} \frac{(\alpha_j \,|\, \alpha_j)}{2} \,, \tag{C.4}$$

which is symmetric by construction. Note that in this case the repeated index is not summed over because it is not contracted.

From (C.3) and (C.4) it is apparent that we must first determine the norms of the simple roots before inner products on Δ can be computed. To that end we reshuffle the defining equation for the Cartan matrix to obtain

$$(\alpha_i \,|\, \alpha_i) = \frac{A_{ij}}{A_{ji}} (\alpha_j \,|\, \alpha_j) \,. \tag{C.5}$$

So once a normalization for one of the simple roots has been chosen, all others are also fixed. A common normalization is to choose $\alpha^2 = 2$ for the longest simple root (i.e. the simple root which has the highest norm). Instead, we will adhere to $\alpha^2 = 2$ for the *shortest* simple root (the simple root with the lowest norm). The latter normalization is particularly convenient for computer-based calculations, because then the root metric B has only integer values.

We now turn to the metric on the weight space. The weight space itself is spanned by the fundamental weights λ^i, which are defined via

$$\lambda^i \bar{\alpha}_j = \delta^i_j \,, \tag{C.6}$$

where the simple coroots $\bar{\alpha}_i$ are given by $\bar{\alpha}_i = \frac{2\alpha_i}{(\alpha_i \,|\, \alpha_i)}$. The basis specified by the fundamental weights is also known as the Dynkin basis. Every weight λ can be expanded on this basis as

$$\lambda = p_i \lambda^i \,. \tag{C.7}$$

The values of the p_i are also known as the Dynkin labels of the weight. The relation between the Dynkin labels and the components of the root is given by

$$p_i = A_{ji} m^j \,. \tag{C.8}$$

As the Dynkin basis is the dual basis of the simple coroots, the metric G on the weight space is the inverse of the simple coroot metric. The latter is given by

$$(\bar{\alpha}_i \,|\, \bar{\alpha}_j) = \frac{2A_{ij}}{(\alpha_i \,|\, \alpha_i)} \,. \tag{C.9}$$

Therefore G is given by

$$G^{ij} \equiv (\lambda_i \,|\, \lambda_j) = \frac{(\alpha_j \,|\, \alpha_j)}{2} \left(A^{-1} \right)^{ij} \,. \tag{C.10}$$

By construction G is symmetric, just like the root metric B.

As explained in [35, 9], the level decomposition of infinite-dimensional Lie algebra entails scanning for subalgebra representations at given levels. The subalgebra representations are defined by their Dynkin labels, and have to satisfy three conditions:

1. The Dynkin labels all have to be integer and non-negative.

2. The associated root labels have to be integers.

3. The length squared of the root must not exceed the maximum value.

The subalgebra is obtained by 'disabling' nodes from the Dynkin diagram. We can then split up the index of the full algebra into $i = (a, s)$, where a runs over the disabled nodes and s over the subalgebra. To see whether condition 2 is satisfied for particular values of p_s, we can invert equation (C.8) in order to obtain

$$m^s = \left(A_{\mathrm{sub}}^{-1} \right)^{ts} (p_t - l^a A_{at}) \,, \tag{C.11}$$

where m^s are the root labels associated to the Dynkin labels p_s, A_{sub} is the Cartan matrix of the subalgebra, and l^a are the levels. Condition 3 may be verified by decomposing (C.3) into its contributions from the deleted nodes and the subalgebra:

$$\alpha^2 = G_{\mathrm{sub}}^{st} \left(p_s p_t - A_{as} A_{bt} l^a l^b \right) + B_{ab} l^a l^b \leq \alpha_{\max}^2 \,. \tag{C.12}$$

Here G_{sub} is the weight metric of the subalgebra, and $\alpha_{\max}^2$ is given by the norm of the longest simple root. Note that for this formula to be valid, we have to make sure that a long (or short) root in the full algebra is also a long (short) root in the subalgebra, which in general is not automatically the case. Luckily we are always free to choose a normalization such that root lengths match.

