Particle Accelerators, 1996, Vol. 54, pp. 107–122
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USING LIE ALGEBRAIC MAPS FOR THE DESIGN AND OPERATION OF COLLIDERS*

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(Received 19 February 1996; in final form 19 February 1996)

Over the past several years the language of Lie algebras has proved very useful in designing and operating linear and circular colliders. We describe how this language has been used in the design and operation of the Stanford Linear Collider (SLC) final focus system, the Final Focus Test Beam (FFTB), and the SLAC PEP-II electron-positron collider. We also discuss applications of Lie algebras to proton collider design.

Keywords: Maps; colliders; Lie algebra.

1 INTRODUCTION

The Lie algebra language for charged particle optics was introduced and developed by A. Dragt[1] and his collaborators and students.[2] They emphasized that maps, described by exponential Lie operators, provide a powerful way to describe accelerator lattices and lattice elements. These maps have the advantage that they are automatically symplectic and have the appropriate number of independent parameters which invariably correspond to significant physical quantities. Maps completely replace trajectories. There are maps for basic elements, maps for subtleties such as edges and fringes, maps which represent element displacements and rotations, maps for entrance beam errors, and maps for beam-line modules as well as complete beam lines. In addition to describing these methods, these workers have fully implemented their ideas in a series of computer codes.[3] For example, the code DESPOT

*Work supported by Department of Energy contract DE-AC03-76SF00515.

developed by E. Forest is being used at SLAC for the design of the PEP-II B-factory lattices.

In applying these methods to accelerator design and operation we have developed a generator-based map composition method utilizing similarity transformations and the Campbell-Baker-Hausdorff (CBH) theorem. This generator-based composition method is a combination of coordinate transformations and standard perturbation theory. It provides insight into the map composition process and can be implemented by symbolic manipulator codes. Analytic formulae may be obtained which describe coefficients of beam-line generator monomials as functions of lattice parameters.

Recently the generator-based map composition method has been enhanced so that it is now a valid alternative to truncated power-series concatenation methods, even for large complex lattices.[4] This method provides a significant advantage in computational speed, facilitating the addition of lattice parameters as map variables.

This paper contains two major sections: (1) the role and meaning of terms in a beam-line generator, including estimation of the size of terms in generators, and (2) applications to linear and circular collider design and operation.

An introduction to Lie algebra notation including a description of the basic electro-magnetic Hamiltonian and its typical simplifications, a description of element maps, and a summary of generator-based map composition methods can be found in SLAC-PUB-6713.[5] The reader can consult Ref. 4 for more details on map composition. Reference 6 contains an earlier exposition of some of these map composition methods.

2 THE INTERPRETATION OF BEAM-LINE GENERATORS

2.1 The Interpretation of Monomials

Monomials in beam-line generators may be specified by the order of the transverse coordinates and by the order of δ. We first look at the monomials which don't contain δ. The linear monomials, x and y cause a step change in p_x and p_y respectively, and could represent a steering corrector kick. The linear monomials p_x and p_y cause a step change in x and y. Since a kick at one beam-line location causes a change in position at another, these four monomials are referred to as steering terms. The terms p_x and p_y can be used to perform coordinate system translations at the entrance and exit of displaced magnets. The terms x and y can also represent a coordinate system rotation (see Ref. 5). For any linear monomials, m_j, $[m_i, m_j]$ is a constant, from which it follows that $\exp[: m_i :]\ \exp[: m_j :] = \exp[: m_i + m_j :]$. Hence steering terms can be combined into one generator or separated, as desired.

The ten quadratic monomials, x^2, xp_x, p_x^2, y^2, yp_y, p_y^2, x_y, xp_y, $p_x y$ and $p_x p_y$ all cause linear coordinate changes. The map produced by any linear combination of these generators can be represented by a matrix (though it does not follow in general that any matrix can be represented by a single generator).[7] The last four terms which involve a variable from each plane, are coupling terms, since if applied to a horizontal coordinate they produce a term linear in a vertical variable, and vice versa. If a general transfer matrix has coupling terms that are small (coordinate planes are not exchanged) then it can be factored into a normal and purely skew matrix, the latter having multiples of the identity in the diagonal 2×2 blocks. It can then be shown that the skew matrix corresponds uniquely to a generator that is a linear combination of the four skew monomials.

