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Hard thermal loops and the entropy of supersymmetric Yang-Mills theories

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Abstract: We apply the previously proposed scheme of approximately self-consistent hard-thermal-loop resummations in the entropy of high-temperature QCD to $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) theories and compare with a (uniquely determined) $R_{[4,4]}$ Padé approximant that interpolates accurately between the known perturbative result and the next-to-leading order strong-coupling result obtained from AdS/CFT correspondence. We find good agreement up to couplings where the entropy has dropped to about 85% of the Stefan-Boltzmann value. This is precisely the regime which in purely gluonic QCD corresponds to temperatures above 2.5 times the deconfinement temperature and for which this method of hard-thermal-loop resummation has given similar good agreement with lattice QCD results. This suggests that in this regime the entropy of both QCD and $\mathcal{N} = 4$ SYM is dominated by effectively weakly coupled hard-thermal-loop quasiparticle degrees of freedom. In $\mathcal{N} = 4$ SYM, strong-coupling contributions to the thermodynamic potential take over when the entropy drops below 85% of the Stefan-Boltzmann value.

Keywords: AdS-CFT Correspondence, Thermal Field Theory, QCD, Supersymmetric gauge theory.

Contents

1. Introduction

Much theoretical effort is presently devoted to the study of QCD in a regime where it appears strongly coupled. This effort is mainly triggered by the beautiful data obtained at the Relativistic Heavy Ion Collider, which reveal that matter produced in high energy nucleus-nucleus collisions is strongly interacting, and may, to a good approximation, be described as an ideal fluid with very low ratio of shear viscosity over entropy density $\eta/\mathcal{S}$ (see, e.g., [1] and references therein).

At the same time, new techniques have become available that allow the treatment of strongly coupled gauge theories. In particular, $\mathcal{N}=4$ supersymmetric Yang-Mills theories (SYM) in the limit of large number of colors N and strong 't Hooft coupling $\lambda \equiv g^2 N$ is now often taken as a model for hot QCD. Through the AdS/CFT conjecture [2, 3] one can make predictions for the otherwise inaccessible strong-coupling regime, in particular real-time quantities (such as transport coefficients) [4–8], where lattice is fraught with large uncertainties [9]. While $\eta/\mathcal{S} \sim (\lambda^2 \log \lambda)^{-1} \gg 1$ at weak coupling [10, 11], the AdS/CFT correspondence gives $\eta/\mathcal{S} = 1/4\pi + O(\lambda^{-3/2})$ at large 't Hooft coupling [12, 13], which has been argued to be a universal result in strongly coupled gauge theories with gravity duals [14–16].

In [17] the thermodynamic entropy of $\mathcal{N}=4$ SYM theory at large 't Hooft coupling and large N has been determined from the AdS/CFT correspondence as[1]

$$\mathcal{S}/\mathcal{S}_0 = \frac{3}{4}\left(1 + \frac{15\zeta(3)}{8}\lambda^{-3/2} + \ldots\right). \tag{1.1}$$

The fact that, in lattice QCD, the value of $\mathcal{S}/\mathcal{S}_0$ for temperatures above $3T_c$ appears to be comparable to this strong-coupling result for $\mathcal{N}=4$ SYM theory has led to the further

[1]In [17] as well as refs. [18–20, 13] this result appears with $(2\lambda)^{-3/2}$ in place of $\lambda^{-3/2}$, which corresponds to a definition of the coupling that differs from the (universally adopted) standard usage of g at weak coupling. The latter requires $\lambda \equiv g^2 N = L^4/\alpha'^2 = 4\pi g_{\rm string} N$, where L is the curvature scale of AdS_5, α' the string tension, and $g_{\rm string}$ the string coupling constant (see also [21]). The comparison of weak and strong coupling results in refs. [18–20, 13, 11] (with the exception of [3]) is therefore incorrect; ref. [22] does have compatible weak and strong coupling results (apart from mixing λ with λ^2), but quotes a then incompatible relation between λ and $g_{\rm string}$.

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suggestion that also for such temperatures QCD is still in a strong-coupling regime. While in this range of temperature the violation of scale invariance in QCD is negligible (in the sense that $(\epsilon - 3P)/\epsilon$ is fairly small), and therefore does not spoil the direct comparison with (conformally invariant) $\mathcal{N} = 4$ SYM theory, we shall argue that such a comparison rather suggests a description in terms of effectively weakly coupled degrees of freedom for both theories.