When using (C.12) to scan for representations, it is important for G_{sub} to only have non-negative entries. If this is not the case, then the root norm α^2 is not a monotonically increasing function of the Dynkin labels p_s at fixed levels l^a, and one might miss representations using a simple scanning algorithm. However, as we always shall be decomposing with respect to (direct products of) finite dimensional subalgebras, G_{sub} will never contain negative entries.
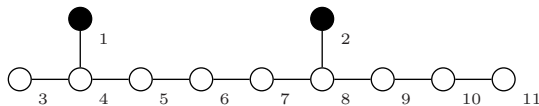
Figure 3: D_8^{+++} decomposed as A_9

l	p_{grav}	m	α^2	d_{grav}	$\mathrm{mult}(\alpha)$	μ	fields
0 0	1 0 0 0 0 0 0 0 1	0 0 -1 -1 -1 -1 -1 -1 -1 -1	2	99	1	1	$\bar{g}_{\mu\nu}$
0 0	0 0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0 0 0	0	1	11	2	$\boldsymbol{0}, \hat{g}_{\mu\nu}$
1 0	0 1 0 0 0 0 0 0 0	1 0 0 0 0 0 0 0 0 0 0	2	45	1	1	$\boldsymbol{2}$
0 1	0 0 0 0 0 1 0 0 0	0 1 0 0 0 0 0 0 0 0 0	2	210	1	1	$^\star \boldsymbol{2}$
1 1	1 0 0 0 0 0 1 0 0	1 1 0 1 1 1 1 1 0 0 0	2	1155	1	1	$^\star g_{\mu\nu}$
1 1	0 0 0 0 0 0 0 1 0	1 1 1 2 2 2 2 2 1 0 0	0	45	8	1	$^\star \boldsymbol{0}$
2 1	0 1 0 0 0 0 0 1 0	2 1 1 2 2 2 2 2 1 0 0	2	1925	1	1	
2 1	1 0 0 0 0 0 0 0 1	2 1 1 3 3 3 3 3 2 1 0	0	99	8	1	
2 1	0 0 0 0 0 0 0 0 0	2 1 2 4 4 4 4 4 3 2 1	-2	1	45	1	top

Table 5: A_9 representations in D_8^{+++}

D. Low level D_8^{+++} decompositions

Here we list the output of SimpLie [38] at low levels, using the various decompositions of D_8^{+++} as indicated by the Dynkin diagram accompanying the tables. The regular subalgebra splits into a part belonging to the gravity line A_n (the white nodes) and a part belonging to the internal duality group G_D (the grey nodes).

In the following tables we respectively list the levels, the Dynkin labels of A_n and G_D, the root labels, the root length, the dimension of the representations of A_n and G, the multiplicity of the root, the outer multiplicity, and the interpretation as a physical field. The deformation– and top-form potentials are indicated by 'de' and 'top', respectively. When the internal group does not exist, we do not list the corresponding columns. In all cases the Dynkin labels of the lowest weights of the representations are given. All tables are truncated at the point when the number of indices of the gravity subalgebra representations exceed the dimension. The order of the levels, Dynkin labels, and root labels as they appear in the tables are determined by the order of the node labels on the Dynkin diagram. This ordering is always first from left to right, then from top to bottom.

The interpretation of the representations at level zero as the graviton is, unlike the p-forms at higher levels, not quite straightforward. The graviton emerges when one combines the adjoint representation of A_n with a scalar coming from one of the disabled nodes, see [34, 9]. We have indicated these parts of the graviton by $\bar{g}_{\mu\nu}$ and $\hat{g}_{\mu\nu}$, respectively.
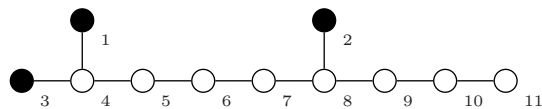
Figure 4: D_8^{+++} decomposed as A_8

l	p_{grav}	m	α^2	d_{grav}	$\text{mult}(\alpha)$	μ	fields
0 0 0	1 0 0 0 0 0 0 1	0 0 0 -1 -1 -1 -1 -1 -1 -1 -1	2	80	1	1	$\bar{g}_{\mu\nu}$
0 0 0	0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0 0 0	0	1	11	3	$\mathbf{0}, \hat{g}_{\mu\nu}$
1 0 0	1 0 0 0 0 0 0 0	1 0 0 0 0 0 0 0 0 0 0	2	9	1	1	$\mathbf{1}$
0 1 0	0 0 0 0 1 0 0 0	0 1 0 0 0 0 0 0 0 0 0	2	126	1	1	$^\star\,\mathbf{2}$
0 0 1	1 0 0 0 0 0 0 0	0 0 1 0 0 0 0 0 0 0 0	2	9	1	1	$\mathbf{1}$
1 1 0	0 0 0 0 0 1 0 0	1 1 0 1 1 1 1 1 0 0 0	2	84	1	1	$^\star\,\mathbf{1}$
1 0 1	0 1 0 0 0 0 0 0	1 0 1 1 0 0 0 0 0 0 0	2	36	1	1	$\mathbf{2}$
0 1 1	0 0 0 0 0 1 0 0	0 1 1 1 1 1 1 1 0 0 0	2	84	1	1	$^\star\,\mathbf{1}$
1 1 1	1 0 0 0 0 1 0 0	1 1 1 1 1 1 1 1 0 0 0	2	720	1	1	$^\star\,g_{\mu\nu}$
1 1 1	0 0 0 0 0 0 1 0	1 1 1 2 2 2 2 2 1 0 0	0	36	8	2	$^\star\,\mathbf{0}$
2 1 1	1 0 0 0 0 0 1 0	2 1 1 2 2 2 2 2 1 0 0	2	315	1	1	
2 1 1	0 0 0 0 0 0 0 1	2 1 1 3 3 3 3 3 2 1 0	0	9	8	1	de
1 1 2	1 0 0 0 0 0 1 0	1 1 2 2 2 2 2 2 1 0 0	2	315	1	1	
1 1 2	0 0 0 0 0 0 0 1	1 1 2 3 3 3 3 3 2 1 0	0	9	8	1	de
2 1 2	0 1 0 0 0 0 1 0	2 1 2 3 2 2 2 2 1 0 0	2	1215	1	1	
2 1 2	1 0 0 0 0 0 0 1	2 1 2 3 3 3 3 3 2 1 0	0	80	8	2	
2 1 2	0 0 0 0 0 0 0 0	2 1 2 4 4 4 4 4 3 2 1	-2	1	45	2	top

