Particle Accelerators
1983 Vol. 13 pp. 67–84
0031-2460/83/1301/0067$06.50/0

CALCULATION OF ISOCHRONOUS FIELDS FOR SECTOR-FOCUSED CYCLOTRONS†

M. M. GORDON

National Superconducting Cyclotron Laboratory, Michigan State University, East Lansing, Michigan 48824, *U.S.A.*

(*Received October 14, 1982*)

During the past 21 years, several different computer programs have been developed at this laboratory for calculating isochronous fields and hence the operational values of trim coil currents. All of these programs utilized a particular set of analytical formulas for obtaining good starting values in the isochronous field calculation. This report first presents these formulas, together with a discussion of their application, and then provides (in an appendix) the detailed analysis from which these formulas were derived.

PREFACE

In the process of designing sector-focused cyclotrons, one generally begins with simple analytical formulas to calculate basic orbit properties, such as the "smooth-approximation" formula for ν_z. When much greater accuracy is required as, for example, in confirming or refining a tentative magnet design, one then uses an equilibrium orbit code based on direct orbit integrations to satisfy this requirement.

Perhaps the only place where very accurate analytical formulas prove extremely useful is in the calculation of the "isochronous field" or the set of trim-coil currents which must produce it. This calculation usually proceeds through an iteration scheme to the final result, and the analytical formulas provide the necessary starting values. Experience has shown that without accurate formulas and the good starting values they provide, the convergence of the iteration process can become tediously slow.

The earliest of the widely known accurate formulas are those due to Smith and Garren,[1] and these formulas were incorporated by Garren[2] into his program for calculating trim-coil currents. Formulas of comparable (if not higher) accuracy were obtained at about the same time by Hagedoorn and Verster[3] and by Parzen.[4]

During the past 21 years, several different programs have been developed and used at this laboratory to calculate trim-coil currents, and among these, the program "Fielder" is perhaps the best documented.[5] All these programs have utilized our own particular formulas for the isochronous-field calculation and the purpose of this report is (at long last) to present these formulas and the analysis upon which they are based. The formulas are discussed in the next section and the analysis then follows as an appendix.

Although this material was developed in 1961, it was not published at that time for reasons which can now only be guessed at. Perhaps we could not spare the time and effort required to make a fair comparison with the other formulas in order to establish

† This material is based upon work supported by the National Science Foundation under Grant No. Phy 80–17605.

which of them were superior. Perhaps we also recognized that our analysis, although rigorous, was rather unorthodox and therefore somewhat difficult to justify.

More recently, in the process of designing superconducting cyclotrons here and in Milan, Bellomo and Resmini[6] chose to incorporate our formulas for the isochronous field into their program for calculating operational values of the trim-coil and main-magnet currents. This choice was based on a preliminary study which indicated that these formulas were significantly more accurate than those of Hagedoorn and Verster, in particular. We should note, however, that superconducting cyclotrons are characterized by relatively low flutter and high spiral, and these characteristics may tend to favor our formulas.

Another, and perhaps the main, reason for publishing this material now is simply that in the process of moving offices, we discovered hidden in the bottom of a desk drawer a set of old multilith stencils on which had been typed the first part of a report presenting in detail the mathematical analysis leading to our formulas. It is precisely this old unfinished report (with some necessary editing) which is printed here as an appendix.

It will be clear from reading this appendix that it contains only the first part of the analysis, namely, the part dealing with the equilibrium-orbit calculation. However, it is just this part which is required for the isochronous-field formulas. The remainder of the analysis dealt with the linear oscillations and nonlinear resonances, and it seems that these parts never reached final form.

It is somewhat surprising that during the past 20 years no one else has carried on or improved the various analyses noted above so as to produce even more accurate analytical formulas. The need for such formulas still exists and present-day computers can obviously handle calculations of far greater complexity than was possible 20 years ago.

ISOCHRONOUS FIELD CALCULATION

The formulas given in this section will be complete in themselves and references to material in the appendix will be made here only as a guide to those seeking background material. These references will be placed in brackets to distinguish them from the (parenthesized) references within this section itself.

1. INPUT DATA

We first define various input parameters which characterize the particular ion under consideration. To represent the charge q and mass m_0, we introduce Q and A' defined by

$$Q = q/e, \qquad A' = m_0/m_1, \tag{1}$$

where e is the charge on the proton and m_1 is a mass unit. In our old programs we took m_1 to be the proton mass, but more recently we have taken m_1 to be the standard atomic mass unit $m_1c^2 = 931.50$ MeV. Note that A' differs slightly from A, the usual mass number, and this difference can often be quite significant.

If v is the ion's velocity, then as usual $\beta = v/c$ and we also have

$$p = \beta/(1 - \beta^2)^{1/2}, \tag{2a}$$

or

$$\beta = p/(1 + p^2)^{1/2}, \tag{2b}$$

where p is the relativistic momentum in m_0c units.

For input and output to this program (and all orbit programs), we use lengths in inches, fields in kilogauss, and energies in MeV. As a result, the only physical constants required internally are: $m_1c^2 = 931.50$ MeV, the speed of light c, and the conversion factor 2.54 cm/inch. In particular, we have

$$\frac{m_1c}{e} = \frac{931.50 \times 10}{2.997925 \times 2.54} \text{ kG} \times \text{in.} \tag{3}$$

for the lumped constant which appears in many equations. Note that it is very important to use exactly the same constants in all orbit programs and related computer routines in order to avoid generating small (and often troublesome) phase-slip errors.

Another parameter required for the input is the ideal orbital frequency $\nu_0 = \nu_{rf}/h$, where ν_{rf} is the assumed rf frequency and h is its harmonic number. We use the angular frequency $\omega_0 = 2\pi\nu_0$, and define the associated length unit a and field unit b by

$$a = c/\omega_0, \tag{4a}$$

$$b = \frac{A'}{Q}\left(\frac{\omega_0}{c}\right)\frac{m_1c}{e}, \tag{4b}$$

so that a, b, and ω_0 are all determined if any one of them is given.

