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NONLINEAR EFFECTS IN THE SINUOUS-INSTABILITY OF THE ELECTRON-ION RING

N. YU. KAZARINOV, A. B. KUZNETSOV, E. A. PERELSTEIN, S. B. RUBIN, and V. F. SHEVTZOV

Joint Institute for Nuclear Research, Dubna, USSR

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Transverse oscillations of the electron–ion ring are considered taking account of the nonlinearity of the polarization forces. The existence of nonlinear stationary waves is pointed out as well as nonlinear stabilization of the sinuous-instability in the small supercriticality regime.

Recently the sinuous-instability of an electron-ion ring, considered by Budker in 1956,[1] is of interest again[2–4] in connection with the development of the collective method of acceleration.[5] There are some restrictions on energy gain and the number of ions accelerated by electron rings.

The linearity of polarization forces at the electron-ion rings due to the displacement of the local centres of mass (the centres of mass of the particles at any azimuth) was assumed in Refs 1–3. The same assumption was made in Ref. 6 where the stabilizing effect of focusing fields was examined. In fact, the polarization forces are of a nonlinear nature. For example, in the case of two elliptical cross-section cylinders with a uniform density, calculations show that the polarization forces have a maximum divergence at the local centres of mass of the order of a half axis. The polarization force pattern for two circular cylinders of equal radii is presented in Figure 1 where S is the ratio of the distance between the local centres of mass to the radius; f is the polarization force in dimensionless units.

We assume that the curvature of the ring and small inhomogeneity of the density do not change the nature of the polarization forces. Their linear growth decreases at distances of order comparable to the small dimensions of the rings.

We shall take into account the weakening of the forces in the case of beams having a plane of symmetry by adding a cubic term into the polarization force expression of Refs. 1, 2, and 6. It is clear that the decrease of the electron-ion ring coupling provides a greater stability of mutual oscillations. In this paper we demonstrate the existence of nonlinear stationary waves due to the nonlinear decrease of polarization forces, and also the nonlinear stabilization of the sinuous instability under certain conditions.

1 NONLINEAR STATIONARY WAVE

Taking into account the decrease of the polarization forces we rewrite the equations for the mutual oscillations of the electron and ion beams[6]

$$\frac{\partial^2 x_1}{\partial t^2} + 2\omega_0 \frac{\partial^2 x_1}{\partial t\, \partial \vartheta} + \omega_0^2 \frac{\partial^2 x_1}{\partial \vartheta^2} + \lambda^2 x_1 = \alpha_1(x - px^3), \quad (1)$$

$$\frac{\partial^2 x_2}{\partial t^2} = -\alpha_2(x - px^3). \quad (2)$$

In these formulae $x = x_2 - x_1$; x_1 and x_2 are the coordinates of the displacement of the local centres of mass from the equilibrium values for the electron and ion beams, respectively, which depend on the time t and the azimuthal coordinate ϑ; ω_0 is the angular frequency of the electrons; α_1, α_2 are the squares of the oscillation frequencies of an electron in the ion field and of an ion in the electron field, respectively; p is a factor related to the nonlinearity of the polarization forces; λ^2 is the focusing field gradient. In the stationary state $x_1 = x_2 = 0$. In the absence of the focusing field we look for the solution of Eqs. (1) and (2) as a nonlinear stationary wave. We suppose that x_1 and x_2 depend only on the variable $z = \omega t - n\vartheta$ where ω is a constant, n is an

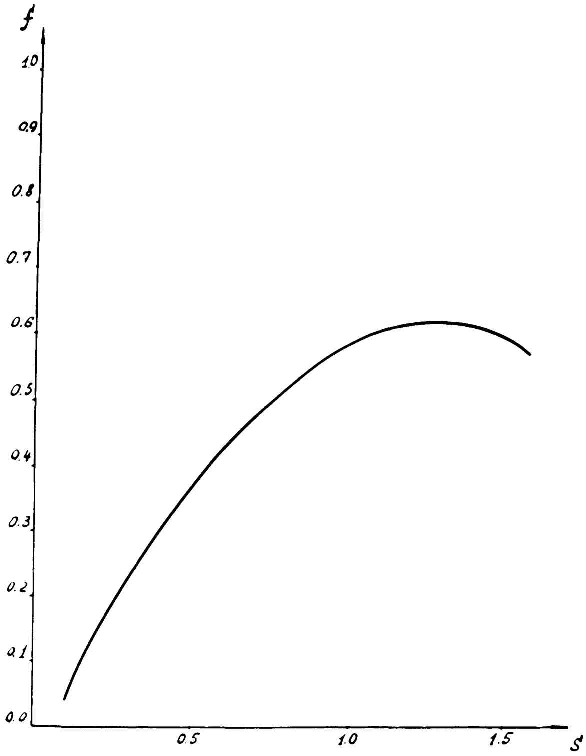

FIGURE 1 The sketch of the polarization force as a function of the distances between the cylinder centres.