Table 6: A_8 representations in D_8^{+++}
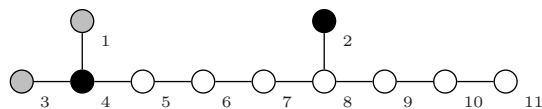
Figure 5: D_8^{+++} decomposed as $A_1 \otimes A_1 \otimes A_7$

l	p_{grav}	p_G	m	α^2	d_{grav}	d_G	mult(α)	μ	fields
0 0	1 0 0 0 0 0 1	0 0	0 0 0 0 -1 -1 -1 -1 -1 -1 -1	2	63	1	1	1	$\bar{g}_{\mu\nu}$
0 0	0 0 0 0 0 0 0	0 2	0 0 -1 0 0 0 0 0 0 0 0	2	1	3	1	1	$\boldsymbol{0}$
0 0	0 0 0 0 0 0 0	2 0	-1 0 0 0 0 0 0 0 0 0 0	2	1	3	1	1	$\boldsymbol{0}$
0 0	0 0 0 0 0 0 0	0 0	0 0 0 0 0 0 0 0 0 0 0	0	1	1	11	2	$\boldsymbol{0}, \hat{g}_{\mu\nu}$
1 0	0 0 0 1 0 0 0	0 0	0 1 0 0 0 0 0 0 0 0 0	2	70	1	1	1	$^\star\, \boldsymbol{2}$
0 1	1 0 0 0 0 0 0	1 1	0 0 0 1 0 0 0 0 0 0 0	2	8	4	1	1	$\boldsymbol{1}$
1 1	0 0 0 0 1 0 0	1 1	0 1 0 1 1 1 1 1 0 0 0	2	56	4	1	1	$^\star\, \boldsymbol{1}$
2 0	1 0 0 0 0 0 1	0 0	0 2 0 0 0 1 2 3 2 1 0	2	63	1	1	1	
0 2	0 1 0 0 0 0 0	0 0	1 0 1 2 1 0 0 0 0 0 0	2	28	1	1	1	$\boldsymbol{2}$
1 2	1 0 0 0 1 0 0	0 0	1 1 1 2 1 1 1 1 0 0 0	2	420	1	1	1	$^\star\, g_{\mu\nu}$
1 2	0 0 0 0 0 1 0	0 2	1 1 0 2 2 2 2 2 1 0 0	2	28	3	1	1	$^\star\, \boldsymbol{0}$
1 2	0 0 0 0 0 1 0	2 0	0 1 1 2 2 2 2 2 1 0 0	2	28	3	1	1	$^\star\, \boldsymbol{0}$
1 2	0 0 0 0 0 1 0	0 0	1 1 1 2 2 2 2 2 1 0 0	0	28	1	8	1	$^\star\, \boldsymbol{0}$
1 3	1 0 0 0 0 1 0	1 1	1 1 1 3 2 2 2 2 1 0 0	2	216	4	1	1	
1 3	0 0 0 0 0 0 1	1 1	1 1 1 3 3 3 3 3 2 1 0	0	8	4	8	2	de
1 4	0 1 0 0 0 1 0	0 0	2 1 2 4 3 2 2 2 1 0 0	2	720	1	1	1	
1 4	1 0 0 0 0 0 1	0 2	2 1 1 4 3 3 3 3 2 1 0	2	63	3	1	1	
1 4	1 0 0 0 0 0 1	2 0	1 1 2 4 3 3 3 3 2 1 0	2	63	3	1	1	
1 4	1 0 0 0 0 0 1	0 0	2 1 2 4 3 3 3 3 2 1 0	0	63	1	8	1	
1 4	0 0 0 0 0 0 0	0 2	2 1 1 4 4 4 4 4 3 2 1	0	1	3	8	1	top
1 4	0 0 0 0 0 0 0	2 0	1 1 2 4 4 4 4 4 3 2 1	0	1	3	8	1	top
1 4	0 0 0 0 0 0 0	0 0	2 1 2 4 4 4 4 4 3 2 1	-2	1	1	45	2	top

