BEAM INSTABILITIES

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Abstract
We start with longitudinal instabilities of bunched beams and introduce first the concept of impedance by approximating a cavity resonance with a RLC-circuit. The response of such a resonator to a pulse excitation gives the wake potential or Green function while a harmonic excitation reveals the concept and properties of the impedance. The interaction of a stationary circulating bunch with an impedance leads to an energy loss and a shift of the incoherent synchrotron frequency. The spectrum of a bunch, executing a synchrotron oscillation, has revolution harmonics with side bands spaced by the synchrotron frequency. The voltage induced by these spectral lines in a narrow band impedance has a memory and can act back later on the same or an other bunch. This can lead to a coupled bunch instability, also called Robinson instability. Its growth rate is determined by the impedance values at the upper and lower side bands. This can be generalized for a more complicated impedance, for the case of many bunches and also for higher modes of longitudinal oscillations. A broad band impedance with only short memory does not cause coupled bunch instabilities but produces some single passage effects like frequency shifts and bunch lengthening. To treat the corresponding instabilities of betatron oscillations we introduce the transverse impedance in which the bean induces a deflecting field. Using the same formalism as for the longitudinal case, we get the growth rate of the transverse instability. The tune dependence on energy deviation, called chromaticity, produces a phase shift of the betatron oscillations between front and back of the bunch which can lead to an instability, called head-tail effect.

1 INTRODUCTION

The motion of a single particle in a storage ring is determined by the external guide fields created by the dipole and quadrupole magnets and the RF-system, and also by initial conditions and synchrotron radiation. The many particles contained in a high intensity beam represent a sizeable charge and current which act as sources of electromagnetic fields called *self fields*. They are modified by the boundary conditions imposed by the beam

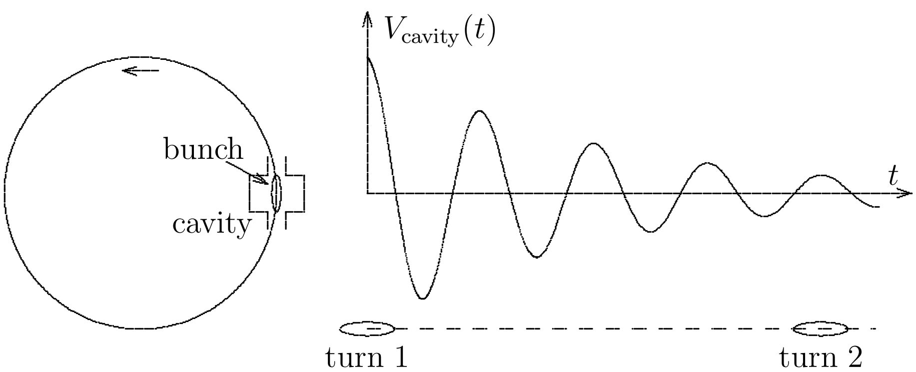


Figure 1: Induced field acting on the bunch the next turn

surroundings (vacuum chambers, cavities, etc.) and act back on the beam. This can lead to a *frequency shift* (change of the betatron or synchrotron frequency), to an increase of a small initial disturbance, an *instability*, or a *change of the particle distribution*, e.g. bunch lengthening. These phenomena are called *collective effects* since they are caused by a common action of the many particles in the beam.

As an introductory example we consider a bunch circulating in a storage ring and going through a passive cavity where it induces electromagnetic fields, Fig. 1. These fields oscillate and decay slowly. In the next turn the bunch might find some field left, having a phase such that a small initial synchrotron oscillation amplitude is increased, leading to an exponentially growing instability.

In most cases the fields created by the beam are small compared to the guide fields and their effects can be treated as a *perturbation*. This is done in three steps:

- First, the motion in the guide field and the stationary particle distribution are established.
- A small disturbance of the bunch from its stationary motion is considered (betatron or synchrotron oscillation). The fields caused by this disturbance are determined taking the boundary condition imposed by the beam surroundings (impedance) into account.
- The effect of these fields on the initial disturbance is investigated. If its amplitude is increased we have an *instability*, if it is decreased we have *damping*, or, if the frequency of the oscillation mode is changed, we have a *frequency shift*.

For the case of small self-fields, considered here, the particle distribution in the bunch is given by external conditions (machine parameter, initial condition, synchrotron radiation) and is usually Gaussian in electron machines. As disturbances of the stationary distribution we consider some modes of oscillation which are orthogonal (independent of each other) and investigate their stability.

Strong self-fields, however, modify the particle distribution and also the modes of oscillation, such that they are no longer independent. A self consistent solution has to be found, which is usually only attempted for the case of bunch lengthening.

We distinguish between *single* and *multi-traversal* collective effects. For the first kind no memory of the induced field over the time interval between the bunch passages is required. An example of a single-traversal effect is bunch lengthening. For multi-traversal effects the impedance needs a memory to make an interaction between bunches or turns possible which can be provided by cavity-like objects with a large quality factor Q.

Finally, we have *longitudinal* effects involving synchrotron oscillations and longitudinal impedances, and *transverse* effects involving betatron oscillations and transverse impedances. In both cases the longitudinal particle distribution (bunch length) is important, because it can be "resolved" by the impedance, while the transverse distribution is usually not resolved and does not affect the instability.

The most important longitudinal single traversal effects are synchrotron frequency shifts and bunch lengthening. In the transverse case the effect of the chromaticity is important which can lead to head-tail instabilities.

2 IMPEDANCES, WAKES AND LONGITUDINAL DYNAMICS

2.1 Cavity resonance

Impedances and wake potentials are treated extensively in the literature, e.g. [1]. We illustrate here some of their essential properties based on the simple example of a cavity resonance.

Cavity-like devices are the most important objects which can cause coupled-bunch mode instabilities, because the induced fields oscillate for a relatively long time and pro-

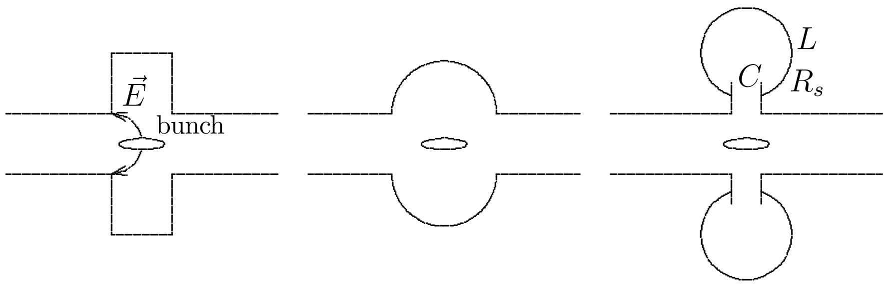
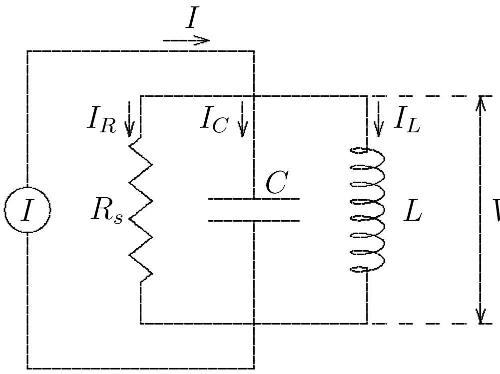


Figure 2: Cavity resembling an RLC-circuit



$$V_R = I_R R_s$$
$$V_C = \frac{1}{C}\int I_C dt$$
$$V_L = L\frac{dI_L}{dt}$$

Figure 3: RLC-circuit equivalent to a cavity resonance

vide a memory over the time interval between bunch passages. Such a cavity can be of a form which resembles an RLC-circuit as shown in Fig. 2, and can be treated as such. The RLC-circuit has a shunt impedance R_s, an inductance L and a capacity C, Fig. 3. In a real cavity these three parameters cannot easily be separated. For this reason we use some other related parameters which can be measured directly: The *resonance frequency* ω_r, the *quality factor* Q and the *damping rate* α:

$$\omega_r = \frac{1}{\sqrt{LC}}\ ,\ \ Q = R_s\sqrt{\frac{C}{L}} = \frac{R_s}{L\omega_r} = R_sC\omega_r\ ,\ \ \alpha = \frac{\omega_r}{2Q}\ ,\ \ C = \frac{Q}{\omega_r R_s}\ ,\ \ L = \frac{R_s}{\omega_r Q}.$$

If this circuit is driven by a current I the voltages across each element are

$$V_R = I_R R_s\,,\ V_C = \frac{1}{C}\int I_C dt\,,\ V_L = L\frac{dI_L}{dt}$$

and have the relations

$$V_R = V_C = V_L = V\ ,\ \ I_R + I_C + I_L = I.$$

Differentiating with respect to t gives

$$\dot{I} = \dot{I}_R + \dot{I}_C + \dot{I}_L = \frac{\dot{V}}{R_s} + C\ddot{V} + \frac{V}{L}.$$

Using $L = R_s/(\omega_r Q)$ and $C = Q/(\omega_r R_s)$ gives the differential equation

$$\ddot{V} + \frac{\omega_r}{Q}\dot{V} + \omega_r^2 V = \frac{\omega_r R_s}{Q}\dot{I}.$$

The solution of the homogeneous equation is a damped oscillation

$$V(t) = \hat{V}\mathrm{e}^{-\alpha\mathrm{t}} \cos\left(\omega_\mathrm{r}\sqrt{1-\frac{1}{4\mathrm{Q}^2}} + \phi\right)$$

or

$$V(t) = \mathrm{e}^{-\alpha\mathrm{t}}\left(\mathrm{A}\cos\left(\omega_\mathrm{r}\sqrt{1-\frac{1}{4\mathrm{Q}^2}}\,\mathrm{t}\right) + \mathrm{B}\sin\left(\omega_\mathrm{r}\sqrt{1-\frac{1}{4\mathrm{Q}^2}}\,\mathrm{t}\right)\right).$$

2.2 Wake potential

We now calculate the response of the *RLC*-circuit shown in Fig. 3, representing a cavity resonance, to a delta function pulse (very short bunch)

$$I(t) = \bar{q}\delta(t)$$

The charge $\bar{q}$ induces a voltage in the capacity

$$V(0^+) = \frac{\bar{q}}{C} = \frac{\omega_r R_s}{Q}\bar{q}.$$

The resulting energy stored in the capacity

$$U = \frac{\bar{q}^2}{2C} = \frac{\omega_r R_s}{2Q}\bar{q}^2 = \frac{V(0^+)}{2}\bar{q} = k_{pm0}\bar{q}^2,$$

must be equal to the energy lost by the charge. Here we introduced the *parasitic mode loss factor* for a point charge

$$k_{pm0} = \frac{U}{\bar{q}^2} = \frac{\omega_r R_s}{2Q}$$

which is the energy loss normalized for the charge $\bar{q}$. The charged capacitor C will discharge first through the resistor R_s and then also through the inductance L

$$\dot{V}(0^+) = -\frac{\dot{\bar{q}}}{C} = -\frac{I_R}{C} = -\frac{1}{C}\frac{V(0^+)}{R_s} = -\frac{\omega_r^2 R_s}{Q^2}\bar{q} = -\frac{2\omega_r k_{pm0}}{Q}\bar{q}.$$

The voltage in this resonance circuit has now the initial conditions

$$V(0^+) = 2k_{pm0}\bar{q} \text{ and } \dot{V}(0^+) = \frac{2\omega_r k_{pm0}}{Q}\bar{q}.$$

We take the solution of the homogeneous differential equation and its derivative

$$\begin{aligned} V(t) &= \mathrm{e}^{-\alpha\mathrm{t}}\left(\mathrm{A}\cos\left(\omega_\mathrm{r}\sqrt{1-\frac{1}{4\mathrm{Q}^2}}\,\mathrm{t}\right) + \mathrm{B}\sin\left(\omega_\mathrm{r}\sqrt{1-\frac{1}{4\mathrm{Q}^2}}\,\mathrm{t}\right)\right) \\ \dot{V}(t) &= \mathrm{e}^{-\alpha\mathrm{t}}\left(\left(-\mathrm{A}\alpha + \mathrm{B}\omega_\mathrm{r}\sqrt{1-\frac{1}{4\mathrm{Q}^2}}\right)\cos\left(\omega_\mathrm{r}\sqrt{1-\frac{1}{4\mathrm{Q}^2}}\,\mathrm{t}\right)\right. \\ &\quad \left. - \left(B\alpha + A\omega_r\sqrt{1-\frac{1}{4Q^2}}\right)\sin\left(\omega_r\sqrt{1-\frac{1}{4Q^2}}\;t\right)\right) \end{aligned}$$

and satisfy the above initial conditions by

$$A = 2k_{pm0}\bar{q} \text{ and } -A\alpha + B\omega_r\sqrt{1-\frac{1}{4Q^2}} = -\frac{2\omega_r k_{pm0}}{Q}\bar{q}.$$

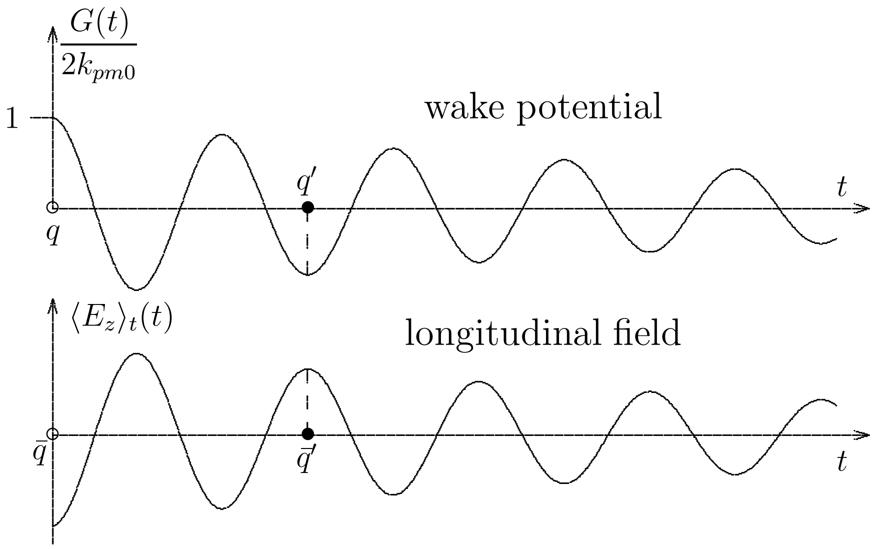


Figure 4: Wake potential and longitudinal field

The voltage in a resonator circuit excited at the time $t = 0$ by a δ-pulse $I(t) = q\delta t$ becomes

$$V(t) = 2\bar{q}k_{pm0}\mathrm{e}^{-\alpha\mathrm{t}}\left(\cos\left(\omega_\mathrm{r}\sqrt{1-\frac{1}{4\mathrm{Q}^2}}\;\mathrm{t}\right) - \frac{\sin\left(\omega_\mathrm{r}\sqrt{1-\frac{1}{4\mathrm{Q}^2}}\;\mathrm{t}\right)}{2\mathrm{Q}\sqrt{1-\frac{1}{4\mathrm{Q}^2}}}\right).$$

This voltage induced by charge $\bar{q}$ at $t = 0$ is seen by a second point charge $\bar{q}'$ traversing the cavity at t and loosing or gaining an energy $U = \bar{q}'V(t)$ as shown in Fig. 4. This energy gain/loss per unit source and probe charges is called point charge *wake potential* or *Green function* $G(t)$. For our resonator (cavity resonance), we have

$$G(t) = 2k_{pm0}\mathrm{e}^{-\alpha\mathrm{t}}\left(\cos\left(\omega_\mathrm{r}\sqrt{1-\frac{1}{4\mathrm{Q}^2}}\;\mathrm{t}\right) - \frac{\sin\left(\omega_\mathrm{r}\sqrt{1-\frac{1}{4\mathrm{Q}^2}}\;\mathrm{t}\right)}{2\mathrm{Q}\sqrt{1-\frac{1}{4\mathrm{Q}^2}}}\right).$$

For a high quality factor, $Q \gg 1$, this simplifies to

$$G(t) \approx 2k_{pm0}\mathrm{e}^{-\alpha\mathrm{t}}\cos\left(\omega_\mathrm{r}\mathrm{t}\right)$$

The wake potential is related to the longitudinal field E_z by a field integral over the object length. Since the field changes during the traversal, this integration has to follow a particle going with nearly the speed of light through the object and taking the momentary field value

$$V = -\int_{z_1}^{z_2} E_z(z,t)dz = -f_t\int_{z_1}^{z_2} E_z(z)dz = -\langle E_z\rangle_t\Delta z$$

with the transit time factor f_t correcting the instantaneous integral over z. We use a wake potential being positive where the particle loses energy consistent with the sign used for resistors.

2.3 Impedance

We use now a *harmonic* excitation of the circuit in Fig. 3 with a current $I = \hat{I}\cos(\omega t)$ which is described by the differential equation

$$\ddot{V} + \frac{\omega_r}{Q}\dot{V} + \omega_r^2 V = -\frac{\omega_r R_s}{Q}\hat{I}\omega\sin(\omega t).$$

The solution of the homogeneous equation is a damped oscillation which disappears after some transient time and we are left with the particular solution of the form $V(t) = A\cos(\omega t) + B\sin(\omega t)$. Inserting this into the differential equation and separating cosine and sine terms gives

$$(\omega_r^2 - \omega^2)A + \frac{\omega_r\omega}{Q}B = 0 \text{ and } (\omega_r^2 - \omega^2)B - \frac{\omega_r\omega}{Q}A = -\frac{\omega_r\omega R_s}{Q}\hat{I}.$$

The voltage induced by the harmonic excitation of the resonator becomes

$$V(t) = \hat{I}R_s\frac{\cos(\omega t) - Q\frac{\omega_r^2-\omega^2}{\omega_r\omega}\sin(\omega t)}{1 - Q^2\left(\frac{\omega^2-\omega_r^2}{\omega_r\omega}\right)^2}.$$

This voltage has a cosine term which is *in phase* with the exciting current. It can absorb energy and is called the *resistive* term. The sine term of the voltage is *out of phase* with the exciting current and does not absorb energy, it is called the *reactive* term. The ratio between the voltage and current is called *impedance*. It is a *function of frequency* ω and has a resistive part $Z_r(\omega)$ and a reactive part $Z_i(\omega)$

$$Z_r(\omega) = R_s\frac{1}{1 + Q^2\left(\frac{\omega_r^2-\omega^2}{\omega_r\omega}\right)^2}\ ,\ Z_i(\omega) = R_s\frac{Q\frac{\omega_r^2-\omega^2}{\omega_r\omega}}{1 + Q^2\left(\frac{\omega_r^2-\omega^2}{\omega_r\omega}\right)^2}.$$

The resonance can be excited either by a current $I(t) = \hat{I}\cos(\omega t)$ or $I(t) = \hat{I}\sin(\omega t)$ resulting in the voltages $V(t)$

$$\begin{aligned} I(t) &= \hat{I}\cos(\omega t) \rightarrow V(t) = \hat{I}\left(Z_r(\omega)\cos(\omega t) - Z_i(\omega)\sin(\omega t)\right), \\ I(t) &= \hat{I}\sin(\omega t) \rightarrow V(t) = \hat{I}\left(Z_r(\omega)\sin(\omega t) + Z_i(\omega)\cos(\omega t)\right). \end{aligned}$$

2.4 Complex notation

We have used a harmonic excitation of the form

$$I(t) = \hat{I}\cos(\omega t) = \hat{I}\frac{e^{j\omega t} + e^{-j\omega t}}{2} \ \ with\ \ 0 \le \omega \le \infty,$$

using positive frequencies only. A complex notation

$$I(t) = \hat{I}e^{j\omega t} \text{ with } -\infty \le \omega \le \infty$$

involving positive and negative frequencies leads to more compact expressions and is often convenient. The real solution can be obtained after, by taking half the sum of the solutions for $e^{\pm\omega t}$. We take the differential equation

$$\ddot{V} + \frac{\omega_r}{Q}\dot{V} + \omega_r^2 = \frac{\omega_r R_s}{Q}\dot{I}$$

of the resonator voltage with the excitation $I(t) = \hat{I}\exp(j\omega t)$ and seek a solution of the form $V(t) = V_0\exp(j\omega t)$, where V_0 is in general complex and get

$$-\omega^2 V_0 e^{j\omega t} + j\frac{\omega_r\omega}{Q}V_0 e^{j\omega t} + \omega_r^2 V_0 e^{j\omega t} = j\frac{\omega_r\omega R_s}{Q}\hat{I}e^{j\omega t}.$$

The impedance, defined as the ratio V/I, is given by

$$Z(\omega) = \frac{V_0}{\hat{I}} = R_s \frac{j\frac{\omega_r \omega}{Q}}{\omega_r^2 - \omega^2 + jQ\frac{\omega_r \omega}{Q}} = R_s \frac{1 - jQ\frac{\omega^2-\omega_r^2}{\omega\omega_r}}{1 + Q^2\left(\frac{\omega^2-\omega_r^2}{\omega\omega_r}\right)^2} = Z_r(\omega) + jZ_i(\omega)$$

and has a real and an imaginary part. For a large quality factor Q the impedance is only large for $\omega \approx \omega_r$ or $|\omega - \omega_r|/\omega_r = |\Delta\omega|/\omega_r \ll 1$ and can be simplified

$$Z(\omega) \approx R_s \frac{1 - j2Q\frac{\Delta\omega}{\omega_r}}{1 + 4Q^2\left(\frac{\Delta\omega}{\omega_r}\right)^2}.$$

The resonator impedance has some specific properties:

$$\begin{aligned} \omega &= \omega_r \rightarrow Z_r(\omega_r) \text{ has a maximum while } Z_i(\omega_r) = 0 \\ |\omega| &< \omega_r \rightarrow Z_i(\omega) > 0 \text{ (inductive)} \\ |\omega| &> \omega_r \rightarrow Z_i(\omega) < 0 \text{ (capacitive)} \end{aligned} \tag{1}$$

and some properties which apply to any impedance or wake potential

$$Z_r(\omega) = Z_r(-\omega) \ , \ Z_i(\omega) = -Z_i(-\omega),$$

$$Z(\omega) = \int_{-\infty}^{\infty} G(t)\mathrm{e}^{-j\omega t} dt \ , \ G(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} Z(\omega)\mathrm{e}^{j\omega t} d\omega, \tag{2}$$

$$t < 0, \rightarrow G(t) = 0 \text{ no fields before particle arrives.} \tag{3}$$

Impedance and Green function are related by a Fourier transform with a factor unity or $1/(2\pi)$ in front of the integral instead of the factor $1/\sqrt{2\pi}$ used elsewhere. Caution; sometimes one uses $I(t) = \hat{I}\mathrm{e}^{-i\omega t}$ instead of $I(t) = \hat{I}\mathrm{e}^{j\omega t}$, this reverses the sign $Z_i(\omega)$.

In Fig. 5 the Green functions and impedances are shown for two resonators of different quality factors.