integer. From the systems (1) and (2) we get

$$\frac{d^2x}{dz^2} + \Omega^2 x - \mu x^3 = 0, \tag{3}$$

where

$$\Omega^2 = \frac{\alpha_1}{(\omega - n\omega_0)^2} + \frac{\alpha_2^2}{\omega^2}, \qquad \mu = p\Omega^2. \tag{4}$$

Equation (3) is the so-called Duffing equation. Its solution is[7]

$$x = C\,\mathrm{sn}[\sigma(z + h), \kappa], \tag{5}$$

where C and σh are the amplitude and the phase, respectively. These values are obtained from the initial conditions. The quantities σ^2 and κ^2 are related to C as follows

$$\kappa^2 = \frac{pC^2}{2 - pC^2}, \tag{6}$$

$$\sigma^2 = \left(1 - \frac{pC^2}{2}\right)\Omega^2. \tag{7}$$

The solution (5) has a physical meaning for $pC^2 \leq 1$. This solution must be azimuthally periodic. Using the periodicity of the elliptic sine with respect to the argument with a period equal to $4K(\kappa)$, where K is an elliptic integral, we obtain the dispersion equation

$$\Omega^2 = \frac{\alpha_1}{(\omega - n\omega_0)^2} + \frac{\alpha_2}{\omega^2} = \frac{4}{\pi^2}\frac{K^2(\kappa)}{1 - \dfrac{pC^2}{2}} \tag{8}$$

The value of ω is defined by this equation.

The right side of (8) increases monotonically from 1 at $C = 0$ to ∞ at $pC^2 = 1$. The patterns of the function $\Omega^2(\omega)$ are illustrated in Figures 2 and 3. There are four points of intersection of the curve $\Omega^2(\omega)$ with the straight line $f(\omega) = 1$ in the first case. It means that four nonlinear stationary waves are possible. The amplitudes of the waves are limited by $0 \leqslant C \leqslant 1/\sqrt{p}$. The minimum of $\Omega^2(\omega)$ is displaced above the straight line $f(\omega) = 1$ in the second case. There are only waves with amplitudes larger than the minimum one defined by the relation

$$\Omega^2(\omega_{\min}) = \frac{4}{\pi^2}\frac{K^2(\kappa_{\min})}{1 - \dfrac{pC_{\min}^2}{2}}. \tag{9}$$

The relative displacement of x in the case $pC^2 \sim \kappa^2 \ll 1$ and $\Omega^2(\omega_{\min}) \simeq 1$ or $\Omega^2(\omega_{\min}) < 1$ takes the form

$$x = C\sin(\omega t - n\vartheta), \tag{10}$$

where ω is the root of the dispersion equation

$$\Omega^2(\omega) = 1 + \tfrac{3}{4}pC^2. \tag{11}$$

If $C \to 0$, the dispersion equation (8) takes the same form as in the linear theory.[1] Thus, we have nonlinear stationary waves with finite amplitudes even if there is instability in the linear theory.

Our results are valid for more complicated cases of real beams if the wave amplitudes are not very large and we may limit ourselves to only the cubic term in the Taylor's series for the polarization forces. In more general cases there exist stationary waves. For example, if the polarization forces are defined by the monotonically increasing potential function $V(x)$, the dispersion equation takes the form

$$\frac{1}{\sqrt{2\pi}}\int_{x_1}^{x_2}\frac{dx}{\sqrt{E - \Omega^2(\omega, n)V(x)}} = 1,$$