Table 7: $A_1 \otimes A_1 \otimes A_7$ representations in D_8^{+++}
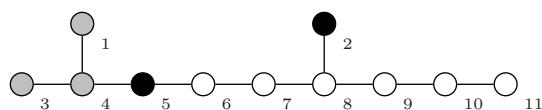
Figure 6: D_8^{+++} decomposed as $A_3 \otimes A_6$

l	p_{grav}	p_G	m	α^2	d_{grav}	d_G	mult(α)	μ	fields
0 0	1 0 0 0 0 1	0 0 0	0 0 0 0 0 -1 -1 -1 -1 -1 -1	2	48	1	1	1	$\bar{g}_{\mu\nu}$
0 0	0 0 0 0 0 0	1 1 0	-1 0 -1 -1 0 0 0 0 0 0 0	2	1	15	1	1	*0*
0 0	0 0 0 0 0 0	0 0 0	0 0 0 0 0 0 0 0 0 0 0	0	1	1	11	2	*0*, $\hat{g}_{\mu\nu}$
1 0	0 0 1 0 0 0	0 0 0	0 1 0 0 0 0 0 0 0 0 0	2	35	1	1	1	* *2*
0 1	1 0 0 0 0 0	0 0 1	0 0 0 0 1 0 0 0 0 0 0	2	7	6	1	1	*1*
1 1	0 0 0 1 0 0	0 0 1	0 1 0 0 1 1 1 1 0 0 0	2	35	6	1	1	* *1*
2 0	0 0 0 0 0 1	0 0 0	0 2 0 0 0 1 2 3 2 1 0	2	7	1	1	1	de
0 2	0 1 0 0 0 0	0 0 0	1 0 1 2 2 1 0 0 0 0 0	2	21	1	1	1	*2*
2 1	1 0 0 0 0 1	0 0 1	0 2 0 0 1 1 2 3 2 1 0	2	48	6	1	1	
2 1	0 0 0 0 0 0	0 0 1	0 2 0 0 1 2 3 4 3 2 1	0	1	6	7	1	top
1 2	1 0 0 1 0 0	0 0 0	1 1 1 2 2 1 1 1 0 0 0	2	224	1	1	1	* $g_{\mu\nu}$
1 2	0 0 0 0 1 0	1 1 0	0 1 0 1 2 2 2 2 1 0 0	2	21	15	1	1	* *0*
1 2	0 0 0 0 1 0	0 0 0	1 1 1 2 2 2 2 2 1 0 0	0	21	1	8	1	* *0*
1 3	1 0 0 0 1 0	0 0 1	1 1 1 2 3 2 2 2 1 0 0	2	140	6	1	1	
1 3	0 0 0 0 0 1	0 2 0	1 1 0 2 3 3 3 3 2 1 0	2	7	10	1	1	de
1 3	0 0 0 0 0 1	2 0 0	0 1 1 2 3 3 3 3 2 1 0	2	7	10	1	1	de
1 3	0 0 0 0 0 1	0 0 1	1 1 1 2 3 3 3 3 2 1 0	0	7	6	8	1	de
1 4	0 1 0 0 1 0	0 0 0	2 1 2 4 4 3 2 2 1 0 0	2	392	1	1	1	
1 4	1 0 0 0 0 1	1 1 0	1 1 1 3 4 3 3 3 2 1 0	2	48	15	1	1	
1 4	1 0 0 0 0 1	0 0 0	2 1 2 4 4 3 3 3 2 1 0	0	48	1	8	1	
1 4	0 0 0 0 0 0	1 1 0	1 1 1 3 4 4 4 4 3 2 1	0	1	15	8	2	top
1 4	0 0 0 0 0 0	0 0 0	2 1 2 4 4 4 4 4 3 2 1	-2	1	1	45	1	top