If T is the ion rotation period in an equilibrium orbit, then we take $\omega = 2\pi/T$, and define the isochronous field to be such that $\omega = \omega_0$ independent of the ion energy over some suitable range of values. Our problem therefore reduces to calculating the average field $B_0(r)$, defined below, which produces this condition over a specified range of values of the radius r.

2. MAGNETIC-FIELD PARAMETERS

The median-plane magnetic field $B_z = -B(r, \theta)$ is characterized by its Fourier series in the form

$$B(r, \theta) = B_0(r) + \sum_{n>0} [G_n(r) \sin(n\theta) + H_n(r) \cos(n\theta)], \tag{5}$$

where the sum extends over $n = N, 2N, 3N, \ldots$, with N being the number of sectors. The term $B_0(r)$ is called the average field and the remaining terms constitute what is known as the "flutter field".

During the design process and before all the magnet parameters are fixed, only tentative or partial values of the Fourier coefficients may be known, and a proper job for the isochronous-field routine is to help in the design of a set of trim coils. When the magnet is finally constructed and the field measurements completed, the data can then be reduced, when necessary, to sets of B_0, G_n, and H_n values as functions of r and the current settings. The job of the isochronous-field routine here is to aid in establishing operational values of the trim-coil currents corresponding to acceleration of various ions to the desired final energies.

Our formulas involve certain dimensionless flutter field parameters defined by

$$K_n(r) = \left(\frac{er}{m_1 c}\right)^2 [G_n^2(r) + H_n^2(r)], \tag{6a}$$

$$K_n'(r) = r\left(\frac{dK_n}{dr}\right). \tag{6b}$$

We also need certain average-field parameters given by

$$M' = 1 + k, \qquad M'' = 3 + 7k + 2k', \tag{7}$$

with k and k' defined by

$$k = \frac{r}{B_0(r)}\frac{dB_0}{dr}, \qquad \text{and,} \quad k' = \frac{r^2}{B_0(r)}\frac{d^2B_0}{dr^2}. \tag{8}$$

These equations for M' and M'' differ slightly from those given in [2D.3], but the difference is not significant for our purposes here.

Before proceeding, we first consider the "zero-flutter" isochronous field for which $G_n = H_n = 0$. In this case, the equilibrium orbit is a circle of radius r and the equations of motion yield

$$\beta = r/a, \tag{9a}$$

$$B_0(r) = b[1 - (r/a)^2]^{-1/2}, \tag{9b}$$

with a and b defined in Eq. (4) above.

If this field is now used to calculate k and k' of Eq. (8) and hence M' and M'' in Eq. (7), we then find

$$M' = 1 + p^2, M'' = 3(1 + p^2)(1 + 2p^2), \tag{10}$$

where, for future purposes, the result has been expressed in terms of p by the use of Eq (2).

Our problem now reduces to finding formulas corresponding to those in Eq. (9) when the flutter field differs from zero.

3. ANALYTICAL FORMULAS

We present first the formula relating $\beta = v/c$, the angular frequency $\omega = 2\pi/T$, and the field parameters evaluated at a specified radius r. This formula is derived in Sec. II.A of the appendix, but here we replace the reference radius r_0 by r, and use the fixed frequency $\omega_0 = 2\pi\nu_0$ as a standard. We then obtain

$$\beta\left(\frac{\omega_0}{\omega}\right) = \frac{r}{a}\left[1 + \frac{\delta_1}{p^2}\right], \tag{11a}$$

where

$$\delta_1 = \frac{1}{8}\left(\frac{Q}{A'}\right)^2 \sum_n \left\{\frac{(2n^2 + 1)K_n}{(n^2 - M')^2}\right\}, \tag{11b}$$

with K_n from (6) and M' from Eq. (7).

The second formula relates p, B_0, and the other field parameters at a given r value. This formula is derived in Sec. II.B of the appendix, and can be conveniently written in the dimensionless form

$$\frac{Q}{A'}\left(\frac{er}{m_1 c}\right) B_0(r) = p\left[1 - \frac{\delta_2}{p^2}\right], \tag{12a}$$

where

$$\delta_2 = \frac{1}{4}\left(\frac{Q}{A'}\right)^2 \sum_n \left\{\frac{K_n + K_n'}{n^2 - M'} + \frac{\frac{1}{2}M''K_n}{(n^2 - M')^2}\right\}, \tag{12b}$$

with the parameters K_n, K_n', M', and M'' given in Eqs. (6) and (7). Here, as in Eq. (11b) above, the sum extends over $n = N, 2N$, etc.

The bracketed quantities in the formulas above are the correction factors resulting from the flutter field. These factors are obtained through a perturbation calcultion and it is important to recognize that the first-order contribution of the flutter field vanishes identically. That is, the values of δ_1/p^2 and δ_2/p^2 are correct to second-order and both are directly proportional to $1/p^2$ as well as $(Q/A')^2$, simply because there are no first-order contributions. Of course, this would no longer be true if still higher order corrections were included.

We should also note that by writing the correction factors in the forms given above, we have explicitly separated the dependence on p from that on r. That is, for a given ion and median-plane field, the quantities δ_1 and δ_2 depend exclusively on r, at least to this (second) order of approximation.

Since the perturbation calculation has been carried out only to second order, the validity of the above formulas requires that

$$\delta_1/p^2 \text{ and } \delta_2/p^2 \ll 1, \tag{13}$$

and the smaller these quantities are, the more accurate will be the results.