where E is an arbitrary constant connected with the

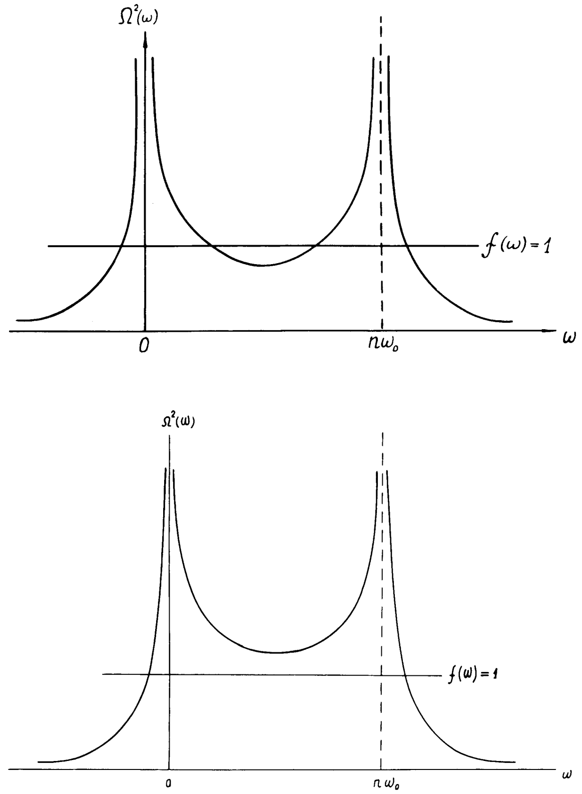

FIGURE 2, 3 The patterns of the function $\Omega^2(\omega)$.

wave amplitude and $x_{1,2}$ are the solutions of the equation

$$E = \Omega^2 V(x).$$

2 INSTABILITY DEVELOPMENT IN THE SMALL SUPERCRITICAL REGIME

Let us consider the development of the sinuous instability in time, taking into account a small nonlinearity of the polarization forces. We consider the case in which only an azimuthal harmonic with number n is excited and the linear theory growth rate γ_n is small in comparison with the real part of the oscillation frequency ω_n (the case of slight supercriticality). In this case we may use the Krylov–Bogolyubov method.[8]

Without focusing forces the above conditions take place when the beam parameters are chosen near the stability region boundary

$$\left(\frac{\alpha_2}{\omega_0^2}\left[1 + \left(\frac{\alpha_1}{\alpha_2}\right)^{1/3}\right]^3 \gtrsim 1\right).$$

In this case only the first harmonic is excited. The possible existence of an unstable harmonic, in the case with focusing forces, was shown in Ref. 6.

Let us introduce new dimensionless variables $u_{1,2}$ into the system (1) and (2),

$$x_{1,2} = a_0 e^{\gamma_n \tau/\omega_0} u_{1,2}(\tau), \tag{12}$$

where $\tau = \omega_0 t$ is the dimensionless time, a_0 is a constant determined by the initial displacement. The functions $u_{1,2}$ satisfy the system of equations

$$\frac{\partial^2 u_1}{\partial \tau^2} + \frac{2\gamma_n}{\omega_0}\left(\frac{\partial u_1}{\partial \tau} + \frac{\partial u_1}{\partial \vartheta}\right) + 2\frac{\partial^2 u_1}{\partial \tau\, \partial \vartheta} + \frac{\partial^2 u_1}{\partial \vartheta^2} + \frac{\gamma_n^2 + \alpha_1 + \lambda^2}{\omega_0^2} u_1 = \frac{\alpha_1}{\omega_0^2} u_2 - \varepsilon \frac{\alpha_1}{\omega_0^2}(u_2 - u_1)^3 e^{2\gamma_n \tau/\omega_0}$$

$$\frac{\partial^2 u_2}{\partial \tau^2} + \frac{2\gamma_n}{\omega_0}\frac{\partial u_2}{\partial \tau} + \frac{\gamma_n^2 + \alpha_2}{\omega_0^2} u_2 = \frac{\alpha_2}{\omega_0^2} u_1 + \varepsilon \frac{\alpha_2}{\omega_0^2}(u_2 - u_1)^3 e^{2\gamma_n \tau/\omega_0} \tag{13}$$

$$\varepsilon = p a_0^2 \ll 1.$$

In the linear approximation ($\varepsilon = 0$) the solution of (13) is found in the form