Table 8: $A_3 \otimes A_6$ representations in D_8^{+++}
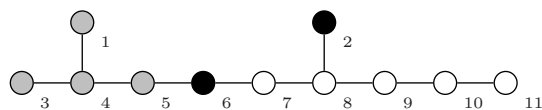
Figure 7: D_8^{+++} decomposed as $D_4 \otimes A_5$

JHEP02(2008)069

l	p_{grav}	p_G	m	α^2	d_{grav}	d_G	mult(α)	μ	fields
0 0	1 0 0 0 1	0 0 0 0	0 0 0 0 0 0 -1 -1 -1 -1 -1	2	35	1	1	1	$\bar{g}_{\mu\nu}$
0 0	0 0 0 0 0	0 0 1 0	-1 0 -1 -2 -1 0 0 0 0 0 0	2	1	28	1	1	**0**
0 0	0 0 0 0 0	0 0 0 0	0 0 0 0 0 0 0 0 0 0 0	0	1	1	11	2	$p=0, \bar{g}_{\mu\nu}$
1 0	0 1 0 0 0	0 0 0 0	0 1 0 0 0 0 0 0 0 0 0	2	15	1	1	1	* **2**
0 1	1 0 0 0 0	0 0 0 1	0 0 0 0 0 1 0 0 0 0 0	2	6	8	1	1	**1**
1 1	0 0 1 0 0	0 0 0 1	0 1 0 0 0 1 1 1 0 0 0	2	20	8	1	1	* **1**
0 2	0 1 0 0 0	0 0 0 0	1 0 1 2 2 2 1 0 0 0 0	2	15	1	1	1	**2**
2 1	0 0 0 0 1	0 0 0 1	0 2 0 0 0 1 2 3 2 1 0	2	6	8	1	1	de
1 2	0 0 0 1 0	0 0 1 0	0 1 0 0 1 2 2 2 1 0 0	2	15	28	1	1	* **0**
1 2	1 0 1 0 0	0 0 0 0	1 1 1 2 2 2 1 1 0 0 0	2	105	1	1	1	$^\star g_{\mu\nu}$
1 2	0 0 0 1 0	0 0 0 0	1 1 1 2 2 2 2 2 1 0 0	0	15	1	8	1	* **0**
2 2	1 0 0 0 1	0 0 1 0	0 2 0 0 1 2 2 3 2 1 0	2	35	28	1	1	
2 2	0 1 0 1 0	0 0 0 0	1 2 1 2 2 2 2 2 1 0 0	2	189	1	1	1	
2 2	0 0 0 0 0	0 0 0 2	0 2 0 0 0 2 3 4 3 2 1	2	1	35	1	1	top
2 2	1 0 0 0 1	0 0 0 0	1 2 1 2 2 2 3 2 1 0	0	35	1	8	1	
2 2	0 0 0 0 0	0 0 1 0	0 2 0 0 1 2 3 4 3 2 1	0	1	28	7	1	top
2 2	0 0 0 0 0	0 0 0 0	1 2 1 2 2 2 3 4 3 2 1	-2	1	1	43	2	top
1 3	1 0 0 1 0	0 0 0 1	1 1 1 2 2 3 2 2 1 0 0	2	84	8	1	1	
1 3	0 0 0 0 1	1 1 0 0	0 1 0 1 2 3 3 3 2 1 0	2	6	56	1	1	de
1 3	0 0 0 0 1	0 0 0 1	1 1 1 2 2 3 3 3 2 1 0	0	6	8	8	1	de
1 4	1 0 0 0 1	0 0 1 0	1 1 1 2 3 4 3 3 2 1 0	2	35	28	1	1	
1 4	0 1 0 1 0	0 0 0 0	2 1 2 4 4 4 3 2 1 0 0	2	189	1	1	1	
1 4	0 0 0 0 0	0 2 0 0	1 1 0 2 3 4 4 4 3 2 1	2	1	35	1	1	top
1 4	0 0 0 0 0	2 0 0 0	0 1 1 2 3 4 4 4 3 2 1	2	1	35	1	1	top
1 4	1 0 0 0 1	0 0 0 0	2 1 2 4 4 3 3 2 1 0	0	35	1	8	1	
1 4	0 0 0 0 0	0 0 1 0	1 1 1 2 3 4 4 4 3 2 1	0	1	28	8	1	top
1 4	0 0 0 0 0	0 0 0 0	2 1 2 4 4 4 4 4 3 2 1	-2	1	1	45	1	top