For a given median-plane field, Eq. (12) constitutes an explicit formula giving p as a function of r. This simple quadratic equation can be solved directly to yield

$$p = \frac{1}{2}[p_0 + (p_0^2 + 4\delta_2)^{1/2}], \tag{14}$$

where $p = p_0$ is the obvious solution of Eq. (12) when $\delta_2 = 0$. Substitution of this p value into Eqs. (2) and (11) then yields (ω_0/ω), and hence the phase slip as a function of r (or the energy).

However, we should prefer instead to consider Eqs. (11) and (12) as providing two relations between the three quantities p, B_0, and (ω_0/ω) at each value of r. To do this, we must first consider the flutter field and hence K_n and K_n', as both given and fixed.

Second, we must remove the implicit dependence of δ_1 and δ_2 on B_0 by replacing M' and M'' in Eqs. (11b) and (12b) by explicit functions of r and p. This can be

accomplished simply by setting M' and M'' to their zero-flutter values, such as those given in Eq. (10) above, a procedure which is justified inasmuch as our formulas will still be correct to second order.

We should note that the terms in the series expressions above for δ_1 and δ_2 fall off at least as fast as $1/n^2$ as n increases. Clearly, these series converge much more rapidly than the Fourier series (5) for the flutter field itself, and as a result, the sums in Eqs. (11b) and (12b) can be terminated at a much lower value of n. A maximum n value $n_{\max} = 10N$ should suffice in all cases, and we often find that $n_{\max} = 5N$ to be quite adequate.

As a final note, we should also point out that the formulas (11) and (12) are completely independent of the spiral. That is, they involve the flutter field only through the parameters K_n and K_n' and, as can be seen from Eq (6), these quantities depend on the amplitudes but not the phases of the Fourier coefficients G_n and H_n characterizing the flutter field. However, experience shows that this conclusion, though a good approximation, is not exactly true. Indeed, one finds that the third-order corrections to the above formulas do depend on the spiral.

4. PROCEDURE

Suppose now that we wish to calculate $B_0(r)$ over a specified range $r_1 \leq r \leq r_2$ so that the angular frequency $\omega = \omega_0$ for the ion orbits within this range. We assume, at least to begin with, that the flutter parameters K_n and K_n' in Eqs. (11) and (12) are known functions of r and; moreover, do not depend on B_0.

Within δ_1 and δ_2 given above, we first replace M' and M'' by their values for zero flutter given in Eq. (10), namely,

$$M' = 1 + p^2, \qquad M'' = 3(1 + p^2)(1 + 2p^2), \tag{15}$$

with p from (2). Actually, this M' should be more accurate than the approximation suggests since for an isochronous field, it is well known that $\nu_r^2 \simeq 1 + p^2$, and we would expect that $M' \simeq \nu_r^2$ here on general grounds.

For each r value, we now set $\omega = \omega_0$ in Eq. (11a) and solve this equation for β by using the following scheme. We first calculate $\beta_0 = r/a$, and then set

$$M' = 1 + p_0^2 = (1 - \beta_0^2)^{-1}, \tag{16}$$

and hence complete the calculation of δ_1 in Eq. (11b). This δ_1 is then fixed during the remainder of the β calculation.

To improve the β value, we use the iteration formula obtained from Newton's method

$$\beta_{j+1} = \frac{\beta_0(1 - \delta_1 + 3\delta_1/\beta_j^2)}{1 + 2\beta_0\,\delta_1/\beta_j^3}, \tag{17}$$

with $j = 0,1,2$, etc. These iterations are continued until $|\beta_{j+1} - \beta_j| < 10^{-6}$, say, and then we finally set $\beta_{j+1} = \beta$.

Although we have never considered it to be necessary, some improvement could presumably be obtained by recalculating δ_1 for each value of β_j within the above loop. One should first check whether the resultant gain in accuracy is enough to warrant the extra computer time.

Having obtained β for the given value of r, we next calculate p from Eq. (2), and hence M' and M'' given in Eq. (15) above. We then evaluate δ_2 in Eq. (12b) and finally obtain $B_0(r)$ from Eq. (12a). The entire calculation is then repeated for each r value within the specified range, $r_1 \leq r \leq r_2$, and the resultant function $B_0(r)$ thereby yields an approximation to the required isochronous field.

Note that if the range of r values includes $r = 0$, then the above procedure should be skipped, and the value $B_0(0) = b$ should be adopted directly, where b is the field unit in Eq. (4b). In addition to giving the correct value, this step will avoid a possible 0/0 division hang-up.

Finally, we should emphasize that although our formulas are relatively good, they are nonetheless only approximately correct. One generally finds that the average field $B_0(r)$ generated by the above procedure does not satisfy the isochronism requirement within the accuracy desired in all cases.

To check the results and, at the same time, to provide a basis for improvements, one needs an equilibrium-orbit code.[7] In addition to ν_r and ν_z, this code provides precise values of the fractional deviation of the orbital frequency $(\omega_0/\omega) - 1$ as a function of the ion's energy E.

REFERENCES

1. L. Smith and A. A. Garren, Lawrence Berkeley Lab, UCRL Rpt. #8598, Jan. 1959.
2. A. A. Garren, Nucl. Instrum. and Methods, **18–19**, 309 (1962).
3. H. L. Hagadoorn and N. F. Verster, Nucl. Instrum. and Methods, **18–19**, 201 (1962).
4. G. Parzen, Ann. of Phys., **15**, 22 (1961).
5. M. M. Gordon and D. A. Johnson, Proc. Sixth Intl. Cyclotron Conf. (Am. Inst. Phys., New York, 1972), 298.
6. G. Bellomo and F. Resmini, IEEE Trans. Nucl. Sci., **NS–26**, 2095 (1979); Nucl. Instrum. and Methods, **180**, 305 (1981).
7. M. M. Gordon and T. A. Welton, Oak Ridge National Lab, ORNL Rpt. #2765, 1959. W. Joho, Swiss Inst. for Nuclear Research, SIN Rpt. #TM–11–07, Feb. 1970.