2.5 Review of the longitudinal dynamics

A particle with a momentum deviation Δp has a different closed orbit which is radially displaced by $\Delta x = D_x \Delta p/p$ with D_x being the dispersion. As a result the orbit length L, the revolution time T_0 and the revolution frequency ω_0 are changed

$$\frac{\Delta L}{L} = \alpha_c \frac{\Delta p}{p} \ , \ \frac{\Delta\omega_0}{\omega_0} = -\frac{\Delta T_0}{T_0} = -\left(\alpha_c - \frac{1}{\gamma^2}\right)\frac{\Delta p}{p} = -\eta_c \frac{\Delta p}{p}$$

with α_c being the momentum compaction and $\eta_c = \alpha_c - 1/\gamma^2$. There is a transition energy $E_T = m_0 c^2 \gamma_T$ with $\gamma_T = 1/\alpha_c^2$ for which the dependence of the revolution frequency on momentum (or energy) changes sign

$$\begin{aligned} E &> E_T \rightarrow \frac{1}{\gamma^2} < \alpha_c \rightarrow \eta_c > 1 \rightarrow \omega_0 \text{ decreases with } \Delta E \\ E &< E_T \rightarrow \frac{1}{\gamma^2} > \alpha_c \rightarrow \eta_c < 1 \rightarrow \omega_0 \text{ increases with } \Delta E. \end{aligned}$$

We will assume in most cases that the particles are ultra-relativistic in which case $\Delta p/p \approx \Delta E/E = \epsilon$ and $\eta_c \approx \alpha_c$.

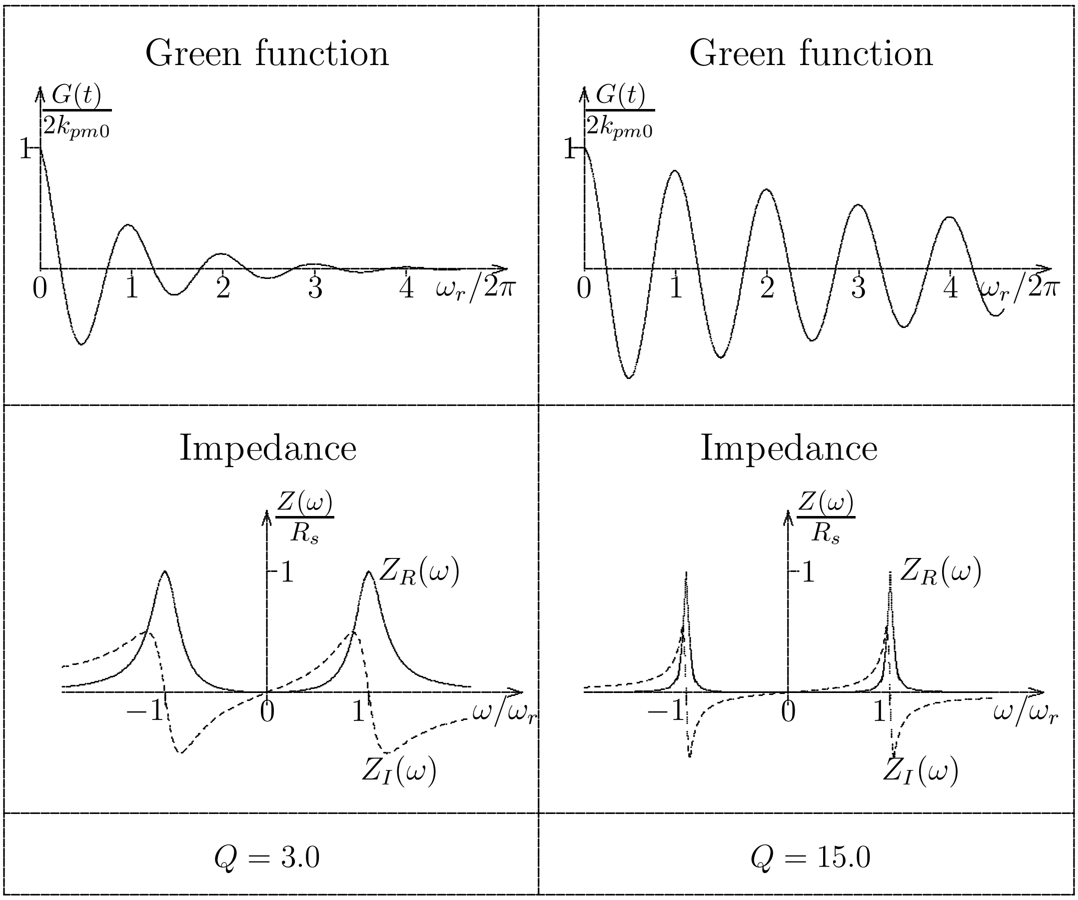


Figure 5: Green function and impedance of a resonance

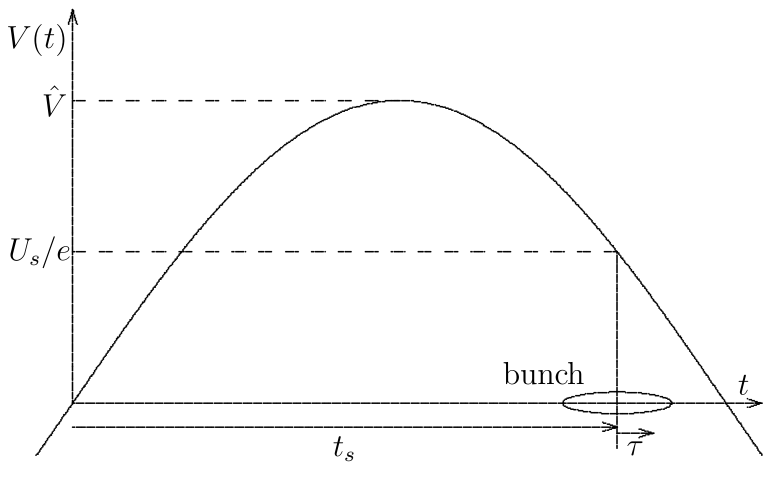


Figure 6: Longitudinal beam dynamics

In the presence of an RF-system and an energy loss per turn U due to synchrotron radiation or an impedance, a circulating particle has according to Fig. 6 each turn a gain or loss δE in energy of

$$\delta E = e\hat{V}\sin(h\omega_0(t_s + \tau)) - U$$

or in relative energy $\delta E/E = \delta\epsilon$

$$\frac{\delta E}{E} = \delta\epsilon = \frac{e\hat{V}\sin(\omega_0 h(t_s+\tau))}{E} - \frac{U}{E}.$$

with t_s being the synchronous arrival time of the particle in the cavity and $\tau = t - t_s$ the deviation from it. We introduce the synchronous phase angle $\phi_s = \omega_0 h t_s$ and assume $\tau \ll T_0$ which allows us to develop the trigonometric function

$$\delta\epsilon = \frac{e\hat{V}\sin(\phi_s)}{E} + \frac{\omega_0 h e\hat{V}\cos\phi_s}{E}\tau - \frac{U}{E}.$$

The energy gain per turn is usually very small, $\delta E \ll E$, and we can make a smooth approximation

$$\delta\epsilon = \frac{\delta E}{E} = \dot{\epsilon}T_0 = \dot{\epsilon}\frac{2\pi}{\omega_0}$$

$$\dot{\epsilon} = \frac{\omega_0 e\hat{V}\sin\phi_s}{2\pi E} + \frac{\omega_0^2 h e\hat{V}\cos\phi_s}{2\pi E}\tau - \frac{\omega_0}{2\pi}\frac{U}{E}. \tag{4}$$

The energy loss U suffered by a particle is in general a function of its deviations ϵ and τ from the nominal energy and synchronous time and can be developed to first order as

$$U(\epsilon,\tau) \approx U_0 + \frac{\partial U}{\partial E}\Delta E + \frac{\partial U}{\partial t}\tau.$$

This leads to an expression for the time derivative of the energy loss

$$\dot{\epsilon} = \frac{\omega_0 e\hat{V}\sin\phi_s}{2\pi E} + \frac{\omega_0^2 h e\hat{V}\cos\phi_s}{2\pi E}\tau - \frac{\omega_0}{2\pi}\frac{U_0}{E} - \frac{\omega_0}{2\pi}\frac{dU}{dE}\epsilon - \frac{\omega_0}{2\pi}\frac{1}{E}\frac{dU}{dt}\tau.$$

To have equilibrium for the synchronous particle, $\epsilon = 0$, $\tau = 0$, we must have

$$U_0 = e\hat{V}\sin\phi_s.$$

With this and using $\dot{\tau} = \omega_0\Delta T_0/2\pi = \eta_c\epsilon$ we get a system of two first order differential equations

$$\begin{aligned}\dot{\epsilon} &= \omega_0^2\frac{he\hat{V}\cos\phi_s}{2\pi E}\tau - \frac{\omega_0}{2\pi}\frac{dU}{dE}\epsilon - \frac{\omega_0}{2\pi}\frac{1}{E}\frac{dU}{dt}\tau \\ \dot{\tau} &= \eta_c\epsilon\,.\end{aligned}$$

They can be combined into one second-order equation

$$\ddot{\tau} + \frac{\omega_0\eta_c}{2\pi}\frac{dU}{dE}\dot{\tau} - \frac{\omega_0^2 h\eta_c e\hat{V}\cos\phi_s}{2\pi E}\tau - \frac{\omega_0\eta_c}{2\pi E}\frac{dU}{dt}\tau = 0$$

which describes a damped oscillation. Using the unperturbed synchrotron frequency ω_{s0} and the damping rate α_s

$$\omega_{s0}^2 = -\omega_0^2\frac{h\eta_c e\hat{V}\cos\phi_s}{2\pi E}\ ,\ \alpha_s = \frac{1}{2}\frac{\omega_0\eta}{2\pi}\frac{dU}{dE}\,, \tag{5}$$

seeking a solution of the form $e^{j\omega t}$, with complex ω and assuming $\alpha_s \ll \omega_{s0}$ we get

$$-\omega^2 + j2\omega\alpha_s + (\omega_{s0}^2 + \frac{\omega_0}{2\pi}\frac{\eta_c}{E}\frac{dU}{dt}) = 0$$

$$\omega = j\alpha_s \pm \sqrt{(\omega_{s0}^2 + \frac{\omega_0}{2\pi}\frac{\eta_c}{E}\frac{dU}{dt}) - \alpha_s^2} \approx j\alpha_s \pm (\omega_{s0} + \frac{1}{2}\frac{\omega_0}{2\pi}\frac{\eta_c}{\omega_{s0}E}\frac{dU}{dt}) \, .$$

Calling

$$\Delta\omega_r = \frac{1}{2}\frac{\omega_0}{2\pi}\frac{\eta_c}{\omega_{s0}E}\frac{dU}{dt}$$

gives

$$\epsilon = A\left(\mathrm{e}^{(-\alpha_s + j(\omega_{s0}+\Delta\omega_r)t} + B\mathrm{e}^{(-\alpha_s - j(\omega_{s0}+\Delta\omega_r)t}\right) .$$

For the initial conditions $\epsilon(t) = \hat{\epsilon}$, $\dot{\epsilon}(0) = -\alpha_s\hat{\epsilon}$ we get $A = B = \hat{\epsilon}/2$ and

$$\epsilon(t) = \hat{\epsilon}\ \mathrm{e}^{-\alpha_s t}\cos((\omega_{s0} + \Delta\omega_r)t).$$

In the absence of any energy loss U we have

$$\epsilon(t) = \hat{\epsilon}\cos(\omega_{s0}t + \phi)$$

with

$$\omega_{s0}^2 = -\omega_0^2\frac{h\eta_c e\hat{V}\cos\phi_s}{2\pi E}$$

In order to get a stable oscillation we need $\omega_{s0}^2 > 0$ which leads to the conditions

$$E > E_T\ \eta_c < 0\ \rightarrow \cos\phi_s < 0\ ,\ \ E < E_T\ \eta_c > 0\ \rightarrow \cos\phi_s > 0.$$

For stability in the presence of an energy loss U we need in addition

$$\alpha_s = \frac{1}{2}\frac{\omega_0\eta}{2\pi}\frac{dU}{dE} > 0.$$

In other words, the energy loss U has to increase for a positive energy deviation of the particle.

3 A STATIONARY BUNCH INTERACTING WITH AN IMPEDANCE

3.1 Spectrum of a stationary bunch

We consider now a bunch in a single traversal with the current $I(t)$ time and $\tilde{I}(\omega)$ in frequency domain

$$\tilde{I}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} I(t)\mathrm{e}^{-j\omega t}dt. \tag{6}$$

We assume the bunch form to be symmetric

$$I(-t) = I(t)$$

which leads to a Fourier transform having only a real part and being symmetric in ω

$$\tilde{I}(-\omega) = \tilde{I}(\omega).$$

This assumption is not necessary but is here used for convenience to reduce the number of terms which have to be carried along in some calculations. Since, in most practical applications, the bunches are to a good approximation symmetric, this represents a minor restriction which could easily be removed. The current of a bunch with Gaussian distribution as a function of time and frequency is illustrated in Fig. 7 and given by the expressions

$$I(t) = \frac{\bar{q}}{\sqrt{2\pi}\sigma_t}\mathrm{e}^{-\frac{t^2}{2\sigma_t^2}}\ \ ;\ \ \tilde{I}(\omega) = \frac{\bar{q}}{\sqrt{2\pi}}\mathrm{e}^{-\frac{\omega^2}{2\sigma_\omega^2}} \tag{7}$$

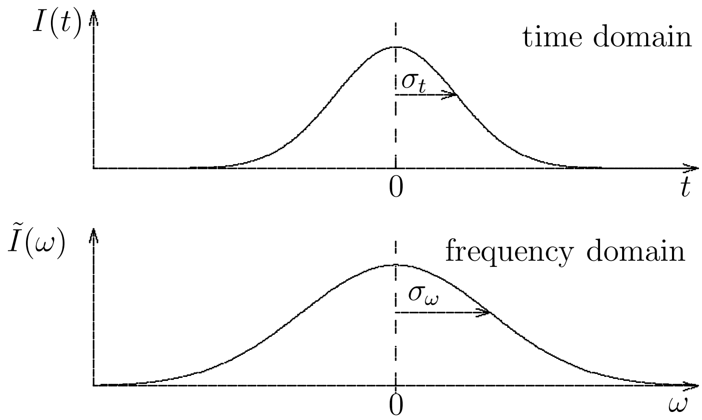


Figure 7: Single passage of a bunch in time and frequency domain

where $\bar{q} = N_b e$ is the total charge of the N_b particles in a bunch. The RMS width of the bunch and its spectrum are σ_t and σ_ω which are related by

$$\sigma_t = \frac{1}{\sigma_\omega}.$$

Next we investigate the case of a circulating bunch having repetitive passages at a given location with frequency $\omega_0 = 2\pi/T_0$. For a stationary bunch, having no synchrotron oscillations, the observed current can be written in this, slightly unusual form

$$I_k(t) = \sum_{k=-\infty}^{\infty} I(t - kT_0). \tag{8}$$

which is not convenient for applications. Since the current is periodic it is natural to express it in a Fourier series using either a complex notation with positive and negative frequencies or trigonometric functions involving positive frequencies only

$$I_k(t) = \sum_{-\infty}^{\infty} I_p \mathrm{e}^{jp\omega_0} = I_0 + 2\sum_{1}^{\infty} I_p \cos(p\omega_0 t) \tag{9}$$

with

$$I_p = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} I(t)\mathrm{e}^{-jp\omega_0 t} = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} I(t)\left(\cos(p\omega_0 t) - j\sin(p\omega_0 t)\right) dt \tag{10}$$

where the sine term vanishes for our symmetric bunch passages. The bunch current component at zero frequency is just its average value

$$I_0 = \langle I \rangle = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} I(t)dt = \frac{\bar{q}}{T_0}. \tag{11}$$

The multiple bunch passage is illustrated in Fig. 8 in time domain on the top, and in frequency domain in the middle using positive and negative frequencies and on the bottom with positive frequencies only. For the latter the current components are twice as large except for the one at zero frequency.

Comparing the Fourier transform (6) with the terms of the Fourier series (10) we find the relation

$$I_p = \frac{\omega_0}{\sqrt{2\pi}}\tilde{I}(p\omega_0).$$

For a Gaussian bunch (7) we get

$$I_p = I_0 \mathrm{e}^{-\frac{p^2\omega_0^2}{2\sigma_\omega^2}}.$$

At low frequencies $p\omega_0 \ll \sigma_\omega$ we have $I_p \approx I_0$.

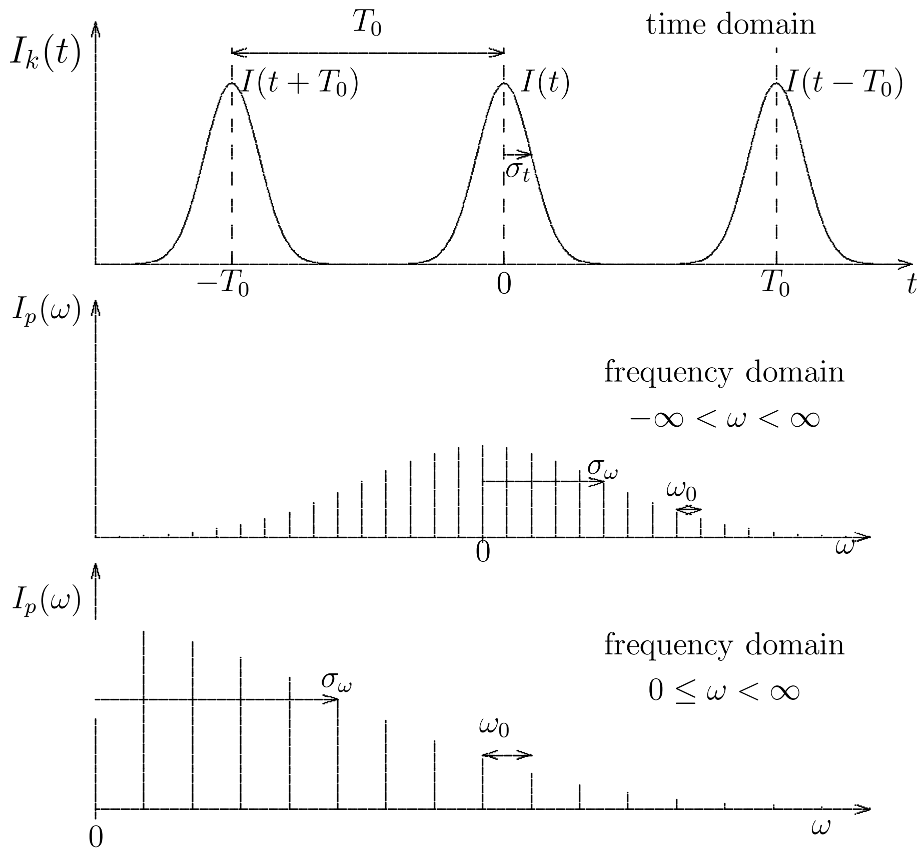


Figure 8: Multiple passage of a bunch in time and frequency domain

3.2 Voltage induced by the stationary bunch

In the presence of a cavity resonance, or any *general impedance* $Z(\omega)$, the circulating stationary bunch induces a voltage. Using the Fourier series (9) of the bunch current we have to multiply each frequency component with the corresponding impedance

$$V_k(t) = \sum_{p=-\infty}^{\infty} I_p Z(p\omega_0) e^{jp\omega_0 t} = \sum_{p=-\infty}^{\infty} I_p (Z_r(p\omega_0) + jZ_i(p\omega_0) e^{jp\omega_0 t}.$$

By combining positive and negative frequencies and observing the symmetry conditions $Z_r(-\omega) = Z_r(\omega)$, $Z_i(-\omega) = -Z_i(\omega)$ and the fact that $Z(0) = 0$, we get a real expression

$$V_k(t) = 2 \sum_{p=1}^{\infty} I_p \left(Z_r(p\omega_0) \cos(p\omega_0 t) - Z_i(p\omega_0) \sin(p\omega_0 t) \right). \tag{12}$$

We calculate the induced voltage $\langle V \rangle$, averaged over all particles in the bunch

$$\langle V \rangle = \frac{1}{I_0 T_0} \int_{-T_0/2}^{T_0/2} I_k(t) V_k(t) dt.$$

With the expressions (9) for the current and (12) for the voltage we get

$$\langle V \rangle = \frac{4}{I_0 T_0} \sum_{p=1}^{\infty} \sum_{p'=1}^{\infty} I'_p I_p$$
$$\left(Z_r(p\omega_0 \int_{-T_0/2}^{T_0/2} \cos(p'\omega_0 t) \cos(p\omega_0 t) dt - Z_r(p\omega_0) \int_{-T_0/2}^{T_0/2} \cos(p'\omega_0 t) \sin(p\omega_0 t) dt \right).$$

The first integral vanishes except for $p' = p$ in which case it has the value $T_0/2$, and the second integral always vanishes. This leads to

$$\langle V\rangle = \frac{1}{I_0}\sum_{p=-\infty}^{\infty}|I_p|^2 Z_r(p\omega_0) = \frac{2}{I_0}\sum_{p=1}^{\infty}|I_p|^2 Z_r(p\omega_0). \quad (13)$$

Only the resistive impedance $Z_r(\omega)$ contributes to this average voltage while the voltages induced in the reactive impedance $Z_i(\omega)$ averages out.

We will also need the average voltage slope

$$\left\langle\frac{dV}{dt}\right\rangle = \frac{1}{I_0 T_0}\int_{-T_0/2}^{T_0/2} I_k(t)\frac{dV_k(t)}{dt}dt.$$

In this case the contribution induced in the resistive impedance averages out to zero and the average voltage slope is determined by the reactive impedance only. Using the same method as above for the average voltage we obtain the averaged voltage slope

$$\left\langle\frac{dV}{dt}\right\rangle = -\frac{\omega_0}{I_0}\sum_{p=-\infty}^{\infty}p|I_p|^2 Z_i(p\omega_0) = -\frac{2\omega_0}{I_0}\sum_{p=1}^{\infty}p|I_p|^2 Z_i(p\omega_0).$$

3.3 Energy loss per turn of a stationary circulating bunch

The energy W_b lost by the whole circulating stationary bunch in one turn due to the impedance $Z(\omega)$ can be obtained from the average voltage (13)

$$W_b = \bar{q}\langle V\rangle = \frac{2\bar{q}}{I_0}\sum_{p=1}^{\infty}|I_p|^2 Z_r(p\omega_0)$$

where $q = eN_b$ is the total charge of the bunch. The average energy loss U per particle in the bunch is

$$U = \frac{W_b}{N_b} = \frac{2e}{I_0}\sum_{p=1}^{\infty}|I_p|^2 Z_r(p\omega_0) = \frac{2T_0}{N_b}\sum_{p=1}^{\infty}|I_p|^2 Z_r(p\omega_0).$$

We can normalize the loss W_b by the square of charge (the charge inducing the voltage and the same charge suffering an energy loss) to get the so-called parasitic mode loss factor of a bunch

$$k_{pm} = \frac{W_b}{\bar{q}^2} = \frac{U}{e\bar{q}} = \frac{2T_0}{\bar{q}^2}\sum_{p=1}^{\infty}|I_p|^2 Z_r(p\omega_0) = \frac{T_0}{\bar{q}^2}\sum_{p=-\infty}^{\infty}|I_p|^2 Z_r(p\omega_0)..$$

This parameter depends on the bunch length. For a short bunch the spectrum extends to higher frequencies. The parameter k_{pm} is therefore expected to increase with decreasing bunch length.

