Published by Institute of Physics Publishing for SISSA

Received: *June 2, 2007*
Accepted: *July 30, 2007*
Published: *August 20, 2007*

Direct integration of the topological string

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Abstract: We present a new method to solve the holomorphic anomaly equations governing the free energies of type B topological strings. The method is based on direct integration with respect to the non-holomorphic dependence of the amplitudes, and relies on the interplay between non-holomorphicity and modularity properties of the topological string amplitudes. We develop a formalism valid for any Calabi-Yau manifold and we study in detail two examples, providing closed expressions for the amplitudes at low genus, as well as a discussion of the boundary conditions that fix the holomorphic ambiguity. The first example is the non-compact Calabi-Yau underlying Seiberg-Witten theory and its gravitational corrections. The second example is the Enriques Calabi-Yau, which we solve in full generality up to genus six. We discuss various aspects of this model: we obtain a new method to generate holomorphic automorphic forms on the Enriques moduli space, we write down a new product formula for the fiber amplitudes at all genus, and we analyze in detail the field theory limit. This allows us to uncover the modularity properties of SU(2), $\mathcal{N} = 2$ super Yang-Mills theory with four massless hypermultiplets.

Keywords: Topological Strings, String Duality, Differential and Algebraic Geometry.

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1. Introduction

Topological string theory has played an important role in the quest for a better understanding of both physical and mathematical aspects of string theory. There are two different topological string theories related to each other by mirror symmetry, and known as the A and B-model. They are obtained from an $\mathcal{N} = 2$ superconformal field theory, twisted in two distinct ways to become type A or type B topological sigma models that are then coupled to gravity. The physical relevance of these theories lies in their intimate connection to type II superstring theory. In particular, the topological string on a given Calabi-Yau manifold computes higher derivative F-terms in the 4d effective action of the corresponding type II theory. From a mathematical point of view, the topological string partition function provides a generating functional for Gromov-Witten invariants in enumerative geometry.

It is therefore desirable to solve the topological string on a given Calabi-Yau manifold, that is to say, to compute all the topological amplitudes $F^{(g)}$ in the genus expansion of the partition function. While this problem is completely solved for the case of non-compact toric Calabi-Yau manifolds thanks to the techniques of localization and the topological vertex, it remains a challenge for the compact case. One of the main tools in solving topological string theory, which also applies to compact Calabi-Yau manifolds, is the holomorphic anomaly equations for the B-model found in [7]. In this work we present a new approach to solving these equations. We make use of the fact that for each Calabi-Yau manifold there exists a target space symmetry group which provides a symmetry of the topological partition function [1] and thereby drastically reduces the space of candidate solutions. The topological string amplitudes $F^{(g)}$ turn out to be polynomials in a finite set of generators which transform in a particularly simple way under the space-time symmetry group. Moreover, it can be shown that all non-holomorphic dependence in these amplitudes arises through a very special set of generators that are suitable generalizations of the non-holomorphic Eisenstein function $E_2(\tau, \bar{\tau})$. The remaining generators are holomorphic. Keeping track of these non-holomorphic contributions we will be able to directly integrate the holomorphic anomaly equations. This method turns out to be very efficient and gives us rich new information about the remaining holomorphic generators. A similar approach to the holomorphic anomaly equations was sketched in [7], in the analysis of toroidal orbifolds. For the quintic Calabi-Yau manifold a more complicated method was outlined in [62]. Other related approaches have been used before in [31, 32] to analyze

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rational elliptic surfaces, and in [49, 50] to study noncritical strings and $\mathcal{N} = 4$ super Yang-Mills theory.

The direct integration of the holomorphic anomaly equations can be performed for a generic Calabi-Yau manifold, as we will show in the final section of this work. However, in order to fully exploit the interplay of the holomorphic anomaly with the space-time symmetry, we will intensively discuss specific examples. To illustrate the general ideas we first study the local Calabi-Yau manifold associated to the Seiberg-Witten curve. Here the target-space symmetry group is a subgroup of $Sl(2,\mathbb{Z})$ and the generating modular functions are well-known.

Applying these methods to a compact Calabi-Yau manifold is far more involved. In the main part of the paper we will focus on the specific example of the Enriques Calabi-Yau [23], arguably the simplest Calabi-Yau compactification with nontrivial topological string amplitudes [41, 48]. This manifold can be obtained as the free quotient $(\mathrm{K3}\times\mathbb{T}^2)/\mathbb{Z}_2$, where $\mathbb{Z}_2$ acts as the Enriques involution on the K3 fibers. The target space duality group of the Enriques Calabi-Yau is shown to be the discrete group $Sl(2,\mathbb{Z}) \times O(10,2,\mathbb{Z})$, with the factors corresponding to the $\mathbb{T}^2$ base and Enriques fiber, respectively. The generating modular forms for $Sl(2,\mathbb{Z})$ are well-known, therefore we will be particularly concerned with the contributions from the Enriques fiber and specially their mixing with the $\mathbb{T}^2$ base.

After integrating the holomorphic anomaly equations the only problem remaining is to fix the holomorphic ambiguities, i.e. the boundary conditions in the integration of the equations. These ambiguities are constrained by information coming from boundaries of the moduli space where the $F^{(g)}$ are known explicitly. In the Enriques case one can use the fiber limit, where all amplitudes can be determined by heterotic-type II duality [41], and a field theory limit where the manifold degenerates to give rise to SU(2), $N_f = 4$ Seiberg-Witten theory. By making use of these boundary conditions we determine the full topological string amplitudes up to genus 6, improving in this way previous results in [41]. As a bonus of our analysis, we clarify the modularity properties of the conformal $N_f = 4$ theory and its gravitational corrections described in [52]. At present the available boundary conditions are not enough to completely solve topological string theory on the Enriques Calabi-Yau, but we provide efficient tools to analyze the amplitudes at all genus with the method of direct integration.

The organization of this paper is as follows. In section 2 we review the derivation of the holomorphic anomaly equations. Section 3 gives a first simple example of the method of direct integration and the fixing of holomorphic ambiguities by application to Seiberg-Witten theory. Section 4 reviews what will be our main focus, the Enriques Calabi-Yau. We introduce modular and automorphic forms which will be relevant later and discuss the topological amplitudes on the Enriques fiber. Also an all-genus product formula for the fiber partition function will be introduced. Section 5 constitutes the core of this work. We show explicitly how one can solve for $F^{(g)}$ up to genus six and present the general recursive formalism. Furthermore, boundary conditions and a reduced Enriques model where part of the moduli space is blown down are investigated. In section 6 we analyze the field theory limit corresponding to $N_f = 4$ SYM and we relate it in detail to the Enriques Calabi-Yau. In section 7 we present a formalism for direct integration on generic Calabi-Yau manifolds.

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Section 8 contains conclusions and an outlook on further directions of investigation. Appendix A reviews some special geometry. Appendix B collects some useful formulae for theta functions and modular forms. Appendix C reviews the heterotic computation of the amplitudes in [46, 41] and presents improved formulae for their antiholomorphic dependence. Finally, appendix D presents the holomorphic anomaly equations on the so-called big moduli space.

2. The holomorphic anomaly equations

In this section we will briefly recall some basics about topological string theory to set the stage for the following sections and to fix our conventions. This will force us to introduce some world-sheet notations and techniques. However, for the rest of this work we will mostly need only the explicit form of the holomorphic anomaly equations. For a more detailed introduction to topological string theory the reader might want to consult references [30, 60, 51, 44, 39].

Type II string theory on Calabi-Yau threefold Y yields a superconformal field theory with left and right moving $(2,2)$ supersymmetry on the world-sheet. This structure admits two topological string theories: the A-and the B-model. The key quantity in these topological theories is their all genus partition function

$$Z = \exp \sum_{g=0}^{\infty} g_s^{2g-2} F^{(g)} . \tag{2.1}$$

This formal expansion in the string coupling g_s contains the topological string amplitudes $F^{(g)}$ for maps from genus g Riemann surfaces into a target Calabi-Yau manifold. The topological string amplitudes of the A– and B-model are identified by mirror symmetry, which maps one theory on Y to its dual on the corresponding mirror Calabi-Yau.