$$u_1 = \varphi_1 \cos \psi_n + \varphi_2 \sin \psi_n$$

$$\psi_n = \frac{\omega_n}{\omega_0}\tau - n\vartheta, \tag{14}$$

$$u_2 = \varphi_3 \cos \psi_n + \varphi_4 \sin \psi_n$$

where the quantities φ_i ($i = 1, 2, 3, 4$) are the arbitrary nontrivial solution of the algebraic system

$$[\gamma_n^2 + \alpha_1 + \lambda^2 - (\omega_n - n\omega_0)^2]\varphi_1 + 2\gamma_n(\omega_n - n\omega_0)\varphi_2 - \alpha_1 \varphi_3 = 0 \tag{15.1}$$

$$-2\gamma_n(\omega_n - n\omega_0)\varphi_1 + [\gamma_n^2 + \alpha_1 + \lambda^2 - (\omega_n - n\omega_0)^2]\varphi_2 - \alpha_1 \varphi_4 = 0 \tag{15.2}$$

$$-\alpha_2 \varphi_1 + (\gamma_n^2 + \alpha_2 - \omega_n^2)\varphi_3 + 2\gamma_n \omega_n \varphi_4 = 0 \tag{15.3}$$

$$-\alpha_2 \varphi_2 - 2\gamma_n \omega_n \varphi_3 + (\gamma_n^2 + \alpha_2 - \omega_n^2)\varphi_4 = 0 \tag{15.4}$$

The dispersion equation is obtained from the existence condition for a nontrivial solution of the system.

The values γ_n and ω_n are found from the dispersion equation

$$\frac{\alpha_1}{(\omega_n - n\omega_0)^2 - \gamma_n^2 - \lambda^2} + \frac{\alpha_2}{\omega_n^2 - \gamma_n^2} + \frac{4\gamma_n^2\omega_n(\omega_n - n\omega_0)}{[(\omega_n - n\omega_0)^2 - \gamma_n^2 - \lambda^2](\omega_n^2 - \gamma_n^2)} = 1 \quad (16)$$

$$(\omega_n - n\omega_0)(\gamma_n^2 - \omega_n^2 + \alpha_2) + \omega_n[\gamma_n^2 - (\omega_n - n\omega_0)^2 + \alpha_1 + \lambda^2] = 0$$

which are the same as that of the linear theory.[6] Assuming combinative frequencies as resulting from the superposition of two stable and one damping oscillation modes do not coincide with ω_n, we look for the solution of the nonlinear system (13) following the Krylov–Bogolyubov method

$$u_1 = \varphi_1(\varepsilon\tau)\cos\psi_n + \varphi_2(\varepsilon\tau)\sin\psi_n + \varepsilon\tilde{u}_1(\psi_n, t)$$
$$u_2 = \varphi_3(\varepsilon\tau)\cos\psi_n + \varphi_4(\varepsilon\tau)\sin\psi_n + \varepsilon\tilde{u}_2(\psi_n, t) \quad (17)$$

where φ_1, φ_2, φ_3 and φ_4 are slowly varying functions.

Let us substitute (17) into (13) under the conditions

$$\frac{d\varphi_1}{d\tau}\cos\psi_n + \frac{d\varphi_2}{d\tau}\sin\psi_n = 0;$$
$$\frac{d\varphi_3}{d\tau}\cos\psi_n + \frac{d\varphi_4}{d\tau}\sin\psi_n = 0$$

and keep only the first order terms proportional to ε.

If the condition $\gamma_n/\omega_n \ll 1$ holds, the exponentials do not change essentially during the rapid oscillation time of the functions $\tilde{u}_1$ and $\tilde{u}_2$. Multiplying the equations obtained by $\cos\psi_n$ and $\sin\psi_n$ we average them over the rapid time at fixed ϑ. This averaging corresponds to that over ψ_n.