That such a description can be successful should not be taken as implying that the system as a whole is literally weakly coupled (typically $\lambda \gtrsim 1$), nor that its properties can be obtained through a straightforward expansion in powers of λ. In fact, the weak-coupling expansion for the thermodynamical potential of $\mathcal{N} = 4$ SYM theory is presently known to order $\lambda^{3/2}$ [18, 23, 22, 20],

$$\mathcal{S}/\mathcal{S}_0 = 1 - \frac{3}{2\pi^2}\lambda + \frac{\sqrt{2}+3}{\pi^3}\lambda^{3/2} + \ldots \tag{1.2}$$

but the extremely poor convergence of this expansions seems to limit its usefulness to very small values of the coupling, where the deviation from the ideal-gas result is only of the order of a few percent (see figure 2 below). In particular, this precludes any comparison with the corresponding results at strong coupling. This situation is indeed very much like in real QCD, where the perturbative series for the thermodynamic potentials of hot QCD [24 − 27] shows convergence only for temperatures more than 10^5 times the deconfinement temperature where the coupling is so small that deviations from ideal-gas thermodynamics are minute.

However, the same impasse appears also in such simple theories as unbroken massless $\mathrm{O}(N)$ ϕ^4 theories, where in the limit $N \to \infty$ the entropy is given by an interaction-free expression for quasi-particles with thermal masses determined by a one-loop gap equation [28], while the corresponding perturbative expansion is ill behaved except for very small values of the (scalar) 't Hooft coupling. As this example shows, the failure of finite-temperature perturbation theory in nonabelian gauge theories is not necessarily due to specifically nonabelian nonperturbative effects, but largely due to an incomplete resummation of screening effects (at least as far as thermodynamical quantities dominated by hard degrees of freedom, such as the entropy, are concerned).[2] For scalar theories, a corresponding reorganization of perturbation theory has been proposed in refs. [30, 31] using simple effective mass terms, and generalized to gauge theories in refs. [32, 33] by replacing the simple mass term by the gauge-invariant hard-thermal-loop (HTL) effective action [34 − 36]. However, this method generally suffers from artificial ultraviolet divergences and moreover does not take into account that the HTL effective action is applicable at hard momentum scales only for small virtuality.

[2] Of course, nonabelian gauge theories do have specific nonperturbative phenomena such as confinement in the chromomagnetostatic sector, characterized by the scale g^2T. Depending on the quantity under consideration, the latter can show up in rather low orders of perturbation theory [29], but in the thermodynamic quantities pressure or entropy, they appear only at 4-loop order, parametrically suppressed by 6 powers of the coupling g. We are primarily interested in locating the regime where a weak-coupling expansion makes sense (sufficiently above the deconfinement transition in the case of QCD) and where we expect such contributions of order g^6 to be comparatively small.

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In refs. [37 − 39], three of us have proposed a different scheme to resum HTL effects which is manifestly ultraviolet finite and applies the (non-local) HTL corrections only in kinematical regimes where these are applicable at weak coupling. When applied to QCD, we have found reasonable agreement with available lattice data (see figure 1) for temperatures down to about 3 times the deconfinement temperature,[3] suggesting that there the entropy of the quark-gluon plasma can be understood in terms of weakly interacting HTL quasiparticles, the main effect of the interactions being incorporated into a renormalization of the quasiparticle properties.

In the following, we shall recapitulate our HTL resummation scheme for the entropy and extend our results to SYM theories, where they can be compared with the strong-coupling result of eq. (1.1). In ref. [42], we have recently tested our approach successfully in the exactly solvable case of large-N_f gauge theories, which is however essentially Abelian. The results from the AdS/CFT correspondence deal with the other extreme of infinite number of colors.