We will now briefly recall the B-model definitions of the free energies $F^{(g)}$. The B-model describes constant maps from a world-sheet Riemann surface Σ_g to points in the Calabi-Yau space Y. Therefore, the B-model definition of the $F^{(g)}$ involves only the integration over the moduli space $\overline{\mathcal{M}}_g$ of the world-sheet and not over the moduli space of maps. More precisely, let us denote by $(\underline{m}, \overline{m})$ coordinates on $\overline{\mathcal{M}}_g$ and abbreviate the correlators of the world-sheet CFT by $\langle \cdot \rangle_g$. The free energies $F^{(g)}$ are then defined by

$$F^{(g)} = \langle 1 \rangle_g = \int_{\overline{\mathcal{M}}_g} \langle \prod_{k=1}^{3g-3} \beta^k \bar{\beta}^k \rangle_g \; [d\underline{m} \wedge d\overline{m}] . \tag{2.2}$$

Here we inserted the operators $\beta^k = \int_{\Sigma_g} G^- \mu^k$ and their complex conjugates to obtain the correct measure on the moduli space. β^k and $\bar{\beta}^k$ contain the the world-sheet Beltrami differentials $\mu^k \in H^1(T\Sigma_g)$ and the world-sheet supersymmetry generators $G^-, \bar{G}^-$. The contraction of $[d\underline{m} \wedge d\overline{m}]$ with the $(\beta^k, \bar{\beta}^k)$ factor is antisymmetric due to the presence of $G^-, \bar{G}^-$ and yields a top form on the complex $3g-3$ dimensional moduli space $\overline{\mathcal{M}}_g$. The fact that one has to integrate only over the moduli space of the world-sheet makes the

B-model far simpler to solve than the A-model. Therefore, it is often easier to use the B-model and the mirror map to determine A-model quantities.

From the point of view of the four-dimensional effective action, one is interested in the dependence of the $F^{(g)}$ on the complex moduli $t^i, \bar{t}^{\bar{i}}$ in the vector multiplets. These parametrize marginal deformations, which in the B-model correspond to complex structure deformation of the Calabi-Yau manifold. Infinitesimally the world-sheet action is perturbed by the $t^i, \bar{t}^{\bar{i}}$ as follows

$$S = S_0 + t^i \int_{\Sigma_g} \mathcal{O}_i^{(2)} + \bar{t}^{\bar{i}} \int_{\Sigma_g} \bar{\mathcal{O}}_{\bar{i}}^{(2)} \,, \tag{2.3}$$

where the sums run over $i = 1, \ldots, h^1(Y, TY) = h^{(2,1)}(Y)$. Here the marginal two-form operators are obtained using the descent equations as

$$\mathcal{O}_i^{(2)} = \{G_0^-, [\bar{G}_0^-, \mathcal{O}_i^{(0)}]\} \mathrm{d}z\mathrm{d}\bar{z} \,, \qquad \overline{\mathcal{O}_{\bar{i}}^{(2)}} = \{G_0^+, [\bar{G}_0^+, \bar{\mathcal{O}}_{\bar{i}}^{(0)}]\} \mathrm{d}z\mathrm{d}\bar{z} \,, \tag{2.4}$$

where G_0^+, G_0^- are the zero modes of the twisted world-sheet supersymmetries G^+, G^-. In these equations we denoted by $\mathcal{O}_i^{(0)}$ the zero-form cohomological operators, which are in one-to-one correspondence with the $H^1(Y, TY)$ cohomology of the target space.

From the point of view of the target space Calabi-Yau the complex fields $t^i, \bar{t}^{\bar{i}}$ provide a set of local coordinates on the moduli space of complex structure deformations $\mathcal{M}$. This space is shown to be a special Kähler manifold with Kähler potential

$$K(t, \bar{t}) = -\log i \int_Y \Omega(t) \wedge \bar{\Omega}(\bar{t}) \,, \tag{2.5}$$

where $\Omega(t)$ is the holomorphic three-form on Y varying holomorphically with a change of the complex structure. $\Omega(t)$ is only unique up to rescalings by a holomorphic function and hence should be viewed as a section of the line bundle $\mathcal{L}$ over the moduli space $\mathcal{M}$. In appendix A we review how the special geometry of $\mathcal{M}$ can be entirely encoded by a single holomorphic section of $\mathcal{L}^2$, the prepotential $F^{(0)} = \mathcal{F}(t)$. From a world-sheet point of view one does not obtain $F^{(0)}$ directly, but rather finds the three-point function

$$C_{ijk}^{(0)} = \langle \mathcal{O}_i^{(0)} \mathcal{O}_j^{(0)} \mathcal{O}_k^{(0)} \rangle_g = -\int_Y \Omega(t) \wedge \partial_i \partial_j \partial_k \Omega(t) \,, \tag{2.6}$$

where ∂_i are derivatives with respect to t^i.

At higher genus a more involved world-sheet analysis can be applied to investigate the properties of the higher $F^{(g)}$. It turns out that the higher genus topological string amplitudes $F^{(g)}$ are not holomorphic, but rather fulfill specific holomorphic anomaly equations. These equations are recursive in the genus and determine the anti-holomorphic derivative of $F^{(g)}$. Therefore, even if the genus zero data are given they determine $F^{(g)}$ only up to a holomorphic ambiguity. We will now briefly state the essential features and results of the work of Bershadsky, Cecotti, Ooguri and Vafa [7], who have shown that

i.) The $F^{(g)}$ transform as section of $\mathcal{L}^{2-2g}$ with the connection (A.3).

ii.) The topological B-model correlation functions

$$C_{i_1 \ldots i_n}^{(g)} = \begin{cases} \langle \int_{\Sigma_g} \mathcal{O}_{i_1}^{(2)} \cdots \int_{\Sigma_g} \mathcal{O}_{i_n}^{(2)} \rangle_g = D_{i_1} \ldots D_{i_n} F^{(g)} & \text{for } g \geq 1 \\ \langle \mathcal{O}_{i_1}^{(0)} \mathcal{O}_{i_2}^{(0)} \mathcal{O}_{i_3}^{(0)} \int_{\Sigma_g} \mathcal{O}_{i_4}^{(2)} \cdots \int_{\Sigma_g} \mathcal{O}_{i_n}^{(2)} \rangle_g = D_{i_4} \ldots D_{i_n} C_{i_1 i_2 i_3}^{(0)} & \text{for } g = 0 \end{cases} \tag{2.7}$$

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can be obtained using the covariant derivatives (A.3) and obey

$$C^{(g)}_{i_1\dots i_n} = 0 \quad \text{for} \quad 2g-2+n \le 0 \; . \tag{2.8}$$

iii.) The anti-holomorphic derivative $\partial_{\bar\imath} = \frac{\partial}{\partial \bar t^i}$ of the $F^{(g)}$,

$$\bar\partial_{\bar\imath} F^{(g)} = \int_{\overline{\mathcal{M}}_g} \bar\partial_{\bar\imath}\mu_g = \int_{\overline{\mathcal{M}}_g} \partial_m \bar\partial_{\bar m}\lambda_{\bar\imath,g} = \int_{\partial\overline{\mathcal{M}}_g} \lambda_{\bar\imath,g}, \tag{2.9}$$

receives only contributions from the complex codimension one locus in the moduli space of Riemann surfaces corresponding to world-sheets which are degenerate with lower genus components. These boundary contributions can be worked out and yield recursive equations for the $F^{(g)}$. For $g>1$ one gets

$$\bar\partial_{\bar\imath} F^{(g)} = \frac{1}{2}\bar C^{(0)jk}_{\bar\imath}\left(D_j D_k F^{(g-1)} + \sum_{r=1}^{g-1} D_j F^{(r)} D_k F^{(g-r)}\right) \tag{2.10}$$

and for $g=1$ a generalisation of the Quillen anomaly

$$\partial_i \bar\partial_{\bar\jmath} F^{(1)} = \frac{1}{2} C^{(0)}_{ikl}\bar C^{(0)kl}_{\bar\jmath} - \left(\frac{\chi}{24}-1\right) G_{i\bar\jmath} \; . \tag{2.11}$$

Here we defined

$$\bar C^{(0)kl}_{\bar\jmath} = e^{2K} G^{k\bar k} G^{l\bar l} \bar C^{(0)}_{\bar\jmath\bar k\bar l} \, , \tag{2.12}$$

where $G_{k\bar k} = \partial_k \bar\partial_{\bar k} K$ is the Weil-Petersson metric of the Kähler potential (2.5).

These are the recursive holomorphic anomaly equations, which we want to integrate directly in this paper. Note that there is no holomorphic anomaly at genus zero. $C^{(0)}_{ijk}$ has no world-sheet moduli dependence, hence no boundaries, and is therefore holomorphic. The genus zero data thus have to be provided from the outset. They can be determined from the period integrals of the manifold Y.