Finally we get the system of the equations

$$\frac{d\varphi_1}{d\tau} = \frac{\omega_0}{2(\omega_n - n\omega_0)}\left\{-\frac{2\gamma_n}{\omega_0^2}(\omega_n - n\omega_0)\varphi_1 + \frac{1}{\omega_0^2}[\gamma_n^2 + \lambda^2 + \alpha_1 - (\omega_n - n\omega_0)^2]\varphi_2 - \frac{\alpha_1}{\omega_0^2}\varphi_4 + \frac{3}{4}\varepsilon\frac{\alpha_1}{\omega_0^2}e^{2\gamma_n\tau/\omega_0}(\varphi_4 - \varphi_2) \times [(\varphi_3 - \varphi_1)^2 + (\varphi_4 - \varphi_2)^2]\right\}$$

$$\frac{d\varphi_2}{d\tau} = -\frac{\omega_0}{2(\omega_n - n\omega_0)}\left\{\frac{1}{\omega_0^2}[\gamma_n^2 + \lambda^2 + \alpha_1 - (\omega_n - n\omega_0)^2]\varphi_1 + \frac{2\gamma_n}{\omega_0^2}(\omega_n - n\omega_0)\varphi_2 - \frac{\alpha_1}{\omega_0^2}\varphi_3 + \frac{3}{4}\varepsilon\frac{\alpha_1}{\omega_0^2}e^{2\gamma_n\tau/\omega_0}(\varphi_3 - \varphi_1) \times [(\varphi_3 - \varphi_1)^2 + (\varphi_4 - \varphi_2)^2]\right\}$$

$$\frac{d\varphi_3}{d\tau} = \frac{\omega_0}{2\omega_n}\left\{-\frac{\alpha_2}{\omega_0^2}\varphi_2 - \frac{2\gamma_n\omega_n}{\omega_0^2}\varphi_3 + \frac{\gamma_n^2 + \alpha_2 - \omega_n^2}{\omega_0^2}\varphi_4 - \frac{3}{4}\varepsilon\frac{\alpha_2}{\omega_0^2}e^{2\gamma_n\tau/\omega_0} \times (\varphi_4 - \varphi_2)[(\varphi_3 - \varphi_1)^2 + (\varphi_4 - \varphi_2)^2]\right\}$$

$$\frac{d\varphi_4}{d\tau} = -\frac{\omega_0}{2\omega_n}\left\{-\frac{\alpha_2}{\omega_0^2}\varphi_1 + \frac{2\gamma_n\omega_n}{\omega_0^2}\varphi_4 + \frac{\gamma_n^2 + \alpha_2 - \omega_n^2}{\omega_0^2}\varphi_3 - \frac{3}{4}\varepsilon\frac{\alpha_2}{\omega_0^2}e^{2\gamma_n\tau/\omega_0} \times (\varphi_3 - \varphi_1)[(\varphi_3 - \varphi_1)^2 + (\varphi_4 - \varphi_2)^2]\right\}. \quad (18)$$

If we introduce the new functions a and χ

$$\varphi_1 = a_1 e^{-\gamma_n\tau/\omega_0}\cos\chi_1 \qquad \varphi_3 = a_2 e^{-\gamma_n\tau/\omega_0}\cos\chi_2$$
$$\varphi_2 = a_1 e^{-\gamma_n\tau/\omega_0}\sin\chi_1 \qquad \varphi_4 = a_2 e^{-\gamma_n\tau/\omega_0}\sin\chi_2 \quad (19)$$

the system (18) can be represented in the form

$$\frac{da_1}{d\tau} = \frac{\alpha_1 a_2}{2\omega_0(\omega_n - n\omega_0)}\sin(\chi_1 - \chi_2)\{1 - g \times [a_1^2 + a_2^2 - 2a_1a_2\cos(\chi_1 - \chi_2)]\} \quad (20.1)$$

$$\frac{da_2}{d\tau} = -\frac{\alpha_2 a_1}{2\omega_0\omega_n}\sin(\chi_1 - \chi_2)\{1 - g \times [a_1^2 + a_2^2 - 2a_1a_2\cos(\chi_1 - \chi_2)]\} \quad (20.2)$$

$$\frac{d\chi_1}{d\tau} = -\frac{1}{2\omega_0(\omega_n - n\omega_0)}\left\{\gamma_n^2 + \lambda^2 + \alpha_1 - (\omega_n - n\omega_0)^2 - \frac{\alpha_1 a_2}{a_1}\cos(\chi_1 - \chi_2) - g\alpha_1\left[1 - \frac{a_2}{a_1}\cos(\chi_1 - \chi_2)\right] \times [a_1^2 + a_2^2 - 2a_1a_2\cos(\chi_1 - \chi_2)]\right\} \quad (20.3)$$