If the impedance is broad band and does not contain resonances of bandwidth smaller than the revolution frequency, the above sum can be approximated by an integral

$$k_{pm} \approx \frac{1}{\bar{q}^2}\int_{-\infty}^{\infty}|\tilde{I}(\omega)|^2 Z_r(\omega)d\omega.$$

The above relation is often used to measure the resistive impedance of a ring. By observing the change of the synchronous phase ϕ_s with current the energy loss U, and therefore the loss factor k_{pm}, can be determined from the relation

$$U = e\hat{V}\sin\phi_s.$$

This gives a a convolution between impedance and power spectrum of the bunch. By doing this experiment for different bunch lengths, we get information of the impedance.

3.4 Incoherent synchrotron frequency shift

We take now the case of a stationary bunch in the presence of an impedance $Z(\omega) = Z_r(\omega) + jZ_i(\omega)$. As we saw before, the bunch induces an average voltage in the resistive part of the impedance

$$\langle V\rangle = \frac{2}{I_0}\sum_{p=1}^{\infty}|I_p|^2 Z_r(p\omega_0) \tag{14}$$

and an averaged voltage slope in the reactive part

$$\left\langle\frac{dV}{dt}\right\rangle = -\frac{2\omega_0}{I_0}\sum_{p=1}^{\infty}p|I_p|^2 Z_i(p\omega_0), \tag{15}$$

both being independent of the energy deviation ϵ. We have to include these voltages in the equation of the synchrotron motion

$$\dot{\epsilon} = \frac{e\hat{V}\sin\phi_s\omega_0}{2\pi E} + \frac{\omega_0^2 he\hat{V}\cos\phi_s}{2\pi E}\tau - \frac{\omega_0}{2\pi}\frac{e\langle V\rangle}{E} - \frac{\omega_0 e}{2\pi E}\left\langle\frac{dV}{dt}\right\rangle\tau.$$

With the condition $e\hat{V}\sin\phi_s = e\langle V\rangle$ we find

$$\begin{aligned}\dot{\epsilon} &= \omega_0^2\frac{he\hat{V}\cos\phi_s}{2\pi E}\tau + \frac{\omega_0 e}{2\pi E}\left\langle\frac{dV}{dt}\right\rangle\tau\\ \dot{\tau} &= \eta_c\epsilon\,,\end{aligned}$$

or, combined into a second-order equation,

$$\ddot{\epsilon} - \left(\frac{\omega_0^2 h\eta_c e\hat{V}\cos\phi_s}{2\pi E} + \frac{\eta_c}{E}\frac{\omega_0}{2\pi}e\left\langle\frac{dV}{dt}\right\rangle\right)\epsilon = 0.$$

$$\begin{aligned}\omega_s^2 &= \omega_{s0}^2 - \frac{2\omega_0^2\eta_c e}{2\pi E I_0}\sum_{p=1}^{\infty}p|I_p|^2 Z_i(p\omega_0)\\ &= \omega_{s0}^2\left(1 + \frac{2}{h\hat{V}\cos\phi_s I_0}\sum_{p=1}^{\infty}p|I_p|^2 Z_i(p\omega_0)\right).\end{aligned} \tag{16}$$

where we used the unperturbed synchrotron frequency given in (5). There is a shift of the incoherent synchrotron frequency. For a small effect this shift can be expressed as

$$\begin{aligned}\frac{\Delta\omega_{si}}{\omega_{s0}} &= \frac{1}{I_0 h\hat{V}\cos\phi_s}\sum_{p=1}^{\infty}pI_p^2 Z_i(p\omega_0)\\ &= \frac{1}{2I_0 h\hat{V}\cos\phi_s}\sum_{p=-\infty}^{\infty}pI_p^2 Z_i(p\omega_0).\end{aligned} \tag{17}$$

For a predominately inductive impedance $Z_i(\omega) > 0$ this frequency shift is negative above transition energy $\cos\phi_s < 0$ and positive bellow transition energy. The longitudinal focusing is reduced in the first case and increased in the second case. This leads to a change of the bunch length being to first order for electrons

$$\frac{\Delta\sigma_s}{\sigma_s} = -\frac{\Delta\omega_{si}}{\omega_{s0}}$$

and for protons

$$\frac{\Delta\sigma_s}{\sigma_s} = -\sqrt{\frac{\Delta\omega_{si}}{\omega_{s0}}}$$

We have taken an average slope to calculate this tune shift. In reality the induced voltage is not linear and will make this incoherent tune shift amplitude dependent leading to a spread in synchrotron frequencies.

4 ROBINSON INSTABILITY, QUALITATIVE

4.1 Introduction

The interaction of a bunch executing a synchrotron oscillation with a narrow cavity resonance can lead to a growing amplitude, called *Robinson instability*, [2]. We will treat it here in some detail because it can be generalized to describe all multi-turn instabilities in storage rings. In order to gain some understanding of the physics involved we start with some qualitative treatment and proceed later to the quantitative investigation which involves some lengthy derivations.

4.2 Qualitative treatment

4.2.1 Modulation of the revolution frequency of an oscillating bunch

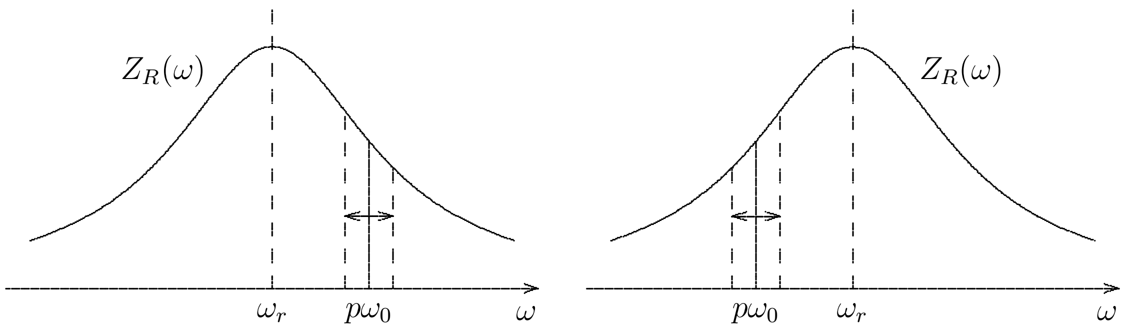


Figure 9: Qualitative treatment of the Robinson instability

We consider a single bunch circulating in a storage ring with revolution frequency ω_0. Its harmonic $p\omega_0$ excites a narrow cavity with resonance frequency $\omega_r \approx p\omega_0$ and impedance $Z(\omega)$ of which we consider only the resistive part Z_r as shown in Fig. 9.

The revolution frequency ω_0 of the circulating bunch depends on its relative energy deviation $\Delta E/E = \epsilon$

$$\frac{\Delta\omega_0}{\omega_0} = -\eta_c\frac{\Delta E}{E} = -\eta_c\epsilon \text{ or } \omega_0 = \omega_0\left(1 - \eta_c\epsilon\right).$$

While the bunch is executing a coherent dipole mode oscillation $\epsilon(t) = \hat{\epsilon}\cos(\omega_s t)$ its revolution frequency is modulated. *Above transition* the revolution frequency ω_0 is *small* when the *energy is high* and ω_0 is *large* when the *energy is small.* If the cavity is tuned to a resonant frequency slightly smaller than the revolution frequency harmonic $\omega_r < p\omega_0$, as shown in Fig. 9 on the left, the bunch sees a higher impedance and *loses more energy* when it has an *energy excess* and it *loses less energy* when it has a *lack of energy.* This leads to a *damping* of the oscillation. If $\omega_r > p\omega_0$ this is reversed, as shown in Fig. 9 on the right, and leads to an *instability.* Below transition energy the dependence of the revolution frequency is reversed which changes the stability criterion.

4.2.2 Effect of the fields induced by the side bands

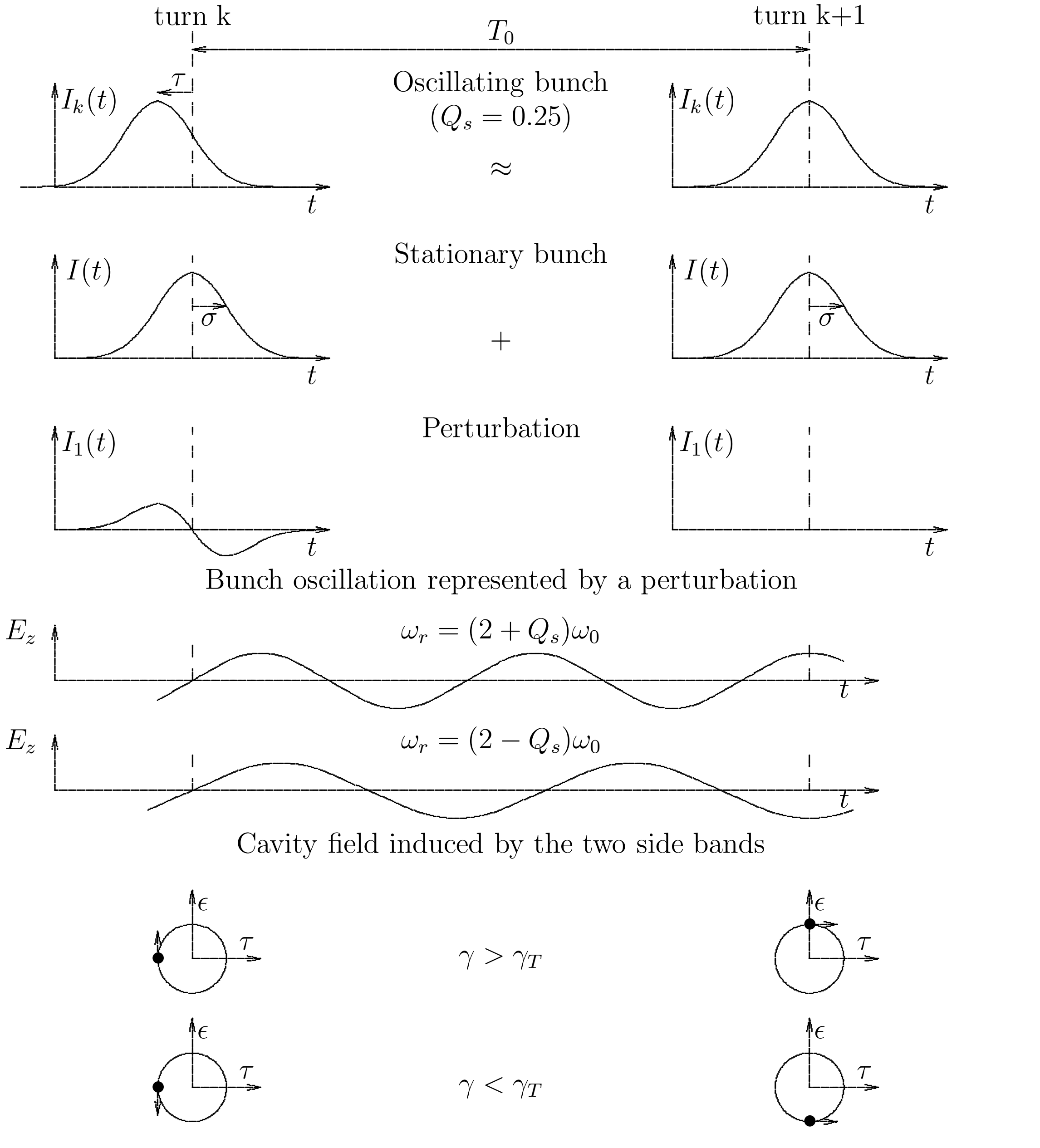


Figure 10: Qualitative understanding from the voltages induced by the two side bands

The simple picture of energy exchange between the oscillating beam and the narrow band impedance and the resulting stability condition illustrates the underlying physics. However, it can not easily be extended to a quantitative treatment. The use of revolution frequency which changes in time represents a mixture of time and frequency domain which is not easily treated by the standard mathematical methods. A bunch executing a synchrotron oscillation is presented in frequency domain by a spectrum consisting of harmonics $p\omega_0$ of the revolution frequency with side bands spaced by $\pm\omega_s$ around them. This will be discussed later in detail.

The oscillating bunch creates frequency components of the current at the carrier $p\omega_0$ with side bands at $\pm\omega_s$ which excite the cavity resonance. The latter is assumed to be sufficiently narrow such that only one value of p has to be considered. We take as an example a bunch with $Q_s = \omega_s/\omega_0 = 0.25$ and show its oscillation on top of Fig. 10 on two successive turns. This oscillation can be presented as a stationary bunch plus a perturbation. This perturbation induces a voltage in the cavity impedance which will act back on the bunch. It is shown in the center for $p = 2$ and the frequencies $\omega = (2 \pm Q_s)\omega_0$ corresponding to the upper or lower side band. After one turn the first one results in a positive and the lower frequency gives in a negative field. At the bottom the bunch motion is presented in the phase space coordinates ϵ and τ. Taking the first case $\gamma > \gamma_T$ above transition energy the bunch has a positive energy deviation after one turn. The field induced by the upper side band is positive leading to an increase of this energy deviation and therefore to a growing oscillation. The field due to the lower side band is negative and reduces the energy deviation leading to damping of the oscillation. Below transition energy, $\gamma < \gamma_T$ the bunch rotates in phase space in the opposite direction which reverses the stability condition. Obviously the special value $Q_s = 0.25$ was chosen to make the stability situation already visible after one turn. For a more realistic smaller value for Q_s the oscillation would have to be followed over several turns making the picture more complicated.

5 ROBINSON INSTABILITY, QUANTITATIVE

5.1 Spectrum of an oscillating bunch

We consider a bunch which executes a rigid synchrotron oscillation with frequency $\omega_s = \omega_0 Q_s$. This means that the bunch as a whole executes this oscillation without changing its longitudinal distribution. It results in a modulation of its passage time t_k at a cavity in successive turns k as illustrated in Fig. 11

$$t_k = kT_0 + \tau_k\ ,\quad \tau_k = \hat{\tau}\cos(2\pi Q_s k),$$

where k is the revolution number and $\hat{\tau}$ the amplitude of the modulation. The current represented by this oscillating bunch is given in time domain by

$$I_k(t) = \sum_{k=-\infty}^{\infty} I(t - kT_0 - \tau_k) = \sum_{k=-\infty}^{\infty} I(t - kT_0 - \hat{\tau}\cos(2\pi Q_s k)). \tag{18}$$

This resembles much a phase oscillation and we expect a spectrum having side bands at $\pm\omega_0 Q_s$ of the revolution harmonics $p\omega_0$. However, here the modulation does not occur with respect to time t but to the turn number k. This makes a minute difference which could be neglected without much loss in accuracy. However, we will use here the correct treatment which will lead to a result being easier to compute.

We assume the oscillation to be small $\hat{\tau} \ll T_0$ and consider it as a perturbation making the approximation

$$I_k(t) = \sum_{k=-\infty}^{\infty} I(t - kT_0 - \tau_k) \approx \sum_{k=-\infty}^{\infty}\left(I(t - kT_0) - \frac{dI(t - kT_0)}{dt}\tau_k\right)$$

as illustrated on the top of Fig. 10. The form of this expression is not very useful for application. The current $I_k(t)$ is not periodic but it consists of a periodic function with a modulation and can be expressed as Fourier series giving a spectrum having lines at $p\omega_0$ with side bands at $\pm Q_s\omega_0$ around them. As mentioned before the modulation occurs not in time but with respects to turns k which makes the following calculation somewhat complicated.

To express the approximate equation describing an oscillating bunch in a more transparent way we need two properties of the Fourier transform.

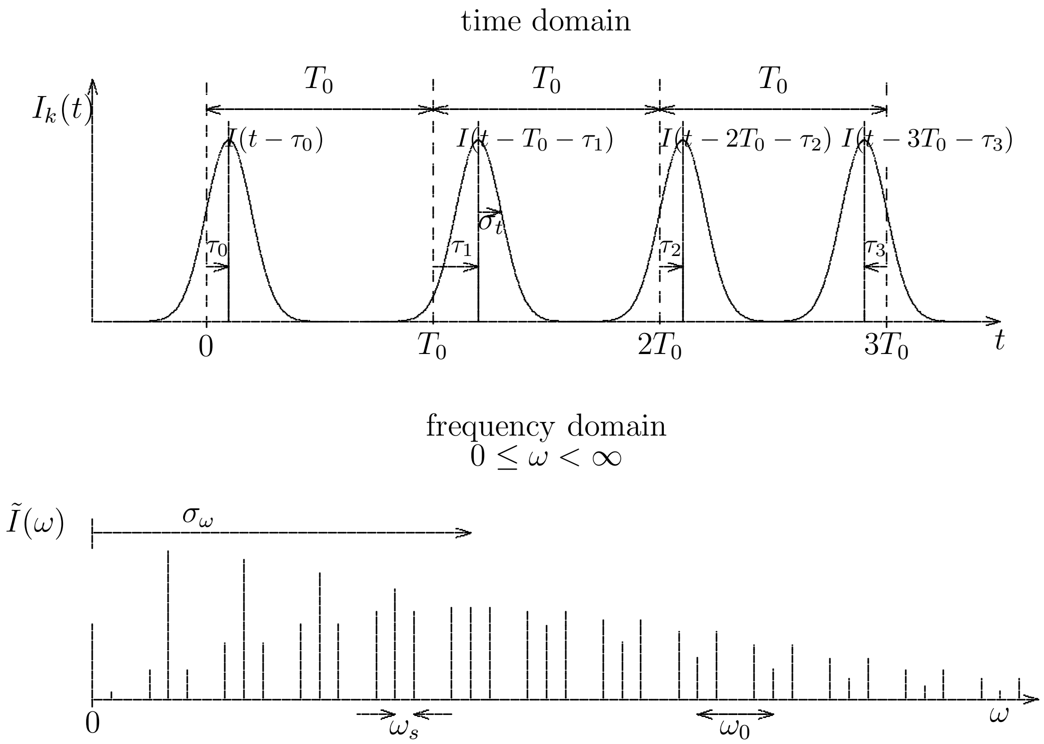


Figure 11: Oscillating bunch in time and frequency domain

First, the shift theorem relates the Fourier transform of a time delayed function to the one of the function itself

$$
\begin{aligned}
\tilde{f}(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \mathrm{e}^{-j\omega t} dt \\
\tilde{f}_\tau(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t-\tau) \mathrm{e}^{-j\omega t} dt = \frac{\mathrm{e}^{-j\omega\tau}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t-\tau) \mathrm{e}^{-j\omega(t-\tau)} d(t-\tau) = \mathrm{e}^{-j\omega\tau} \tilde{f}(\omega).
\end{aligned}
$$

The delay introduces a phase factor $\exp(-j\omega\tau)$.

Second, the Fourier transform of the time derivative of a function can be obtained with an integration by parts and related to the one of the function itself

$$
\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{dI(t)}{dt} \mathrm{e}^{-j\omega t} dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} U'V dt = \frac{1}{\sqrt{2\pi}} \left(UV|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} UV' \right) dt
$$

with $U'(t) = dI(t)/dt,\ \ U(t) = I(t),\ \ V(t) = \exp(-j\omega t),\ \ V'(t) = -j\omega \exp(-j\omega t)$. Using also $\dot{I}(\pm\infty) = 0$ we get for the Fourier transform of the derivative

$$
\frac{\widetilde{dI}}{dt}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{dI(t)}{dt} \mathrm{e}^{-j\omega t} dt = j\omega \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} I(t) \mathrm{e}^{-j\omega t} dt = j\omega \tilde{I}(\omega).
$$

With this we obtain the Fourier transform of the current (18) representing an oscillating bunch

$$
\begin{aligned}
\tilde{I}_k(\omega) &= \tilde{I}(\omega) \sum_{k=-\infty}^{\infty} \mathrm{e}^{-j\omega k T_0} \left(1 - j\omega\hat{\tau} \cos(2\pi Q_s k)\right) \\
&= \tilde{I}(\omega) \sum_{k=-\infty}^{\infty} \left[\mathrm{e}^{-j\omega k T_0} - j\frac{\omega\hat{\tau}}{2} \left(\mathrm{e}^{-jkT_0(\omega-\omega_s)} + \mathrm{e}^{-jkT_0(\omega+\omega_s)} \right) \right]
\end{aligned}
$$

with $\omega_s = \omega_0 Q_s$ being the synchrotron frequency. It consists of lines at revolution frequency harmonics $p\omega_0$ caused by the stationary bunch motion and of side bands caused by the bunch oscillation. This is expected since this oscillation resembles a phase modulation. The sums over exponentials appearing above ad up to infinite if the exponent is of the form $j2\pi n$, with n being an integer, and average out to zero otherwise. This leads to the relation between sum over $\exp(jkx)$ and a repetitive δ-function (comb function)

$$\sum_{k=-\infty}^{\infty} \mathrm{e}^{jkx} = \sum_{k=-\infty}^{\infty} \mathrm{e}^{-jkx} = 2\pi \sum_{p=-\infty}^{\infty} \delta(x - 2\pi p)$$

Using also the property $\delta(ax) = \delta(x)/a$ of the δ-function gives

$$\tilde{I}_k(\omega) = \omega_0 \tilde{I}(\omega) \sum_{p=-\infty}^{\infty} \left[\delta(\omega - p\omega_0) - j\frac{\omega\hat{\tau}}{2} \left(\delta(\omega - p\omega_0 - \omega_s) + \delta(\omega - p\omega_0 + \omega_s) \right) \right]. \quad (19)$$

We get this current in time domain by an inverse Fourier transform

$$I_k(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{I}_k(\omega) \mathrm{e}^{j\omega t} d\omega \quad (20)$$

giving the current of a rigid bunch oscillation

$$I_k(t) = \frac{\omega_0}{\sqrt{2\pi}} \sum_{p=-\infty}^{\infty} \left[\tilde{I}(p\omega_0) \mathrm{e}^{jp\omega_0 t} - \right.$$
$$\left. j\frac{\omega_0 \hat{\tau}}{2} \left((p - Q_s) \tilde{I}((p - Q_s)\omega_0) \mathrm{e}^{j(p-Q_s)\omega_0 t} + (p + Q_s) \tilde{I}((p + Q_s)\omega_0) \mathrm{e}^{j(p+Q_s)\omega_0 t} \right) \right].$$

To make the equation more compact we introduce the abbreviations for the frequencies, current components and impedances at the harmonics $p\omega_0$ and their side bands

$$\begin{array}{rclrcl} \omega_p & = & p\omega_0 & \omega_{(p\pm Q)} & = & (p \pm Q_s)\omega_0 \\ I_p & = & \frac{\omega_0}{\sqrt{2\pi}} \tilde{I}(p\omega_0) & I_{(p\pm Q)} & = & \frac{\omega_0}{\sqrt{2\pi}} \tilde{I}((p \pm Q)\omega_0) \\ Z_p & = & Z(p\omega_0) & Z_{(p\pm Q)} & = & Z((p + Q)\omega_0) \\ Z_{rp} & = & Z_r(p\omega_0) & Z_{r(p\pm Q)} & = & Z_r((p \pm Q)\omega_0) \\ Z_{ip} & = & Z_i(p\omega_0) & Z_{i(p\pm Q)} & = & Z_i((p \pm Q)\omega_0) \end{array} \quad (21)$$

This gives the current of the oscillating bunch in complex and, by combining terms with positive and negative values of p, with $\tilde{I}(\omega) = \tilde{I}(-\omega)$, also in real presentation

$$\begin{array}{rcl} I_k(t) & = & \sum_{p=-\infty}^{\infty} \left[I_p \mathrm{e}^{j\omega_p t} - j\frac{\hat{\tau}}{2} \left((p - Q)\omega_0 I_{(p-Q)} \mathrm{e}^{j(p-Q)\omega_0 t} + (p + Q)\omega_0 I_{(p+Q)} \mathrm{e}^{j(p+Q)\omega_0 t} \right) \right]. \\ I_k(t) & = & I_0 + 2 \sum_{\omega>0} \left[I_p \cos(p\omega_0 t) + \right. \\ & & \left. \frac{\omega_0 \hat{\tau}}{2} \left((p - Q_s) I_{(p-Q)} \sin((p - Q)\omega_0 t) + (p + Q_s) I_{(p+Q)} \sin((p + Q)\omega_0 t) \right) \right]. \end{array} \quad (22)$$

The latter spectrum is shown at the bottom of Fig. 11.