It is further shown in ref. [7] that (2.10) can be integrated recursively. With an iterative procedure of complexity growing exponentially with the genus, one rewrites (2.10) as

$$\partial_{\bar k} F^{(g)}(t,\bar t) = \bar\partial_{\bar k} \Gamma^{(g)}(\hat\Delta^{ij}, \hat\Delta^i, \hat\Delta, C^{(r<g)}_{i_1\dots i_n}) \, , \tag{2.13}$$

and integrates it to

$$F^{(g)}(t,\bar t) = \Gamma^{(g)}(\hat\Delta^{ij}, \hat\Delta^i, \hat\Delta, C^{(r<g)}_{i_1\dots i_n}) + f^{(g)}(t) \; . \tag{2.14}$$

Here $\Gamma^{(g)}$ is a functional of some propagators $\hat\Delta^{ij}, \hat\Delta^i, \hat\Delta$ and the lower genus vertices $C^{(r)}_{i_1\dots i_n}$ with $r<g$. The holomorphic ambiguity $f^{(g)}(t)$ arises as an integration constant. To prove that the functional $\Gamma^{(g)}$ exists at every genus, [7] show that it is the disconnected Feynman graph expansion of an auxiliary action with the above vertices and propagators, whose partition function fulfills a master equation equivalent to (2.10) and (2.11). The propagators can be defined using the genus zero data as follows. Since

$$\bar D_{\bar\imath} \bar C^{(0)}_{\bar\jmath\bar k\bar l} = \bar D_{\bar\jmath} \bar C^{(0)}_{\bar\imath\bar k\bar l} \tag{2.15}$$

one can integrate

$$\bar{C}^{(0)}_{\bar{j}\bar{k}\bar{l}} = -\frac{1}{2}e^{-2K}\bar{D}_{\bar{\imath}}\bar{D}_{\bar{j}}\bar{\partial}_{\bar{k}}\hat{\Delta} \tag{2.16}$$

as

$$G_{\bar{\imath}j}\hat{\Delta}^j = \frac{1}{2}\bar{\partial}_{\bar{\imath}}\hat{\Delta}\,, \qquad G_{\bar{\imath}k}\hat{\Delta}^{kj} = \bar{\partial}_{\bar{\imath}}\hat{\Delta}^j\,, \qquad \bar{C}^{(0)jk}_{\bar{\imath}} = \bar{\partial}_{\bar{\imath}}\hat{\Delta}^{jk}\,. \tag{2.17}$$

Note that the propagators are defined by these equations only up to holomorphic ambiguities arising in the integration steps. Fixing these ambiguities directly affects the definition of the holomorphic functions $f^{(g)}(t)$ in (2.14). It turns out that a preferred choice for this ambiguity is provided by relating the propagators in a canonical way to $F^{(1)}(t,\bar{t})$ [1].

The combinatorics of the Feynman graph expansion are useful to establish some general properties of the $F^{(g)}$, but its complexity grows exponentially with the genus. However, the $F^{(g)}$ are invariant under space-time modular transformations which are a symmetry of the full string compactification. As we will discuss later, they generically admit a split into a universal factor times a modular form. Here the weights of the modular forms grow linearly with the genus. Since the ring of modular forms is finitely generated, the complexity of modular invariant expressions grows only polynomially with the genus. The method of direct integration that we develop in this paper uses this connection with modular forms such that its complexity also grows only polynomially with the genus. It has the advantage that the modular properties of the amplitudes are manifest in all steps of the derivation.

3. Solving Seiberg-Witten theory by direct integration

Local Calabi-Yau geometries provide simple and instructive examples for the interplay between holomorphicity and modular invariance in topological string theory. In this section we will explain the key features using the simplest example, namely the local Calabi-Yau corresponding to SU(2) Seiberg-Witten theory with no matter [54]. In section 3.1 we first recall the geometry of Seiberg-Witten theory. We show that all genus zero data can be expressed in terms of a finite set of holomorphic modular forms. All higher amplitudes $F^{(g)}$ are invariant under the modular group. In section 3.2 we directly integrate the holomorphic anomaly equations, determining all $F^{(g)}$ up to a holomorphic modular ambiguity. Modularity restricts this ambiguity so much that simple boundary conditions set by the effective action near special points in the moduli space allow one to reconstruct all $F^{(g)}$. We review such a convenient set of boundary conditions in section 3.3. The general philosophy presented in this section will be later applied to the more complicated case of compact Calabi-Yau manifolds.

3.1 The Seiberg-Witten geometry

Seiberg-Witten theory with no matter [54] can be obtained in the A-model as a limit of the local Calabi-Yau geometry $\mathcal{O}(-2,-2) \to \mathbb{P}^1 \times \mathbb{P}^1$ [36]. The mirror B-model geometry of this limit is the Seiberg-Witten elliptic curve $\mathcal{E}$

$$y^2 = (x-u)(x-\Lambda^2)(x+\Lambda^2)\,, \tag{3.1}$$

whose modular group is $\Gamma(2)$. This subgroup of $Sl(2,\mathbb{Z})$ acts on the period integrals

$$t = \int_a \lambda\,, \qquad\qquad t_D = \int_b \lambda\,, \tag{3.2}$$

where $\lambda = \frac{\sqrt{2}}{2\pi}\frac{y}{x^2-1}\mathrm{d}x$ is the Seiberg-Witten meromorphic differential. In the limit described above, λ is obtained as a reduction of the holomorphic $(3,0)$ form of the Calabi-Yau manifold. Rigid special geometry guarantees the existence of a prepotential $F^{(0)} = \mathcal{F}(t)$ with the properties

$$t_D = \frac{\partial \mathcal{F}}{\partial t}\,, \qquad\qquad \tau = -\frac{1}{4\pi}\frac{\partial^2 \mathcal{F}}{\partial^2 t}\ . \tag{3.3}$$

These conditions are obtained as the rigid limit of the special geometry relations presented in appendix A. Note that τ is precisely the complex structure parameter of the torus and hence parametrizes the upper half-plane. In particular, $\mathrm{Im}\tau > 0$ is guaranteed by the Riemann inequality consistent with the fact that $\mathrm{Im}\tau$ is the gauge kinetic coupling function of Seiberg-Witten theory. Moreover, a modular transformation acts on τ as

$$\tau \ \mapsto \ \frac{a\tau + b}{c\tau + d}\ . \tag{3.4}$$

The genus zero data are functions of τ and transform in a particularly simple way under (3.4). They can be expressed in terms of a finite set of modular generators, which we will specify in the following.

A modular function $f(\tau)$ of weight m is defined to transform as $f(\tau) \mapsto (c\tau+d)^m f(\tau)$ under (3.4). Focusing on the modular group of the Seiberg-Witten curve, we note that the ring of modular functions of $\Gamma(2)$ can be expressed as powers of the Jacobi θ-functions. Relevant properties of the Jacobian θ-functions are summarized in appendix B. We introduce two generators

$$K_2 = \vartheta_3^4 + \vartheta_4^4, \qquad\qquad K_4 = \vartheta_2^8\,, \tag{3.5}$$

which are of modular weight two and four respectively. The modular transformation properties follow from (B.3). K_2, K_4 generate the graded ring of holomorphic modular forms $\mathcal{M}_*(\Gamma(2))$ of $\Gamma(2)$, which we will also denote by $\mathbb{C}[K_2, K_4]$. It turns out to be useful to also introduce

$$h = K_2\,, \qquad\qquad E_4 = \frac{1}{4}(K_2^2 + 3K_4)\ . \tag{3.6}$$

As we will see when we develop the method of direct integration, it is natural to take h, E_4 as the generators of the ring $\mathcal{M}_*(\Gamma(2))$.

Let us now express the genus zero data in terms of modular forms. The connection with the geometry of the Seiberg-Witten curve is given by the following relation

$$u(\tau) = \frac{K_2}{\sqrt{K_4}}\ . \tag{3.7}$$

The combination $z(\tau) = 1/u^2(\tau)$ is modular invariant and can be viewed as the analog of the mirror map for this non-compact Calabi-Yau manifold. The analog of the holomorphic triple coupling is

$$C \equiv C^{(0)}_{ttt} = \frac{\partial \tau}{\partial t} = \frac{32 K_4^{1/4}}{K_2^2 - K_4} \tag{3.8}$$

Note that C^2 is a form of weight -6 under the modular transformations in $\Gamma(2)$. The modular group $\Gamma(2)$ also determines the periods t, t_D as weight 1 objects[1]

$$t(\tau) = \frac{E_2(\tau) + K_2(\tau)}{3K_4^{1/4}(\tau)}, \qquad t_D(\tau_D) = -i\frac{2E_2(\tau_D) - K_2(\tau_D) - 3K_4^{1/2}(\tau_D)}{3\Big(2K_2(\tau_D) - 2K_4^{1/2}(\tau_D)\Big)^{1/2}}, \tag{3.9}$$

where $\tau_D = -\frac{1}{\tau}$ and E_2 is the second Eisenstein series defined in (B.9). It is natural to give the periods in the above parameters. In the electric phase of Seiberg-Witten theory the $q = e^{2\pi i\tau}$ series converges and t is the physical expansion parameter, while in the magnetic phase the $q_D = e^{2\pi i\tau_D}$ series converges and t_D is the physical expansion parameter. Of course $t_D(\tau)$ and $t(\tau_D)$ can be obtained by performing an S-duality transformation on E_2 and the Jacobi theta functions.