2. Quasiparticle entropy and HTL resummation

Expressed in terms of (self-consistently) dressed propagators (D and S for bosonic and fermionic ones) and dressed self-energies (Π and Σ, respectively), the entropy admits the following (exact) representation [43, 38, 39]:

$$\begin{aligned}\mathcal{S} = & -\mathrm{tr} \int \frac{d^4k}{(2\pi)^4} \frac{\partial n(\omega)}{\partial T} \Big[\mathrm{Im} \log D^{-1} - \mathrm{Im}\, \Pi \, \mathrm{Re}\, D \Big] \\ & -2\, \mathrm{tr} \int \frac{d^4k}{(2\pi)^4} \frac{\partial f(\omega)}{\partial T} \Big[\mathrm{Im} \log S^{-1} - \mathrm{Im}\, \Sigma \, \mathrm{Re}\, S \Big] + \mathcal{S}' \end{aligned} \tag{2.1}$$

(n and f are Bose-Einstein and Fermi-Dirac distributions, respectively), where $\mathcal{S}'$ is a 3-loop order quantity that, loosely speaking, describes residual interactions of the quasiparticles: the bulk of the interactions have been incorporated in the spectral data determining the properties of these quasiparticles through the dressed propagators.

In practice, some approximations are needed in order to evaluate the quantities which appear in eq. (2.1). Our approximation scheme is based on a skeleton expansion truncated at two loop order, whereby $\mathcal{S}' = 0$. Furthermore, the exact expressions for the self-energies that would result from solving self-consistent gap equations are approximated by their hard-thermal loop expressions, in a way that is described below. This further approximation has the virtue of leading to gauge invariant results, which would otherwise not be the case in such an approach.

In perturbation theory, HTL self energies provide the relevant leading-order corrections for soft momenta $\omega, k \sim gT$ as well as for hard momenta, provided the latter have small virtuality. One of the merits of the entropy expression (2.1) in comparison with the usual loop-wise expansion of the pressure (whose first derivative with respect to temperature also yields the entropy) is that the contributions of order g^2 and g^3 are generated by either

[3] Similar agreement can be obtained by suitable renormalization-scale optimizations of the three-loop result from dimensional reduction [40, 41].

soft momenta or hard momenta with small virtuality, for which the HTL approximation is meaningful.

Indeed, to leading order in g^2, all interaction effects are encoded in the (asymptotic) thermal masses of hard excitations and summarized by the simple formula [38, 39]

$$\mathcal{S}_2 = -T \left\{ \sum_b \frac{m^2_{\infty(b)}}{12} + \sum_f \frac{m^2_{\infty(f)}}{24} \right\}, \tag{2.2}$$

where the sum is over all bosonic and fermionic degrees of freedom. The asymptotic thermal masses can be read from the HTL self energies, for which they provide the scale and which read explicitly

$$\hat{\Pi}_T = m^2_{\infty(g)} + \frac{\omega^2 - k^2}{2k^2} \Pi_L, \tag{2.3}$$

$$\hat{\Pi}_L = 2m^2_{\infty(g)} \left(1 - \frac{\omega}{2k} \log \frac{\omega + k}{\omega - k}\right) \tag{2.4}$$

for the spatially transverse and longitudinal gauge bosons and

$$\hat{\Sigma}_\pm = \frac{m^2_{\infty(f)}}{2k} \left(1 - \frac{\omega \mp k}{2k} \log \frac{\omega + k}{\omega - k}\right) \tag{2.5}$$

for the parts of the fermionic self energy with chirality equal/opposite to helicity, while for scalars there is no momentum dependence, $\hat{\Pi}_s \equiv m^2_{\infty(s)}$.

Specializing to unbroken SYM theories (without matter in the fundamental representation), for the possible choices of $\mathcal{N}$ we have a number of adjoint scalar (n_s) and fermionic (n_f) degrees of freedom in addition to the two polarizations of the gauge bosons as given by table I ($\mathcal{N}$=0 corresponding to non-supersymmetric pure-glue QCD). The asymptotic thermal masses of the gauge bosons, adjoint scalars, and adjoint fermions turn out to be given by

$$m^2_{\infty(s)} = m^2_{\infty(g)} = \frac{2 + n_s + n_f/2}{12} \lambda T^2, \tag{2.6}$$

$$m^2_{\infty(f)} = \frac{2 + n_s}{8} \lambda T^2, \tag{2.7}$$

which coincide when supersymmetry is realized (whereas only $m_{\infty(g)}$ is present anyway when $\mathcal{N}$=0).