5.2 Voltage induced by an oscillating bunch

We calculate the voltage induced by the current $I_k(t)$ in an impedance $Z(\omega)$. The Fourier transform of this voltage is given by

$$\tilde{V}_k(\omega) = \tilde{I}_k(\omega)Z(\omega).$$

and the corresponding expression in time domain is obtained from (22) in complex and real notation

$$V_k(t) = \sum_{p=-\infty}^{\infty}\Big[I_pZ_p\mathrm{e}^{j\omega_pt} \tag{23}$$
$$-j\frac{\omega_0\hat{\tau}}{2}\Big((p-Q_s)I_{(p-Q)}Z_{(p-Q)}\mathrm{e}^{j(p-Q)\omega_0t} + (p+Q_s)I_{(p+Q)}Z_{(p+Q)}\mathrm{e}^{j(p+Q)\omega_0t}\Big)\Big]$$

$$V_k(t) = 2\sum_{\omega>0}\Big[I_p\left(Z_{rp}\cos(\omega_pt) - Z_{ip}\sin(\omega_pt)\right) + \tag{24}$$
$$\frac{\omega_0\hat{\tau}}{2}\Big((p-Q_s)I_{(p-Q)}Z_{r(p-Q)}\sin((p-Q)\omega_0t) + (p+Q_s)I_{(p+Q)}Z_{r(p+Q)}\sin((p+Q)\omega_0t) +$$
$$(p-Q_s)I_{(p-Q)}Z_{i(p-Q)}\cos((p-Q)\omega_0t) + (p+Q_s)I_{(p+Q)}Z_{i(p+Q)}\cos((p+Q)\omega_0t)\Big)\Big]$$

The real notation (24) can also been obtained from the complex one (23) by combining terms with positive and negative values of p and observing the symmetry relations $\tilde{I}(\omega) = \tilde{I}(-\omega)$, $Z_r(\omega) = Z_r(-\omega)$ and $Z_i(\omega) = -Z_i(-\omega)$. The current of an oscillating bunch and the voltage induced in the resistive and reactive part of a narrow band impedance are shown in Fig. 12 in frequency domain.

This voltage $V_k(t)$ has been induced in the impedance by the bunch current over many turns. We calculate now its effect on the bunch itself in a *single traversal* during turn k and calculate the resulting energy exchange ΔW of the whole rigid bunch

$$\Delta W = \int_{-\infty}^{\infty} I(t-kT_0-\tau_k)V_k(t)dt \approx \int_{-\infty}^{\infty}\left(I(t-kT_0)V_k(t) - \tau_k\frac{dI(t-kT_0)}{dt}V_k(t)\right)dt.$$

Since here voltage and current can be in phase or out of phase with respect to each other ΔW has to be understood as a generalized energy transfer which might contain a reactive part. We wrote this single traversal integral as one with infinite limits since due to the finite bunch length $I(t)$ vanishes outside an interval smaller than $\pm T_0/2$. For the voltage $V_k(t)$ we can use either the complex (23) or the real (24) notation. We chose the first and encounter integrals of the form

$$\int_{-\infty}^{\infty} I(t-kT_0)\mathrm{e}^{jp\omega_0t}dt = \sqrt{2\pi}\tilde{I}(-p\omega_0) = \frac{2\pi}{\omega_0}I_p$$
$$\int_{-\infty}^{\infty} I(t-kT_0)\mathrm{e}^{j(p-Q)\omega_0t}dt = \sqrt{2\pi}\mathrm{e}^{-jk2\pi Q_s)}\tilde{I}(-(p-Q)\omega_0) = \frac{2\pi}{\omega_0}\mathrm{e}^{-k2\pi Q_s}I_{(p-Q)}$$
$$\int_{-\infty}^{\infty} I(t-kT_0)\mathrm{e}^{j(p+Q)\omega_0t}dt = \sqrt{2\pi}\mathrm{e}^{jk2\pi Q_s)}\tilde{I}(-(p+Q)\omega_0) = \frac{2\pi}{\omega_0}\mathrm{e}^{jk2\pi Q_s}I_{(p+Q)}$$
$$\int_{-\infty}^{\infty} \frac{dI(t-kT_0)}{dt}\tau_k\mathrm{e}^{j\omega_pt}dt = -j\sqrt{2\pi}\omega_p\tau_k\tilde{I}(\omega_p) = -j\frac{2\pi}{\omega_0}\tau_k\omega_pI_p.$$

We neglect terms of higher order than linear in $\hat{\tau}$ or τ_k and get for the generalized energy exchange during the turn k

$$\Delta W = T_0\sum_{p=-\infty}^{\infty}\Big[|I_p|^2Z_{rp} - 2\tau_k\omega_p|I_p|^2Z_p + \tag{25}$$
$$j\frac{\omega_0\hat{\tau}_k}{2}\Big((p-Q_s)|I_{(p-Q)}|^2Z_{(p-Q)}\mathrm{e}^{-jk2\pi Q_s} + (p+Q_s)|I_{(p+Q)}|^2Z_{(p+Q)}\mathrm{e}^{jk2\pi Q_s}\Big)\Big]$$

Collecting terms with positive and negative values for p and satisfying the symmetry relation for the impedance we can express this in real notation

$$\begin{aligned}\Delta W &= 2T_0 \sum_{\omega>0} \Bigg[|I_p|^2 Z_{rp} - \frac{\omega_0 \hat{\tau}}{2} \Big(2p|I_p|^2 Z_{ip} \cos(2\pi Q_s) + \\ &\qquad \Big((p-Q_s)|I_{(p-Q)}|^2 Z_{i(p-Q)} + (p+Q_s)|I_{(p+Q)}|^2 Z_{i(p+Q)} \Big) \cos(2\pi Q_s) - \\ &\qquad \Big((p-Q_s)|I_{(p-Q)}|^2 Z_{r(p-Q)} - (p+Q_s)|I_{(p+Q)}|^2 Z_{r(p+Q)} \Big) \sin(2\pi Q_s) \Big) \Bigg] \end{aligned} \tag{26}$$

We started with a synchrotron motion expressed as a function of turns

$$\tau_k = \hat{\tau} \cos(2\pi Q_s k).$$

We make now a smooth approximation and express this motion as a function of time $2\pi Q_s k \approx \omega_s t$ with $\omega_s = \omega_0 Q_s$ and get for the synchrotron motion

$$\hat{\tau} \cos(2\pi Q_s k) \approx \hat{\tau} \cos(\omega_s t) = \tau(t) \quad , \quad \hat{\tau} \sin(2\pi Q_s k) \approx \hat{\tau} \sin(\omega_s t) = -\frac{\dot{\tau}(t)}{\omega_s}.$$

We divide the energy loss ΔW of the whole bunch by the total bunch charge $\bar{q} = T_0 I_0$ to get the average voltage per particle $\langle V \rangle$ due to the impedance

$$\begin{aligned}\langle V \rangle &= \frac{\Delta W}{T_0 I_0} = \frac{2}{I_0} \sum_{\omega>0} |I_p|^2 Z_{rp} \\ &\qquad - \frac{\dot{\tau}\omega_0}{\omega_s I_0} \sum_{\omega>0} \Big((p-Q)|I_{(p-Q)}|^2 Z_{r(p-Q)} - (p+Q)|I_{(p+Q)}|^2 Z_{r(p+Q)} \Big) \\ &\quad - \frac{\tau\omega_0}{I_0} \sum_{\omega>0} \Big[2pI_p^2 Z_{ip} - \Big((p-Q)|I_{(p-Q)}|^2 Z_{r(p-Q)} + (p+Q)|I_{(p+Q)}|^2 Z_{r(p+Q)} \Big) \Big] \\ &= \langle V \rangle_0 + \frac{\dot{\tau}\omega_0}{\omega_s} \langle V \rangle_r + \tau\omega_0 \langle V \rangle_i \end{aligned} \tag{27}$$

with

$$\begin{aligned}\langle V \rangle &= \frac{2}{I_0} \sum_{\omega>0} |I_p|^2 Z_{rp} = \frac{1}{I_0} \sum_{p=-\infty}^{\infty} |I_p|^2 Z_{rp} \\ \langle V \rangle_r &= -\frac{1}{I_0} \sum_{\omega>0} \Big((p-Q)|I_{(p-Q)}|^2 Z_{r(p-Q)} - (p+Q)|I_{(p+Q)}|^2 Z_{r(p+Q)} \Big) \\ &= \frac{1}{I_0} \sum_{p=-\infty}^{\infty} (p+Q)|I_{(p+Q)}|^2 Z_{r(p+Q)} \qquad (28) \\ \langle V \rangle_i &= \frac{1}{I_0} \sum_{\omega>0} \Big[-2p|I_p|^2 Z_{ip} + \Big((p-Q)|I_{(p-Q)}|^2 Z_{i(p-Q)} + (p+Q)|I_{(p+Q)}|^2 Z_{i(p+Q)} \Big) \Big] \\ &= \frac{1}{I_0} \sum_{p=-\infty}^{\infty} \Big[-p|I_p|^2 Z_{ip} + (p+Q)|I_{(p+Q)}|^2 Z_{i(p+Q)} \Big] \end{aligned}$$

This induced average voltage has a first term $\langle V \rangle_0$ which is independent of the oscillation and leads to an energy loss of the stationary bunch we treated before. The next term $\dot{\tau}\omega_0/\omega_s \langle V \rangle_r$ is proportional to $\dot{\tau}$, which leads to a growth or damping of the oscillation as will be shown later. The last term $\tau\omega_0\langle V \rangle_i$ is proportional to τ and can lead to a change of frequency. The first part of this term depends only on the impedance at the revolution harmonics $p\omega_0$. This voltage is induced by the stationary bunch and leads to an incoherent frequency shift we discussed before.

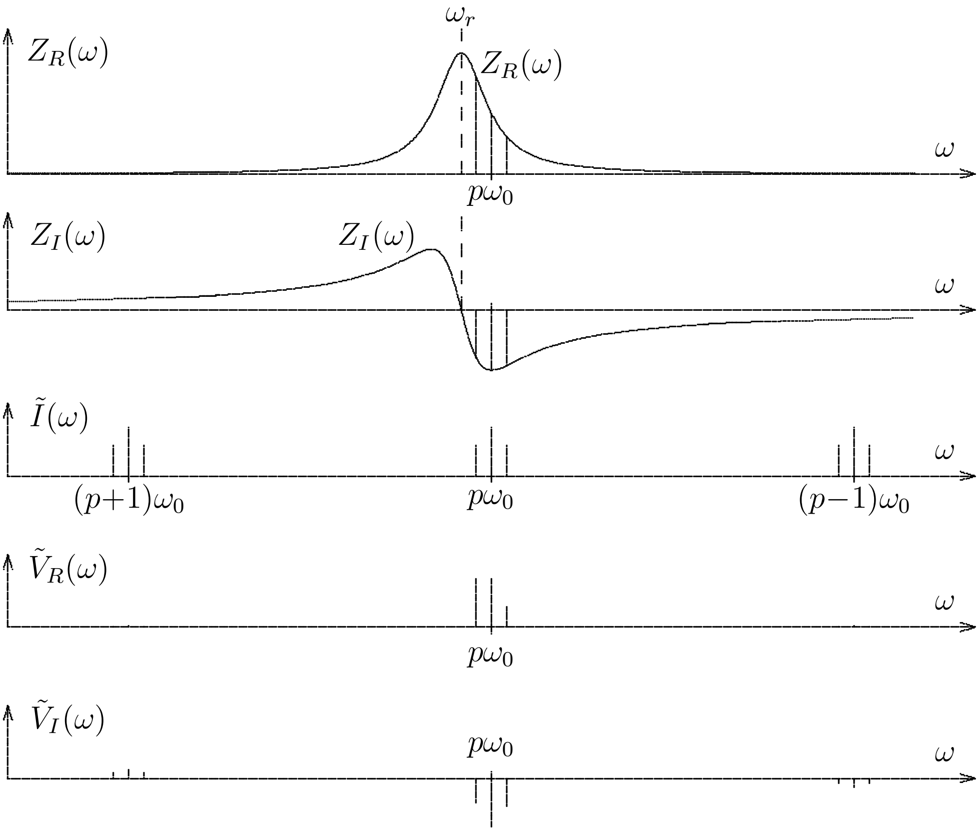


Figure 12: Voltage induced by an oscillating bunch in a narrow band impedance

5.3 Robinson instability due to a narrow cavity resonance

We consider now first the interaction of the oscillating bunch with a single cavity resonance which is sufficiently narrow such that only one revolution harmonic p with its side band pair induce a voltage as shown in Fig. 12. In this case the above equation does not contain a summation but only a single value for p.

The bunch executes a synchrotron oscillation which is approximately described as $\tau = \hat{\tau}\cos(\omega_s t)$ and produces side bands to the revolution frequency harmonics of the bunch. The average voltage

$$\langle V\rangle = \langle V\rangle_0 + \frac{\dot{\tau}\omega_0}{\omega_s}\langle V\rangle_r + \tau\omega_0\langle V\rangle_i$$

seen by the bunch, due to its interaction with this impedance, is now given by a single summation index p of the expression (27). It contributes to the energy loss of the particle in the bunch and we include this induced voltage in the equation (4) for the energy gain and loss.

$$\begin{aligned}
\dot{\epsilon} &= \frac{\omega_0 e\hat{V}\sin\phi_s}{2\pi E} + \frac{\omega_0^2 he\hat{V}\cos\phi_s}{2\pi E}\tau - \frac{\omega_0 e}{2\pi E}\langle V\rangle \\
\dot{\tau} &= \eta_c\epsilon.
\end{aligned}$$

Using the equilibrium condition

$$e\hat{V}\sin\phi_s = e\langle V\rangle_0 = \frac{2I_p^2 Z_{rp}}{I_0}$$

and combining the two equations gives

$$\ddot{\tau} = -\frac{\omega_0^2 \eta_c e}{2\pi E \omega_s} \langle V \rangle_r \dot{\tau} + \left(\frac{\omega_0^2 \eta_c h e \hat{V} \cos\phi_s}{2\pi E} - \frac{\omega_0^2 \eta_c e}{2\pi E} \langle V \rangle_i \right) \tau.$$

With the unperturbed synchrotron frequency ω_{s0}

$$\omega_{s0}^2 = -\omega_0^2 \frac{\eta_c h e \hat{V} \cos\phi_s}{2\pi E}$$

we get the second-order equation

$$\ddot{\tau} + \frac{\omega_{s0}}{2h\hat{V}\cos\phi_s} \langle V \rangle_r \dot{\tau} + \omega_{s0}^2 \left(1 - \frac{1}{h\hat{V}\cos\phi_s} \langle V \rangle_i \right) \tau = 0.$$

Its solution is an oscillation

$$\epsilon = \hat{\epsilon} e^{-\alpha_s t} \cos(\omega_s t + \phi)$$

with damping or growing rate α_s and frequency square ω_s^2

$$\alpha_s = \frac{\omega_{s0}}{2h\hat{V}\cos\phi_s} \langle V \rangle_r \,, \quad \omega_s^2 = \omega_{s0}^2 \left(1 - \frac{1}{h\hat{V}\cos\phi_s} \langle V \rangle_i \right). \tag{29}$$

We express the average voltage component $\langle V \rangle_r$ and $\langle V \rangle_i$ with their expressions (28) taking only a single value of the harmonics p and get

$$\alpha_s = \frac{\omega_{s0} \left((p+Q_s)|I_{(p+Q)}|^2 Z_{r(p+Q)} - (p-Q_s)|I_{(p-Q)}|^2 Z_{r(p-Q)} \right)}{2 I_0 h \hat{V} \cos\phi_s} \tag{30}$$

$$\omega_s^2 = \omega_{s0}^2 \left(1 + \frac{2p I_p^2 Z_{ip}}{I_0 h \hat{V} \cos\phi_s} - \frac{\left((p+Q_s)|I_{(p+Q)}|^2 Z_{i(p+Q)} + (p-Q_s)|I_{(p-Q)}|^2 Z_{i(p-Q)} \right)}{I_0 h \hat{V} \cos\phi_s} \right).$$

The growth rate of the Robinson instability is given by the difference of the resistive impedance at the upper and lower synchrotron side band, Fig. 13. Above transition energy we have $\cos\phi_s < 0$ and $\alpha_s > 0$, i.e. stability if $Z_{r(p-Q)} > Z_{r(p+Q)}$ as we found already from qualitative arguments.

The RF-cavity itself has a narrow-band impedance around $h\omega_0$ which can drive an instability. Since the bunch length is usually much shorter than the RF wavelength we have $I_{(p+Q)} \approx I_{(p-Q)} \approx I_p = I_h \approx I_0$ so that

$$\alpha_s \approx \frac{\omega_{s0} I_0 (Z_{r(p+Q)} - Z_{r(p-Q)})}{2\hat{V}\cos\phi_s}.$$

The shifted synchrotron frequency shift (30), due to the reactive part of the impedance, has a second term which only depends on the impedance at the revolution harmonic $p\omega_0$ and not on the one at the side bands. It is present also in the absence of a coherent motion and produces a change of the incoherent synchrotron frequency which will be discussed later in more detail

$$\omega_{si}^2 = \omega_{s0}^2 \left(1 + \frac{2p|I_p|^2 Z_p}{I_0 h \hat{V} \cos\phi_s} \right).$$

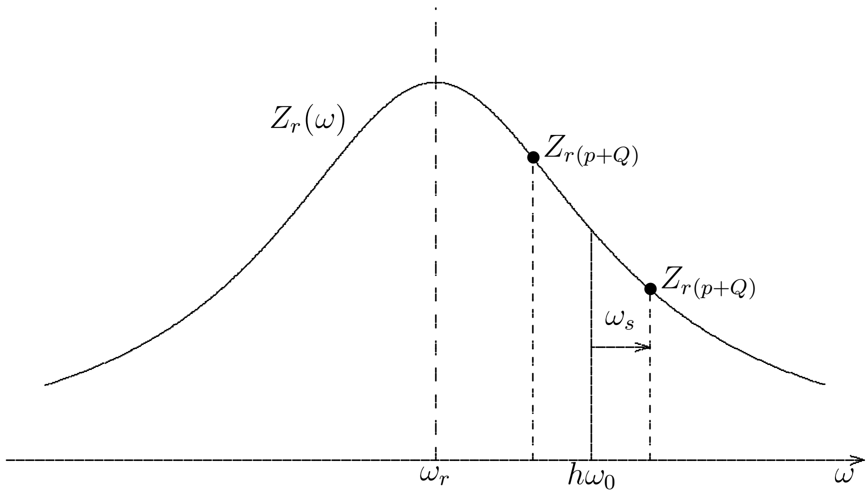


Figure 13: Quantitative treatment of the Robinson instability

The coherent synchrotron motion produces a further shift compared to ω_{si}

$$\omega_s^2 = \left(\omega_{si}^2 - \omega_{s0}^2 \frac{\Big((p+Q_s)|I_{(p+Q)}|^2 Z_{i(p+Q)} + (p-Q_s)|I_{(p-Q)}|^2 Z_{i(p-Q)}\Big)}{I_0 h\hat{V}\cos\phi_s}\right).$$

For a small effect, the shift of the coherent frequency with respect to the incoherent one $\Delta\omega_r = \omega_s - \omega_{si}$ is given by

$$\frac{\Delta\omega_s}{\omega_{s0}} \approx -\frac{\Big((p+Q_s)|I_{(p+Q)}|^2 Z_{i(p+Q)} + (p-Q_s)|I_{(p-Q)}|^2 Z_{i(p-Q)}\Big)}{2I_0 h\hat{V}\cos\phi_s}.$$

5.4 General impedance

So far we assumed a narrow, resonant type impedance which covers the side band pair $(p \pm Q_s)\omega_0$ of a single harmonic p of the revolution frequency. If the impedance is more broad it can cover several side band pairs as shown in Fig. 14. The oscillating bunch induces now voltages in each such side band which have to be included to calculate their effect on the bunch. The growth rate and frequency are now obtained from (29) by using the complete expression (28) for the average voltage $\langle V\rangle_r$.

This gives the growth (or damping) rate of the instability containing a kind of convolution between power spectrum and impedance expressed with positive frequencies only and both side bands

$$\alpha_s = \frac{\omega_0 Q_{s0}}{2hI_0\hat{V}\cos\phi_s}\sum_{\omega>0}\Big((p+Q_s)|I_{(p+Q)}|^2 Z_{(p+Q)} - (p-Q_s)|I_{(p-Q)}|^2 Z_{(p-Q)}\Big)$$

as shown in Fig. 14, or with positive and negative frequencies but only upper side bands

$$\begin{aligned}\alpha_s &= \frac{\omega_0 Q_{s0}}{2hI_0\hat{V}\cos\phi_s}\sum_{p=-\infty}^{\infty}(p+Q_s)|I_{(p+Q)}|^2 Z_{r(p+Q)}) \qquad (31)\\ &= \frac{\omega_0^2 Q_{s0}}{4\pi hI_0\hat{V}\cos\phi_s}\sum_{p=-\infty}^{\infty}(p+Q_s)\omega_0\left|\tilde{I}((p+Q_s)\omega_0)\right|^2 Z_r(\omega_0(p+Q_s)\omega_0)\end{aligned}$$

as shown in Fig. 15.