3.2 Direct integration

Having discussed the genus zero geometry, let us now turn to the higher genus free energies $F^{(g)}$ and their holomorphic anomaly. Starting with $F^{(1)}$, we note that the holomorphic anomaly equation (2.11) specializes to

$$\partial_t \partial_{\bar{t}} F^{(1)} = \frac{1}{2} C_{\bar{t}}^{(0)\, tt} C_{ttt}^{(0)} \ . \tag{3.10}$$

where the indices are raised with the Weil-Petersson metric $G_{t\bar{t}} = 2\mathrm{Im}\tau$. This equation integrates immediately to

$$F^{(1)} = -\frac{1}{2} \log \mathrm{Im}\tau - \log|\Phi(\tau)|\,, \tag{3.11}$$

where $\partial\tau/\partial t$ is evaluated using (3.8). The holomorphic object $\Phi(\tau)$ is the ambiguity at genus one. It is determined from modular constraints and the physical requirement that $F^{(1)}$ should only be singular at the discriminant of $\mathcal{E}$. Note that under a modular transformation (3.4) one finds that $\mathrm{Im}\tau \ \mapsto \ |c\tau + d|^{-2}\mathrm{Im}\tau$. Together with the invariance of $F^{(1)}$ this implies that $\Phi(\tau)$ must be a modular form of weight 1. The only modular form of weight 1 which has only poles at the discriminant of $\mathcal{E}$ is the square of the η function given in (B.7). This fixes the ambiguity at genus one as $\Phi(\tau) = \eta^2(\tau)$.

At genus one the non-holomorphic dependence was induced through the appearance of $\mathrm{Im}\tau$. As dictated by the holomorphic anomaly equations, all higher $F^{(g)}$ also depend on $\bar{t}$. We now show that this dependence arises through the propagator $\hat{\Delta}^{tt}$ only. $\hat{\Delta}^{tt}$ is obtained in the local limit of (2.17) and thus obeys

$$\partial_{\bar{t}} \hat{\Delta}^{tt} = C_{\bar{t}}^{(0)\, tt} \ . \tag{3.12}$$

All other propagators vanish in this limit. To integrate this condition, we first multiply both sides in (3.12) by $C_{ttt}^{(0)}$. The result is easily compared to the holomorphic anomaly

[1] They can be calculated likewise using the Picard-Fuchs equation.

equation (3.10) of $F^{(1)}$. Changing derivatives by inserting $\partial\tau/\partial t = C^{(0)}_{ttt}$ one evaluates with the help of (B.12)

$$\hat{\Delta}^{tt} = 2\partial_\tau F^{(1)}(\tau,\bar{\tau}) = -\frac{1}{12}\widehat{E}_2(\tau,\bar{\tau})\,, \qquad \partial_\tau = (2\pi i)^{-1}\frac{\partial}{\partial\tau} \tag{3.13}$$

The occurrence of the non-holomorphic extension of the second Eisenstein series $E_2(\tau)$

$$\widehat{E}_2(\tau,\bar{\tau}) = E_2(\tau) - \frac{3}{\pi \mathrm{Im}\tau}\,. \tag{3.14}$$

is forced by modular invariance. Since $F^{(1)}(\tau,\bar{\tau})$ is a modular function of weight zero, its derivative must be a modular form of weight 2 which is not holomorphic. The only form with these properties is the almost holomorphic form $\widehat{E}_2(\tau,\bar{\tau})$. This form is the canonical, almost holomorphic extension of the second Eisenstein series E_2, where E_2 is the unique holomorphic quasimodular form of weight 2 transforming as

$$E_2(\tau) \quad\mapsto\quad (c\tau+d)^2 E_2(\tau) - \frac{6}{\pi} ic(c\tau+d) \tag{3.15}$$

under a modular transformation (3.4). The shift in the transformation of the anholomorphic piece in (3.14) cancels precisely the shift in (3.15). More generally the ring $\hat{\mathcal{M}}_*$ of almost holomorphic forms of $\Gamma(2)$ is generated as $\mathbb{C}[\widehat{E}_2, h, \Delta]$.

Using the propagator and general properties of the Feynman graph expansion one can extract the fact that the higher genus $F^{(g)}$ are weight 0 forms with the structure

$$F^{(g)}(\tau,\bar{\tau}) = C^{2g-2}\sum_{k=0}^{3g-3}\widehat{E}_2^k(\tau,\bar{\tau})c_k^{(g)}(\tau)\,, \qquad g>1\,, \tag{3.16}$$

where we defined $C = C^{(0)}_{ttt}$. Modular invariance implies then that the holomorphic forms $c_k^{(g)}(\tau)$ are modular of weight $6(g-1)-2k$ in $\mathbb{C}[h,\Delta]$. We will show next that all forms $c_k^{(g)}(\tau)$ with $k>0$ are very easily determined by direct integration of the holomorphic anomaly equation. The form $c_0^{(g)}(\tau)$ is not determined in this way and corresponds to a holomorphic modular ambiguity.

In order to analyze the holomorphic anomaly equations in the local case, it turns out to be very useful to discuss some general properties related to modular transformations. Let us first discuss how derivatives transform under the modular transformation (3.4). Denoting by f_k a modular form of weight k it is elementary to check that its derivative transforms under (3.4) as

$$\partial_\tau f_k \quad\mapsto\quad (c\tau+d)^{k+2}\partial_\tau f_k + \frac{k}{2\pi i}c(c\tau+d)^{k+1}f_k\,. \tag{3.17}$$

Similarly, we can evaluate $\partial_t f_k = C^{-1}\partial_\tau f_k$, where as above $C = C^{(0)}_{ttt}$. In order to cancel the shift in (3.17) we will now introduce covariant derivatives. There are two possible ways to achieve this.[2] Firstly, one can cancel the shift against the shift of $(\mathrm{Im}\tau)^{-1}$ and set

$$D_t f_k = \left(\partial_t - \frac{kC}{4\pi\mathrm{Im}\tau}\right)f_k\,, \qquad D_\tau f_k = \left(\partial_\tau - \frac{k}{4\pi\mathrm{Im}\tau}\right)f_k\,. \tag{3.18}$$

[2]We thank Don Zagier for explaining us several manipulations involved in the following.

Here D_t is the covariant derivative to the Weil-Petersson metric $G_{t\bar{t}}$ and D_τ is the so-called Mass derivative. D_t maps almost holomorphic forms of $\Gamma(2)$ of weight k into almost holomorphic forms of weight $k-1$, while D_τ increases the weight from k to $k+2$. Note that both derivatives in (3.18) are non-holomorphic due to the appearance of $\mathrm{Im}\tau$. There is however a second possibility to cancel the shift (3.17) which is manifestly holomorphic. More precisely, one can cancel the shift against the shift (3.15) of $E_2(\tau)$ and define

$$\hat{D}_t f_k = \left(\partial_t - \frac{1}{12}kCE_2\right) f_k\,, \qquad \hat{D}_\tau f_k = \left(\partial_\tau - \frac{1}{12}kE_2\right) f_k\,. \tag{3.19}$$

In this case $\hat{D}_\tau$ is known as the Serre derivative. Both $\hat{D}_t$ and $\hat{D}_\tau$ are holomorphic. They map holomorphic modular forms of weight k to holomorphic modular forms of weight $k-1$ and $k+2$ respectively. It is easy to check that the following identity holds

$$D_t f_k = \hat{D}_t f_k + \frac{1}{12}kC\widehat{E}_2\, f_k\,, \qquad D_\tau f_k = \hat{D}_\tau f_k + \frac{1}{12}k\widehat{E}_2\, f_k\,. \tag{3.20}$$

These equations also imply that whenever f_k is holomorphic all the non-holomorphic dependence of $D_t f_k$ and $D_\tau f_k$ lies in a term involving the propagator. In other words, once again all anti-holomorphic dependence arises through the propagator $\widehat{E}_2$ only. The generalizations of the modular derivatives (3.18) and (3.19) will reappear in later sections of this work. For the Enriques Calabi-Yau they are given in (A.3),(4.11) and (5.31), while in the general discussion of compact Calabi-Yau manifolds they appear in (7.4),(7.18) and (7.42).