Writing

$$\mathcal{S}/\mathcal{S}_0 = 1 + a_2 \frac{\lambda}{\pi^2} + a_3 \frac{\lambda^{3/2}}{\pi^3} + \ldots, \tag{2.8}$$

the coefficient a_2 is determined by these results for the asymptotic thermal masses through (2.2) as tabulated in table 1 for the possible values of $\mathcal{N}$.

At low momenta $p \lesssim \lambda^{1/2} T$, the degeneracy of the thermal masses of the various excitations is broken, but to leading order they are determined by the structure of the HTL effective action in a universal fashion. This gives different momentum dependent HTL masses for gauge bosons and fermions, while the HTL mass of scalar particles is momentum

$\mathcal{N}$	n_s	n_f	$m_\infty^2/\lambda T^2$	g_*/N_g	a_2	a_3	λ_*
0	0	0	1/6	2	−5/16	$5/4\sqrt{3}$	1.85
1	0	2	1/4	15/4	−3/8	$1/\sqrt{2}$	2.78
2	2	4	1/2	15/2	−3/4	$1+1/\sqrt{2}$	1.91
4	6	8	1	15	−3/2	$3+\sqrt{2}$	1.14

Table 1: Number of adjoint scalar (n_s) and fermionic (n_f) degrees of freedom for different $\mathcal{N}$, and the resulting common asymptotic thermal mass for all excitations. Also given are the effective numbers of freedom as measured by $g_* \equiv \mathcal{S}_0/(2\pi^2 T^3/45)$; the coefficients a_2 and a_3 in the perturbative expression for $\mathcal{S}/\mathcal{S}_0$; and the values $\lambda = \lambda_*$ where $\mathcal{S}_3 = |\mathcal{S}_2|$.

independent. For gauge bosons, the static Debye mass differs from the asymptotic thermal mass according to $m_D^2 = 2m_{\infty(g)}^2$, while the plasma frequency in the gauge boson propagator is given by $\omega_{pl}^2 = \frac{2}{3}m_{\infty(g)}^2$ and the fermionic plasma frequency by $M_f^2 = \frac{1}{2}m_{\infty(f)}^2$.

Perturbation theory at finite temperature is made ill-behaved by the appearance of so-called plasmon terms $\propto \lambda^{3/2}$, reflecting the existence of collective phenomena,

$$\frac{\mathcal{S}_3}{N_g} = \frac{1}{3\pi}\left\{m_D^3 + n_s m_{\infty(s)}^3\right\} = \frac{2\sqrt{2}+n_s}{3\pi}m_\infty^3, \tag{2.9}$$

where $N_g = N^2 - 1$. For the different $\mathcal{N}$ this yields the coefficients a_3 in (2.8) as tabulated in table 1. The plasmon term renders $\mathcal{S}/\mathcal{S}_0$ a nonmonotonic function of λ, which after going through a minimum attains again the Stefan-Boltzmann value at the uncomfortably low values $\lambda = \lambda_*$ as given in table 1, the minimum occurring at $\lambda = \frac{4}{9}\lambda_*$. Clearly, strict perturbation theory breaks down for such couplings.

However, as the example of O($N \to \infty$) $\lambda\phi^4$ theory shows [44], such a breakdown of perturbation theory even occurs in an essentially free theory for quasiparticles ($\mathcal{S}' = 0$ in this theory) when the coupling constant dependence of the entropy expression (2.1) through thermal masses is expanded out. In refs. [37–39] we have therefore proposed to evaluate the thermal self energies in HTL theory to leading and next-to-leading order and to keep the entropy expression in eq. (2.1) unexpanded in the coupling constant.

Approximating thus Π and Σ in (2.1) by the leading-order HTL expressions (2.3)–(2.5) turns out to take into account all of $\mathcal{S}_2$. However, only a quarter of the plasmon term $\mathcal{S}_3$ is due to the entropy of the soft excitations in the HTL approximation, the remaining three quarters arise from next-to-leading order corrections of the asymptotic masses of the various excitations. These corrections, which are due to the coupling of the hard degrees of freedom with the soft collective excitations, are momentum dependent even for hard momenta $k \sim T$, but to order $\lambda^{3/2}$ only their average, weighted according to

$$\bar{\delta}m_\infty^2 = \frac{\int dk\, k\, n'(k)\, \mathrm{Re}\,\delta\Pi(\omega = k)}{\int dk\, k\, n'(k)}, \tag{2.10}$$

is relevant (with a similar expression for the fermionic case).