Since the growth rate depends on the difference in resistive impedance between the upper and lower side band, a smooth broad band impedance will not result in a strong instability. This is consistent with the time domain picture which demands a memory of the fields between bunch passages

We also find the synchrotron frequency for this broad band impedance by using the summation in (30)

$$\omega_s^2 = \omega_{si}^2 - \frac{\omega_{s0}^2}{I_0 h \hat{V}\cos\phi_s}\sum_{p=1}^{\infty}\Big((p+Q_s)|I_{(p+Q)}|^2 Z_{i(p+Q)} + (p-Q_s)|I_{(p-Q)}|^2 Z_{i(p-Q)}\Big) \quad (32)$$

with the incoherent synchrotron frequency ω_{si} now given by

$$\omega_{si}^2 = \omega_{s0}^2\left(1+\frac{1}{I_0 h\hat{V}\cos\phi_s}\sum_{p=-\infty}^{\infty} p|I_p|^2 Z_{ip}\right)\ ,\quad \frac{\Delta\omega_{si}}{\omega_{s0}} \approx \frac{1}{2I_0 h\hat{V}\cos\phi_s}\sum_{p=-\infty}^{\infty} p|I_p|^2 Z_{ip}. \quad (33)$$

It should be noted that this incoherent frequency was derived before (16) for a stationary bunch in the presence of a reactive impedance.

Assuming a small effect due to the impedance we get for the coherent synchrotron frequency shift $\Delta\omega_r = \omega_s - \omega_{si}$

$$\begin{aligned}\Delta\omega_r &= -\frac{\omega_0 Q_{s0}}{2I_0 h\hat{V}\cos\phi_s}\sum_{\omega>0}\Big((p+Q_s)|I_{(p+Q)}|^2 Z_{i(p+Q)} + (p-Q_s)|I_{(p-Q)}|^2 Z_{i(p-Q)}\Big)\\ &= -\frac{\omega_0 Q_{s0}}{2I_0 h\hat{V}\cos\phi_s}\sum_{p=-\infty}^{\infty}(p+Q_s)|I_{(p+Q)}|^2 Z_{i(p+Q)} \qquad (34)\\ &= -\frac{\omega_0^2 Q_{s0}}{4\pi h I_0\hat{V}\cos\phi_s}\sum_{p=-\infty}^{\infty}(p+Q_s)\omega_0\left|\tilde{I}((p+Q_s)\omega_0)\right|^2 Z_i((p+Q_s)\omega_0)\end{aligned}$$

A broad band impedance changes little between the side bands and we can approximate $Z_{i(p-Q)} \approx Z_{i(p+Q)} \approx Z_{ip}$. Furthermore, also the current components are about the same at these three frequencies $I_{(p-Q)} \approx I_{(p+Q)} \approx I_p$ and $Q_s \ll p$. In this case we can approximate the coherent frequency shift $\Delta\omega_r$

$$\frac{\Delta\omega_r}{\omega_{s0}} \approx -\frac{1}{2I_0 h\hat{V}\cos\phi_s}\sum_{p=-\infty}^{\infty} p|I_p|^2 Z_{ip} \approx -\frac{\Delta\omega_{si}}{\omega_{s0}}. \quad (35)$$

For a broad band reactive impedance the incoherent and coherent frequency shift are of opposite sign but of about the same magnitude. This results in a coherent frequency being not or only little different from the unperturbed one, $\omega_s \approx \omega_{s0}$, but in a separation between coherent and incoherent frequencies.

5.5 Complex notation

Some times the growth rate α_s given in (31) and coherent frequency shift $\Delta\omega_r$ given in (34) are combined into a complex frequency shift

$$\Delta\omega = \Delta\omega_r + j\alpha_s = \frac{\omega_{s0}}{2I_0 h\hat{V}\cos\phi_s}\sum_{p=-\infty}^{\infty}(p+Q_s)|I_{(p+Q)}|^2(jZ_{r(p+Q)} - jZ_{i(p+Q)})$$

to obtain a more compact formula. This frequency shift is put into the general solution of the synchrotron oscillation in the presence of an impedance

$$\tau(t) = \hat{\tau}e^{j\omega t} = \hat{\tau}e^{j(\omega_{si}+\Delta\omega)t} = \hat{\tau}e^{j(\omega_{si}+\Delta\omega_r+j\alpha_s)t}.$$

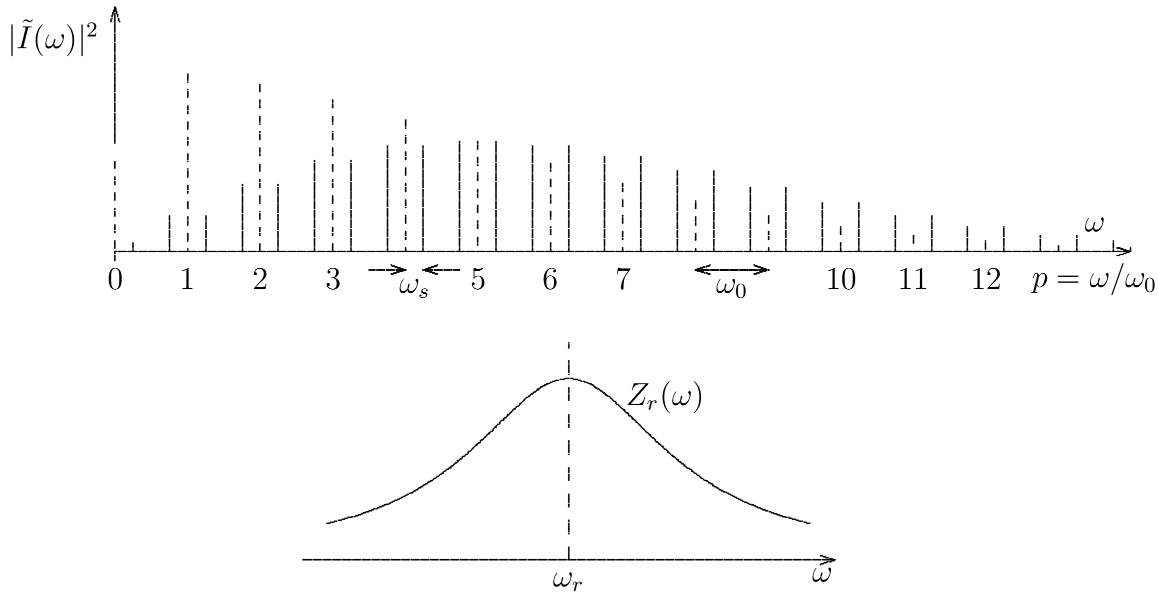


Figure 14: Convolution of power spectrum and general impedance using positive frequencies with upper and lower side bands

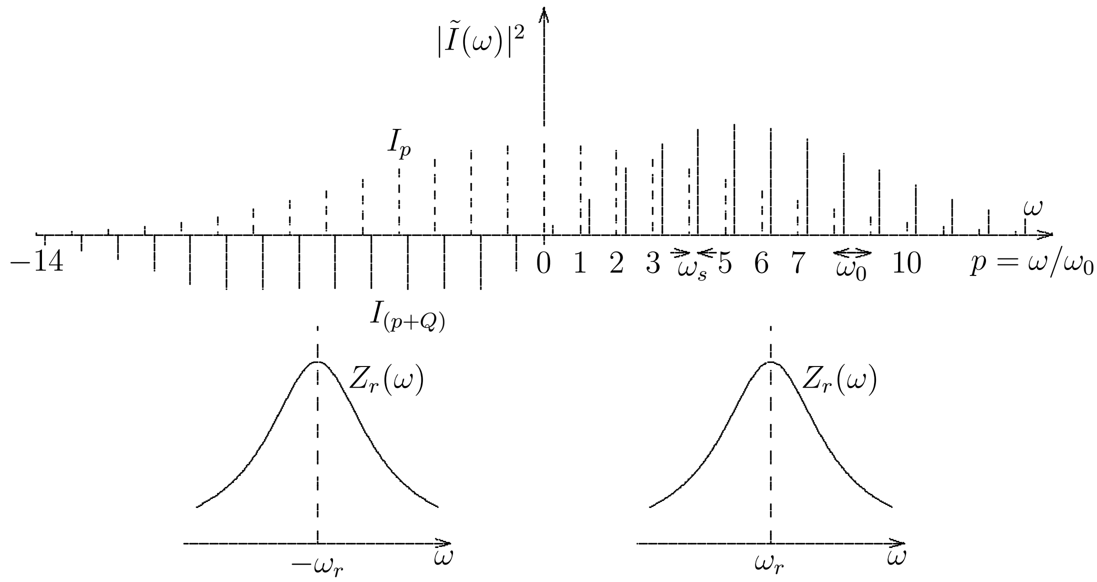


Figure 15: Convolution of power spectrum and a general impedance using positive and negative frequencies with upper side bands only

Combining the solution for positive and negative frequencies $e^{\pm\omega t}$ we get

$$\tau(t) = \hat{\tau} e^{-\alpha_s t} \cos((\omega_{si} + \Delta\omega_r)t).$$

Using also the complex impedance $Z = Z_r + jZ_i$ we can express the complex frequency shift

$$\Delta\omega = \Delta\omega_r + j\alpha_s = j\frac{\omega_{s0}}{2I_0 h\hat{V}\cos\phi_s}\sum_{p=-\infty}^{\infty}(p+Q_s)|I_{(p+Q)}|^2 Z_{(p+Q)}$$

which contains growth rate and coherent frequency shift in a compact form.

5.6 Many bunches

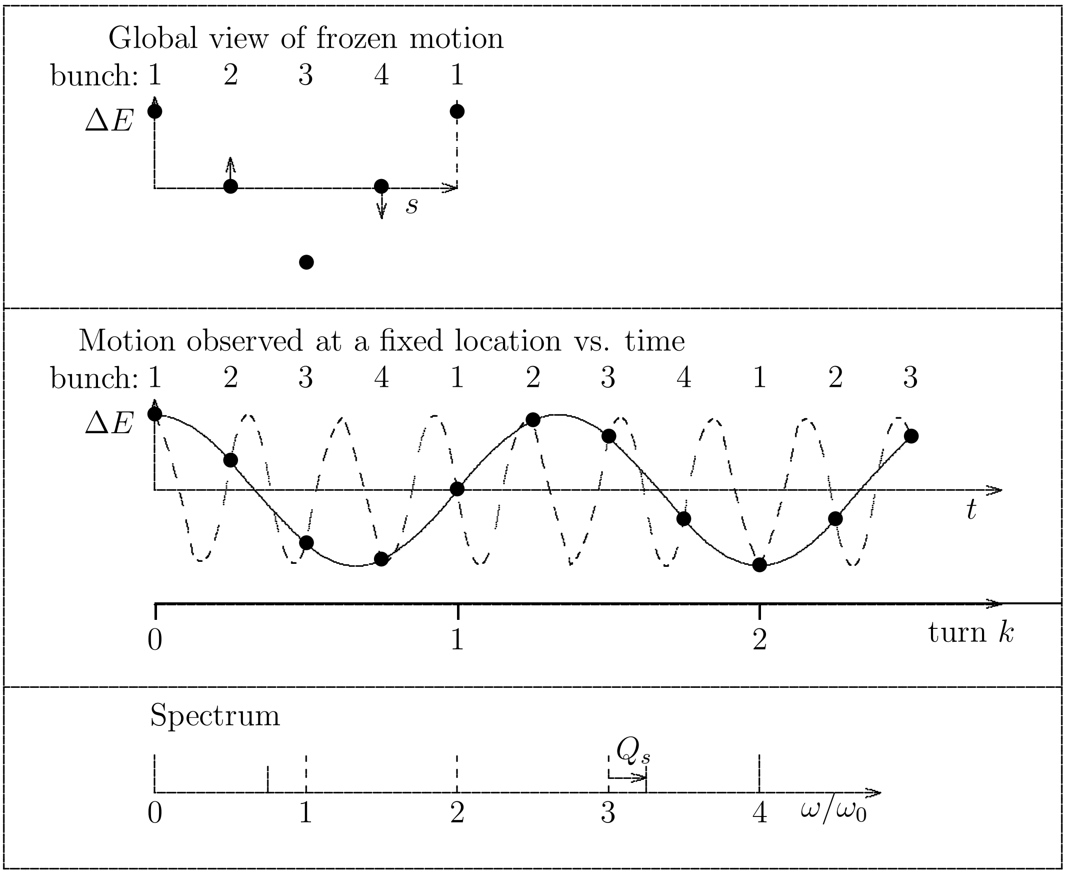


Figure 16: Robinson instability extended to many bunches

With M circulating, equidistant bunches there are M independent modes of coupled bunch oscillations, labeled with $0 \leq n \leq M-1$ being related to the oscillation phase difference $\Delta\phi$ between adjacent bunches $n = \Delta\phi/(2\pi M)$. Each mode n has one pair of side bands in each frequency range of $M\omega_0$

$$\omega_{(p\pm n,Q)} = \omega_0(pM \pm (n + Q_s))$$

The growth rate of each mode n is given by a sum over the impedance differences of each side band pair, [4, 5].

$$\alpha_s = \frac{\omega_s}{2hI_0\hat{V}\cos\phi_s\omega_0}\sum_p\Big(\omega_{(p+n,Q)}I^2_{(p+n,Q)}Z_{r(p+n,Q)} - \omega_{(p-n,Q)}I^2_{p-n,Q)}Z_{r((p-n,Q))}\Big)\,.$$

In contrast to the one bunch case, the side bands of a multi-bunch mode n can be separated by more than a revolution frequency. Even for a relatively broad band impedance there can now be a significant difference in impedance at these frequencies resulting in a large growth rate. Again, this is consistent with the time domain picture that the memory of the impedance has to last now only for a bunch spacing and not for a revolution time. This is illustrated at the bottom of Fig. 16 where the values $M = 4$, $n = 1$ and $Q_s = \omega_s/\omega_0 = 0.25$ have been chosen as example. It is interesting that for the modes $n = 0$ and $n = 2$ the side bands are close together making an instability less likely.

5.7 Bunch shape oscillations

So far, we considered only dipole oscillations where the bunch makes a rigid oscillation around the nominal phase without changing the form. There are higher modes of oscillation, called bunch-shape oscillations, which can be classified as quadrupole ($m = 2$), sextupole ($m = 3$), octupole ($m = 4$), etc. modes with frequencies

$$\omega_{p\pm} = \omega_0(pM \pm (n + mQ_s)).$$

Each mode has a spectrum with side bands at a distance $m\omega_s$ from the revolution harmonics. Again, to calculate the stability of these modes we have to sum the products of impedance times the square of the current components over these side bands.

5.8 Further generalization of the Robinson instability

We have assumed that the effect of the impedance is relatively weak such that the changes in synchrotron frequency and growth rate are small compared to the synchrotron frequency itself. For very narrow-band cavities with high shunt impedance, e.g. superconducting cavities, this might no longer be true. In this case we have to evaluate the impedance not at the unperturbed side band ω_{s0} but at the shifted synchrotron frequency ω_s. Furthermore, if we are interested in the growth rate we have to consider the cavity impedance for a growing oscillation which is different as soon as the growth time of the oscillation becomes comparable to the filling time of the cavity. Taking this into account one arrives at a 4th-order equation for the frequency shift and growth rate resulting in a more general stability criterion, often called the second Robinson criterion [2].

We have considered stability only for the case of an infinitesimally small oscillation and we have calculated its initial growth or damping time. If, however, the oscillation amplitude becomes large, some non-linear effects should be included. The modulation index of the phase oscillation will become large leading to side bands at twice the synchrotron frequency. They have to be included in the sum over the impedance contributions. This can lead to a situation where the beam is unstable for small oscillation amplitudes but becomes stable again at large amplitudes. In practice, such cases have bunches oscillating with finite but more or less constant amplitudes [6, 7].

6 BUNCH LENGTHENING

6.1 Broadband impedance

A ring impedance consists often of many resonances with frequencies ω_r, shunt impedance R_s and quality factors Q. At low frequencies, $\omega < \omega_r$, their impedances are mainly inductive

$$Z(\omega) = R_s \frac{1 - jQ\frac{\omega^2-\omega_r^2}{\omega\omega_r}}{1 + \left(Q\frac{\omega^2-\omega_r^2}{\omega\omega_r}\right)^2} \approx j\frac{R_s\omega}{Q\omega_r} + ...$$

The sum impedance at low frequencies of all these resonances divided by the mode number $n = \omega/\omega_0$ is called

$$\left|\frac{Z}{n}\right|_0 = \sum_k \frac{R_{sk}\omega_0}{Q_k\omega_{rk}} = L\omega_0.$$

with L being the inductance.

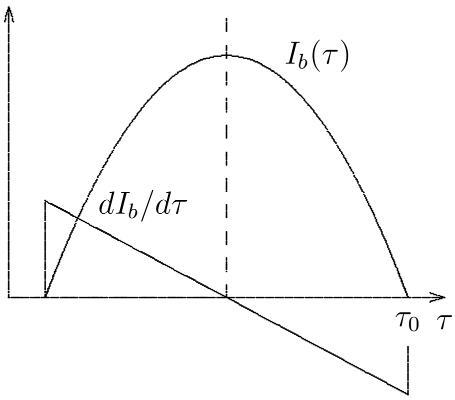


Figure 17: Current and its derivative of a parabolic bunch

6.2 Synchrotron frequency shift and potential well lengthening of a parabolic bunch

A bunch with current $I_b(t)$ induces a voltage $V_i = -LdI_b/dt$ which is added to the RF-voltage

$$V(t) = \hat{V}\sin(h\omega_0 t) - L\frac{dI_b}{dt}.$$

Developing around t_s, calling $\tau = t - t_s$, $\phi_s = h\omega_0 t_s$ and using a parabolic bunch, shown in Fig. 17, of half length τ at the base, average current I_0, peak current $\hat{I}$ of the form

$$I_b(\tau) = \hat{I}\left(1 - \frac{\tau^2}{\tau_0^2}\right) = \frac{3\pi I_0}{2\omega_0\tau_0}\left(1 - \frac{\tau^2}{\tau_0^2}\right)\ ,\quad \frac{dI_b}{d\tau} = -\frac{3\pi I_0 \tau}{\omega_0\tau_0^3}$$

and Fourier transform

$$\tilde{I}(\omega) = \frac{6\pi I_0}{\sqrt{2\pi}\omega_0}\frac{\sin(\tau_0\omega) - \tau_0\omega\cos(\tau_0\omega)}{(\tau_0\omega)^3}.$$

The total voltage becomes

$$V(\tau) = \hat{V}\sin\phi_s + \hat{V}\cos\phi_s h\omega_0\tau\left(1 + \frac{3\pi|Z/n|_0 I_0}{h\hat{V}\cos\phi_s(\omega_0\tau_0)^3}\right).$$

It has a linear dependence on τ and leads to a new synchrotron frequency given by

$$\omega_s^2 = \omega_0^2\frac{h\eta_c\hat{V}\cos\phi_s}{2\pi E}\left(1 + \frac{3\pi|Z/n|_0 I_0}{h\hat{V}\cos\phi_s(\omega_0\tau)^3}\right) = \omega_{s0}^2\left(1 + \frac{3\pi|Z/n|_0 I_0}{h\hat{V}\cos\phi_s(\omega_0\tau)^3}\right). \qquad (36)$$

Assuming a small change of the synchrotron frequency $\omega_s = \omega_{s0} + \Delta\omega_s$ we make a linear approximation to the above equation

$$\frac{\Delta\omega_s}{\omega_{s0}} \approx \frac{3\pi|Z/n|_0 I_0}{2h\hat{V}\cos\phi_s(\omega_0\tau_0)^3}. \qquad (37)$$

Above transition energy, $\cos\phi_s < 0$, the inductive impedance reduces the synchrotron frequency of the particles inside the bunch; below transition energy, $\cos\phi_s > 0$, this frequency is increased.

We compare this result with the incoherent frequency shift obtained earlier with the Robinson formalism (33). We use the relation $I_p = \omega_0 \tilde{I}(p\omega_0)/\sqrt{2\pi}$, replace the line spectrum with a continuous one and the sum by an integral

$$\frac{\Delta\omega_{si}}{\omega_{s0}} = \frac{1}{2I_0 h\hat{V}\cos\phi_s}\sum_{p=-\infty}^{\infty} p|I_p|^2 Z_{ip} \approx \frac{1}{4\pi I_0 h\hat{V}\cos\phi_s}\int_{-\infty}^{\infty}\omega|\tilde{I}(\omega)|^2 Z_i(\omega)d\omega.$$

We assume an inductive impedance which can be expressed as $Z_i(\omega) = \omega L = |Z/n|\omega/\omega_0$ and get for the frequency shift

$$\frac{\Delta\omega_{si}}{\omega_{s0}} = \frac{|Z/n|}{4\pi\omega_0 I_0 h\hat{V}\cos\phi_s}\int_{-\infty}^{\infty}\omega^2|\tilde{I}(\omega)|^2 d\omega.$$

Using also the expression for $\tilde{I}(\omega)$ we get

$$\frac{\Delta\omega_{si}}{\omega_{s0}} = \frac{9I_0|Z/n|}{2h\hat{V}\cos\phi_s(\omega_0\tau_0)^3}\int_{-\infty}^{\infty}\frac{\sin(\tau_0\omega) - \tau_0\omega\cos(\tau_0\omega)}{(\tau_0\omega)^3}d(\tau_0\omega) = \frac{3\pi I_0|Z/n|}{2h\hat{V}\cos\phi_s(\omega_0\tau_0)^3}$$

which agrees with (37).

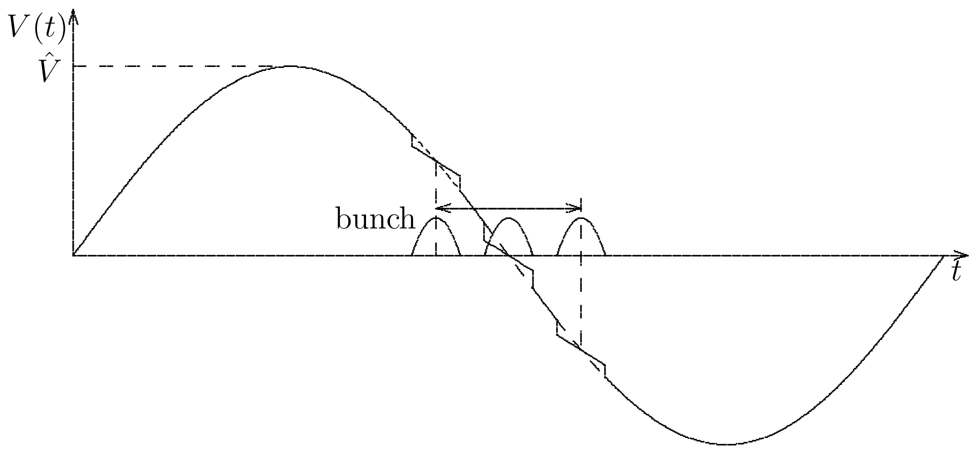


Figure 18: Vanishing frequency shift of a coherent bunch oscillation

The above frequency change (decrease for $\gamma > \gamma_T$, increase for $\gamma < \gamma_T$) applies only to the incoherent motion of individual particles. The coherent dipole (rigid bunch) mode is not affected since it carries the induced voltage with it, as shown in Fig. 18. This separates the coherent synchrotron frequency from the incoherent distribution and leads to a loss of Landau damping. We found this result already before, (35), but here we give a more physical explanation for it.