Here we will us the covariant derivatives (3.18) and (3.19) to rewrite the holomorphic anomaly equations (2.10). Firstly, we will apply modularity and the fact that all non-holomorphic dependence arises through the propagator $\widehat{E}_2(\tau,\tau)$ to convert anti-holomorphic derivatives into derivatives with respect to $\widehat{E}_2$. Using (3.20) we will be able to carefully keep track of the $\widehat{E}_2$ dependence in the holomorphic anomaly equations. Eventually, a solution will be simply obtained by direct integration of a polynomial in $\widehat{E}_2$.

To begin with, note that the holomorphic anomaly equations specialize in the local limit to

$$\partial_{\bar{t}} F^{(g)} = \frac{1}{2} C^{(0)tt}_{\bar{t}} \left(D_t \partial_t F^{(g-1)} + \sum_{r=1}^{g-1} \partial_t F^{(r)} \partial_t F^{(g-r)} \right)\,. \tag{3.21}$$

Using the fact that all non-holomorphic dependence arises only through the propagator $\widehat{E}_2(\tau,\bar{\tau})$, this equation can be rewritten as

$$\frac{\partial F^{(g)}}{\partial \widehat{E}_2} = \frac{1}{48} \left(D_t \partial_t F^{(g-1)} + \sum_{r=1}^{g-1} \partial_t F^{(r)} \partial_t F^{(g-r)} \right)\,. \tag{3.22}$$

Here we used (3.12) to substitute $C^{(0)tt}_{\bar{t}}$ with the derivative $\partial_{\bar{t}}\widehat{E}_2$, which then cancels with the same factor arising on the left-hand side of this equation. Let us now manipulate the right-hand side of (3.25) and split off the derivative of $F^{(1)}$ in the second term

$$\frac{\partial F^{(g)}}{\partial \widehat{E}_2} = \begin{cases} \frac{1}{48}\left(D_t \partial_t F^{(1)} + (\partial_t F^{(1)})^2 \right) & g=2\,, \\ \dfrac{1}{48}\left((D_t + 2\partial_t F^{(1)})\partial_t F^{(g-1)} + \displaystyle\sum_{r=2}^{g-2} \partial_t F^{(r)} \partial_t F^{(g-r)} \right) & g>2\,, \end{cases} \tag{3.23}$$

where the sum now runs from $r=2$ to $r=g-2$. One then notes that $\partial_t F^{(1)}$ can be replaced by $-\frac{1}{24}C\widehat{E}_2$ by using (3.13). Furthermore, we replace the non-holomorphic derivative D_t with its holomorphic counterpart $\hat{D}_t$ via (3.20). Altogether, one evaluates

$$\frac{\partial F^{(2)}}{\partial \widehat{E}_2} = -\frac{1}{48\cdot 24}\left(\hat{D}_t(C\widehat{E}_2) - \frac{1}{8}(C\widehat{E}_2)^2\right) \tag{3.24}$$

for genus two and for $g>2$

$$\frac{\partial F^{(g)}}{\partial \widehat{E}_2} = \frac{1}{48}\left(\left(\hat{D}_t - \frac{1}{6}C\widehat{E}_2\right)\partial_t F^{(g-1)} + \sum_{r=2}^{g-2}\partial_t F^{(r)}\partial_t F^{(g-r)}\right) . \tag{3.25}$$

We are now in the position to make the dependence on $\widehat{E}_2$ explicit. This can be done by rewriting the right-hand side of (3.25) using (3.19). We also define $\hat{d}_t$ and $\hat{d}_\tau$ as covariant derivatives $D_t, \hat{D}_\tau$ not acting on the propagators $\widehat{E}_2$, such that e.g. $\hat{d}_\tau(\hat{E}_2^k\, c_k^{(r)}) = \hat{E}_2^k\, \hat{D}_\tau c_k^{(r)}$. Applying the chain rule we find

$$\partial_t F^{(r)} = \left[\hat{d}_t + (\hat{D}_t\widehat{E}_2)\partial_{\widehat{E}_2}\right]F^{(r)} = C\left[\hat{d}_\tau - \frac{1}{12}(E_4 + \widehat{E}_2^2)\partial_{\widehat{E}_2}\right]F^{(r)} , \tag{3.26}$$

where (3.14), (3.19) and (B.13) are applied to evaluate the derivative of E_2. The Eisenstein series E_4 arises naturally in rewriting the derivatives. We will therefore work with the ring $\mathbb{C}[\widehat{E}_2, h, E_4]$ introduced in (3.6).

Similarly, we rewrite the second derivative

$$\begin{aligned}\hat{D}_t\partial_t F^{(g-1)} = \frac{1}{12^2}C^2\Big(& 12^2\hat{d}_\tau^2 + 6^2 h\hat{d}_\tau + 2E_4(\widehat{E}_2\partial_{\widehat{E}_2} + \widehat{E}_2^2\partial^2_{\widehat{E}_2}) \\ & -(3h + 12\hat{d}_\tau)\widehat{E}_2^2\partial_{\widehat{E}_2} + 2\widehat{E}_2^3\partial_{\widehat{E}_2} + \widehat{E}_2^4\partial^2_{\widehat{E}_2} \\ & +(-9E_4h + 2h^3 - 12E_4\hat{d}_\tau)\partial_{\widehat{E}_2} + E_4^2\partial^2_{\widehat{E}_2}\Big)F^{(g-1)} ,\end{aligned} \tag{3.27}$$

where we have used that the derivative of C is given by $\hat{D}_\tau C = \frac{1}{4}h\,C$. This is how the holomorphic modular form h defined in (3.6) arises in the direct integration.

We can now actually perform the direct integration. This is done by inserting the expressions (3.26) and (3.27) for $\partial_t F^{(r)}$ and $\hat{D}_t\partial_t F^{(g-1)}$ into the holomorphic anomaly equation (3.25). Replacing all $F^{(r)}$ for $1<r<g$ with their propagator expansion (3.16), it is then straightforward to keep track of the number of propagators $\widehat{E}_2$ in each term of the right-hand side of (3.25). Finally, $F^{(g)}$ is determined up to a $\widehat{E}_2-$independent ambiguity by integrating the resulting polynomial in $\widehat{E}_2$. Without much effort this procedure can be repeated iteratively up to the desired genus.

Note that the equation (3.24) for $F^{(2)}$ is particularly simple to integrate. Using (3.19) and (B.13) one evaluates

$$\hat{D}_t(C\widehat{E}_2) - \frac{1}{8}(C\widehat{E}_2)^2 = \frac{1}{24}C^2\left(-5\widehat{E}_2^2 + 6\widehat{E}_2 h - 2E_4\right) . \tag{3.28}$$

Inserted into (3.25) it is straightforward to integrate this quadratic polynomial in $\widehat{E}_2$ to derive $F^{(2)}$ as

$$F^{(2)}(\tau,\bar{\tau}) = \frac{1}{2\cdot 24^3}C^2\left(\frac{5}{3}\,\widehat{E}_2^3 - 3\,h\widehat{E}_2^2 + 2\,E_4\widehat{E}_2\right) + C^2 c_0^{(2)}, \tag{3.29}$$

where $c_0^{(2)}(h,E_4)$ is the holomorphic ambiguity which can be fixed by additional boundary conditions as we discuss in the next section. For genus up to 7 the expressions for $F^{(g)}$ were calculated in [33] using the Feynman graph expansion. The direct integration using (3.25) provides a far more effective method to solve Seiberg-Witten theory and confirms the results of [33]. Furthermore, the modular properties of the expressions are manifest at each step. As we will discuss in the later sections, similar constructions will provide us with a powerful tool to determine the set of candidate modular generators for more complicated Calabi-Yau manifolds. In particular, holomorphic modular forms are needed to parametrize the holomorphic ambiguity. In case we know the ring of holomorphic modular forms, fixing the ambiguity reduces to a determination of a finite set of numerical factors at each genus. For Seiberg-Witten theory this can be done systematically, as we will discuss in the next section.