Monomials that are of order three in the transverse variables represent sextupole like terms, since the generator for a sextupole contains x^3 and xy^2. A linear transformation of these monomials can produce terms like $x^2 p_x$, xp_x^2, p_x^3 from x^3, and xyp_y, xp_y^2, px_y^2, $p_x y p_y$, $p_x p_y^2$ from xy^2. The skew sextupole has y^3 and yx^2 and linear transformations produce a similar set of monomials with x and y interchanged. In general the skew multipole monomials have the powers of the vertical coordinates adding to an odd number while for the normal multipoles this sum is even.

Monomials that are of order four in the transverse variables represent octupole and skew octupole like terms, monomials that are of order five in the transverse variables represent decapole and skew decapole like terms, and so on. Fifth-order terms play an important role in final focus systems.

We also arrange monomials with nonzero δ according to the power of the transverse variable. Generator monomials that are linear in the transverse variable and have a single power of δ are called linear dispersion terms since they produce a steering in the transverse coordinates that is linear in δ. Generator monomials that are linear in the transverse variable and have a power of δ greater than one are called higher order dispersion terms. Monomials that are quadratic in the transverse variable and have a single power of δ are called linear chromaticity terms since they produce a change that is linear in the transverse coordinates and linear in δ. Monomials that are quadratic in the transverse variable and have a power of δ greater than one are called nonlinear chromaticity terms. Monomials that are cubic in the transverse variable and have a nonzero power of δ are called chromatic sextupole terms, and so on. The resonance terms in the one-turn generator which are derived from a chromatic sextupole monomial will drive sidebands to sextupole resonances.

2.2 Normalized Variables

The design of a beam-line module usually involves assumptions about the beam distribution through the module. These assumptions are contained in the specification of the β functions. Using the β functions (or beam distribution) one can describe a symplectic map of this distribution to a circular distribution. There is some arbitrariness because a rotation of the final circular distribution leaves it unchanged. Courant and Snyder defined a convention by taking the choice:

$$x = \beta_x^{1/2}\hat{x}, \qquad p_x = \beta_x^{-1/2}(\hat{p}_x - \alpha_x\hat{x}) \tag{1}$$

where $\hat{x}$ and $\hat{p}_x$ are referred to as normalized variables. Forest[8] has introduced the concept of a normalized beam line (which he calls the Floquet line) which runs parallel to the real beam-line so that these transformation can be viewed as transformations from the normalized beam line to the real line. In the normalized beam line the beam distribution remains circular and merely rotates as it proceeds along the line according to the phase advance. The map from the normalized line to the real line can be viewed as taking place in two steps. The first step is a phase space tilt defined by $x' = \hat{x}$ and $p_x' = \hat{p}_x - \alpha_x\hat{x}$, which is given by the generator $G = 1/2\alpha_x x^2$. The second step is a scaling $x = \beta_x^{1/2}x'$ and $p_x = \beta_x^{-1/2}p_x'$, which has a generator $G = -1/2\ln\beta_x x p_x$.

If there is coupling in the main beam line, the transformation to the normalized line can be made in three steps. The first step is a pure skew transformation with generator $G = a\,xy + b\,xp_y + c\,p_x y + d\,p_x p_y$. This generator can be determined uniquely from the linear matrix along the beamline by factoring it into a normal times a pure skew matrix. This corresponds to the usual convention for rings based on the one-turn matrix. For a functioning beamline one could also find these parameters by measuring the co-variance matrix, sometimes called the Σ matrix.[9]

These transformations from the normalized beam line can be used to represent general mismatching into a beamline, for example, to answer questions concerning the ability of a beam line to match and compensate a variety of incoming beam conditions without giving rise to unacceptable nonlinear aberrations.