APPENDIX

1. EQUATION OF MOTION

Consider the motion of a particle of fixed momentum p and charge e in the median-plane magnetic field $B(r, \theta)$, where r and θ are the usual polar coordinates. The differential equation for the trajectory $r = r(\theta)$ of this particle is

$$\frac{d}{d\theta}\frac{\dot{r}}{\sqrt{r^2 + \dot{r}^2}} = \frac{r}{\sqrt{r^2 + \dot{r}^2}} - S(r, \theta), \qquad S(r, \theta) = (e/p)rB(r, \theta), \tag{1.1}$$

where dots are used to denote derivatives with respect to θ. This equation is equivalent to the familiar equation for the radius of curvature of the trajectory, $p = eB\rho$, when ρ is expressed in polar coordinates.

The above differential equation can be derived from a Lagrangian $L = L(r, \dot{r}, \theta)$ given by

$$L = (r^2 + \dot{r}^2)^{1/2} - \int S\, dr. \tag{1.2}$$

According to Hamilton's Principle $\int L d\theta$ is stationary; this integral contains two terms, the first being the arc length of the trajectory, and the second being (e/p) times the flux swept out by the particle's radius vector. The canonical momentum p_r for this case is given by

$$P_r = \frac{\partial L}{\partial \dot{r}} = \frac{\dot{r}}{\sqrt{r^2 + \dot{r}^2}}. \tag{1.3}$$

so that this p_r is then the actual radial momentum divided by p. The Hamiltonian H for this system can be derived from the equation

$$H = H(r, p_r, \theta) = \dot{r}p_r - L,$$

and the result is

$$H = -r(1 - p_r^2)^{1/2} + \int S\, dr. \tag{1.4}$$

The differential equation for the trajectory (1.1) can then be written in canonical form. The foregoing equations give the various traditional formulations of the problem in polar coordinates. The difficulties involved in developing solutions of these equations are in large measure connected with the complicated non-linear involvement of the derivatives.

In most applications of practical importance, the quantity $(\dot{r}/r)^2$ is small, that is to say, the orbits are not badly non-circular, and also fairly well centered. With this essential assumption, the Lagrangian (1.2) is expanded as

$$L = r\left[1 + \frac{1}{2}\left(\frac{\dot{r}}{r}\right)^2 - \frac{1}{8}\left(\frac{\dot{r}}{r}\right)^4 + \frac{1}{16}\left(\frac{\dot{r}}{r}\right)^6 - \cdots\right] - \int S\, dr,$$

and regrouped into two parts, $L = L_0 + \Delta L$, where

$$L_0 = \frac{1}{2}\frac{\dot{r}^2}{r} - \int (S - 1)\, dr,$$

$$\Delta L = -\frac{1}{8}\frac{\dot{r}^4}{r^3} + \frac{1}{16}\frac{\dot{r}^6}{r^5} - \cdots \tag{1.5}$$

Thus, L_0 must be the dominant part of L, and ΔL can be considered as a perturbation. With this in mind, a new variable q is introduced by

$$r = q^2, \tag{1.6}$$

which simplifies the form of L_0. Making this substitution and dividing L by 4 (which does not, of course, alter the equations of motion; actually, L given above differs from the "physical" Lagrangian by a factor p), the new Lagrangian becomes:

$$L_0 = \frac{1}{2}\dot{q}^2 - \int V\, dq,$$

$$V(q, \theta) = \frac{1}{2}q(S - 1),$$

$$\Delta L = -\frac{1}{2}\frac{\dot{q}^4}{q^2} + \frac{\dot{q}^6}{q^4} - \cdots \tag{1.7}$$

Using only the main term L_0 in the Lagrangian, the differential equation for $q = q(\theta)$ becomes simply

$$\ddot{q} = -V(q, \theta). \tag{1.8}$$

It is this equation which will form the basis of the discussion in this report; when necessary, additional correction terms to this equation will be obtained from ΔL of Eq. (1.7). Eq. (1.8) gives good results for equilibrium orbits and requires only minor correction for linear-oscillation theory, but needs more extensive correction for the nonlinear theory. Since the terms in ΔL are of fourth and higher order in the derivatives, the corrections to Eq. (1.8) are of third and higher order. If Eq. (1.8) is rewritten in terms of $r(\theta)$, the result is

$$\frac{\ddot{r}}{r} = \frac{1}{2}\left(\frac{\dot{r}}{r}\right)^2 + 1 - (e/p)rB(r, \theta), \tag{1.9}$$

which differs from the equation used by G. Parzen (Ref. 4) in the inclusion of the second-order term $\frac{1}{2}(\dot{r}/r)^2$ here.

The transformation to the variable q can be made canonical in form by introducing the corresponding momentum p_q defined as

$$p_r = 2p_q/q. \tag{1.10}$$

Substituting the new variables into H of (1.4) and dividing by 4 (Jacobian), we find the new Hamiltonian $H(q, p_q, \theta)$

$$H = (q/2)^2\left[1 - \left(1 - \left(\frac{2p_q}{q}\right)^2\right)^{1/2}\right] + \int V\,dq.$$

As in the case of the Lagrangian, H can be separated into two parts $H_0 + \Delta H$ given by

$$H_0 = \frac{1}{2}p_q{}^2 + \int V\,dq$$

$$\Delta H = \frac{1}{2}\frac{p_q{}^4}{q^2} + \frac{p_q{}^6}{q^4} + \cdots \tag{1.11}$$

The differential Eq. (1.8) for $q(\theta)$ follows directly from H_0. The discussion in this report does not make use of the canonical formalism; however, all the results obtained can be derived within such a framework (see Ref. 3).