$$\frac{d\chi_2}{d\tau} = -\frac{1}{2\omega_0\omega_n}\left\{\gamma_n^2 + \alpha_2 - \omega_n^2 - \alpha_2\frac{a_1}{a_2}\cos(\chi_1 - \chi_2) - ga_2\left[1 - \frac{a_1}{a_2}\cos(\chi_1 - \chi_2)\right] \times [a_1^2 + a_2^2 - 2a_1a_2\cos(\chi_1 - \chi_2)]\right\},$$

$$g = 0.75\varepsilon. \quad (20.4)$$

From equations (20.1) and (20.2) the first integral follows

$$a_1^2 - Sa_2^2 = C, \quad (21)$$

where $S = (\alpha_1/\alpha_2)(\omega_n/n\omega_0 - \omega_n)$, C is an arbitrary constant.

For $g = 0$, when only one mode is excited, we have $C = 0$. We suppose $C = 0$ always.

Noting the right parts of equations (20) include only combinations $\chi_1 - \chi_2 = \chi$ we get with Eqs. (20)–(21)

$$\frac{da}{d\tau} = -\frac{\alpha_2\sqrt{S}}{2\omega_0\omega_n}a\sin\chi \times \{1 - ga^2(1 + S - 2\sqrt{S}\cos\chi)\}$$
$$\frac{d\chi}{d\tau} = \frac{1}{\omega_0}\left\{\delta - \frac{\alpha_2\sqrt{S}}{\omega_n}\cos\chi - \frac{\alpha_2}{2\omega_n}ga^2(1 + \sqrt{S} - 2\sqrt{S}\cos\chi)^2\right\}, \quad (22)$$

where

$$\delta = \frac{\gamma_n^2 - \omega_n^2 + \alpha_2}{\omega_n}, a = a_2.$$

The system (22) has integral

$$\cos\chi = \frac{\sqrt{C_1 + 2g\Delta a^2} - 1 + g(1 + S)a^2}{2g\sqrt{S}a^2}. \quad (23)$$

Here C_1 is a constant ($C_1 \to 1$ with $g \to 0$) and $\Delta = 2\omega_n\delta/\alpha_2 - (1 + S)$.

The maximum of the amplitude a occurs with $\cos\chi = -1$ and it is equal ($n = 1$, $\lambda = 0$)

$$a_{max} = \left\{\frac{\Delta + (1 + \sqrt{S})^2 + \{[\Delta + (1 + \sqrt{S})^2]^2 - (C_1 - 1)(1 + \sqrt{S})^4\}^{1/2}}{g(1 + \sqrt{S})^4}\right\}^{1/2}. \quad (24)$$

In the absence of a focusing force the amplitude a_{max} approximately coincides with the minimum possible amplitude of the nonlinear stationary wave. In the case $\alpha_1/\alpha_2 \ll 1$

$$\omega_1 \simeq \omega_0\left[1 - \frac{1}{2}\left(\frac{\alpha_1}{2\alpha_2}\right)^{1/3}\right];$$
$$\gamma_1 \simeq \frac{\sqrt{3}}{2}\omega_0\left(\frac{\alpha_1}{2\alpha_2}\right)^{1/3} \quad (25)$$

and

$$a_{max} = \frac{4}{\sqrt{\varepsilon}}\left(\frac{\alpha_1}{2\alpha_2}\right)^{1/6}. \quad (26)$$

The minimum amplitude of the nonlinear stationary wave is $2^{5/6}$ times smaller than the amplitude of (26).

When $\alpha_1 = \alpha_2 (S = 1)$ and $C_1 = 1 + g^2\Delta^2$, then

$$a_{max} = \left\{\frac{\Delta + 4 + [(\Delta + 4)^2 + 16g^2\Delta^2]^{1/2}}{16g}\right\}^{1/2}. \quad (27)$$

For example integral curves of the system (22), when $\alpha_1/\omega_0^2 = \alpha_2/\omega_0^2 = 0.1257$, $\omega_1/\omega_0 = 0.5$, $\gamma_1/\omega_0 = 2.165 \cdot 10^{-2}$, $g = 7.5 \cdot 10^{-3}$, $\Delta = -3.9703$, are represented in Fig. 4. The curve 1 corresponds to $C_1 = 1$, the curve 2 to $C_1 = 1 + g^2\Delta^2$, 3 to $C_1 = 1 + 2g^2\Delta^2$, 4 to $C_1 = 1 + 3g^2\Delta^2$.