An important advantage of normalized variables is that $\hat{x}$ and $\hat{p}_x$ have the same dimension. Furthermore, once normalized variables are introduced, it is possible to introduce action-angle variables by $\hat{x} = (2J_x)^{1/2}\cos\theta_x$ and $\hat{p}_x = -(2J_x)^{1/2}\sin\theta_x$. The transform to action-angle variables in a generator can be accomplished with a linear transformation[10]

$$\hat{x} = \frac{(h_x^+ + h_x^-)}{2}, \quad \text{and} \quad \hat{p}_x = \frac{(h_x^+ - h_x^-)}{2i}. \tag{2}$$

After these transformations, a polynomial $G(x, p_x, y, p_y)$ becomes a polynomial of the form

$$\tilde{G}(h_x^+, h_x^-, h_y^+, h_y^-) = \sum a_{pqrs}\, h_x^{+p}\, h_x^{-q}\, h_y^{+r}\, h_y^{-s} \tag{3}$$

Since $\tilde{G}$ must be real, we must have $a_{pqrs} = a^*_{qpsr}$. The significance of terms in this generator will be discussed below in the section on circular ring generators.

2.3 Linac Beam-line Generators

By collapsing the beam-line product using truncated power-series concatenations and subsequently carrying out a Dragt-Finn[11] factorization procedure, or alternatively by employing generator-based composition methods, it is possible to obtain a beam-line map in the form

$$M = T\Lambda\ \exp(:G:), \tag{4}$$

where T is a steering map, Λ is the linear map through the beam line, and $\exp(:G:)$ is a general nonlinear map. Following the rule of "first-things-first" $\exp(:G:)$ must be understood to act on the linear coordinates at the end of the beamline, and hence $\exp(:G:)$ may be interpreted as an adjustment at the end of the beamline which incorporates all the non-linearities of the design line.

To function properly, whether considering a final focus system whose purpose is to achieve a small spot size, or a ring, whose purpose is to achieve a large dynamic aperture and a favorable environment for beam-beam tail dynamics, accelerators must be almost linear. Thus even though there may be occasional large departures from linearity, the final beam-line map will be close to linear, leading to the expectation that G is not a large generator.

To define the strength of a generator monomial, it is best to begin by transforming to the normalized beamline. We will call the transformed monomial $\hat{G}_j$. The strength of $\hat{G}_j$ can be based on its first Poisson brackets:

$$\Delta\hat{x}_j = [\hat{G}_j, \hat{x}], \quad \Delta\hat{p}_{xj} = [\hat{G}_j, \hat{p}_x] \quad \Delta\hat{y}_j = [\hat{G}, \hat{y}_j], \quad \Delta\hat{p}_{yj} = [\hat{G}_j, \hat{p}_y]. \tag{5}$$

For a linear collider, since only the spot size is of significance, it is appropriate to use a measure

$$m_{Gx}^2 = \frac{\Delta\hat{x}^2}{\varepsilon_x}, \quad m_{Gy}^2 = \frac{\Delta\hat{y}^2}{\varepsilon_y}. \tag{6}$$

Usually one uses the average of m_G^2 over a Gaussian beam distribution. Given this definition, an aberration of dimensionless strength m_G will cause an increase of the spot size given by

$$\bar{\sigma}_x\bar{\sigma}_y = \sigma_x\sigma_y(1 + m_{Gx}^2)^{1/2}(1 + m_{Gy}^2)^{1/2}. \tag{7}$$

Each monomial can be interpreted as an independent aberration, and indeed the low-order monomials produce "orthogonal" effects, where orthogonal means that, when computing the rms using a Gaussian distribution, each term gives contributions which add in quadrature. This has the pleasant consequence that the square of the beam size will vary as a parabola as each aberration strength is changed, and thus elements that contribute to one and only one aberration can be incorporated in the beam line for the purpose of tuning to the minimum spot size. Higher order terms are not necessarily orthogonal, but orthogonal combinations can be constructed. For example x^3 is not orthogonal to x, but $x^3 - 3x\sigma_x^2$ is orthogonal to x, which can be verified by showing $\langle x^3(x^3 - 3x\sigma_x^2)\rangle = 0$ when the average is taken over a Gaussian distribution. Thus, steering should be introduced when tuning a sextupole aberration.