For the minimally supersymmetric case ($\mathcal{N} = 1$), the solution to the asymptotic thermal masses can be taken over from ref. [38], eq. (13), by replacing Casimir factors of the fundamental representation by those of the adjoint one. This shows that the degeneracy of the asymptotic thermal masses in supersymmetric theories carries over to $m_\infty^2 + \bar\delta m_\infty^2$. Generalizing to $\mathcal{N} > 1$ and $n_s \neq 0$, we find

$$\bar\delta m_\infty^2 = -\lambda T m_\infty \frac{2\sqrt{2} + n_s}{4\pi}, \tag{2.11}$$

which is valid for all the cases covered by table I.

Interpreted strictly perturbatively, this leads to tachyonic masses for $\lambda \gtrsim \lambda_*$. However, in scalar $\mathrm{O}(N \to \infty)$ models, where the same difficulty occurs, approximating the true gap equation by one that is quadratic in m gives surprisingly accurate results even for very large coupling. This just corresponds to putting

$$m_\infty^2 = m_\infty^2|_{\mathrm{HTL}} - \lambda T m_\infty \frac{2\sqrt{2} + n_s}{4\pi}, \tag{2.12}$$

where the notation $m_\infty^2|_{\mathrm{HTL}}$ is now used for the leading-order quantities in eqs. (2.6)–(2.7), and m_∞ is defined as the solution to the above equation.[4]

In ref. [39] we have proposed to employ such quadratic gap equations for correcting the mass of hard excitations by using (2.12) as the prefactor in the HTL self energies (2.3) and (2.4), but only when the momentum is hard as specified by a dividing scale $\Lambda = \sqrt{2\pi T m_D c_\Lambda}$. This is to take into account that soft excitations are known to have rather different corrections in gauge theories (e.g. the Debye mass squared at next-to-leading order receives positive and moreover logarithmically enhanced corrections to order $\lambda^{3/2}$ [29], which equally applies for the adjoint scalars[5]). The introduction of Λ means that for soft excitations the leading-order HTL masses are kept untouched.[6] As part of the theoretical error estimate, c_Λ is being varied in the range $\frac{1}{2} \ldots 2$. In QCD, an even greater uncertainty comes however from the dependence of the coupling on the renormalization scale $\bar\mu_{\mathrm{MS}}$ whose central value we choose as $2\pi T$, which is close to the optimal value considered in ref. [45].

3. Results and discussion

For comparison with our new results for $\mathcal{N} = 4$ SYM which we display further below, we start by exhibiting the results of our corresponding previous calculation in pure glue QCD. In figure 1 the result obtained in ref. [39] is reproduced, where the band marked "NLA" corresponds to the HTL entropy evaluated to next-to-leading order according to the above procedure with $c_\Lambda = \frac{1}{2} \ldots 2$ and $\bar\mu_{\mathrm{MS}} = 2\pi T$. In the larger band bounded by dashed lines, the renormalization scale is also varied, $\bar\mu_{\mathrm{MS}} = \pi T \ldots 4\pi T$. (The lines marked $c_\Lambda = 0$

[4] Since this procedure may look rather *ad-hoc* at this point, let us emphasize that, in the case of the scalar theory at least, it corresponds to a meaningful approximation to the actual gap equation required by the condition of self-consistency [39].

[5] The term proportional to $\log\lambda$ in the NLO correction to the adjoint scalars' screening mass is in fact identical to that of the Debye mass, which in the static case is likewise carried by an adjoint scalar, namely A_0.

[6] Repeating the procedure of ref. [39], the separation scale is only introduced for bosonic modes which are solely responsible for the plasmon effect.