3.3 Boundary conditions

To systematically fix the $c_0^{(g)}$ we have to understand the boundary behavior of the $F^{(g)}$. As it is well known, there are three distinguished regions in the moduli space of pure SU(2) $\mathcal{N}=2$ SYM which correspond to the geometrical singularities of $\mathcal{E}$. We will parametrize the moduli space by the vacuum expectation value $u=\langle \mathrm{Tr}\Phi^2\rangle$ of the scalar Φ in the $\mathcal{N}=2$ vector multiplet. The first region occurs at $u\sim\frac{1}{2}t^2\to\infty$, and it corresponds physically to the semiclassical regime. The monopole region occurs near $u\to\Lambda^2$, where a magnetic monopole of charge $(e,m)=(0,1)$ becomes massless and the electric SU(2) theory with gauge coupling $\mathrm{Im}\tau$ is strongly coupled. At the point $u\to-\Lambda^2$ a dyon of charge $(e,m)=(-1,1)$ becomes massless. However, this point is identified with the monopole point by a $\mathbb{Z}_2$ exact quantum symmetry. For this reason there are no independent boundary conditions at $u\to-\Lambda^2$ and we focus on $u\to\Lambda^2$ and $u\sim\infty$. In both cases the elliptic curve acquires a node, i.e. a local singularity of the form $\xi^2+\eta^2=(u\pm\Lambda^2)$, where a cycle of $\mathbb{S}^1$ topology shrinks. In string theory, a point in the moduli space where a node in the target geometry develops is called a conifold point.

The natural physical parameter in the magnetic monopole region $u\to\Lambda^2$ is t_D. We get first a convergent expansion for the $F^{(g)}$ in the variable $q_D=\exp(2\pi i\tau_D)$ for $\tau_D=-\frac{1}{\tau}\to i\infty$, which corresponds to $t_D\to 0$. This is obtained by an S- transformation of the modular expressions for the $F^{(g)}(\tau,\bar{\tau})$ such as (3.29), which converge in the semiclassical region. The holomorphic magnetic expansions $\mathcal{F}_D^{(g)}(\tau_D)$ can be obtained by formally taking the limit $\bar{\tau}_D\to\infty$, while keeping τ_D fixed. Finally we obtain the expansion in t_D by inverting (3.9). In these magnetic expansions, a gap structure was observed near the monopole (or conifold) point [33]. One finds that the leading behavior of $\mathcal{F}_D^{(g)}(\tau_D)$ is of the form

$$\mathcal{F}_D^{(g)} = \frac{B_{2g}}{2g(2g-2)\tilde{t}_D^{2g-2}} + k_1^{(g)}\tilde{t}_D + \mathcal{O}(\tilde{t}_D^2)\,, \tag{3.30}$$

where the B_n are the Bernoulli numbers and we used a rescaled variable $\tilde{t}_D = i\frac{t_D}{2}$. The knowledge of the leading coefficients and the absence of the remaining $2g-3$ sub-leading negative powers in the $\tilde{t}_D$ expansion imposes $2g-2$ conditions. Since $\dim M_{6g-3}(\Gamma(2)) = \left[\frac{3g-1}{2}\right]$ this overdetermines the $c_0^{(g)}$, e.g. for $g=2$ we find $c_0^{(2)} = -\frac{1}{2\cdot 24^3}\left(\frac{1}{2}E_4\, h + \frac{1}{30}h^3\right)$. It is very easy to integrate (3.25) using (3.26), (3.27) and the gap condition, which fixes the ambiguity to arbitrary genus. This solves the theory completely. One finds moreover a pattern in the first subleading term in the magnetic expansion

$$k_1^{(g)} = \frac{((2g-3)!!)^3}{g!2^{7g-2}} \; . \tag{3.31}$$

The gap can be explained by using the embedding of Seiberg-Witten theory into type IIA string theory compactified on a suitable Calabi-Yau manifold. The most generic singularity of a d complex dimensional manifold is a node where an $\mathbb{S}^d$ shrinks. The codimension one locus in the moduli space where this happens is called the conifold. It was argued in [56, 58] that at the conifold a RR-hypermultiplet becomes massless. This hypermultiplet is charged and couples to the $\mathrm{U}(1)$ vector multiplets. Its one loop effect on the kinetic terms of the vector multiplets in the effective action is captured by the local expansion of $F^{(0)}$ [56]. A gravitational one-loop effect yields the moduli dependence of the R_+^2 term in the effective action and is given by local expansion $F^{(1)}$ [58]. Using further one-loop arguments it was shown that the $F^{(g)}$, which capture the moduli dependence of the coupling of the self-dual part of the curvature to the self-dual part of the graviphoton $R_+^2\, F_+^{2g-2}$, have the following gap structure

$$F^{(g)}_{\text{conifold}} = \frac{(-1)^{g-1}B_{2g}}{2g(2g-2)t_D^{2g-2}} + \mathcal{O}(t_D^0), \tag{3.32}$$

where t_D is a suitable coordinate transverse to the conifold divisor [34]. The Seiberg-Witten gauge theory embedded in type IIA string theory inherits this structure, and the massless hypermultiplet at the conifold is identified as a monopole becoming massless at the monopole point. In this way, (3.32) explains the field theory result (3.30) and extends it to the full supergravity action.

Once the Seiberg-Witten amplitudes $F^{(g)}$ have been determined in terms of modular functions, these can be expanded around every point in the moduli space. For example, in the semiclassical regime $\tau \to i\infty$, $u \to \infty$ one finds the holomorphic amplitudes

$$\mathcal{F}^{(g)} = \frac{(-1)^g B_{2g}}{g(2g-2)(2t)^{2g-2}} + \frac{l^{(g)}_{2g+6}}{t^{2g+6}} + \mathcal{O}(t^{2g+10}) \; . \tag{3.33}$$

The higher order terms in this expansion correspond to gauge theory instantons and have been computed in [52].

4. A first look at the Enriques Calabi-Yau

In this section we review some basic properties of topological string theory on the Enriques Calabi-Yau. We begin by reviewing the $N=2$ special geometry of the classical moduli space of Kähler and complex structure deformations in section 4.1. The first world-sheet instanton corrections arise from genus one Riemann surfaces as shown in refs. [23, 29, 41]. The holomorphic higher genus free energies, restricted to the K3 fiber, can be also derived by using heterotic-type II duality [41]. We briefly summarize these results in section 4.2. In understanding and deriving the expression for the full $F^{(g)}$ an important hint is given by their transformation properties under the symmetry group of the full topological string theory on the Enriques Calabi-Yau. More precisely, generalizing the results of the previous section, one expects that all $F^{(g)}$ are built out of functions transforming in a particularly simple way under the group $Sl(2,\mathbb{Z}) \times O(10,2,\mathbb{Z})$. In section 4.4 we review some essentials about these modular and automorphic functions and forms.

4.1 Special geometry of the classical moduli space

The Enriques Calabi-Yau can be viewed as the first non-trivial generalization of the product space $\mathbb{T}^2 \times$ K3. It is defined as the orbifold $(\mathbb{T}^2 \times \mathrm{K3})/\mathbb{Z}_2$, where $\mathbb{Z}_2$ acts as a free involution [23]. This involution inverts the coordinates of the torus and acts as the Enriques involution on the K3 surface. The cohomology lattice of $\mathbb{T}^2 \times$ K3 takes the form [5]

$$\Gamma^{6,22} = \Gamma^{2,2} \oplus [\Gamma^{1,1} \oplus E_8(-1)]_1 \oplus [\Gamma^{1,1} \oplus E_8(-1)]_2 \oplus \Gamma^{1,1}_g \oplus \Gamma^{1,1}_s \,, \tag{4.1}$$

where the inner products on the sublattices $E_8(-1)$ and $\Gamma^{1,1}$ are given by

$$(C^{\alpha\beta}) = -C_{E_8} \,, \qquad (C^{ij}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \,. \tag{4.2}$$

with $\alpha, \beta = 1, \ldots, 8$ and $i, j = 1, 2$. Here C_{E_8} is the Cartan matrix of the exceptional group E_8. The lattice (4.1) splits into $H^1(\mathbb{T}^2) \oplus H_1(\mathbb{T}^2) = \Gamma^{2,2}$ and $H^*(K3) = \Gamma^{4,20}$. Under heterotic-type II duality it can be identified with the Narain lattice of the heterotic compactification on $\mathbb{T}^6$. The $\mathbb{Z}_2$ involution on the Enriques Calabi-Yau acts on the five terms of the lattice (4.1) as [23][3]

$$|p_1, p_2, p_3, p_4, p_5\rangle \to e^{\pi i \delta \cdot p_5} | -p_1, p_3, p_2, -p_4, p_5\rangle \,, \tag{4.3}$$

where p_i is an element of the i-th term in (4.1) and we denoted $\delta = (1, -1) \in \Gamma^{1,1}_s$.