Monomials in $\hat{G}$ that have a magnitude $m_G = 0.2$ will cause a 2% increase in the spot size. By looking at the composition of G using the generator-based process outlined in Ref. 5, the source of terms in G can be identified. For example, p_y^2 will signify a displaced waist, and will come from a mis-powered quad or an offset beam at a sextupole. A term δp_y^2 will signify residual chromaticity, and could come from mis-powered sextupoles. A term δp_y will signify residual dispersion, which will come from an offset beam in the final doublet or in a sextupole, and so on.

2.4 Circular Ring Generators

The significance of terms in the circular ring generator differ from those of the linac in that the beam passes millions of times through the same beamline. Small terms in the generator, if "resonant", can be very important. In action-angle variables the action, also known as the Courant-Snyder invariant, remains almost constant from turn-to-turn, while the angle changes by a discrete almost constant amount. A resonant condition occurs when a term in the generator remains unchanged under this repeated lock-step change of angle. To extract this information it is required to transform to an action-angle resonance basis as indicated above in the section on normalized variables. The terms in this sum with $p = q$ and $r = s$ have no θ dependence. For these terms:

$$\Delta J_x = [\hat{G}_{p=q,r=s}, J_x] = 0, \qquad \text{and} \qquad \Delta\theta_x = [\hat{G}_{p=q,r=s}, \theta_x] = f(J_x, J_y) \tag{8}$$

Hence, to first order, these terms incrementally change θ independent of initial θ. Since, in effect, they change the one-turn tune-shifts as a function of amplitude, these terms are called "tune-shift-with-amplitude" terms.

The remaining terms, called "resonance" terms, have oscillating θ dependence, and their importance depends upon whether a resonant conditions is satisfied. Adding together terms with their complex conjugate yields terms of the form

$$\begin{aligned}
&(2J_x)^{\left(\frac{p+q}{2}\right)}(2J_y)^{\left(\frac{r+s}{2}\right)}\{\,\mathrm{Re}\,(a_{pqrs})\cos[(p-q)\theta_x + (r-s)\theta_y] \\
&- \;\mathrm{Im}\,(a_{pqrs})\sin[(p-q)\theta_x + (r-s)\theta_y]\} \qquad (9)\\
&= (2J_x)^{\left(\frac{p+q}{2}\right)}(2J_y)^{\left(\frac{r+s}{2}\right)}|a_{pqrs}|\cos[(p-q)\theta_x + (r-s)\theta_y + \varphi_{pqrs}]
\end{aligned}$$

A term is resonant when for some integer n it is valid that

$$(p-q)\Delta\theta_x + (r-s)\Delta\theta_y = 2\pi n \tag{10}$$

We note in passing that the recurrent cycle of linear map followed by a nonlinear generator map, is similar to the beam-beam problem which consists of a linear map followed by the beam-beam kick. The beam-beam kick, derived from a potential, is here replaced by a generator "potential" which is a function of both position and momentum variables. Many of the analytical tools developed for beam-beam studies do go through for weak generators, especially the description of resonance behavior.

For a ring the definition for the strength of a generator must be based on all four of its first Poisson brackets. One useful measure is the maximum or rms value of

$$m_{G_j}^2 = \frac{\left(\Delta\hat{x}_j^2 + \Delta\hat{p}_{xj}^2\right)}{\varepsilon_x} + \frac{\left(\Delta\hat{y}_j^2 + \Delta\hat{p}_{yj}^2\right)}{\varepsilon_y} \tag{11}$$

over some region of phase space. For purposes of estimating resonant strengths for PEP-II we have made tables[13] of the maximum of m_G for each resonance on an ellipse in amplitude space defined by

$$\frac{2J_x}{\varepsilon_x} + \frac{2J_y}{\varepsilon_y} = N^2. \tag{12}$$

Since we are interested in the behavior of the dynamic aperture in the region near 12σ we take $N = 12$. Because of damping, if a particle is going to get lost it will get lost in about one-fifth of a damping time, hence the dynamic aperture for PEP-II is determined in about 1000 turns. Resonant terms with coefficients that have maximum m_G less than 10^{-3} can be of no importance. We have found that resonances with maximum $m_G = .02$ can have important effects. Inspection of such resonance strengths for all resonance terms in the beam-line generator can give a global picture of the behavior of a lattice (see Ref. 12, these proceedings, for further details).