1A. *Definition of Field Quantities*

The median-plane magnetic field is split into two parts

$$B(r, \theta) = B_0(r) + B_f(r, \theta), \tag{1A.1}$$

where $B_0(r)$ is the average field and B_f is the flutter field; that is,

$$\langle B_f(r, \theta)\rangle = 0, \tag{1A.2}$$

where angular brackets denote an average over θ with r fixed. As a result, the flutter field can be expressed as a Fourier series in either of the forms

$$\begin{aligned} B_f(r, \theta) &= \sum_{n>0} [G_n(r) \sin n\theta + H_n(r) \cos n\theta] \\ &= \sum_{n>0} B_n(r) \cos n [\theta - \zeta_n(r)], \end{aligned} \tag{1A.3}$$

where B_n is the amplitude of the nth harmonic, and $\theta = \zeta_n(r)$ is the equation of the "spiral" line associated with this harmonic; for a field with N sectors, the summation extends over the values $n = N, 2N, 3N, \ldots$. For analytical simplicity, the complex form of the Fourier Series will be used here; that is,

$$B_f(r, \theta) = \sum_{n>0} \frac{1}{2} Q_n(r) X_n, \tag{1A.4}$$

where X_n will be used throughout as an abbreviation for $\exp(in\theta)$, and now, the summation extends over $n = \pm N, \pm 2N, \ldots$; and the value of Q_n is given by

$$\begin{aligned} Q_n &= H_n - iG_n \\ &= B_n \exp(-in\zeta_n) \end{aligned} \tag{1A.5}$$

in terms of the real field quantities defined above. Since B_f is real, it follows that

$$Q_{-n} = \bar{Q}_n, \tag{1A.6}$$

where the bar on a quantity is used to designate complex conjugate. The use of this complex notation simplifies the analysis and the resultant formulae; in all cases, these formulae can be reduced to real form by the usual procedures with the above definitions.

The function $V(q, \theta)$ of Eqs. (1.7), (1.8) will also be split into two parts

$$\begin{aligned} V(q, \theta) &= M(q) + F(q, \theta) \\ M(q) &= \frac{1}{2}\left(\frac{q}{q_0}\right)\left[\frac{erB_0(r)}{p} - 1\right] \\ F(q, \theta) &= \frac{1}{2}\left(\frac{q}{q_0}\right)\left[\frac{erB_f(r, \theta)}{p}\right], \end{aligned} \tag{1A.7}$$

where it is understood that $r = q^2$ here and q_0 is a "reference" value of q (which will be explained in the next section), introduced to make these quantities explicitly dimensionless. The flutter term $F(q, \theta)$ is then expressed as a Fourier series:

$$F(q, \theta) = \sum F_n(q) X_n,$$

where $$F_n(q) = \frac{1}{2}\left(\frac{q}{q_0}\right)\left[\frac{erQ_n(r)}{2p}\right],$$

and $$F_{-n} = \bar{F}_n,\ F_0 = 0, \tag{1A.8}$$

with $Q_n(r)$ given by (1A.5) above.

2. EQUILIBRIUM ORBITS

The first problem to be dealt with is that of calculating the equilibrium orbits associated with the given magnetic field; the accuracy of this calculation then determines the accuracy to which other orbit characteristics can be calculated. The equilibrium orbit is a simple closed orbit having the same symmetry as the field; that is, for this orbit $r(\theta)$ or $q(\theta)$ is a periodic solution of the equation of motion having the same N-fold periodicity as the sector structure of the magnetic field. Other closed orbits can exist, in addition to the equilibrium orbit, but these so-called "fixed-point" orbits have different periodicity. For example, in an N-sector cyclotron at low energies, there are $2N$ such fixed-point orbits which close in one revolution; these are, however, asymmetric (off-center) orbits.

The equilibrium orbit is specified by a dimensionless variable $y(\theta)$ defined by

$$q(\theta) = q_0[1 + y(\theta)], \tag{2.1}$$

where $q(\theta)$, and hence $y(\theta)$, is a periodic function having period $2\pi/N$; and q_0 is a constant reference value of q which is related to what will be called here the "reference radius" r_0 through the equation $r_0 = {q_0}^2$. As will be shown, it is advantageous to define q_0 by

$$\begin{aligned} \langle q(\theta)\rangle &\equiv q_0, \\ \langle y(\theta)\rangle &= 0, \end{aligned} \tag{2.2}$$

that is, q_0 is the mean value of $q(\theta)$ for the equilibrium orbit. Other definitions which have been used for a "reference radius" r_0 are: (a) that $2\pi r_0$ is the total arc length of the equilibrium orbit; (b) that $p = er_0B_0(r_0)$, where $B_0(r)$ is the average magnetic field; or (c) that r_0 is the mean radius of the equilibrium orbit. Each of these definitions, including the one chosen here, Eq. (2.2), has its advantages, and its corresponding disadvantages. As will be seen in the next section, the resultant values of r_0 differ from each other only in second order. There is, however, a greater analytical simplicity resulting from the definition (2.2) to be used here.

The reference value q_0, or the corresponding reference radius $r_0 = {q_0}^2$, increases uniformly with p and covers the entire range of r values of the given field. It is q_0 (or r_0) rather than p, which is chosen as the independent variable here in all calculations. The reason for this choice is just that it is simpler to determine p from a given q_0 and the values of the field parameters evaluated at $q = q_0$, than to do the reverse. The basic input data for calculation are the Fourier components of the field (1A.3) given as a function of r; at a particular $r = r_0 = {q_0}^2$, there is an equilibrium orbit whose mean q-value is this q_0, and the corresponding p value is then determined.