The Enriques Calabi-Yau has holonomy group $\mathrm{SU}(2) \times \mathbb{Z}_2$. This implies that type II string theory compactified on the Enriques Calabi-Yau will lead to a four-dimensional theory with $\mathcal{N} = 2$ supersymmetry. Nevertheless, due to the fact that it does not have the full SU(3) holonomy of generic Calabi-Yau threefolds, various special properties related to $\mathcal{N} = 4$ compactification on $\mathbb{T}^2 \times K3$ are inherited.

As an example of the close relation of the Enriques Calabi-Yau to its $\mathcal{N} = 4$ counterpart $\mathbb{T}^2 \times K3$ one notes that the moduli space of Kähler and complex structure deformations are simply cosets. The complex dimensions of these moduli spaces are given by the dimensions $h^{(1,1)}$ and $h^{(2,1)}$ of the cohomologies $H^{(1,1)}$ and $H^{(2,1)}$. They can be determined constructing a basis of $H^{(p,q)}$ of forms of K3 and $\mathbb{T}^2$ invariant under the free involution. One obtains [23]

$$h^{(2,1)} = h^{(1,1)} = 11 \,, \tag{4.4}$$

[3]The effect of the phase factor on the type II side was interpreted as turning on a Wilson line [23].

while $H^{(0,0)}$, $H^{(3,3)}$ as well as $H^{(3,0)}$ are one-dimensional. Moreover, one can show that the Enriques Calabi-Yau is self-mirror and that both the Kähler and complex structure moduli spaces are given by the coset

$$\mathcal{M} = \frac{Sl(2,\mathbb{R})}{\mathrm{SO}(2)} \times \mathcal{N}_8\,, \tag{4.5}$$

where

$$\mathcal{N}_s = \frac{O(s+2,2)}{O(s+2)\times O(2)}. \tag{4.6}$$

The actual moduli space is obtained after dividing $\mathcal{M}$ by the discrete groups $Sl(2,\mathbb{Z}) \times O(10,2;\mathbb{Z})$. $\mathcal{M}$ is a simple example of a special Kähler manifold. We will discuss its properties in the following.

It is a well-known fact that the geometric moduli space of a Calabi-Yau manifold consists of two special Kähler manifolds corresponding to Kähler and complex structure deformations. A summary of some of the basic definitions and identities of special geometry can be found in appendix A. Essentially all information is encoded in one holomorphic function, the prepotential $\mathcal{F}$. Let us for concreteness consider the moduli space of Kähler structure deformations of the Enriques Calabi-Yau which is of the form (4.5). Denoting by $\hat{\omega}$ the harmonic $(1,1)$-form in the $\mathbb{T}^2$-base and by ω_a the $(1,1)$ forms in the Enriques fiber, we obtain complex coordinates S, t^a by expanding the combination

$$J + iB_2 = \ S\,\hat{\omega} + t^a\,\omega_a\,, \qquad\qquad a = 1,\ldots,10\,, \tag{4.7}$$

where J is the Kähler form on the Enriques Calabi-Yau and B_2 is the NS-NS two-form. Note that in our conventions $\mathrm{Re}\,S > 0$ and $\mathrm{Re}\,t^a > 0$ such that the world-sheet instantons arise as series in $q_S = e^{-S}$ and $q_{t^a} = e^{-t^a}$ in the large radius expansion. We note that these complexified Kähler parameters t^a can be regarded as a parametrization of the coset $\mathcal{N}_8$. The parametrization we are using here is the one suitable for the conventional large radius limit and corresponds to what was called in [41] the geometric reduction. In terms of (4.7), the prepotential takes the form

$$\mathcal{F} = -\frac{i}{2} C_{ab} t^a t^b S\ . \tag{4.8}$$

For the Enriques Calabi-Yau the cubic expression for the genus zero free energy $F^{(0)} = \mathcal{F}$ is exact and world-sheet instanton corrections will only arise at higher genus. This is precisely the reason for the simple form (4.5) of the moduli space. The symmetric matrix C_{ab} in (4.8) encodes the intersections in the Enriques fiber E such that

$$C_{ab} = \int_E \omega_a \wedge \omega_b\ . \tag{4.9}$$

The inverse matrix $C^{ab} \equiv C^{-1\,ab}$ can be calculated explicitly and coincide in an appropriate basis with the intersection matrix of the $\mathbb{Z}_2$ invariant lattice of the second and the third factor in (4.1), i.e.

$$\Gamma_E = \Gamma^{1,1} \oplus E_8(-1)\,, \qquad\qquad (C^{ab}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \times (-C_{E_8}). \tag{4.10}$$

Here C_{E_8} is the Cartan matrix of the exeptional group E_8. The lattice Γ_E is identified with the second cohomology group of the Enriques surface.

The prepotential for the Enriques Calabi-Yau encodes the classical geometry of the moduli space (4.5). The Kähler potential is derived using equation (A.8) to be of the form

$$K = -\log\Big[Y(S+\bar{S})\Big]\,, \qquad Y = \frac{1}{2}C_{ab}(t^a+\bar{t}^a)(t^b+\bar{t}^b)\;. \tag{4.11}$$

Note that K as given in (A.8) contains a term $-\log|X^0|^2$, with X^0 being the fundamental period. Such a term can be removed by a Kähler transformation $K \to K - f - \bar{f}$, where f is a holomorphic function, such that our expression (4.11) corresponds to a certain Kähler gauge. In general, all objects we will consider below are sections of a line bundle $\mathcal{L}$ which parametrizes such holomorphic rescalings $V \to e^f V$. As an example e^{-K} is a section of $\mathcal{L}\otimes\bar{\mathcal{L}}$. Such Kähler transformations do not change the Kähler metric which is obtained by evaluating the holomorphic and anti-holomorphic derivative of K. The Kähler metric splits into two pieces

$$G_{S\bar{S}} = \frac{1}{(S+\bar{S})^2}\,, \qquad G_{a\bar{b}} = -\frac{C_{ab}}{Y} + \frac{C_{ac}(t+\bar{t})^c C_{bd}(t+\bar{t})^c}{Y^2}\,, \tag{4.12}$$

with all other components vanishing. The Christoffel symbols for this metric are easily evaluated to be

$$\Gamma^S_{SS} = 2K_S\,, \qquad \Gamma^c_{ab} = K_e C^{ed}\hat{\Gamma}^c_{ab|d}\,, \tag{4.13}$$

where K_S and K_a are the first derivatives of the Kähler potential (4.11) and we have defined

$$\hat{\Gamma}^b_{ac|d} = \Big(\delta^b_c C_{ad} + \delta^b_a C_{cd} - \delta^b_d C_{ac}\Big)\;. \tag{4.14}$$

It is also easy to derive the holomorphic Yukawa couplings $C^{(0)}_{ijk}$ defined in (A.12). In coordinates S, t^a one uses the prepotential (4.8) to show

$$C^{(0)}_{Sab} = C_{ab}\;. \tag{4.15}$$

In general $C^{(0)}_{Sab}$ is a section of $\mathcal{L}^2\otimes\mathrm{Sym}^3(T^*\mathcal{M})$. In the case of the Enriques Calabi-Yau it is constant in the Kähler gauge and coordinates chosen above, and covariantly constant in a general gauge. The covariant derivative, acting on a section of $\mathcal{L}^m\otimes\bar{\mathcal{L}}^n$, is (A.3)

$$D_a = \partial_a + mK_a, \qquad D_{\bar{a}} = \partial_{\bar{a}} + nK_{\bar{a}}, \tag{4.16}$$

and includes the Christoffel symbols when acting on tensors. Applied to $C^{(0)}_{Sab}$ one shows

$$D_c C^{(0)}_{abS} = -\Gamma^d_{ca}C_{db} - \Gamma^d_{cb}C_{ad} + 2\partial_c K C_{ab} = 0\,, \tag{4.17}$$

which vanishes by means of the equation (4.13) for the Christoffel symbols. A similar equation holds for the covariant derivative $D_S C^{(0)}_{abS}$, showing that $C^{(0)}_{abS}$ is indeed covariantly constant. Once again, this special property of the Yukawa couplings is immediately traced back to the fact that the prepotential $\mathcal{F}$ receives no instanton corrections.