3 APPLICATIONS TO LINEAR COLLIDERS

3.1 Sextupole Alignment in Final Focus Systems

There are many aberrations to tune in order to achieve the minimum spot size in a linear collider. Because of beam-line errors, the knobs to tune aberrations may not be orthogonal. For example, if the beam centroid is off-axis in the chromatic correction sextupoles, then when the strength of a sextupole pair is changed to adjust the chromaticity compensation (δp_y^2 or δp_y^2), feeddown aberrations such as the waist (p_y^2 or p_x^2), skew quad ($p_x p_y$), and dispersion (δp_x or δp_y) can change.[14] Attempts to tune these effects can lead in a circle. Since the coupling control and waist knobs for the SLC final focus system were fairly good at changing one aberration only, and steering through the final triplet was able to produce a dispersion knob, it was possible to untangle the sextupole chromaticity knob by changing the sextupole pair knob a rather large amount and measuring the change in each of the other aberrations. Data analysis could determine the change of beam orbit through the sextupoles that would orthogonalize the sextupole chromaticity knob. The process could be iterated to achieve reliable orthogonalization, and smaller spot sizes were indeed achieved.[15]

Interestingly, when the FFTB was commissioned, it was noted by observations based on an independent wire alignment system, that the sextupole mover motion was very reliable and repeatable. Instead of eliminating the closed orbit through the sextupoles, sextupole motion was used to generate orthogonal tuning knobs for the waist, dispersion, and skew quad aberrations.[16] This was possible in this case because the energy spread of the beam was so small that the chromaticity knob was rather insensitive and did not need to be finely adjusted.

The tolerance on the stability of the orbit position at the sextupoles is severe in the FFTB (≈ 1 μm) and very severe in the next linear collider (NLC) (≈ 0.2 μm). What must actually remain stable is the sum of the beam positions in the two sextupoles of each sextupole pair. A local feedback system can be installed to maintain this alignment if BPMs, that are stable for many minutes, can be constructed to measure orbit changes with the requisite resolution.[17]

3.2 SLC Final Focus System Upgrade

Based on Lie algebra generator-based beam-line composition methods, an analytic formula was derived for the beam-line generator[18] which could predict the spot size as a function of the incoming β-match. The result was confirmed by measurements. The residual aberration dominating the vertical spot size was identified to be $\delta^2 p_y^2$. The coefficient of this term is very sensitive to the phase advance between the sextupole pair and the final triplet. If the first order chromaticity cancellation is not perfectly "in phase", a $\delta^2 p_y^2$ term is generated. Thus an upgrade was conceived that would insert another quad in the final telescope to change this phase and eliminate this aberration. The upgrade was performed, and the vertical spot size has been reduced, as predicted, from slightly more than 1 μm to less than 0.5 μm.[19]

The next aberrations which limit spot size are octupole-like terms which arise as a result of the interleaved sextupoles. A set of three octupoles was designed which on paper would yield even smaller spot sizes. However there is another limit, based on the energy change of particles passing through the final telescope when they emit synchrotron radiation, that is about 0.4 μm so not much more can be gained.

As part of the upgrade the upper transformer was also redesigned to permit more flexible and reliable control of the β match and coupling removal.[20] There are eight parameters to completely control a β match, four coupling knobs plus vertical and horizontal α and β. Only two of the coupling knobs impact the spot size so there are six parameters to control. Multiknobs (which will depend on existing beam-line element strengths) that change one parameter without changing the others can be determined by a Lie generator-based analysis. Multiknobs based on these formulae have been successfully installed and operated.