Table 9: $D_4 \otimes A_5$ representations in D_8^{+++}
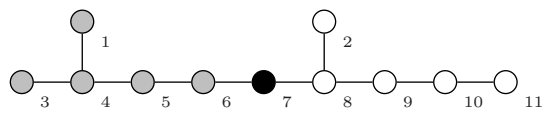
Figure 8: D_8^{+++} decomposed as $D_5 \otimes A_5$

l	p_{grav}	p_G	m	α^2	d_{grav}	d_G	mult(α)	μ	fields
0	0 0 0 0 0	0 0 0 1 0	-1 0 -1 -2 -2 -1 0 0 0 0 0	2	1	45	1	1	$\boldsymbol{0}$
0	1 0 0 0 1	0 0 0 0 0	0 -1 0 0 0 0 0 -1 -1 -1 -1	2	35	1	1	1	$\bar{g}_{\mu\nu}$
0	0 0 0 0 0	0 0 0 0 0	0 0 0 0 0 0 0 0 0 0 0	0	1	1	11	1	$\hat{g}_{\mu\nu}$
1	0 1 0 0 0	0 0 0 0 1	0 0 0 0 0 0 1 0 0 0 0	2	15	10	1	1	$\boldsymbol{2},^{\star}\boldsymbol{2}$
2	0 0 0 1 0	0 0 0 1 0	0 1 0 0 0 1 2 2 1 0 0	2	15	45	1	1	$^{\star}\boldsymbol{0}$
2	1 0 1 0 0	0 0 0 0 0	1 0 1 2 2 2 2 1 0 0 0	2	105	1	1	1	$^{\star}g_{\mu\nu}$
3	1 0 0 0 1	0 0 1 0 0	0 1 0 0 1 2 3 3 2 1 0	2	35	120	1	1	
3	0 1 0 1 0	0 0 0 0 1	1 1 1 2 2 2 3 2 1 0 0	2	189	10	1	1	
3	0 0 0 0 0	0 0 0 1 1	0 2 0 0 0 1 3 4 3 2 1	2	1	320	1	1	top
3	1 0 0 0 1	0 0 0 0 1	1 1 1 2 2 2 3 3 2 1 0	0	35	10	8	1	
3	0 0 0 0 0	0 0 0 0 1	1 2 1 2 2 2 3 4 3 2 1	-2	1	10	43	1	top

Table 10: $D_5 \otimes A_5$ representations in D_8^{+++}
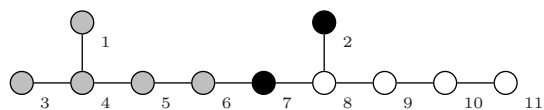
Figure 9: D_8^{+++} decomposed as $D_5 \otimes A_4$

l	p_{grav}	p_G	m	α^2	d_{grav}	d_G	mult(α)	μ	fields
0 0	0 0 0 0	0 0 0 1 0	-1 0 -1 -2 -2 -1 0 0 0 0 0	2	1	45	1	1	**0**
0 0	1 0 0 1	0 0 0 0 0	0 0 0 0 0 0 0 -1 -1 -1 -1	2	24	1	1	1	$\bar{g}_{\mu\nu}$
0 0	0 0 0 0	0 0 0 0 0	0 0 0 0 0 0 0 0 0 0 0	0	1	1	11	2	$p=0, \hat{g}_{\mu\nu}$
1 0	1 0 0 0	0 0 0 0 0	0 1 0 0 0 0 0 0 0 0 0	2	5	1	1	1	**1**
0 1	1 0 0 0	0 0 0 0 1	0 0 0 0 0 0 1 0 0 0 0	2	5	10	1	1	**1**
1 1	0 1 0 0	0 0 0 0 1	0 1 0 0 0 0 1 1 0 0 0	2	10	10	1	1	***1**
0 2	0 1 0 0	0 0 0 0 0	1 0 1 2 2 2 2 1 0 0 0	2	10	1	1	1	***1**
1 2	0 0 1 0	0 0 0 1 0	0 1 0 0 0 1 2 2 1 0 0	2	10	45	1	1	***0**
1 2	1 1 0 0	0 0 0 0 0	1 1 1 2 2 2 2 1 0 0 0	2	40	1	1	1	* $g_{\mu\nu}$
1 2	0 0 1 0	0 0 0 0 0	1 1 1 2 2 2 2 2 1 0 0	0	10	1	8	1	***0**
2 2	0 0 0 1	0 0 0 1 0	0 2 0 0 0 1 2 3 2 1 0	2	5	45	1	1	de
2 2	1 0 1 0	0 0 0 0 0	1 2 1 2 2 2 2 2 1 0 0	2	45	1	1	1	
1 3	0 0 0 1	0 0 1 0 0	0 1 0 0 1 2 3 3 2 1 0	2	5	120	1	1	de
1 3	1 0 1 0	0 0 0 0 1	1 1 1 2 2 2 3 2 1 0 0	2	45	10	1	1	
1 3	0 0 0 1	0 0 0 0 1	1 1 1 2 2 2 3 3 2 1 0	0	5	10	8	1	de
2 3	1 0 0 1	0 0 1 0 0	0 2 0 0 1 2 3 3 2 1 0	2	24	120	1	1	
2 3	0 0 0 0	0 0 0 1 1	0 2 0 0 0 1 3 4 3 2 1	2	1	320	1	1	top
2 3	0 1 1 0	0 0 0 0 1	1 2 1 2 2 2 3 3 1 0 0	2	75	10	1	1	
2 3	0 0 0 0	0 0 1 0 0	0 2 0 0 1 2 3 4 3 2 1	0	1	120	7	1	top
2 3	1 0 0 1	0 0 0 0 1	1 2 1 2 2 2 3 3 2 1 0	0	24	10	8	2	
2 3	0 0 0 0	0 0 0 0 1	1 2 1 2 2 2 3 4 3 2 1	-2	1	10	43	2	top
1 4	1 0 0 1	0 0 0 1 0	1 1 1 2 2 3 4 3 2 1 0	2	24	45	1	1	
1 4	0 0 0 0	1 1 0 0 0	0 1 0 1 2 3 4 4 3 2 1	2	1	210	1	1	top
1 4	0 0 0 0	0 0 0 1 0	1 1 1 2 2 3 4 4 3 2 1	0	1	45	8	1	top
1 4	0 1 1 0	0 0 0 0 0	2 1 2 4 4 4 4 3 1 0 0	2	75	1	1	1	
1 4	1 0 0 1	0 0 0 0 0	2 1 2 4 4 4 4 3 2 1 0	0	24	1	8	1	
1 4	0 0 0 0	0 0 0 0 0	2 1 2 4 4 4 4 4 3 2 1	-2	1	1	45	1	top