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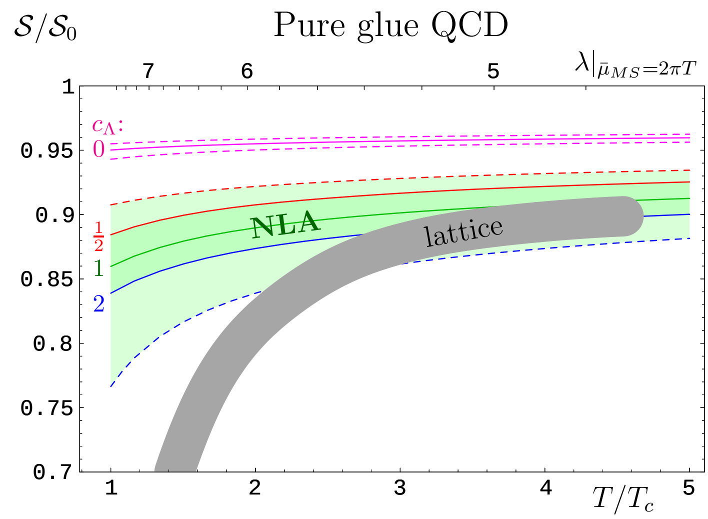

Figure 1: The HTL resummed entropy density of pure glue QCD, divided by its Stefan-Boltzmann value as calculated in a next-to-leading approximation (NLA) according to ref. [39]. The full lines are evaluated with renormalization scale $\bar\mu_{\rm MS} = 2\pi T$ and different c_Λ (see text). The upper and lower dashed lines correspond to renormalization scales $\bar\mu_{\rm MS} = 4\pi T$ and πT, respectively. The dark band represents the continuum-extrapolated lattice results of ref. [46].

correspond to also rescaling the HTL masses of soft excitations by the same factor as the asymptotic thermal mass, which, as we argued above, does not do justice to the former.) These results are compared with the continuum-extrapolated lattice results of ref. [46] where the width of the dark band is meant as a rough indicator of the lattice errors (the more recent results of ref. [47] would be centered about the upper limit of this band). For temperatures larger than about 3 times the phase transition temperature T_c one finds a remarkably good agreement between our NLA approximation and the lattice results, which is perhaps even better than one would have the right to expect (given the relatively large values of the corresponding coupling 'constant', $g^2_{\rm QCD} = \lambda/3$, with $\lambda|_{\bar\mu_{\rm MS}=2\pi T}$ as indicated on the top of figure 1).

The strong-coupling results provided by AdS/CFT correspondence now give us an opportunity to test our procedure in the case of $\mathcal{N} = 4$ SYM theory. While no lattice results are available, the weak-coupling and the strong-coupling results to order $\lambda^{3/2}$ and $\lambda^{-3/2}$, respectively, together should give an idea of the true result. To make this a bit more quantitative, we shall construct a Padé approximant[7] which reproduces both, the weak-coupling and the strong-coupling results, to the order they are known by a rational function of $\lambda^{1/2}$ or, equivalently $\lambda^{-1/2}$. The fact that at both zero and infinite coupling a finite result is obtained requires that this rational function has polynomials of equal degree in numerator and denominator. It turns out that there is just sufficient information to fix

[7] Padé approximations have already been considered in ref. [22], however only for the weak-coupling part. The two variants considered in ref. [22] turn out to coincide numerically rather closely with the NLA results for $c_\Lambda = 1$ and $1/2$ (see figure 2 for these latter results).

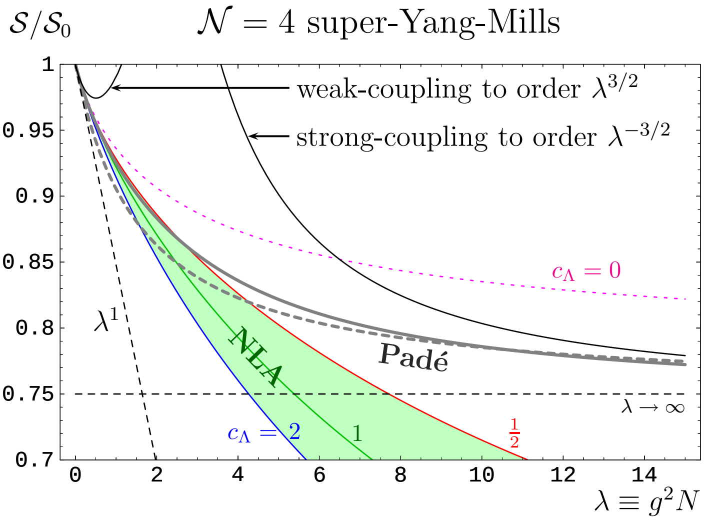

Figure 2: Weak and strong coupling results for the entropy density of $\mathcal{N} = 4$ SYM theory together with the NLA results obtained in analogy to QCD (cp. figure 1), but as a function of λ, which here is a free parameter. The dashed and full heavy gray lines represent the Padé approximants $R_{[1,1]}$ and $R_{[4,4]}$ which interpolate between weak and strong coupling results to leading and next-to-leading orders, respectively.