From Eq. (2.2), it follows that $y(\theta)$ can be expanded into a Fourier series having zero average value. Thus

$$y(\theta) = \sum A_n X_n;$$
$$A_0 \equiv 0; \qquad A_{-n} = \bar{A}_n; \tag{2.3}$$

where A_n are the unknown Fourier coefficients, $X_n \equiv \exp(in\theta)$ as before, and the unmarked summation implies n covers all positive and negative values: $n = \pm N, \pm 2N, \ldots$. The differential equation for $y(\theta)$, which follows from Eq. (1.8) and Eq. (2.1), is then

$$\ddot{y} = -V(y, \theta)$$
$$= -M(y) - F(y, \theta), \tag{2.4}$$

where it is to be understood that these functions are in their dimensionless forms given in Eq. (1A.7) and evaluated at $q = q_0(1 + y)$. Since the equilibrium orbit is periodic, the right-hand side of this equation must average to zero, that is

$$\langle V(y, \theta) \rangle \equiv 0, \tag{2.5}$$

and it is this equation which is used in the next section to determine p as a function of q_0. The proper differential equation for y is then really

$$\ddot{y} = -V(y, \theta) + \langle V(y, \theta) \rangle, \tag{2.6}$$

with the advantage that a solution for y can be obtained in advance of determining p. Although the above equation is approximate, the same principles apply to the exact equation; furthermore, for equilibrium-orbit calculations, this equation is quite adequate.

In all cases of practical interest, the magnitude of y is sufficiently small compared with unity that $V(y, \theta)$ can be expanded in a rapidly converging series in powers of y. The differential equation for y then becomes

$$\ddot{y} = -V(\theta) - yV'(\theta) - \frac{1}{2}y^2V''(\theta) - \cdots$$
$$= -M - F(\theta) - yM' - yF'(\theta) - \frac{1}{2}y^2M'' - \frac{1}{2}y^2F''(\theta) - \cdots, \tag{2.7}$$

where it is to be understood that primes on V, F, M, or the F_n of (1A.8) denote dimensionless derivatives with respect to q as, for example,

$$V'(q, \theta) = q_0 \frac{\partial}{\partial q} V(q, \theta),$$
$$V''(q, \theta) = {q_0}^2 \frac{\partial^2}{\partial q^2} V(q, \theta), \text{ etc;} \tag{2.8}$$

futhermore, whenever the q is not explicitly shown in the functional dependence of all these quantities, it is to be understood that they are evaluated at $q = q_0$, such being the

case for all quantities appearing in Eq. (2.7). As noted above, the average value of the right-hand side of Eq. (2.7) must be subtracted off; thus, for example, the first term $(-M)$ can be immediately dropped, since it is a constant.

If $F(q, \theta)$ were zero, then the equilibrium orbit would be a circle of radius $r_0 = {q_0}^2$; then the required solution of the above differential equation would be $y = 0$, since the average of y must be zero. This being so, a straightforward procedure for obtaining a solution of the differential equation is a perturbation technique wherein F, wherever it occurs, is replaced by λF

$$\ddot{y} = -\lambda F(\theta) - yM' - y\lambda F'(\theta) - \cdots, \tag{2.9}$$

where λ is a parameter introduced solely for the convenience of identifying successive orders in an expansion in the flutter. The value of λ covers the range $0 \leq \lambda \leq 1$, and when the calculation is completed, λ is set equal to unity. The value of $y(\theta)$ is then expanded in a power series in λ:

$$y(\theta) = \lambda y_1(\theta) + \lambda^2 y_2(\theta) + \cdots, \tag{2.10}$$

where the zero-order term is absent since, as noted above, in the absence of F, y is zero. Inserting this series into the differential equation above, and equating the coefficients of a given power of λ on each side, the result is

$$\begin{aligned} \ddot{y}_1 + M'y_1 &= -F(\theta), \\ \ddot{y}_2 + M'y_2 &= -y_1F'(\theta) - \frac{1}{2}M''{y_1}^2, \end{aligned} \tag{2.11}$$

etc. where it is understood again that the average value of the rightside of each equation is to be subtracted off. The result is a system of consecutive simple differential equations for $y_k(\theta)$. In order to obtain the correct third and higher order equations, it would be necessary to add correction terms from ΔL of Eq. (1.7); the first and second order equations are correct as they stand.

These equations can be further simplified by introducing the Fourier series for $F(q, \theta)$ from Eq. (1A.8) and for $y(\theta)$ from Eq. (2.3); the coefficients A_n are likewise expanded in a power series in λ, so that

$$\begin{aligned} A_n &= \lambda {A_n}^{(1)} + \lambda^2 {A_n}^{(2)} + \cdots \\ y_k(\theta) &= \sum {A_n}^{(k)} X_n. \end{aligned} \tag{2.12}$$

Inserting these back into Eq. (2.11), and equating coefficients of X_n (harmonic balancing) in each case, the results are

$$\begin{aligned} (n^2 - M'){A_n}^{(1)} &= F_n, \\ (n^2 - M'){A_n}^{(2)} &= \sum_m \left[{f_m}' + \frac{1}{2}M''{A_m}^{(1)} \right] A^{(1)}_{n-m}, \end{aligned} \tag{2.13}$$

where the sum over m covers all positive and negative values, with no contribution coming from $m = 0$ or $m = n$. The result is a set of consecutive algebraic equations which can be solved in sequence to the order of accuracy desired. Note that F_n, ${F_n}'$ are

simple constants, like M, M' and M'' [see Eq. (2.8)]. The requirement noted above that the average of the right side of Eq. (2.11) be subtracted off, is automatically taken care of here simply by omitting the equations for $n = 0$.

For most purposes, it is sufficient to use only the first-order calculation of A_n above. For a cyclotron, where ν_z is small, F and its derivatives are not very large; each succeeding order in A_n involves an additional factor of F_n (or its derivatives) and at least one additional factor of $(n^2 - M')^{-1}$, and so the convergence is rapid. Even more important is the fact that given A_n to first order plus the condition $A_0 \equiv 0$, it is then possible to calculate most of the important machine parameters correct to second order. In what follows, it will therefore be assumed that

$$A_n = \frac{F_n}{n^2 - M'}, \tag{2.14}$$

which is the first-order expression for A_n given by Eq. (2.13), and the superscript will be dropped. Note that if the main field harmonic $n = N$ predominates over that of $n = 2N$, then the second-order value of A_n for $n = 2N$ can exceed the first-order result.