The reduction of longitudinal focusing leads also to a change of the bunch length. For protons, with negligible emitted synchrotron radiation, the phase space area is conserved, $\sigma_s\sigma_\epsilon =$ constant. This gives a relation between the change in bunch length and synchrotron frequency and a linearized bunch lengthening (37)

$$\frac{\tau_0}{\tau_{00}} = \sqrt{\frac{\omega_{s0}}{\omega_s}}, \quad \frac{\Delta\tau_0}{\tau_{00}} \approx -\frac{3\pi|Z/n|_0 I_0}{4h\hat{V}\cos\phi_s(\omega_0\tau_0)^3}.$$

For electrons, the energy spread is determined and fixed by synchrotron radiation, leading to the corresponding relations

$$\frac{\tau_0}{\tau_{00}} = \frac{\omega_{s0}}{\omega_s}, \quad \frac{\Delta\tau_0}{\tau_{00}} \approx -\frac{3\pi|Z/n|_0 I_0}{2h\hat{V}\cos\phi_s(\omega_0\tau_0)^3}.$$

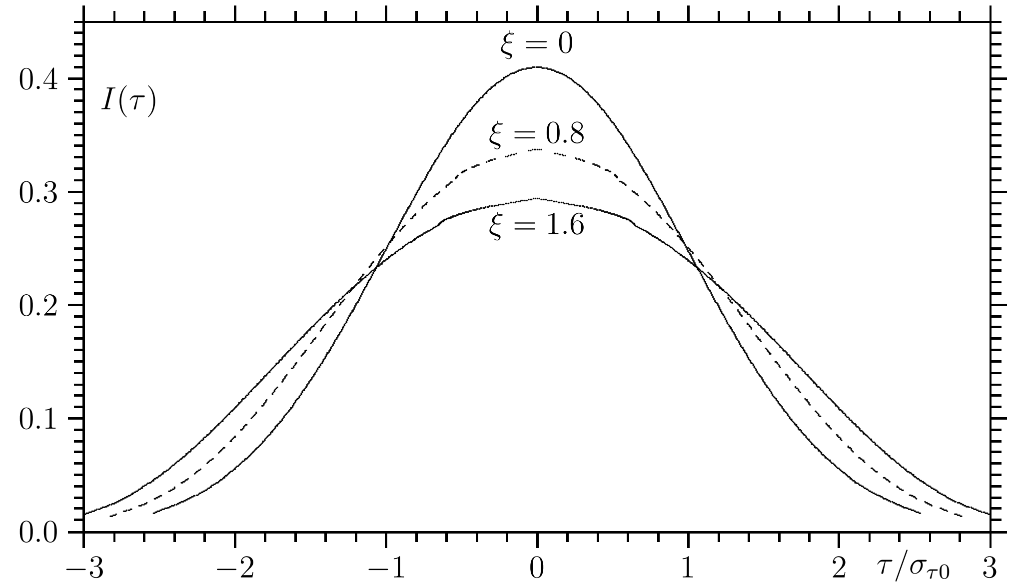


Figure 19: Potential well lengthening of a bunch with Gaussian energy spread

If the effect is stronger we have to go back to the accurate expression (36) for the change of synchrotron frequency which leads to a 4th-order equation for protons

$$\left(\frac{\tau_0}{\tau_{00}}\right)^4 + \frac{3\pi|Z/n|_0 I_0}{h\hat{V}\cos\phi_s(\omega_0\tau_{00})^3}\left(\frac{\tau_0}{\tau_{00}}\right) - 1 = 0$$

and to a 3rd-order expression for electrons

$$\left(\frac{\tau_0}{\tau_{00}}\right)^3 - \frac{\tau_0}{\tau_{00}} + \frac{3\pi|Z/n|_0 I_0}{h\hat{V}\cos\phi_s(\omega_0\tau_{00})^3} = 0$$

6.3 Potential well lengthening of a bunch with a Gaussian energy distribution

The above bunch lengthening expressions are based on a parabolic bunch form and are therefore only approximations for electrons which have Gaussian bunches at vanishing impedance. An inductance leads to bunch lengthening but contrary to a parabolic bunch the Gaussian form is altered. A self consistent distribution for electron bunches with a Gaussian energy distribution can be obtained [8] leading to an implicit and transcendent equation

$$I(\phi)e^{\xi I(\phi)/I_0(0)} = I(0)e^{\xi I(0)/I_0(0)}e^{-\phi^2/2\sigma_{\phi 0}^2} \qquad (38)$$

where $\sigma_{\phi 0}$ is the RMS bunch length expressed in RF-phase and $I_0(0)$ the peak current, both in the absence of impedance, while ξ is a parameter giving the strength of the effect

$$\sigma_{\phi 0} = \left(\frac{Q_s}{\alpha_c h}\right)^2 \sigma_\epsilon\ , \quad I_0(0) = \frac{\sqrt{2\pi}hI_0}{\sigma_{\phi 0}}\ , \quad \xi = \frac{\sqrt{2\pi}h^2 I_0|Z/n|}{\hat{V}\cos\phi_s\sigma_{\phi 0}^3}.$$

The above equation (38) determines the self-consistent current distribution $I(\phi)$ for a Gaussian energy distribution in the presence of an inductive impedance. It has to be solved numerically and the bunch form is plotted in Fig. 19 for 3 values of the strength parameter ξ.

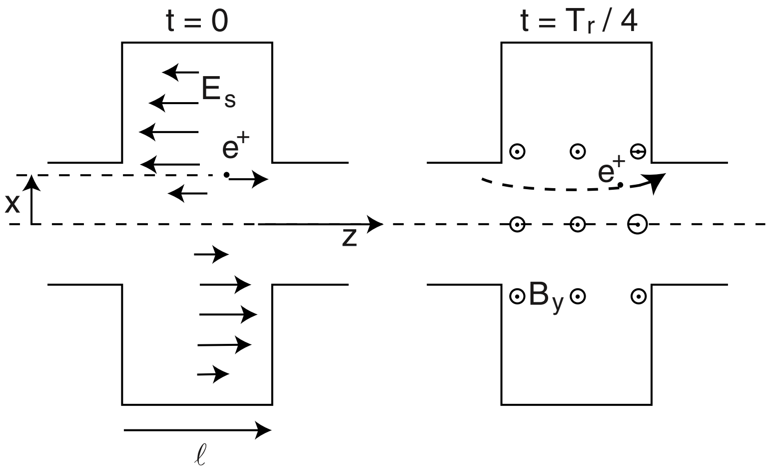


Figure 20: Transverse impedance

7 TRANSVERSE INSTABILITIES

7.1 Transverse impedance

A transverse impedance is excited by the longitudinal bunch motion and produces a deflection field. It is illustrated in Fig. 20 where a positive charge e^+ goes through a cavity of resonant frequency ω and excites a mode (dipole mode) having a longitudinal field E_z with a transverse gradient $\partial E_z/\partial x$. Since E_z vanishes on axis this dipole mode is only excited by a bunch with a transverse off-set giving a dipole moment $I_b \Delta x$. After 1/4 oscillation the longitudinal electric field E_z is converted into a transverse magnetic field B_y which deflects the beam in the x-direction. Maxwell's equation in differential and integral form

$$\dot{\vec{B}} = -\mathrm{curl}\vec{E}\ ,\ \ \int \dot{\vec{B}} d\vec{a} = -\oint \vec{E} d\vec{s}$$

gives

$$E = E_z = \frac{\partial \hat{E}}{\partial x} x \cos(\omega t)\ \rightarrow\ B = B_y = \frac{1}{\omega}\frac{\partial \hat{E}}{\partial x}\sin(\omega t)$$

To describe a general deflecting field we define a transverse impedance, Z_T or $Z_\perp$, in analogy to the longitudinal one [1]

$$Z_T(\omega) = j\frac{\int\left(\vec{E}(\omega) + [\vec{v}\times\vec{B}(\omega)]\right)_T ds}{Ix(\omega)} = -\frac{\omega\int\left(\vec{E}(\omega) + [\vec{v}\times\vec{B}(\omega)]\right)_T ds}{I\dot{x}(\omega)}.$$

using

$$x = \hat{x}e^{j\omega t}\ ,\ \ \dot{x} = j\omega\hat{x}e^{j\omega t}.$$

The presentation of the impedance definition on the left relates the deflecting field to an exciting dipole moment. If the two, the transverse excursion and force, are in phase there is no energy transfer to the transverse motion, therefore the factor 'j' in front. However, if deflecting field and transverse velocity are in phase there is energy transfer which is made clear in the second presentation on the right.

In our cavity mode the dipole moment Ix induces first a longitudinal field which indicates that the dipole mode has also a longitudinal impedance Z_L. An excitation at a distance x_0 gives a gradient of $\partial E_z/\partial x$ which is related to Ix_0 by a factor k

$$\frac{\partial E_z}{\partial x} = kIx_0 \text{ and } E_z(x) = \frac{\partial E_z}{\partial}x = kIx_0x\ \ ,\ \ E_z(x_0) = kIx_0^2.$$

The longitudinal impedance of this mode is

$$Z_L(x_0) = -\frac{\int E_z(x_0)dz}{I} = kx_0^2\ell$$

ℓ is the cavity length. With Maxwell's equation

$$\int \dot{\vec{B}} d\vec{a} = -\oint \vec{E} d\vec{s}$$

we obtain a relation between the electric field gradient and the deflecting magnetic field it is transformed into

$$\dot{B}_y x\ell = -x\ell \frac{\partial E_z}{\partial x}.$$

With $I(t) = \hat{I}e^{j\omega t}$ we get for the fields

$$\dot{B}_y(t) = \hat{B}j\omega e^{j\omega t} = -\frac{\partial E_z(t)}{\partial x} = -\frac{\partial \hat{E}_z}{\partial x}e^{j\omega t}\ , \quad B_y(t) = B_y e^{j\omega t} = j\frac{1}{\omega}\frac{\partial E_z(t)}{\partial x} = j\frac{1}{\omega}\frac{\partial \hat{E}_z}{\partial x}e^{j\omega t}.$$

With this we have a relation between the electric field gradient and the deflecting magnetic field which can be applied to the two impedances

$$Z_T(\omega) = j\frac{\int \left(\vec{E}(\omega) + [\vec{v}\times\vec{B}(\omega)]\right)_T ds}{Ix(\omega)} = -j\frac{B_y c\ell}{Ix_0} = \frac{c}{\omega}k\ell = \frac{2c}{\omega}\frac{d^2 Z_L}{dx^2}.$$

Our transverse impedance is related to the second derivative of the longitudinal belonging to the *same mode*. From this we get the symmetry relations

longitudinal	:	$Z_r(-\omega)$	$=$	$Z_r(\omega)$	$Z_i(-\omega)$	$=$	$-Z_i(\omega)$
transverse	:	$Z_{Tr}(-\omega)$	$=$	$-Z_{Tr}(\omega)$	$Z_{Ti}(-\omega)$	$=$	$Z_{Ti}(\omega)$

While the above accurate relation applies to the same mode of oscillation there exists also an approximate relation for the two impedances belonging to different modes. Taking some *average of different oscillation modes* in the same vacuum chamber of radius b and ring circumference of $2\pi R$ one obtains the approximate but very useful relation [9]

$$Z_T(\omega) \approx \frac{2R}{b^2}\frac{Z(\omega)}{(\omega/\omega_0)}$$

7.2 Transverse dynamics

The transverse focusing provided by the quadrupoles keeps the beam in the vicinity of the nominal orbit. A particle executes a betatron motion around this orbit. This motion has the form of an oscillation which is not harmonic but has a phase advance per unit length which varies around the ring. Often this is approximated by a smooth focusing given by

$$\ddot{x} + \omega_0^2 Q_x^2 x = 0$$

with ω_0 being the revolution frequency and Q_x the horizontal tune, i.e. the number of betatron oscillation executed per turn.

A stationary observer, or impedance, sees the particle position x_k only at one location each turn k as indicated by the points in Fig. 21, and has no information of what the particle does in the rest of the ring . Therefore, we have no information about the integer

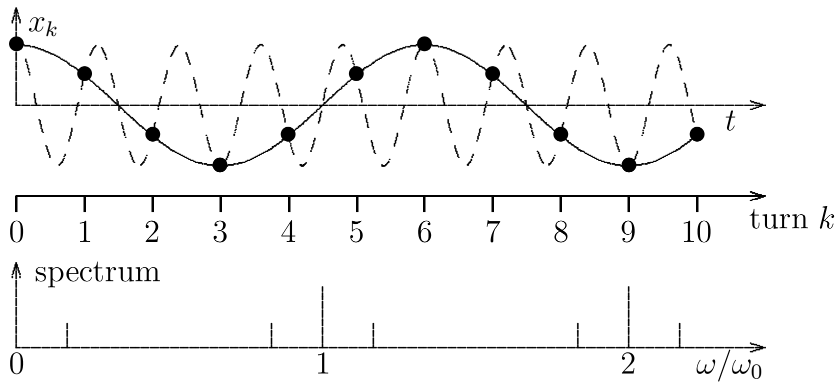


Figure 21: Betatron oscillation observed at one location in time and frequency domain

part of the tune Q_x = integer + q but only about the fractional part q with determines the phase

$$x_k = \hat{x}\cos(2\pi qk)\ ,\ \ x'_k = -\frac{\hat{x}}{\beta_x}\sin(2\pi qk).$$

We observe this motion as a function of turn k. We can make a harmonic fit, i.e. a Fourier analysis, Fig. 21. For a single bunch circulating in the machine we find at the revolution harmonic $p\omega_0$ an upper and lower side band. The distance of the side band is given by the fractional part q because the integer part cannot be observed. For a very short bunch these side bands will extend up to very high frequencies, for longer bunches they will get smaller and vanish with increasing frequencies. A transverse impedance (or a position monitor) is sensitive to the dipole moment $I \cdot x$ of the current and does not see the revolution harmonics.

In general the betatron tune depends on the momentum deviation Δp of a particle which is quantified by the chromaticity

$$\Delta Q = Q'\frac{\Delta p}{p} \approx Q'\frac{\Delta E}{E}.$$

A finite chromaticity will influence the motion of a particle executing at the same time betatron and synchrotron oscillations and make certain modes of oscillations complicated. This will be discussed later while in this and the next section we assume $Q' = 0$.

8 TRANSVERSE INSTABILITIES WITH $Q' = 0$

8.1 Qualitative treatment

We consider a positive charge e^+ going at $t = 0$ through a cavity and exciting a deflecting mode as with frequency $\omega_r = 2\pi/T_r$, as illustrated in Fig. 22. At first, $t = 0$, this mode consists of a longitudinal field with a gradient $\partial E_z/\partial x$ which is later, at $t = T_r/4$, converted into a magnetic field $B = -B_y$, pointing in the negative y-direction. A positive charge going in the z-direction will obtain a Lorentz force in the positive x-direction. After a further quarter cavity oscillation, at $t = T_r/2$, we have again a longitudinal electric field with a gradient but of opposite sign compared to the beginning. At $t = 3T_r/4$ this will be converted into a magnetic field pointing in the positive y-direction. The Lorentz force on a positive charge going in the z-direction is now in the negative x-direction. The interaction of a bunch with this cavity depends on the relation between its fractional tune q and the frequency of the cavity. For the latter also only its fractional part is of importance, as an integer number k' of oscillations executed while the bunch is not in the cavity, has no influence.

We discuss now the interaction between the bunch and the cavity and make some simple choices to facilitate the illustration. For the fractional tune we take $q = 1/4$. For

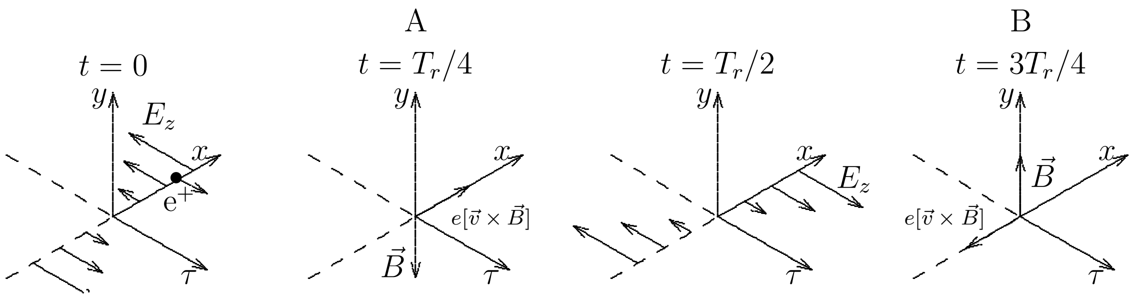


Figure 22: Illustration of a bunch interacting with a deflecting cavity mode

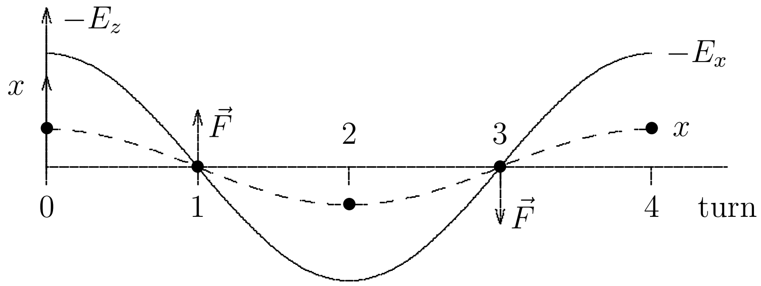


Figure 23: Interaction with the cavity tuned to the upper side band

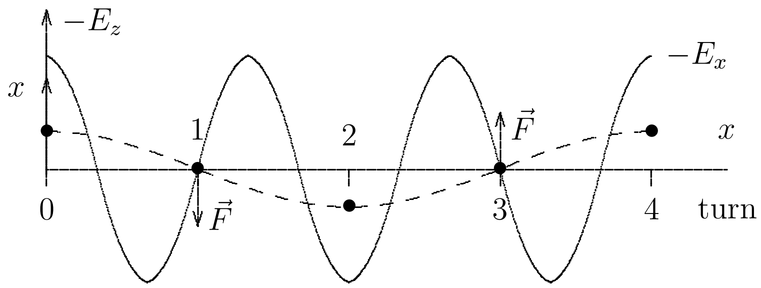


Figure 24: Interaction with the cavity tuned to the lower side band

the cavity frequency we consider two cases: First, $\omega_r = (k' + 1/4)\omega_0$, in which case the bunch having excited the cavity will find it after one turn in the situation 'A' as shown in Fig. 23. Here, the Lorentz force is opposite to the particle velocity and reduces the oscillation which leads to damping. Second, $\omega_r = (k'+3/4)\omega_0 = (k'+1-1/4)\omega_0$, the bunch finds the cavity after one turn in the situation 'B' shown in Fig. 24 where the Lorentz force is in the direction of the transverse particle velocity and increases the oscillation which leads to an instability. As a result we find for one circulating bunch stability if the cavity is tuned to the upper side-band.

The resistive impedance at the upper side band damps, the one at the lower side band excites the oscillation. If we have a more general impedance extending over several side bands $\omega_0(p+q)$ and $\omega_0(p-q)$ we expect that the growth or damping rate of the oscillation is given by an expression of the form

$$\frac{1}{\tau_s} \propto \sum_p \left(|I_{(p+q)}|^2 Z_{Tr}(\omega_{(p+q)}) - |I_{(p-q)}|^2 Z_{Tr}(\omega_{(p-q)}) \right) \text{ with } \omega_{(p\pm q)} = \omega_0\,(p \pm q)$$

where $I_{p\pm}$ is the Fourier component of the beam current at the upper or lower side band. It appears here as the square I_p^2 since the instability is driven by the energy transfer from the longitudinal to the transverse motion.
We can estimate some properties of the proportionality factor missing in the above equation. The product $I_p^2 Z_T = P/y$ represents a power transfer per unit length. To get a

growth rate we have to divide this by the energy of the bunch having N_b particles which can be related to the average current of the bunch $I_0 = eN_b\omega_0/2\pi$

$$\frac{1}{\tau_s} \propto \frac{P}{m_0 c^2 \gamma N_0} = \frac{e\omega_0 P}{2\pi m_0 c^2 \gamma I_0}.$$

8.2 Quantitative treatment

We consider a transverse impedance $Z_T(\omega)$ which interacts with a bunch executing a rigid transverse oscillation with a tune $Q_x = \text{integer} + q$. For convenience we assume the impedance to be in a symmetry point with $\beta'_x = 0$. We consider now a transverse rigid bunch executing a betatron oscillation with the center-of-mass position and angle at the impedance location as a function of turn number k of the form

$$x_k = \hat{x}\cos(2\pi qk)\ ,\quad x'_k = -\frac{\hat{x}}{\beta_x}\sin(2\pi qk).$$

This motion in time (turn) and frequency domain is shown in Fig. 21.

To get the fields induced in the impedance we also have to consider the longitudinal distribution and motion of the bunch treated before. In a single traversal it is (6)

$$I(t)\ ,\quad \tilde{I}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} I(t)\mathrm{e}^{-j\omega t}dt.$$

and illustrated in Fig. 7. For a stationary circulating bunch the current in time domain expressed directly or as a Fourier series is according to (8) and (9)

$$I_k(t) = \sum_{-\infty}^{\infty} I(t - kT_0) = I_0 + 2\sum_{p=1}^{\infty} I_p\cos(p\omega_0 t).$$

and shown in Fig. 8.