uniquely all the coefficients in the $R_{[4,4]}$ approximant

$$R_{[4,4]} = \frac{1 + \alpha\lambda^{1/2} + \beta\lambda + \gamma\lambda^{3/2} + \delta\lambda^2}{1 + \bar{\alpha}\lambda^{1/2} + \bar{\beta}\lambda + \bar{\gamma}\lambda^{3/2} + \bar{\delta}\lambda^2}. \tag{3.1}$$

The fact that the weak coupling result has no term of order $\lambda^{1/2}$, while the strong-coupling result approaches $3/4$ at large couplings, where it has no terms of order $\lambda^{-1/2}$ or λ^{-1}, completely constrains the coefficients in the denominator of (3.1) in terms of those in the numerator:

$$\bar{\alpha} = \alpha, \quad \bar{\beta} = \frac{4}{3}\beta, \quad \bar{\gamma} = \frac{4}{3}\gamma, \quad \bar{\delta} = \frac{4}{3}\delta. \tag{3.2}$$

The remaining four independent coefficients are then uniquely fixed by the results quoted above for the weak-coupling expansion to order $\lambda^{3/2}$ and, respectively, the strong-coupling expansion to order $\lambda^{-3/2}$, together with the expectation that the subsequent term in the latter should be suppressed by at least a factor of λ^{-1}. One thus finds

$$\begin{aligned} \alpha &= \frac{2(9 + 3\sqrt{2} + \gamma\pi^3)}{9\pi}, & \beta &= \frac{9}{2\pi^2}, \\ \gamma &= \frac{2}{15\zeta(3)}, & \delta &= \frac{2}{15\zeta(3)}\alpha\,. \end{aligned} \tag{3.3}$$

Gratifyingly, the resulting rational function has no pole at positive values of $\lambda^{1/2}$ and it interpolates monotonically between zero and infinite coupling, as commonly expected [3]. In figure 2, the Padé approximant (3.1) to the weak- and strong-coupling results is represented by the heavy gray line marked "Padé". In order to get an idea of the convergence of successive Padé approximants, the simpler Padé approximant $R_{[1,1]}$ which only reproduces the weak-coupling result to order λ^1 and the constant limit at infinite coupling is also shown (dashed gray line). This comparison suggests that the true result should be not too far from the $R_{[4,4]}$ result, which we thus adopt for assessing the HTL-resummed (NLA) entropy discussed above. Evaluating the latter precisely in the same way as we did in QCD (but with less uncertainty, since because of the finiteness of the $\mathcal{N}=4$ SYM theory there is no ambiguity from renormalization-scale dependence), we obtain a result that for[8] c_Λ between 1/2 and 2 agrees remarkably well with the Padé results up to a coupling $\lambda \sim 3$, whereas the ordinary weak-coupling result to order $\lambda^{3/2}$ fails for all $\lambda \gtrsim 0.5$.

After this comparison within $\mathcal{N}=4$ SYM, we may also try to compare the situation between the latter theory and QCD. At the top of figure 1 we have indicated the values of $\lambda = 3g^2$ for (purely gluonic) QCD, when renormalized with two-loop running coupling at the scale $\bar{\mu}_{MS} = 2\pi T$ and for $T_c = 1.14\Lambda_{\rm QCD}$. In refs. [48, 8, 11] (see also ref. [21]), where recently other weak and strong coupling results of $\mathcal{N}=4$ SYM theory have been compared, it has been argued that values of λ in QCD ($\lambda_{\rm QCD}$) and in $\mathcal{N}=4$ SYM ($\lambda_{\rm SYM}$) should be related such that comparable thermal effects like screening masses arise. Now, (leading-order perturbative) thermal masses in $\mathcal{N}=4$ SYM theory are 6 times as large as in purely gluonic QCD, just reflecting the much higher number of degrees of freedom for a given color number N. So, by itself, this comparison would suggest a factor of 6 of difference between $\lambda_{\rm QCD}$ and $\lambda_{\rm SYM}$. However, instead of the asymptotic thermal masses, one could also compare the normalization of $\mathcal{S}_0$, the ratio $\mathcal{S}_2/\mathcal{S}_0$ or $\mathcal{S}_3/\mathcal{S}_2$ as done in table I, so that the reduction of $\lambda_{\rm SYM}$ over $\lambda_{\rm QCD}$ could be argued to be anywhere between 1/6 and 1/1.6.