Table 11: $D_5 \otimes A_4$ representations in D_8^{+++}
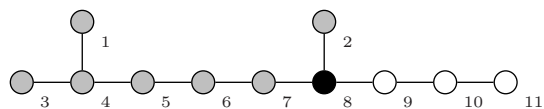
Figure 10: D_8^{+++} decomposed as $D_6 \otimes A_1 \otimes A_3$

l	p_{grav}	p_G	m	α^2	d_{grav}	d_G	$\text{mult}(\alpha)$	μ	fields
0	0 0 0	0 0 0 0 0 1 0	-1 0 -1 -2 -2 -2 -1 0 0 0 0	2	1	66	1	1	**0**
0	1 0 1	0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 -1 -1 -1	2	15	1	1	1	$\bar{g}_{\mu\nu}$
0	0 0 0	0 2 0 0 0 0 0	0 -1 0 0 0 0 0 0 0 0 0	2	1	3	1	1	**0**
0	0 0 0	0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0 0 0	0	1	1	11	1	$\hat{g}_{\mu\nu}$
1	1 0 0	0 1 0 0 0 0 1	0 0 0 0 0 0 0 1 0 0 0	2	4	24	1	1	**1**, ***1**
2	0 1 0	0 0 0 0 0 1 0	0 1 0 0 0 0 1 2 1 0 0	2	6	66	1	1	***0**
2	2 0 0	0 0 0 0 0 0 0	1 1 1 2 2 2 2 2 0 0 0	2	10	1	1	1	$^\star g_{\mu\nu}$
2	0 1 0	0 2 0 0 0 0 0	1 0 1 2 2 2 2 2 1 0 0	2	6	3	1	1	***0**
3	0 0 1	0 1 0 0 1 0 0	0 1 0 0 0 1 2 3 2 1 0	2	4	440	1	1	de
3	1 1 0	0 1 0 0 0 0 1	1 1 1 2 2 2 2 3 1 0 0	2	20	24	1	1	
3	0 0 1	0 1 0 0 0 0 1	1 1 1 2 2 2 2 3 2 1 0	0	4	24	8	1	de
4	0 0 0	0 0 0 0 1 0 1	0 2 0 0 0 1 2 4 3 2 1	2	1	2079	1	1	top
4	1 0 1	0 0 0 1 0 0 0	0 2 0 0 1 2 3 4 2 1 0	2	15	495	1	1	
4	0 0 0	0 2 0 1 0 0 0	0 1 0 0 1 2 3 4 3 2 1	2	1	1485	1	1	top
4	1 0 1	0 0 0 0 0 0 2	1 2 1 2 2 2 2 4 2 1 0	2	15	77	1	1	
4	0 2 0	0 0 0 0 0 1 0	1 2 1 2 2 2 3 4 2 0 0	2	20	66	1	1	
4	1 0 1	0 2 0 0 0 1 0	1 1 1 2 2 2 3 4 2 1 0	2	15	198	1	1	
4	1 0 1	0 0 0 0 0 1 0	1 2 1 2 2 2 3 4 2 1 0	0	15	66	8	1	
4	0 0 0	0 2 0 0 0 1 0	1 1 1 2 2 2 3 4 3 2 1	0	1	198	8	1	top
4	0 0 0	0 0 0 0 0 1 0	1 2 1 2 2 2 3 4 3 2 1	-2	1	66	43	2	top
4	2 1 0	0 0 0 0 0 0 0	2 2 2 4 4 4 4 4 1 0 0	2	45	1	1	1	
4	0 2 0	0 2 0 0 0 0 0	2 1 2 4 4 4 4 4 2 0 0	2	20	3	1	1	
4	1 0 1	0 2 0 0 0 0 0	2 1 2 4 4 4 4 4 2 1 0	0	15	3	8	1	
4	1 0 1	0 0 0 0 0 0 0	2 2 2 4 4 4 4 4 2 1 0	-2	15	1	44	2	
4	0 0 0	0 2 0 0 0 0 0	2 1 2 4 4 4 4 4 3 2 1	-2	1	3	45	1	top