The dipole moment of an oscillating bunch at turn k and as function of t is

$$D_k = x_k I_k\ ,\ D_k(t) = \hat{x}\sum_{k=-\infty}^{\infty}\cos(2\pi qk)I(t - kT_0) \tag{39}$$

To express this in a series we form the Fourier transform of $I_k(t)$

$$\begin{aligned}\tilde{I}_k(\omega) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\sum_{k=-\infty}^{\infty} I(t - kT_0)\mathrm{e}^{-j\omega t}dt \\ &= \frac{1}{\sqrt{2\pi}}\sum \mathrm{e}^{-k\omega T_0}\int_{-\infty}^{\infty} I(t - kT_0)\mathrm{e}^{-j\omega(t-kT_0)}dt = \tilde{I}(\omega)\sum_{k=-\infty}^{\infty}\mathrm{e}^{-jk\omega T_0}\end{aligned}$$

The Fourier transform of the dipole moment is

$$\begin{aligned}\tilde{D_x}(\omega) &= \hat{x}\tilde{I}(\omega)\sum_{-\infty}^{\infty}\cos(2\pi qk)\mathrm{e}^{-jk\omega T_0} \\ &= \frac{\hat{x}\tilde{I}(\omega)}{2}\sum_{-\infty}^{\infty}\left[\mathrm{e}^{-jk(\omega T_0 + 2\pi q)} + \mathrm{e}^{-jk(\omega T_0 - 2\pi q)}\right]\end{aligned}$$

The sums become ∞ if the exponent is of form $2\pi p$ and vanish otherwise

$$\sum_{k=-\infty}^{\infty}\mathrm{e}^{-jkx} = 2\pi\sum_{p=-\infty}^{\infty}\delta(x - 2\pi p)\ \text{ and }\ \delta(ax) = \frac{1}{a}\delta(x)\ \text{gives}$$

$$\tilde{D}_k(\omega) = \hat{x}\frac{\omega_0\tilde{I}(\omega)}{2}\sum_{-\infty}^{\infty}[\delta(\omega-(p-q)\omega_0)+\delta(\omega-(p+q)\omega_0)] \tag{40}$$

The inverse Fourier transform gives the oscillating dipole in time domain

$$D_k(t) = \frac{\omega_0\hat{x}}{2\sqrt{2\pi}}\sum_{-\infty}^{\infty}\left[\tilde{I}((p+q)\omega_0)\mathrm{e}^{j((p+q)\omega_0 t)}+\tilde{I}((p-q)\omega_0)\mathrm{e}^{j((p-q)\omega_0 t)}\right]$$

Using

$$(p+q)\omega_0 = (p+q)\omega_0,\ (p-q)\omega_0 = (p-q)\omega_0,\ I_{(p\pm q)} = \frac{\omega_0}{\sqrt{2\pi}}\tilde{I}((p\pm q)\omega_0)\ \text{ gives}$$

$$D_k(t) = \frac{\hat{x}}{2}\sum_{p=-\infty}^{\infty}\left[I_{(p+q)}\mathrm{e}^{j(t(p+q)\omega_0)}+I_{(p-q)}\mathrm{e}^{j(t(p-q)\omega_0)}\right].$$

Combining terms $p > 0$ from the first, $p < 0$ from second the part and vice versa, and using $\tilde{I}(\omega) = \tilde{I}(-\omega)$ gives

$$D_k(t) = \hat{x}\sum_{\omega>0}^{\infty}[I_{(p+q)}\cos((p+q)\omega_0 t)+I_{(p-q)}\cos((p-q)\omega_0 t)].$$

A charge e going through the impedance element at turn k is exposed to a transverse force changing its momentum $\Delta p_{ke} = F_T\Delta t \approx F_T\Delta s/c$

$$\Delta p_{ke} = \frac{e}{c}\int\left[\vec{E}(\omega)+[\vec{v}\times\vec{B}(\omega)]\right]_T ds = \frac{-jeD_k(t)Z_T}{c}.$$

We get the momentum change of the *whole* bunch by a convolution between its charge distribution given by the *single traversal current* $I(t)$ and the deflecting field in turn k

$$\begin{aligned}\Delta p_k &= -j\frac{1}{c}\int_{-\infty}^{\infty}I(t)D_k(t+kT_0)Z_T dt\\ &= -j\frac{\hat{x}}{2c}\sum_{p=-\infty}^{\infty}\int_{-\infty}^{\infty}I(t)\left[I_{(p+q)}Z_T((p+q)\omega_0)\mathrm{e}^{j(p+q)\omega_0(t+kT_0)}+\right.\\ &\qquad\left.I_{(p-q)}Z_T((p-q)\omega_0)\mathrm{e}^{j(p-q)\omega_0(t+kT_0)}\right]dt\end{aligned}$$

This contains integrals of the form

$$\int_{-\infty}^{\infty}I(t)\mathrm{e}^{-j(t+kT_0)(p+q)\omega_0}dt = \sqrt{2\pi}\mathrm{e}^{-jT_0k(p+q)\omega_0}\tilde{I}((p+q)\omega_0) = \frac{2\pi}{\omega_0}\mathrm{e}^{-j2\pi qk}I_{(p+q)}$$

giving

$$\Delta p_k = -j\frac{\hat{c}T_0}{2c}\sum_{-\infty}^{\infty}\left[|I_{(p+q)}|^2 Z_T((p+q)\omega_0)\mathrm{e}^{-j2\pi qk}+|I_{(p-q)}|^2 Z_T((p-q)\omega_0)\mathrm{e}^{j2\pi qk}\right].$$

Combining terms $p > 0$ from the first, $p < 0$ from the second part and vice versa, using relations $Z_{Tr}(\omega) = Z_{Tr}(-\omega)$, $Z_{Ti}(\omega) = Z_{Ti}(-\omega)$ gives

$$\Delta p_k = -\frac{T_0}{c}\sum_{\omega>0} \Big[\Big(|I_{(p+q)}|^2 Z_{Tr}((p+q)\omega_0) - |I_{(p-q)}|^2 Z_{Tr}((p-q)\omega_0)\Big)\hat{x}\sin(2\pi qk)$$
$$- \Big(|I_{(p+q)}|^2 Z_{Ti}((p+q)\omega_0) + |I_{(p-q)}|^2 Z_{Ti}((p-q)\omega_0)\Big)\hat{x}\cos(2\pi qk)\Big].$$

using the form of the betatron oscillation we started from

$$x_k = \hat{x}\cos(2\pi qk) \ , \ x_k' = -\frac{\hat{x}}{\beta_x}\sin(2\pi qk) \ , \ \dot{x}_k = cx_k' = -\frac{\hat{x}c}{\beta_x}\sin(2\pi qk)$$

$$\Delta p_k = \frac{T_0}{c^2}\sum_{\omega>0} \Big[\Big(|I_{(p+q)}|^2 Z_{Tr}((p+q)\omega_0) - |I_{(p-q)}|^2 Z_{Tr}((p-q)\omega_0)\Big)\beta_x\dot{x}_k$$
$$+ \Big(|I_{(p+q)}|^2 Z_{Ti}((p+q)\omega_0) + |I_{(p-q)}|^2 Z_{Ti}((p-q)\omega_0)\Big)cx_k\Big].$$

The transverse velocity and angle change with the transverse momentum

$$\Delta x_k' = \frac{\Delta \dot{x}_k}{c} = \frac{\Delta p_k}{N_0 m_0 \gamma c} = \frac{e\Delta p_k}{m_0\gamma c I_0 T_0}$$

$$\Delta\dot{x}_k = \frac{e}{m_0c^2\gamma I_0}\sum_{\omega>0} \Big[\Big(|I_{(p+q)}|^2 Z_{Tr}((p+q)\omega_)) - |I_{(p-q)}|^2 Z_{Tr}((p-q)\omega_0)\Big)\beta_x\dot{x}_k$$
$$+\Big(|I_{(p+q)}|^2 Z_{Ti(p+q)} + |I_{(p-q)}|^2 Z_{Ti(p-q)}\Big)cx_k\Big].$$

The velocity change has a component proportional to velocity and resistive impedance and one proportional to displacement and reactive impedance. The first leads to exponential growth or damping, the second to a change of the betatron frequency.
We start with the first part alone and a smooth approximation and get an acceleration $\ddot{x} = \Delta\dot{x}\omega_0/2\pi$ which we ad to the one due to focusing by beam optics

$$\ddot{x} + 2\alpha_s\dot{x} + Q_x^2\omega_0^2 = 0, \quad \text{solution:} \quad x = x_0 \mathrm{e}^{-\alpha_s t}\cos(Q_x\omega_0 t + \phi) \quad \text{if} \quad a \ll Q_x\omega_0$$

$$\alpha_s = \frac{1}{\tau} = \frac{e\omega_0\beta_x}{4\pi m_0 c^2\gamma I_0}\sum_{\omega>0}\Big(|I_{(p+q)}|^2 Z_{Tr(p+q)} - |I_{(p-q)}|^2 Z_{Tr(p-q)}\Big). \tag{41}$$

using $(p-q)\omega_0 = -(-p+q)\omega_0$ for $p<0$ and $Z_{Tr}(\omega) = -Z_{Tr}(-\omega)$ gives a sum with positive and negative frequencies

$$\alpha_s = \frac{1}{\tau} = \frac{e\omega_0\beta_x}{4\pi m_0 c^2\gamma I_0}\sum_{p=-\infty}^{\infty}|I_{(p+q)}|^2 Z_{Tr(p+q)}. \tag{42}$$

The growth rate is given by a convolution between the power spectrum components and the impedance at the betatron side bands.

The reactive impedance alone gives an angular change $\Delta x_k' = \Delta\dot{x}_k/c$ proportional to x_k. This represents a focusing element of strength

$$\frac{1}{f} = -\frac{\Delta x_k'}{x_k} = -\frac{e}{m_0c^2\gamma I_0}\sum_{\omega_\pm>0}\Big(|I_{(p+q)}|^2 Z_{Ti(p+q)} + |I_{(p-q)}|^2 Z_{Ti(p-q)}\Big)\Big)x_k$$

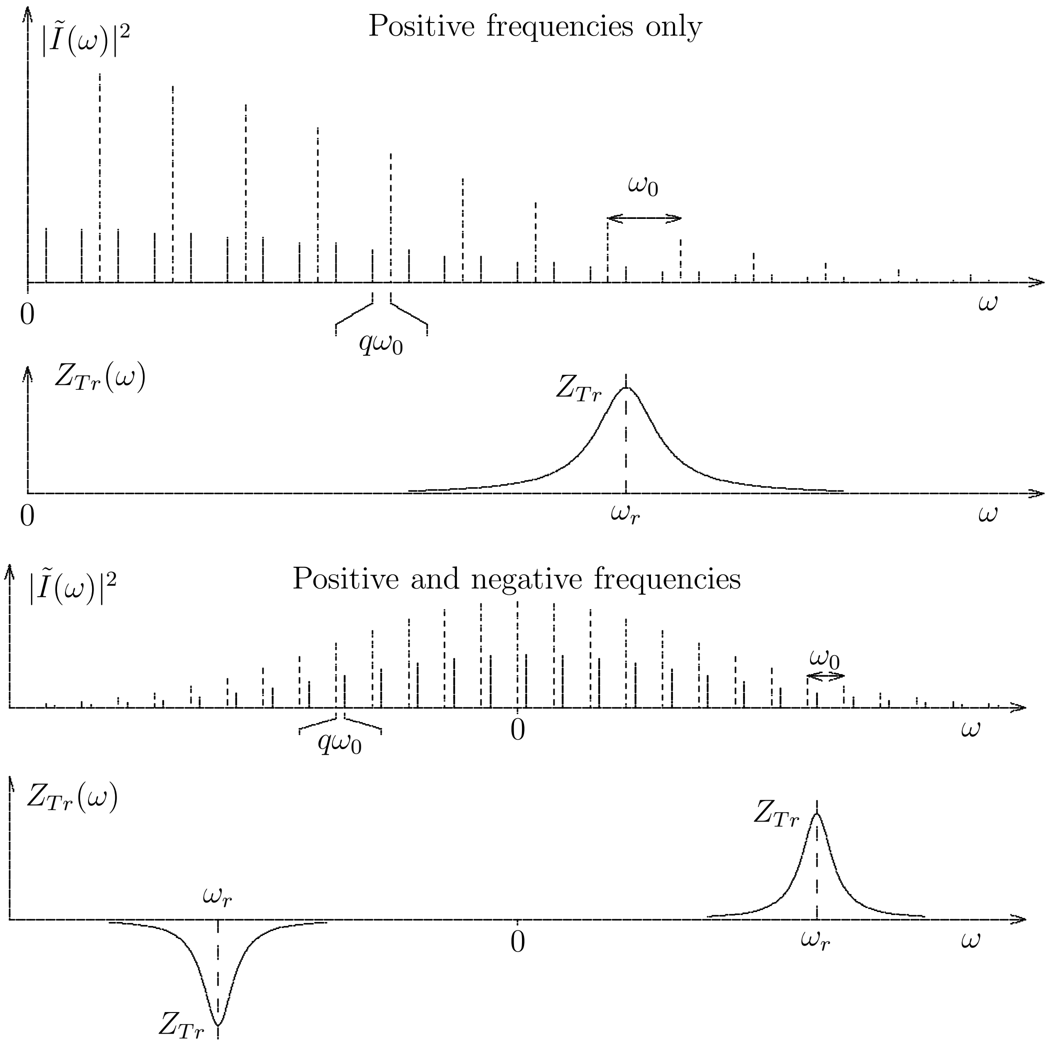


Figure 25: Interaction of a bunch with a narrow band resonance

which results in a change $\Delta Q_x = \beta_x/(4\pi f)$ of tune and $\Delta\omega_\beta = \omega_0 \Delta Q_x$ of betatron frequency

$$\begin{aligned}
\Delta\omega_\beta &= -\frac{e\omega_0\beta_x}{4\pi m_0 c^2 \gamma I_0} \sum_{\omega>0} \left(|I_{(p+q)}|^2 Z_{Ti(p+q)} + |I_{(p-q)}|^2 Z_{Ti(p-q)} \right) \\
&= -\frac{e\omega_0\beta_x}{4\pi m_0 c^2 \gamma I_0} \sum_{p=-\infty}^{\infty} |I_{(p+q)}|^2 Z_{Ti(p+q)}. \qquad (43)
\end{aligned}$$

An inductive impedance $Z_{Ti} > 0$ is defocusing giving negative tune shift.

8.3 Instability due to the resistive impedance

The transverse motion of the bunch is a damped or growing oscillation of the form

$$x = x_0 e^{-\alpha_s t} \cos((Q_x\omega_0 + \Delta\omega_\beta)t + \phi) \text{ if } \alpha_s \ll Q_x\omega_0$$

with the rate given according to (41) by a sum over positive frequencies

$$\alpha_s = \frac{1}{\tau} = \frac{e\omega_0\beta_x}{4\pi m_0 c^2 \gamma I_0} \sum_{\omega>0} \left(|I_{(p+q)}|^2 Z_{Tr(p+q)} - |I_{(p-q)}|^2 Z_{Tr(p-q)} \right).$$

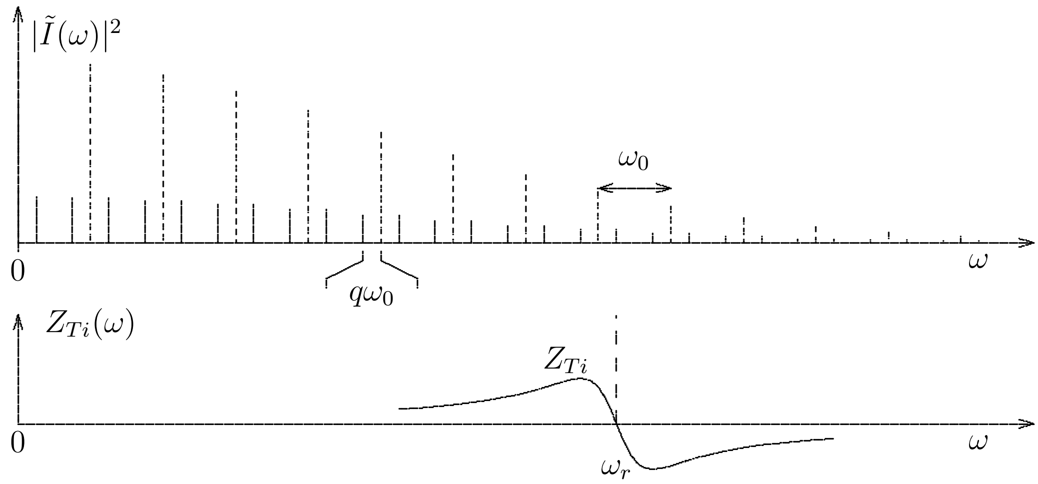


Figure 26: Frequency shift due to a reactive impedance

as shown on the upper part of Fig 25. We can also express the damping or growth rate by a sum over positive and negative frequencies with upper side bands (42)

$$\begin{aligned}\alpha_s &= \frac{e\omega_0\beta_x}{4\pi m_0 c^2\gamma I_0}\sum_{p=-\infty}^{\infty}|I_{(p+q)}|^2 Z_{Tr(p+q)}) \\ &= \frac{e\omega_0^3\beta_x}{8\pi^2 m_0 c^2\gamma I_0}\sum_{p=-\infty}^{\infty}\left|\tilde{I}(\omega_0(p+q))\right|^2 Z_{Tr}(\omega_0(p+q)).\end{aligned}$$

as shown on the lower part of Fig. 25

To drive this instability we need a narrow band impedance with a memory lasting at least for one turn. It is worthwhile to note that the growth rate is proportional to the value of the beta function at impedance. For this reason one often tries to reduce β_x and β_y at the location of unavoidable impedances like RF-cavities. For a distributed impedance we replace the local beta function by its average $\beta_x \approx \langle\beta_x\rangle \approx R/Q_x$ with $R =$ average ring radius.

8.4 Frequency shift due to the reactive impedance

We consider now the change $\Delta\omega_\beta$ of the oscillation

$$x = x_0 e^{-at}\cos((Q_x\omega_0 + \Delta\omega_\beta)t + \phi) \text{ if } a \ll Q_x\omega_0$$

executed by the bunch. According to (43) it is again given by a convolution of the power spectrum of the bunch and the reactive impedance involving positive frequencies with both side bands as shown in Fig. 26, or with both signs of p and upper side bands

$$\begin{aligned}\Delta\omega_\beta &= -\frac{e\omega_0\beta_x}{4\pi m_0 c^2\gamma I_0}\sum_{\omega_\pm>0}\left(|I_{(p+q)}|^2 Z_{Ti(p+q)} + |I_{(p-q)}|^2 Z_{Ti(p-q)}\right) \\ &= -\frac{e\omega_0\beta_x}{4\pi m_0 c^2\gamma I_0}\sum_{p=-\infty}^{\infty}|I_{(p+q)}|^2 Z_{Ti(p+q)} \\ &= -\frac{e\omega_0^3\beta_x}{8\pi^2 m_0 c^2\gamma I_0}\sum_{p=-\infty}^{\infty}\left|\tilde{I}((p+q)\omega_0)\right|^2 Z_{Tr}(\omega_0(p+q)).\end{aligned}$$

The betatron frequency shift can also be caused by a wide band impedance since there is no cancelation between the upper and lower side band. A measurement of this

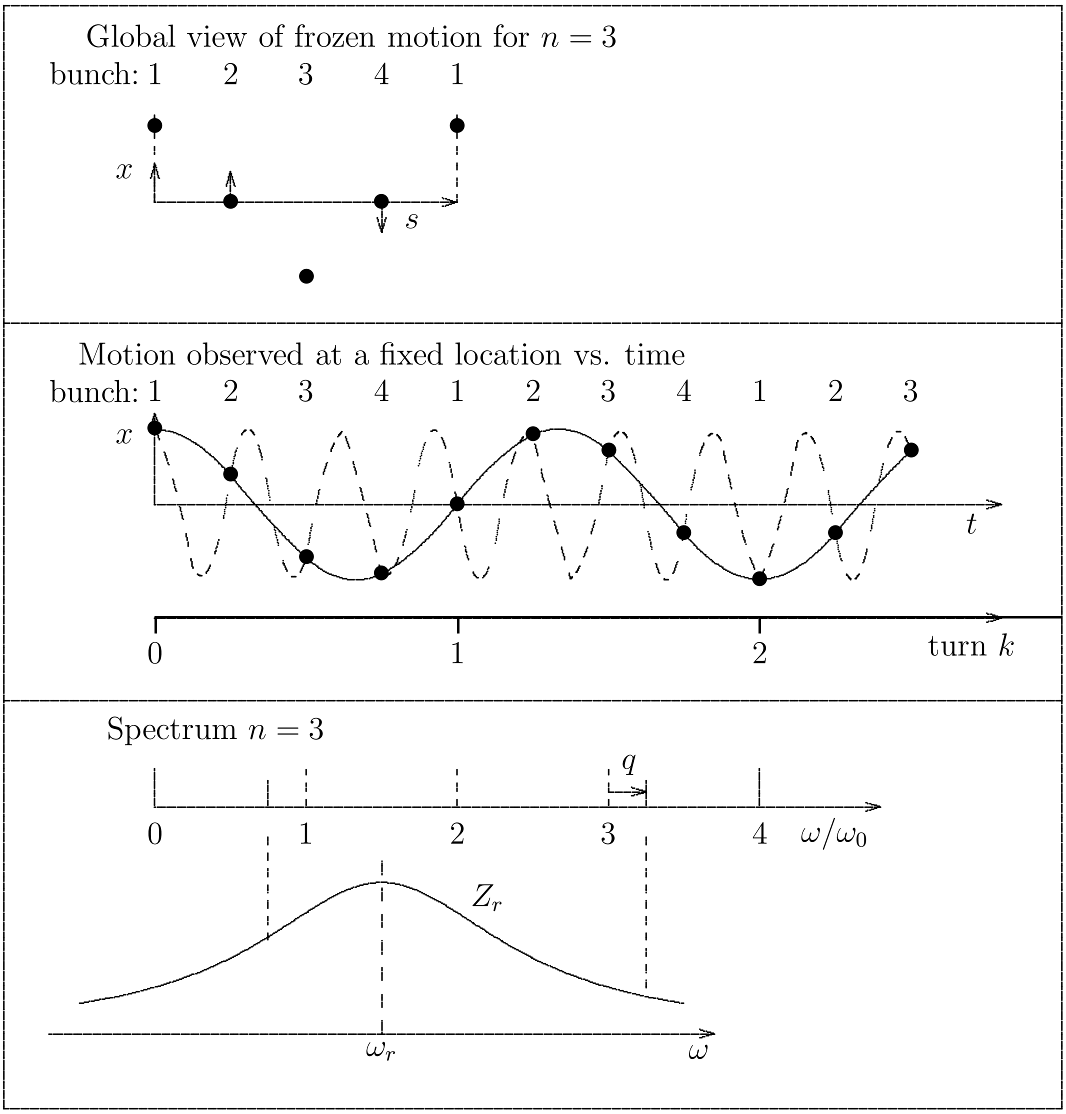


Figure 27: Instability for many bunches

shift is often used to obtain a convolution between the impedance and the bunch spectrum. Doing this for different bunch lengths, some information on the impedance itself can be extracted. This frequency shift acts only on the coherent (center of mass) motion of the bunch and has little influence on the incoherent motion of the individual particles and there frequencies. The reactive impedance can cause a separation between the coherent betatron frequency in the incoherent frequency distribution which can lead to a loss of Landau damping.

8.5 Transverse instability of many bunches

M bunches can oscillate in M different modes $n = M\Delta\phi/(2\pi)$ with $\Delta\phi$ being the phase shift between adjacent bunches. These modes have the frequencies and growth rate

$$\omega_{p\pm} = \omega_0 \left(pM \pm (n+q)\right)$$

$$\alpha_s = \frac{1}{\tau} = \frac{e\omega_0\beta_x}{4\pi m_0 c^2 \gamma I_0} \sum_{\omega>0} \left(|I_{(p+q)}|^2 Z_{Tr(p+q)} - |I_{(p-q)}|^2 Z_{Tr(p-q)} \right).$$

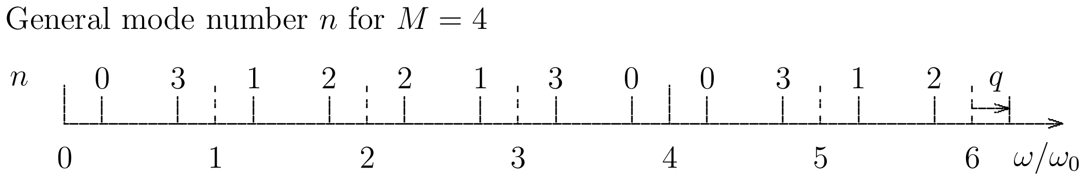


Figure 28: side bands of all modes

9 HEAD-TAIL INSTABILITY

9.1 Head-tail mode oscillations

The longitudinal synchrotron motion in energy and time deviation, ΔE and τ, can also influence the transverse motion. Particles executing a vertical betatron oscillation move at the same time frome the head to the tail of the bunch and vice-versa and go through some deviations ΔE from the nominal energy as shown in Fig. 29. If the chromaticity $Q' = dQ/(dp/p)$ vanishes the betatron tune does not depend on energy and there is no systematic betatron phase shift between head and tail of the bunch as shown on the left of the figure. However, for $Q' \neq 0$ the betatron frequency is different for the positive and negative energy deviation the particle goes through. A particle can accumulate a phase shift going from head to tail via $\Delta E > 0$ which is again lost going back to the head as shown on the right of the figure. For $\gamma > \gamma_T$ it has an excess energy moving from head to tail and an energy lack moving from tail to head. For $Q' > 0$, this gives a phase advance in the first and a phase lag in the second step and vice versa for $Q' < 0$ or $\gamma < \gamma_T$.