There is an opportunity to explore the aberrations of a final focus system by inducing steering at the entrance to the system. By comparing measurements with prediction based on analytic formulae for the spot size as a function of steering, the presence of aberrations other than design aberrations can be detected. Such a scheme was proposed and carried out on the SLC upgrade.[21]

3.3 NLC Final Focus System Design

There are many aberrations that arise in final focus systems, each of which can limit performance. It is very helpful to find analytic formulae for the strength of these aberrations and optimize the system design accordingly. Some final focus system designs for the NLC are longer than a kilometer, and it is essential to understand what is controlling the length. First attempts to find formulae for optimizing length were helpful but not predictive because they did not include tolerance considerations. Recent attempts, with tolerances included, are providing reliable guidelines for choice of system optical functions and module lengths.[22]

4 APPLICATION TO CIRCULAR ELECTRON-POSITRON COLLIDERS

4.1 Monitoring the Design Process

Lattice ring design processes often consist of choosing a lattice configuration and tracking it at several working points to determine the dynamic aperture. Based on these observations, plus perhaps a Fourier analysis of particle trajectories, the lattice is revised and retested. More often, tracking is so computer intensive that only one working tune is investigated, and particle trajectory analysis is performed only occasionally. This process converges very slowly, because:

1. There are many lattice parameters to vary and specify — not only the basic lattice parameters, but also the tolerances for a variety of errors, such as powering, alignment, and multipole strengths.
2. The sensitivity of what is observed is a very coarse indicator.
3. The results are often uncertain. These uncertainties include questions such as
   - Are the simulation code algorithms adequate, and have they been properly implemented?
   - Have the error introduction and correction been carried out as supposed?
   - Do the input files correctly describe the lattice, its errors, and the correction procedures as supposed?

Preparing one-turn map generators to accompany each lattice tracking run can significantly help with items (2) and (3). Visually scanning resonance strengths and comparing them with previous runs can monitor input files and provide more detailed information about lattice performance. Indeed the resonance coefficients give the same kind of information as Fourier analysis of particle tracking, but with much more accuracy and detail.[13]

It is now standard practice in the PEP-II design process to produce a one-turn map generator corresponding to every element-by-element tracking run. Very puzzling results which arose during the design of solenoid compensation schemes resulted in the observation that the $\delta^2 J_x$ and $\delta^2 J_y$ coefficients were not near zero as had been supposed. This led to the discovery that the second-order chromatic behavior was being calculated differently in MAD8.1, where the lattice was designed, than in DESPOT, where the lattice was tracked and map coefficients calculated.

4.2 Map-Aided Lattice Design

For application of Lie algebra maps to circular electron-positron colliders, see Y. Yan,[12] these proceedings.

4.3 Control of Beam-line Generator Parameters

Although we attempt to design lattices in a modular way so that leading aberrations are compensated, at least semi-locally, we have uncovered two cases where changes in straight sections between arcs produced a marked improvement in lattice performance. In the PEP-II low-energy ring (LER), the first tune-plane scans showed significant dips in aperture at tunes corresponding to octupole resonances, especially $4\nu_x$ and $2\nu_x + 2\nu_y$. At this time the lattices of the six arcs had a phase advance of 90° per cell and contained interleaved sextupoles, producing obvious octupole terms. By pairing up four of the six arcs and controlling the phase advances between them, the strength of the octupole resonance was substantially decreased.

In the PEP-II high-energy ring (HER), to our surprise, a reconfiguration of straights to incorporate octupole tune-shift-with-amplitude modules changed the lattice behavior. By looking at resonant generator coefficients, we were able to determine that when the phase of the straight section between arcs was changed the sign of a large chromatic resonance terms ($\delta J_x \cos 2\theta_x$) changed. This conclusion was confirmed by taking the map for the lattice with poorer performance, changing the sign of this term, and doing nPB tracking using the modified generator. We call maps with switched parameters "SWIMS." SWIMS provide a very powerful tool for investigating the impact of each generator coefficient.