Another procedure for calculating $y(\theta)$ is to assume that all nonlinear terms of Eq. (2.7) can be neglected so that the differential equation for $y(\theta)$ becomes

$$\ddot{y} + [M' + F'(\theta)]y = -F(\theta). \tag{2.15}$$

This is an inhomogeneous equation of the Mathieu-Hill variety which contains the first-order and most of the second-order results in Eq. (2.11) above. The technique for solving such an equation will be discussed later (see also Ref. 4). One result is that the first-order A_n given in Eq. (2.14) can be improved if M' is replaced by

$$M' + 2\sum_{n>0} \frac{|F_n'|^2}{n^2 - 4M'}, \tag{2.16}$$

provided $N^2 - 4M' > 2|F_N'|^2$ sufficiently; that is, provided the free oscillation of Eq. (2.15) is not too close to the $N/2$ stop-band, in which case Eq. (2.16) is not accurate.

2A. *Mean Radius and Rotation Period*

From the definition of the preceding section, the radius $r_e(\theta)$ of the equilibrium orbit is given by

$$r_e = q^2 = r_0(1 + y)^2 = r_0(1 + 2y + y^2), \tag{2A.1}$$

where $r_0 = {q_0}^2$ is the reference radius. Since by definition the average of $y(\theta)$ is zero, the mean radius R of the equilibrium orbit is then

$$\begin{aligned} R &= r_0(1 + \langle y^2 \rangle) \\ &= r_0\left(1 + 2\sum_{n>0} |A_n|^2\right), \end{aligned} \tag{2A.2}$$

where the Fourier series for $y(\theta)$ of Eq (2.3) has been inserted. It is clear that a knowledge of A_n to first order in the flutter thus gives R correct to second order.

The rotation period T for a particle in the equilibrium orbit is obtained from the equation for the arc length as

$$\frac{vT}{2\pi} = v/\omega = \langle (r^2 + \dot{r}^2)^{1/2} \rangle$$
$$= \left\langle q^2 + 2\dot{q}^2 - 2\frac{\dot{q}^4}{q^2} + \cdots \right\rangle, \tag{2A.3}$$

where v is the speed and the square root has been expanded as in section 1 above. Inserting $q = q(y)$ and keeping only terms to second order in y the result becomes

$$\frac{vT}{2\pi} = r_0\langle 1 + y^2 + 2\dot{y}^2 \rangle$$
$$= r_0\left[1 + 2\sum_{n>0} (1 + 2n^2)\,|A_n|^2 \right], \tag{2A.4}$$

where the Fourier series for $y(\theta)$ of Eq. (2.3) gives the result in terms of A_n. Here again, Eq. (2.14) for the first order A_n value gives this expression correct to second order.

Inserting $q = q(y)$ and keeping only terms to second order in y the result becomes harmonic acceleration) independent of the energy. In this case, the above equation imposes a condition between the energy, the reference radius r_0, and the field parameters evaluated at this radius. Alternatively, for a given field, the above equation supplies one relation needed to calculate the phase slip as a function of energy (the other relation needed is the momentum equation given below).

2B. *Momentum*

As noted before, the periodicity of the equilibrium orbit gives rise to the condition (2.5), namely,

$$\langle V(y, \theta) \rangle = 0, \tag{2B.1}$$

which then provides a relation between the momentum p, the reference radius r_0, and the field parameters evaluated at this radius. The above relation is not exact; however, an examination of the contribution of ΔL, the error in the Lagrangian given by Eq. (1.7), shows that the error in Eq. (2B.1) is of fourth order in the flutter. An exact relation, which is commonly used, is

$$p = e\,\langle B(r, \theta)(r^2 + \dot{r}^2)^{1/2} \rangle,$$

with $r = r_e(\theta)$, the radius of the equilibrium orbit, being inserted.

The expansion of $V(y, \theta)$ to second order is

$$V(y, \theta) = M + F(\theta) + yM' + yF'(\theta) + \frac{1}{2}y^2M'', \tag{2B.2}$$

and since the second and third terms average to zero, Eq. (2B.1) becomes

$$M + \langle yF' \rangle + \frac{1}{2} M'' \langle y^2 \rangle = 0. \tag{2B.3}$$

Inserting the Fourier series representation for $y(\theta)$ and $F'(\theta)$, the resulting equation for M is then

$$M = -\sum_{n>0} [(F_n' \bar{A}_n + \bar{F}_n' A_n) + M'' |A_n|^2]. \tag{2B.4}$$

For A_n given by Eq. (2.14) correct to first order, this equation is then correct to second order in the flutter.

From the definition of $M(q)$ given in Eq. (1A.7), it follows that

$$\frac{er_0 B_0(r_0)}{p} = 1 + 2M \tag{2B.5}$$

which, in combination with (2B.4), is then the (implicit) equation for p. The explicit value of $p = p(r_0)$ can be obtained through an iteration process. Here, where the calculation is valid only to second order, it is sufficient to set $p = er_0 B_0(r_0)$ in calculating M in Eq. (2B.4), and then solve Eq. (2B.5) for p using this value of M. The resultant value of p will be correct to second order.

2C. *Isochronism*

When the magnetic field is specified, the above procedure will give the relativistic momentum p at any r_0 value. From this the corresponding value of v, the velocity can be calculated and then the rotation period T can be evaluated from Eq. (2A.4). In this way T and hence the phase-slip can be evaluated as a function of r_0 and the particle energy.