The head-tail mode oscillation is shown in Fig. 30. On the left half we have $Q' = 0$. The motion of the bunch $y(t)$ is shown on the very left which consists just of a rigid up-and-down motion. In the next row this motion is multiplied with the bunch current giving the dipole moment $y \cdot I(t)$ which induces the voltage in the transverse impedance. On the right the same quantities are plotted for the case of $Q' \neq 0$ which clearly show the phase shift between head and tail. An experimental verification of this motion has been done [10] and is shown in Fig. 31. For relatively long bunches this mode can be observed directly with a fast position monitor giving a signal being proportional the instantaneous dipole moment $x(t) \cdot I(t)$. Several superimposed traces on the scope are shown, each corresponding to a turn of the oscillating bunch passing through the monitor. On the left we have $Q' = 0$, on the right $Q' > 0$. This figure shows the same behavior as the calculated plotted in Fig. 30.

9.2 Head-tail instability

A broad band impedance is excited by oscillating particles A at the bunch head which in turn excite particles B at the tail with a phase shifted by $\Delta\phi$ compared to the head. Half a synchrotron oscillation later particles B are at the head and while particles A are at the tail oscillating with phase $-\Delta\phi$ compared to B (assuming $Q' = 0$). The excitation by the head has the wrong phase to keep oscillation growing unless $Q' \neq 0$ producing a phase shift during a motion from head to tail or vice versa. The wake field excited by the head of the bunch will affect the tail later. The tail oscillates therefore with a phase lag compared to the tail. To keep the oscillation growing the head particle must undergo a relative phase delay while moving to the tail and the tail particle a relative phase advance moving to the head. We expect a possible instability if $Q' < 0$ for $\gamma > \gamma_T$ or if $Q' > 0$ for $\gamma < \gamma_T$. The 'wiggle' of the head-tail motion is seen by a stationary observer (impedance) as an oscillation with the chromatic frequency ω_ξ which has to be considered in calculating the head-tail instability.

$$\Delta p/p = \Delta\hat{p}/p \sin(\omega_s t) \ , \ \tau = -\hat{\tau}\cos(\omega_s t) \quad \text{with} \quad \hat{\tau} = \frac{\omega_s}{\eta_c}\frac{\Delta\hat{p}}{p}$$

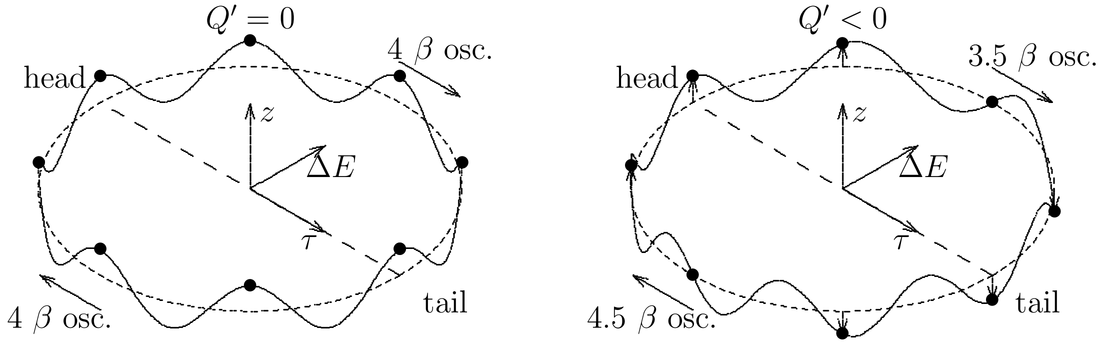


Figure 29: Combined betatron and synchrotron motion

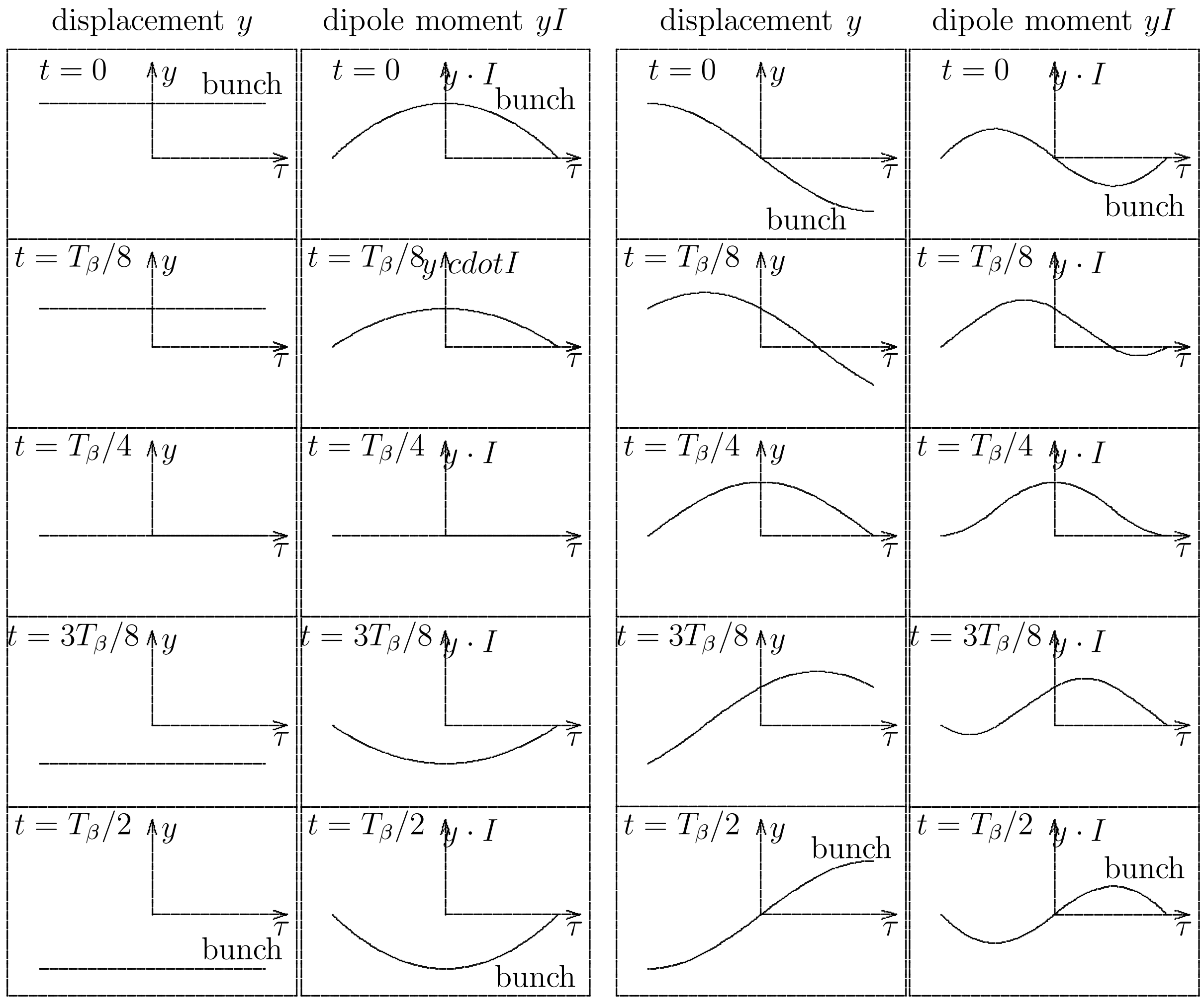
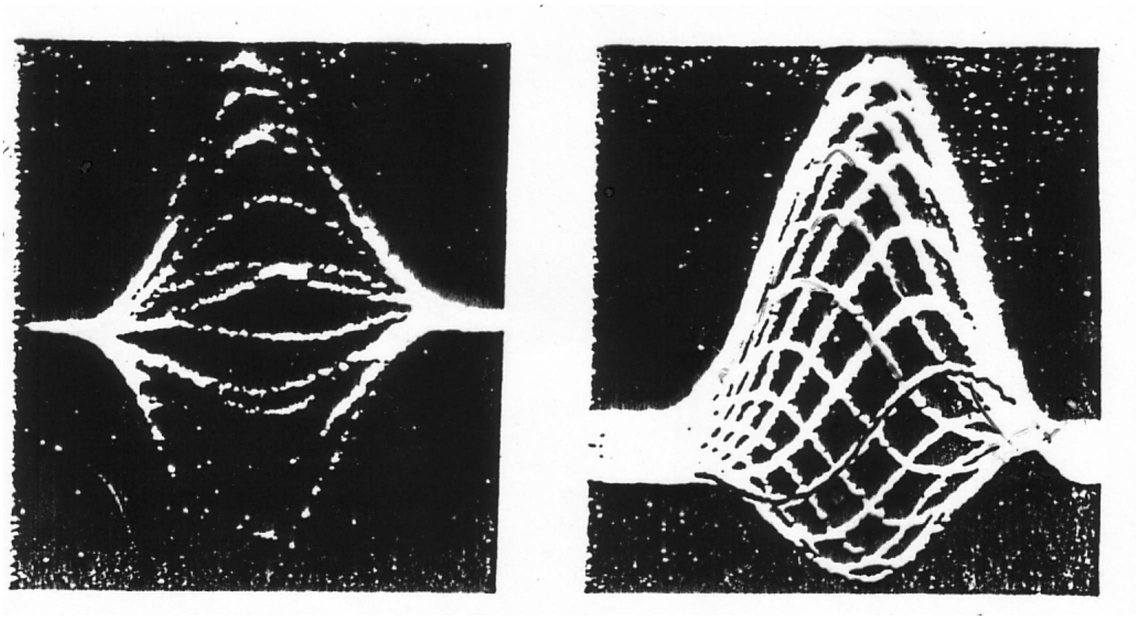


Figure 30: Head-tail mode observed in steps of its period $T_\beta/8$, left: $Q' = 0$, right: $Q' \neq 0$

$\omega_s = Q_s\omega_0$ is the synchrotron frequency and $\eta_c = \alpha_c - 1/\gamma^2$ with α_c = momentum compaction. The relative betatron phase shift of a particle executing part of a synchrotron oscillation is

$$\Delta\phi_\beta = \omega_0 \int_{t_1}^{t_2} \Delta Q dt = \omega_0 Q' \frac{\Delta\hat{p}}{p} \int_{t_1}^{t_2} \sin(\omega_s t) dt$$



$Q' = 0$ $\qquad\qquad$ $Q' > 0$

Figure 31: Head-tail mode $m = 0$ for vanishing and finite chromaticity

$$= \quad -\omega_0 Q' \frac{\Delta\hat{p}}{p} \left(\cos(\omega_s t_2) - \cos(\omega_s t_2)\right) = \frac{\omega_0 Q'}{\eta_c}(\tau_2 - \tau_1)$$

This gives for the chromatic frequency

$$\omega_\xi = \frac{\Delta\phi_\beta}{\Delta\tau} = \frac{\omega_0 Q'}{\eta_c}.$$

This 'wiggle' of the head-tail mode shifts the envelope of the side bands by the chromatic frequency $\omega_\xi = Q'\omega_0/\eta_c$ as shown in Fig. 32. This results in current components

$$I_{(p+q+\xi)} = \frac{\omega_0}{\sqrt{2\pi}}\tilde{I}((p+q)\omega_0 + \omega_\xi)\ , \quad I_{(p-q-\xi)} = \frac{\omega_0}{\sqrt{2\pi}}\tilde{I}((p-q)\omega_0 - \omega_\xi)$$

which can be very different for adjacent side bands. Since now the difference between upper and lower side band is large even a broad band impedance can lead to an instability with growth (or damping) rate [11]

$$\alpha = \frac{e\omega_0\beta_x}{4\pi m_0 c^2 \gamma I_0}\sum_{\omega>0}\left[|I_{(p+q+\xi)}|^2 Z_{Tr}((p+q)\omega_0 + \omega_\xi) - |I_{(p-q-\xi)}|^2 Z_{Tr}((p-q)\omega_0 - \omega_\xi)\right].$$

9.3 Higher head-tail modes

So far we considered a head tail mode in which for vanishing chromaticity all particle move in phase up and down. It is also possible that oscillation of the head and tail oscillate with opposite phase as shown in Fig. 33. Here, the particles move up and down with a phase which depends on their longitudinal position resulting in a difference of π between heat and tail. There are now two modes possible. In one the particle ahead has a phase lag, shown on the left, in the other a phase advance, shown on the right of the figure. The two modes are labeled by $m = \pm 1$. Their frequency is different, in the first case an extra betatron oscillation is subtracted, in the second case added per synchrotron period, resulting in a frequency $\omega_\beta = (p \pm q \pm Q_s)\omega_0$.

The projected position $y(\tau)$ and dipole moment $y \cdot I(\tau)$ of this head-tail mode $m = \pm 1$ is shown in Fig. 34 in steps of $T_\beta/8$. It should be noted that the project position

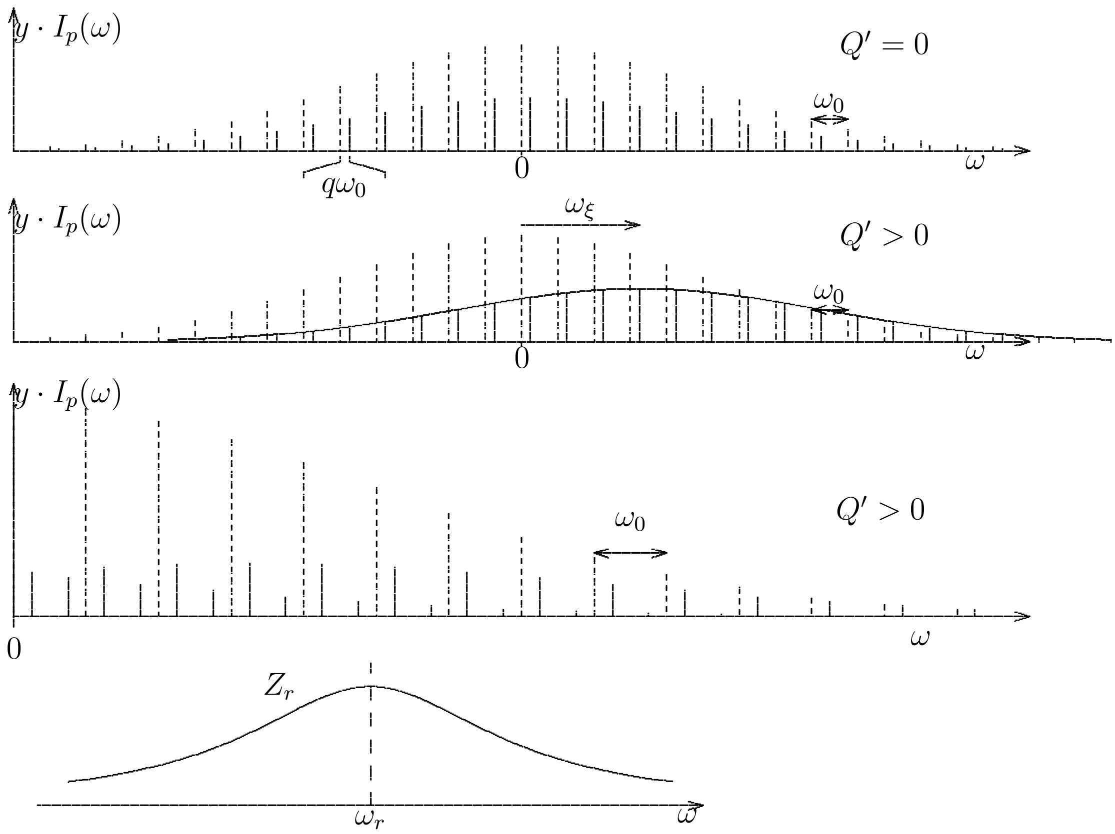


Figure 32: Head-tail mode spectrum; top: $Q' = 0$, middle: $Q' > 0$ positive and negative frequencies, bottom: $Q' > 0$ with positive frequencies only

in the center $\tau = 0$ vanishes always forming a node. Obviously the individual particle still move at this position but their phases are opposite for $\pm\Delta p$ resulting in a vanishing projection.

There are higher head tail modes with the general frequencies

$$\omega_\beta = \omega_0(p \pm q \pm mQ_s)$$

shown in Fig. 35

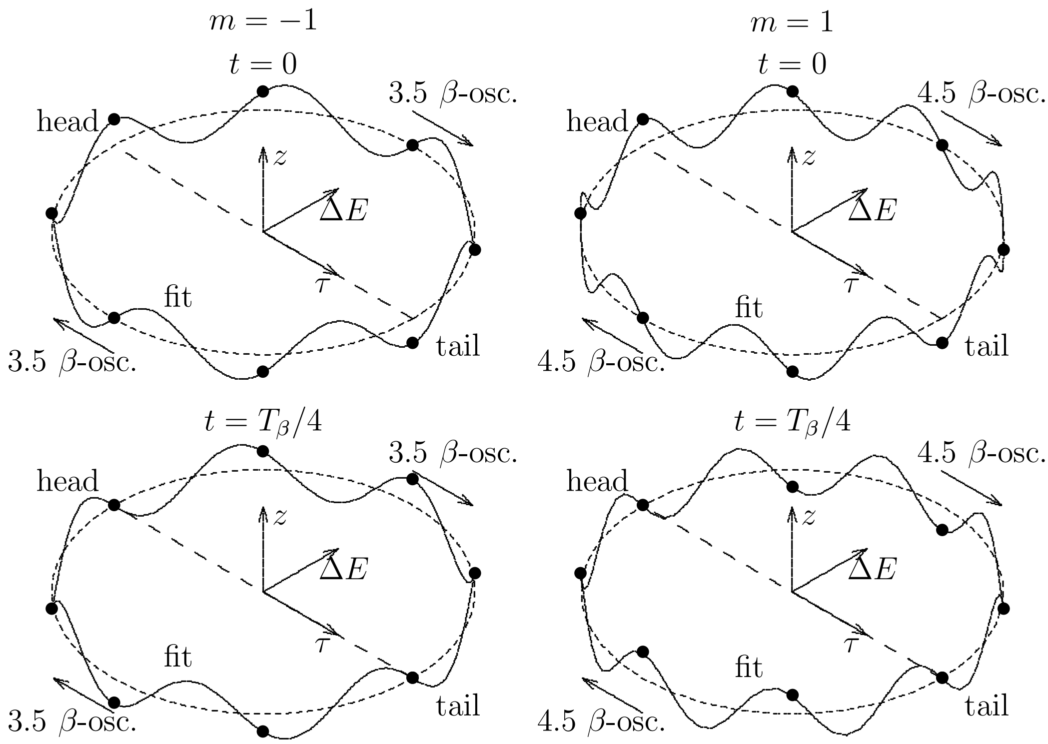


Figure 33: Higher head-tail mode $m = \pm 1$ for $Q' = 0$

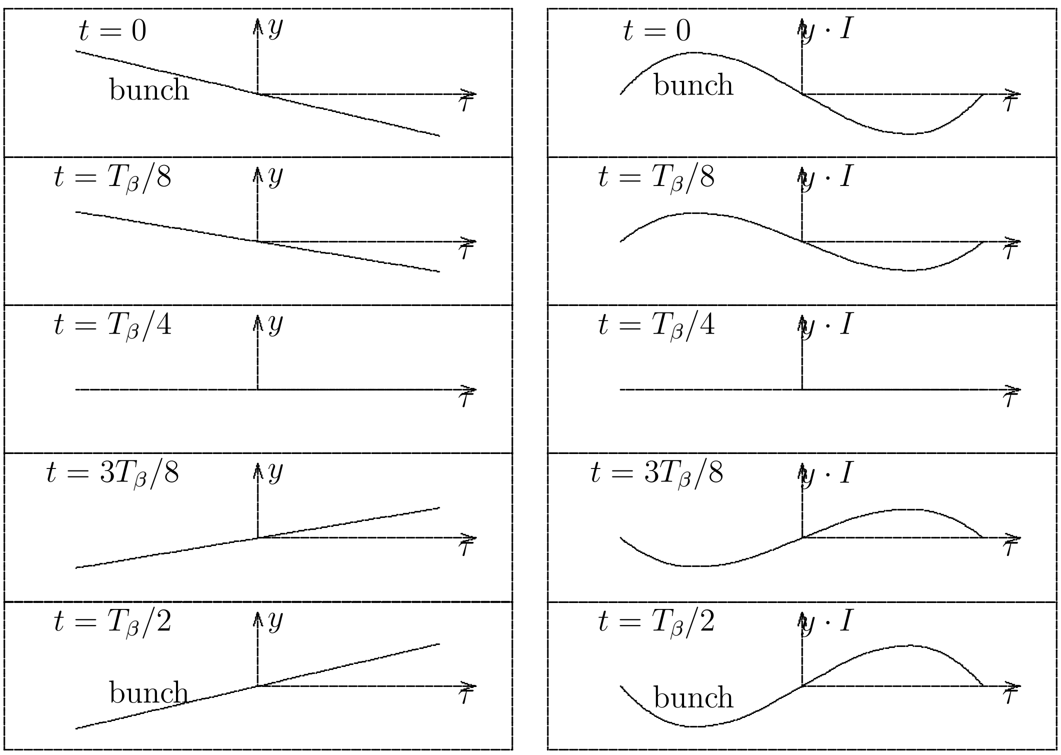


Figure 34: Head-tail mode $m = \pm 1$, $Q' = 0$ seen in steps of $T_\beta = T_0/q$, left displacement $y(\tau)$, right: dipole moment $y \cdot I(\tau)$

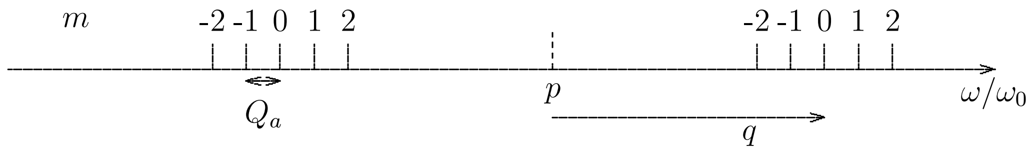


Figure 35: Detailed spectrum of higher head-tail modes $\omega = \omega_0(p \pm q \pm Q_s)$

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