Table 12: $D_6 \otimes A_1 \otimes A_3$ representations in D_8^{+++}
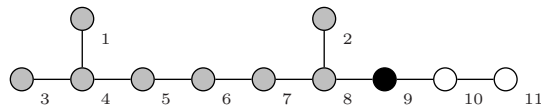
Figure 11: D_8^{+++} decomposed as $D_8 \otimes A_2$

l	p_{grav}	p_G	m	α^2	d_{grav}	d_G	$\text{mult}(\alpha)$	μ	fields
0	0 0	0 0 0 0 0 0 0 1	-1 -1 -1 -2 -2 -2 -2 -2 0 0 0	2	1	120	1	1	**0**
0	1 1	0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 -1 -1	2	8	1	1	1	$\bar{g}_{\mu\nu}$
0	0 0	0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0 0 0	0	1	1	11	1	$\hat{g}_{\mu\nu}$
1	1 0	0 0 0 0 0 0 0 1	0 0 0 0 0 0 0 1 0 0	2	3	120	1	1	***0**
2	0 1	0 0 0 0 0 1 0 0	0 1 0 0 0 0 1 2 2 1 0	2	3	1820	1	1	de
2	2 0	0 0 0 0 0 0 0 1	1 1 1 2 2 2 2 2 0 0	2	6	120	1	1	
2	0 1	0 2 0 0 0 0 0 0	1 0 1 2 2 2 2 2 1 0	2	3	135	1	1	de
2	0 1	0 0 0 0 0 0 0 0	2 2 2 4 4 4 4 4 2 1 0	-2	3	1	44	1	de
3	0 0	0 1 0 0 1 0 0 0	0 1 0 0 0 1 2 3 3 2 1	2	1	60060	1	1	top
3	1 1	0 0 0 1 0 0 0 0	0 2 0 0 1 2 3 4 3 1 0	2	8	8008	1	1	
3	1 1	0 1 0 0 0 0 1 0	1 1 1 2 2 2 2 3 3 1 0	2	8	7020	1	1	
3	0 0	0 1 0 0 0 0 1 0	1 1 1 2 2 2 2 3 3 2 1	0	1	7020	8	1	top
3	1 1	0 0 0 0 0 1 0 0	1 2 1 2 2 2 3 4 3 1 0	0	8	1820	8	1	
3	0 0	0 0 0 0 0 1 0 0	1 2 1 2 2 2 3 4 3 2 1	-2	1	1820	43	1	top
3	3 0	0 0 0 0 0 0 0 1	2 2 2 4 4 4 4 4 3 0 0	2	10	120	1	1	
3	1 1	0 2 0 0 0 0 0 0	2 1 2 4 4 4 4 4 3 1 0	0	8	135	8	1	
3	1 1	0 0 0 0 0 0 0 1	2 2 2 4 4 4 4 4 3 1 0	-2	8	120	44	2	
3	0 0	0 2 0 0 0 0 0 0	2 1 2 4 4 4 4 4 3 2 1	-2	1	135	45	1	top
3	0 0	0 0 0 0 0 0 0 1	2 2 2 4 4 4 4 4 3 2 1	-4	1	120	195	1	top
3	1 1	0 0 0 0 0 0 0 0	3 3 3 6 6 6 6 6 3 1 0	-4	8	1	192	1	

Table 13: $D_8 \otimes A_2$ representations in D_8^{+++}

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