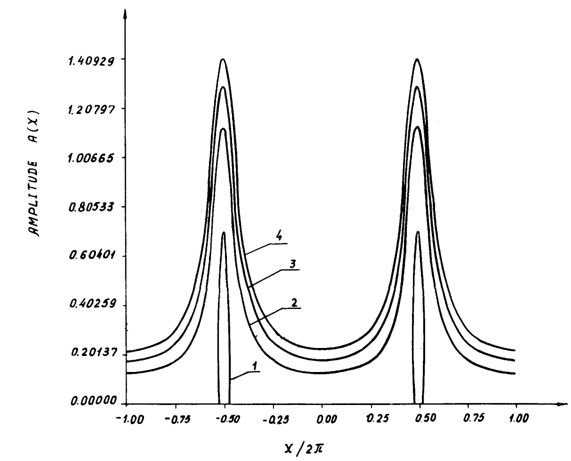

FIGURE 4 The integral curves of the system (22), when $\alpha_1/\omega_0^2 = \alpha_2/\omega_0^2 = 0.1257$, $g = 7.5 \cdot 10^{-3}$.
1—$C_1 = 1$, 2—$C_1 = 1 + g^2\Delta^2$, 3—$C_1 = 1 + 2g^2\Delta^2$, 4—$C_1 = 1 + 3g^2\Delta^2$.