The majority of beam-line generator coefficients must be controlled in the design process by choosing a good bare-lattice design and by specifying tolerances on lattice errors. We have noted in beam-beam halo simulations[23] that halos are a result of resonance streaming and are particularly sensitive to the coefficients of the tune-shift-with-amplitude terms J_x^2, J_y^2, and $J_x J_y$. These terms can arise from long or interleaved sextupoles, from the kinematic correction term in the Hamiltonian of the large angle IP region, or from quadrupole fringes. To gain some operational control over these coefficients, we have installed two nine-octupole modules. Each module has three sets of three octupoles placed in three cells of a 120° FODO array. Sets of three octupoles at 120° or 60° insure that all but the $2\nu_x - 2\nu_y$ resonance terms are cancelled. The 120° provides larger β ratios, hence improves the orthogonality of the families.[24]

5 APPLICATION TO CIRCULAR PROTON COLLIDERS

5.1 Statistical Maps

During the Superconducting Super Collider (SSC) design process, a quantity called smear was used as a basis for evaluating lattices. Forest[25] pointed out that this could be calculated with Lie algebraic methods. Bengtsson *et al.*[26] extended the formalism to include lattice alignment errors, and implemented it using a symbolic manipulator. He performed a detailed analysis of the smear tracking runs to get not only the smear, but also the resonance coefficients.

Even more interesting, it was possible to calculate the rms distribution of smear that would be observed in an ensemble of lattices given a specification of the rms of the multipole and alignment errors. In a similar way, generator-based map composition methods provide an ability to look at the statistical behavior of maps.[27]

5.2 Fast Computation and Tracking of Maps

A generator-based method, using a third-order CBH formula, was proposed and partially implemented by the SSC design group.[28,29] The methods described in Ref. 5 provides a very accurate and fast alternative method to calculate maps for circular proton colliders. It is interesting to note in this connection, that Yan,[30] using implicit map methods, was able to establish that only terms through 5th or 6th-order in the one-turn beam-line generator were necessary to explain the long term dynamic aperture of the SSC.

It has been established that the maps are sufficiently accurate. A method for rapid symplectic map tracking, now referred to as kick factorization, was proposed by us,[31] implemented by Forest, and tested by Zimmermann *et al.*[32] The results of Zimmermann using a random phase between kicks were not particularly encouraging, as the speed was only a factor of two better than power-series tracking, with less accuracy. Recently, the method has been substantially improved by Abel *et al.*[33]

6 SUMMARY

The role and meaning of monomials in a beam-line generator, their transformation into a resonance basis and methods for estimating their size and importance have been described. Several applications of these methods in the design and operation of linear and circular colliders were presented. This language has provided an enhanced understanding of single-particle dynamics in complex situations, with consequent impact on machine operation, and has additionally provided important tools for simulating lattices, monitoring the design process, and providing information on lattice performance to guide the design process.

Results of a generator-based map composition show that:

- Most contributions to the generator come from a transparent addition process (first-order CBH), especially true for weak generators such as fringe fields and multipole errors
- The contributions of a generator that are modified by the presence of strong elements elsewhere in the lattice can be quantitatively determined through use of similarity transformations
- Accurate calculations can be carried out using one or at most two Poisson brackets to include the interaction of weak lattice elements

In a generator-based map composition, there need be no interconnection to the order of transverse variables, as in power-series methods. Map construction speed using generator-based methods can be accurate and very fast even for very high-order maps. Because all maps are expressed through generators, symplecticity is guaranteed at every step. Map construction processes, such as statistical map constructions, are possible. A generator-based map composition avoids:

- finding symplectic integrators for all elements,
- using truncated power-series concatenation to combine elements, and
- performing Dragt-Finn factorization to get the beam-line generator.

The beam-line or one-turn map written in the form of a product of a linear transformation and an exponential map

$$M = \Lambda \exp[: G :]$$

contains all the information needed to specify the single-particle dynamics of a lattice, and as such can be considered the "specification" or fingerprint of a lattice. For circular electron colliders, an nPB tracking algorithm allows the significance of parameters contained in the high-order generator G to be probed. For circular proton colliders, a kick factorization algorithm appears promising.

Acknowledgements

I am deeply indebted to Karl Brown for his constant encouragement. I thank Etienne Forest for introducing me to Lie algebras and adding new features to DESPOT when we wanted to try them out on the PEP-II rings. I have enjoyed the collaboration of Ghislain Roy, Yiton Yan, Tong Chen, Yunhai Cai and Frank Zimmermann, and Nick Walker, who implemented these ideas in the SLC final focus operation and upgrade.

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