The above process can be reversed in such a way as to yield a formula (approximate) for $B_0(r)$ such that the isochronism condition is satisfied. To do this, T in Eq. (2A.4) is set equal to the rf period and the equation is solved for $\beta = \beta(r_0)$ as

$$\beta = v/c = (r_0\omega/c)\left[1 + 2\sum_{n>0} (1 + 2n^2)\,|A_n|^2\right], \tag{2C.1}$$

where $\omega = 2\pi/T$. Eq. (2B.5) is then solved for $B_0(r_0)$ to give

$$B_0(r_0) = \frac{b}{\sqrt{1 - \beta^2}} [1 + 2M] \left(\frac{\beta c}{r_0 \omega}\right), \tag{2C.2}$$

where $b = m_0/e$ is the isochronous field value for $\beta = r_0 = 0$. Inserting the value of M from Eq. (2B.4) and the value of β above, the formula for $B_0(r_0)$ is completed.

The formula for $B_0(r_0)$ is an implicit one since M', M'', all of which depend on B_0, are involved therein. Here again the value of $B_0(r_0)$ can be obtained by iteration. The first approximation is the zero-flutter isochronous field obtained by setting all second-order

corrections to β and B_0 equal to zero; the result is

$$\begin{aligned} \beta &= (r_0\omega/c), \\ p/m_0c &= (r_0\omega/c)[1 - (r_0\omega/c)^2]^{-1/2}, \\ B_0(r_0) &= b[1 - (r_0\omega/c)^2]^{-1/2}. \end{aligned} \tag{2C.3}$$

These formulae are then used, as needed, in the calculation of M in Eq. (2B.4) and of the second-order correction to β in Eq. (2C.1). When these results are then inserted into Eq. (2C.2), the formula for $B_0(r_0)$ will be explicit and correct to second order, which is the limit of accuracy of these formulae. The resultant values for the isochronous average field $B_0(r)$ should be adequate for most purposes.

2D. *Field Derivatives*

The equations in the preceding discussion and those to follow involve the quantities $M(q)$, $F_n(q)$, and their derivatives, evaluated at $q = q_0$. The question arises of how to evaluate these quantities starting from a given set of fields, expressed as Fourier coefficients given as functions of r, such as in Eq. (1A.3). A parameter α is introduced which is defined as

$$\alpha = \frac{er_0B_0(r_0)}{p}; \tag{2D.1}$$

then wherever r occurs, it is replaced by $r_0(q/q_0)^2$ so that $M(q)$ and $F_n(q)$ can be written as

$$\begin{aligned} M(q) &= \frac{\alpha}{2}q^3\frac{B_0(r_0q^2)}{B_0(r_0)} - \frac{1}{2}q, \\ F_n(q) &= \frac{\alpha}{4}q^3\frac{Q_n(r_0q^2)}{B_0(r_0)}, \end{aligned} \tag{2D.2}$$

where q/q_0 has been replaced simply by q so that all functions and derivatives are evaluated at $q = 1$ now. In this way, the following list of values is obtained:

$$\begin{aligned} M &= \frac{1}{2}(\alpha - 1), \\ M' &= \frac{1}{2}\alpha(2B_0' + 3) - \frac{1}{2}, \\ M'' &= \alpha(2B_0'' + 7B_0' + 3), \\ M''' &= \alpha(4B_0''' + 24B_0'' + 27B_0' + 3), \end{aligned} \tag{2D.3}$$

where the primes on B_0 represent dimensionless derivatives with respect to r, that is,

$$B_0' = \frac{r_0}{B_0}\frac{dB_0}{dr_0}, \qquad B_0'' = \frac{r_0^2}{B_0}\frac{d^2B_0}{dr_0^2}, \text{ etc.} \tag{2D.4}$$

evaluated at $r = r_0$. By the same process, the following results are also obtained

$$\begin{aligned} F_n &= \frac{\alpha}{4} Q_n{}^0, \\ F_n' &= \frac{\alpha}{4}(2Q_n' + 3Q_n{}^0), \\ F_n'' &= \frac{\alpha}{2}(2Q_n'' + 7Q_n' + 3Q_n{}^0), \\ F_n''' &= \frac{\alpha}{2}(4Q_n''' + 24Q_n'' + 27Q_n' + 3Q_n{}^0), \end{aligned} \tag{2D.5}$$

where, by analogy to (2D.4),

$$Q_n{}^0 = \frac{Q_n}{B_0}; \qquad Q_n' = \frac{r_0}{B_0}\frac{dQ_n}{dr_0}; \qquad Q_n'' = \frac{r_0{}^2}{B_0}\frac{d^2Q_n}{dr_0{}^2} \tag{2D.6}$$

etc, again evaluated at $r = r_0$. The parameter α is carried along during the calculation; it is finally evaluated using Eq. (2B.4). In any second-order expression, it may be replaced by unity, since it differs from this value only in second order.

As a specific example, consider the zero-flutter isochronous field, the pertinent equations for which are given in Eq. (2C.3) above. In terms of p, which here is the momentum in m_0c units, the equation for $M(q)$ is

$$M(q) = \frac{1}{2} q^3(1 + p^2 - p^2q^4)^{-1/2} - \frac{1}{2} q, \tag{2D.7}$$

so that

$$\begin{aligned} M &= 0, \\ M' &= 1 + p^2, \\ M'' &= 3(1 + p^2)(1 + 2p^2), \\ M''' &= 3(1 + p^2)(1 + 16p^2 + 24p^4), \end{aligned} \tag{2D.8}$$

and the expression for $p = (p/m_0c)$ in terms of r_0 is given in Eq. (2C.3). For an isochronous field, these values can be used in evaluating any expression which is otherwise of second order in the flutter.

There is a certain combination of the quantities in Eq. (2D.8) which plays an important role in nonlinear resonance effects. The quantity referred to is given by

$$J_0 \equiv \frac{1}{2} M''' + 6(M')^2 - \frac{5}{6}\frac{(M'')^2}{M'}. \tag{2D.9}$$

Using the values given above for the zero-flutter isochronous field, this quantity is identically zero for all values of p.