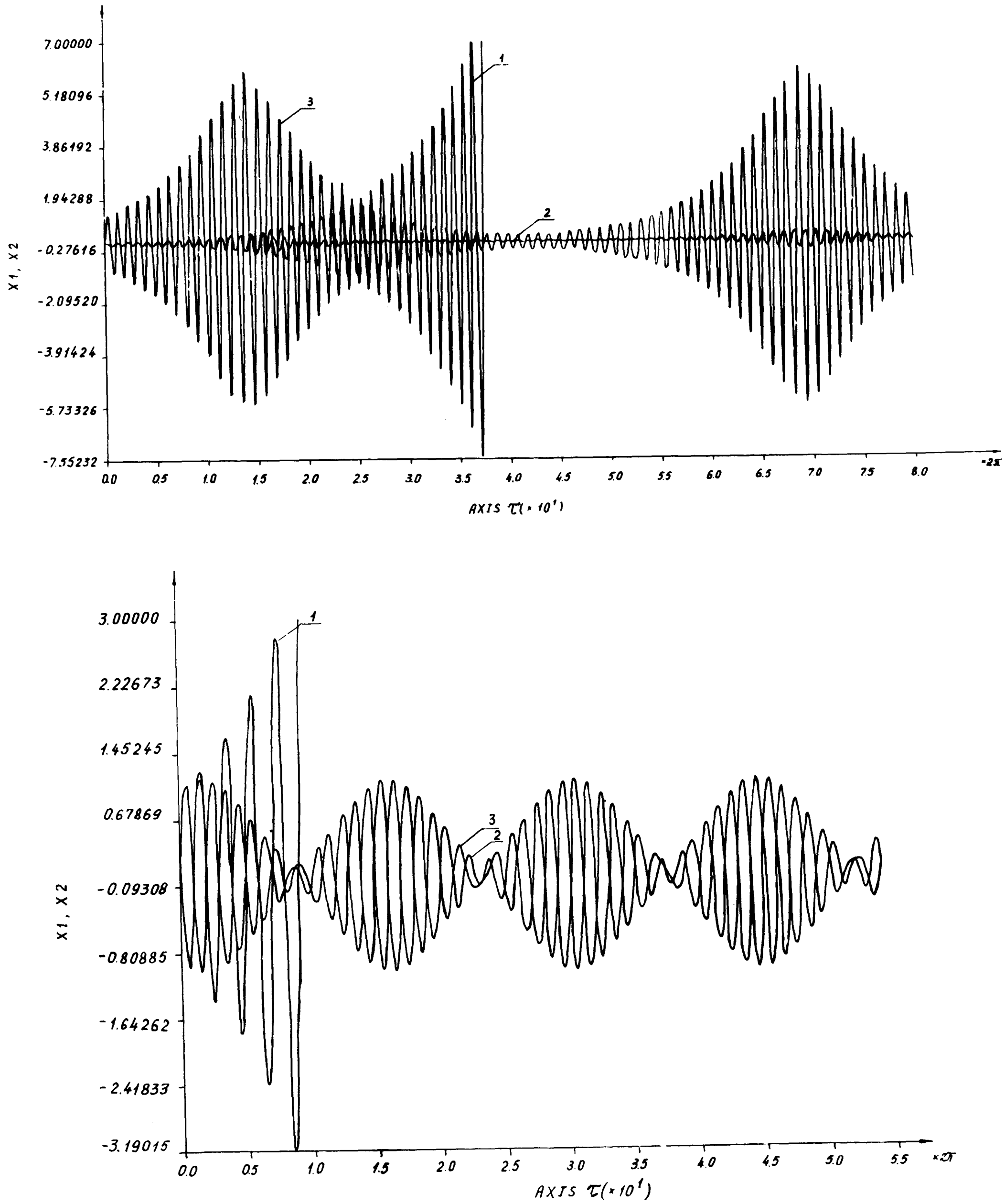

FIGURE 5 The time dependence of the local centres of mass displacements: x_1—the curves 1 ($g = 0$), 2 ($g = 7.5 \cdot 10^{-3}$); x_2—the curve 3 ($g = 7.5 \cdot 10^{-3}$).
1) $\alpha_1/\omega_0^2 = 3.248 \cdot 10^{-5}$, $\alpha_2/\omega_0^2 = 1.002$, $\omega_1/\omega_0 = 0.9875$, $\gamma_1/\omega_0 = 2.165 \cdot 10^{-2}$.
2) $\alpha_1/\omega_0^2 = \alpha_2/\omega_0^2 = 0.1257$, $\omega_1/\omega_0 = 0.5$, $\gamma_1/\omega_0 = 2.165 \cdot 10^{-2}$.

The time dependence of the local centers of mass displacements x_i $(i = 1.2)$ was obtained also in a direct way by the numerical solution of the systems (18), (17), (12). The results of these calculations with parameters

1) $\alpha_1/\omega_0^2 = 3.248 \cdot 10^{-5}, \alpha_2/\omega_0^2 = 1.002,$
$\omega_1/\omega_0 = 0.9875, \gamma_1/\omega_0 = 2.165 \cdot 10^{-2}$ and
2) $\alpha_1/\omega_0^2 = 0.1257, \alpha_2/\omega_0^2 = 0.1257,$
$\omega_1/\omega_0 = 0.5, \gamma_1/\omega_0 = 2{,}165 \cdot 10^{-2}$

are shown respectively in Figs. 5.1 and 5.2. The curves 1 correspond to $g = 0$ and x_1, the curves 2, 3 to $g = 7.5 \cdot 10^{-3}$ for x_1 and x_2, respectively.

From the general expression for the maximum oscillation amplitude (24) one can see that the ratio of the amplitude to the characteristic beam dimension includes the small factor γ_1/ω_1. Therefore this ratio may be sufficiently small.

CONCLUSIONS

The nonlinear weakening of polarization forces due to the finite difference between the local centres of mass of the electron and ion beams, which must exist for real beams, results in additional effects in comparison with the linear sinuous-instability theory. The nonlinear stationary wave appears even in the case of linear instability. At small supercriticality the nonlinear stabilization of the instability is possible in a one-mode approximation. The final amplitudes are small in comparison with the beam dimensions.

The nonlinearity of polarization forces causes a nonlinear oscillation frequency shift for the stable modes of the linear theory, but the amplitudes do not vary in time. The frequency shift calculated by the Krylov–Bogolyubov method coincides with the shift of the nonlinear stationary wave in the first-order approximation.

In the absence of a focusing force unstable oscillations turn into the nonlinear wave. The amplitude of the wave is close to a stationary wave minimum amplitude.

In the presence of a focusing force, instability ceases also at comparatively small amplitudes after a time of the order of the reciprocal increment of the linear theory. Qualitatively the same results were obtained in the experiment[9] where in the presence of the focusing force successive excitations of the single harmonics were observed. Saturation of the oscillations occurs after a very short time and the particles are not lost from the beam. Therefore the beam parameters (intensity of the electron and ion beams) do not pass into the linear stability region because of the nonlinear stabilization of instability.

The case where the parameters are chosen far from a stable region boundary (relative increment is large and many harmonics are excited simultaneously) needs to be considered additionally.

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