In fact, the lattice results for the entropy of (pure glue) QCD and the Padé estimate of the entropy of SYM present yet another possibility for such a comparison, and to that aim we shall focus our attention at larger temperatures in QCD, where $\mathcal{S}/\mathcal{S}_0$ flattens out. (Close to the phase transition the entropy of pure glue QCD becomes very small, whereas $\mathcal{S}/\mathcal{S}_0$ in $\mathcal{N}=4$ SYM approaches 3/4 at very large coupling.) In QCD, $\mathcal{S}/\mathcal{S}_0$ flattens out for $T \geq 3T_c$, corresponding to $\lambda_{\rm QCD} \leq 5.3$. The pure glue 4D lattice results, which are available up to about $5T_c$, have $\mathcal{S}/\mathcal{S}_0$ interpolating between about 0.87 and 0.9 (a few percent larger values are obtained in the lattice simulations of ref. [47]).[9] Taking the Padé approximant (3.1) for $\mathcal{S}/\mathcal{S}_0$ as plausible estimate for the true nonperturbative result in $\mathcal{N}=4$ SYM, a comparison of the two theories then leads one to map the regime $T \geq 3T_c$ of pure glue QCD to couplings $\lambda_{\rm SYM} \leq 2.5$. At such values neither the (strict) weak-coupling nor the strong-coupling expansion is giving reasonable approximations, however

[8] Just as in the case of QCD we find that it is indeed crucial that the correction (2.11) is applied only for hard excitations, as we have done by introducing a separation of hard and soft physics by the scale $\Lambda = \sqrt{2\pi T m_D c_\Lambda}$ and $c_\Lambda \sim 1$, which we take as validation of our procedure proposed in the case of QCD. Note that for this separation to make sense it is essential that $m_D \ll 2\pi T$, i.e., $\lambda \ll 2\pi^2 \sim 20$, which is indeed fulfilled in the range where our NLA result is plotted for $\mathcal{N}=4$ SYM.

[9] The lattice results for QCD that are frequently referred to are usually results with quarks which have not been continuum extrapolated and so typically give somewhat smaller numbers. Reliable continuum extrapolations have so far been done only for purely gluonic QCD. According to ref. [49] the continuum extrapolation of the lattice results with quarks can be expected to increase the measured values to bring them close to the pure glue ratio.

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our ‘NLA’ resummation of perturbation theory is remarkably close to the Padé results. (Looking only at the strong-coupling result to order $\lambda^{-3/2}$ one would be led instead [21] to couplings $\lambda \sim 5.5$ and thus closer to a direct identification of $\lambda_{\rm QCD}$ and $\lambda_{\rm SYM}$. However, for such a value of λ, the NLO result in the strong-coupling expansion deviates already significantly from the LO corresponding result of 3/4, thus suggesting that one should expect large corrections from higher orders, as indeed suggested by the Padé results.)

If $\mathcal{N} = 4$ SYM theory really provides some guidance here, this would indicate that, above $3T_c$, the QCD coupling is not large enough to drive the system into a regime that resembles that of the strong-coupling limit of SYM theories. This value of the coupling is admittedly too large to validate naive perturbative calculations, but not so large as to prevent the simple reorganizations of the perturbative expansion that we have presented here, and which seems to imply that the entropy of both theories for the corresponding couplings can be accounted for in terms of effectively weakly coupled (HTL) quasiparticle degrees of freedom.

Evidently, there is important additional nonperturbative physics at couplings $\lambda \gtrsim 4$ in SYM as there is in QCD for $T \lesssim 2.5T_c$, though in these respective regimes the behavior of the entropy in the two theories is hardly comparable.

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