The space $\mathcal{M}$ has two different types of singular loci in complex codimension one on the moduli space [23, 4] which lead to conformal field theories in four dimensions. The first degeneration comes from the shrinking of a smooth rational curve $e \in \Gamma_E$ with $e^2 = -2$. The shrinking $\mathbb{P}^1$ leads to an SU(2) gauge symmetry enhancement together with a massless hypermultiplet, also in the adjoint representation of the gauge group. We then obtain for this point the massless spectrum of $\mathcal{N} = 4$ supersymmetric gauge theory. In terms of the complexified Kähler parameters introduced in (4.7) this singular locus occurs along

$$t^1 = t^2. \tag{4.18}$$

In order to understand the second singular locus, we first point out that the coset $\mathcal{N}_8$ can be parametrized in many different ways. In [41] it was noticed that there is a parametrization of this coset in terms of some coordinates t_D^a, $a = 1, \cdots, 10$ which are related to what was called there the BHM reduction. By using the formulae in [41] it is easy to see that the coordinates t^a and t_D^a are related by the following simple projective transformation,

$$\begin{aligned} t^1 &= t_D^1 - \frac{1}{4t_D^2}\sum_{i=3}^{10}(t_D^i)^2, \\ t^2 &= \frac{2\pi^2}{t_D^2}, \\ t^i &= -\pi \mathrm{i}\frac{t_D^i}{t_D^2}, \quad i = 3, \cdots, 10. \end{aligned} \tag{4.19}$$

The second singular locus occurs when

$$t_D^1 = t_D^2. \tag{4.20}$$

On this locus one gets as well an SU(2) gauge symmetry enhancement. In addition one gets four hypermultiplets in the fundamental representation of SU(2), and the resulting gauge theory is $\mathcal{N} = 2$, SU(2) Yang-Mills theory with four massless hypermultiplets. In figure 1 we represent schematically the two singular loci in moduli space, related by the projective transformation (4.19). In sections 5 and 6 of this paper we will explore in some detail the field theory limit of the topological string amplitudes and we will verify this picture of the moduli space.

4.2 Genus one and the free energies on the Enriques fiber

So far we have discussed the classical moduli space of the Enriques Calabi-Yau Y. We introduced the prepotential $\mathcal{F}$ which is cubic in the Kähler structure deformations and receives no worldsheet instanton corrections. One expects that such a simple structure will no longer persist at higher genus. This is already true at genus one as was shown in [29, 41]. Heterotic-type II duality can also be used to determine all higher genus free energies on the K3 fibers of the Enriques Calabi-Yau [41]. In this section we will summarize some results of [41] and present a closed expression for the fiber free energies also including the anti-holomorphic dependence.

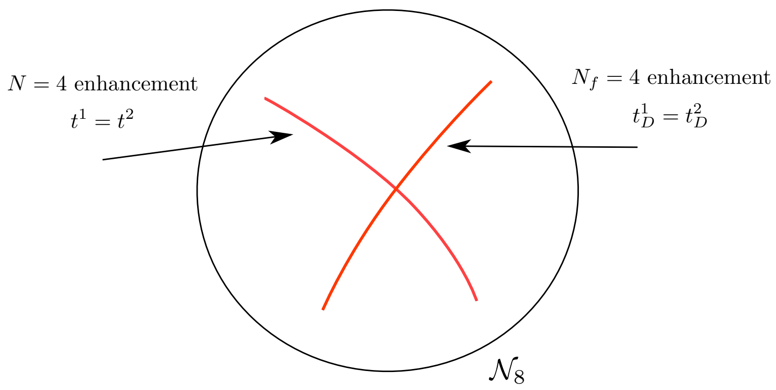

Figure 1: The singular loci in the moduli space $\mathcal{N}_8$, leading to two different gauge theories in the field theory limit.

Let us begin with a brief discussion of the free energies for the Enriques fiber. The fiber limit of the topological string amplitudes corresponds to blowing up the volume of the base space by taking

$$S \;\rightarrow\; \infty\,, \qquad\qquad q_S \equiv e^{-S} \;\rightarrow\; 0\,. \tag{4.21}$$

In what follows we will need to distinguish the full topological string amplitudes $F^{(g)}$ from their fiber limits as well as from their holomorphic limits. We will denote,

$$F_E^{(g)}(t,\bar{t}) = \lim_{S\rightarrow\infty} F^{(g)}(t,\bar{t}) \tag{4.22}$$

and

$$\mathcal{F}_E^{(g)}(t) = \lim_{\bar{t}\rightarrow\infty} F_E^{(g)}(t,\bar{t}). \tag{4.23}$$

The fiber limit $F_E^{(g)}(t,\bar{t})$ can be calculated using heterotic-type II duality [3, 46, 41]. In the heterotic string they are given by a one-loop computation of the form

$$F_E^{(g)}(t,\bar{t}) = \int \mathrm{d}\tau\, \overline{\Theta}^g_\Gamma(\tau, v^+) f_g(\tau,\bar{\tau})/Y^{g-1} \tag{4.24}$$

where Y is defined in (4.11), and $\Theta^g_\Gamma(\tau, v^+)$ is a theta function with an insertion of $2g-2$ powers of the right-moving heterotic momentum. We will not need the precise definitions of Θ^g_Γ and f_g here. However, it is important to note that these amplitudes can be evaluated in closed form by using standard techniques for one-loop integrals. The holomorphic limit (4.23) was determined in [41] and it is given by

$$\mathcal{F}_E^{(g)}(t) = \sum_{r>0} c_g(r^2)\left[2^{3-2g}\mathrm{Li}_{3-2g}(e^{-r\cdot t}) - \mathrm{Li}_{3-2g}(e^{-2r\cdot t})\right], \tag{4.25}$$

where Li_n is the polylogarithm of index n defined as

$$\mathrm{Li}_n(x) = \sum_{d=1}^{\infty} \frac{x^d}{d^n}\,. \tag{4.26}$$

In formula (4.25) we have also set $r^2 = C^{ab} r_a r_b$ and $r \cdot t = r_a t^a$. We will sometimes write

$$r = (n, m, \vec{q}). \tag{4.27}$$

The restriction $r > 0$ means $n > 0$, or $n = 0, m > 0$, or $n = m = 0$, $\vec{q} > 0$. Finally, we need to define the coefficients $c_g(n)$. They can be identified as the expansion coefficients of a particular quasi-modular form

$$\sum_n c_g(n) q^n = -2 \frac{\mathcal{P}_g(q)}{\eta^{12}(2\tau)}, \tag{4.28}$$

with $\mathcal{P}_g(q)$ given by

$$\left(\frac{2\pi\eta^3\lambda}{\vartheta_1(\lambda|\tau)} \right)^2 = \sum_{g=0}^{\infty} (2\pi\lambda)^{2g} \mathcal{P}_g(q). \tag{4.29}$$

The definition of $\eta(\tau)$ and the theta-function $\vartheta_1(\lambda|\tau)$ can be found in appendix B. From the definition (4.29) and the identities summarized in appendix B one also infers that the $\mathcal{P}_g$ are quasimodular forms of weight $2g$ and can be written as polynomials in the Eisenstein series E_2, E_4, E_6. We have for example

$$\mathcal{P}_1(q) = \frac{1}{12} E_2(q) \,, \qquad \mathcal{P}_2(q) = \frac{1}{1440} (5E_2^2 + E_4) \,. \tag{4.30}$$

In general, as we will see in section 5, it is very hard to include the $\mathbb{T}^2$-base in order to obtain the expressions $\mathcal{F}^{(g)}$ for the full Enriques Calabi-Yau. It turns out that only $\mathcal{F}^{(1)}$ factorizes nicely, namely we can write the A-model free energy $\mathcal{F}^{(1)}$ as [29, 41]

$$\mathcal{F}^{(1)}(S,t) = \mathcal{F}^{(1)}_{\text{base}} + \mathcal{F}^{(1)}_E \,, \tag{4.31}$$

where $\mathcal{F}^{(1)}_{\text{base}}$ and $\mathcal{F}^{(1)}_E$ are the contributions from the $\mathbb{T}^2$ base and the K3 fiber. $\mathcal{F}^{(1)}_{\text{base}}$ is the torus free energy given by [6]

$$\mathcal{F}^{(1)}_{\text{base}} = -12 \log \eta(S) \,, \tag{4.32}$$

where $\eta(S)$ is defined in (B.7), while

$$\mathcal{F}^{(1)}_E = -\frac{1}{2} \log \Phi(t), \tag{4.33}$$

where $\Phi(t)$ is the infinite product

$$\Phi(t) = \prod_{r>0} \left(\frac{1 - e^{-r \cdot t}}{1 + e^{-r \cdot t}} \right)^{2c_1(r^2)} . \tag{4.34}$$

This infinite product first appeared in the work of Borcherds [10]. As we will discuss in more detail later on, $\Phi(t)$ is the key example of a holomorphic automorphic form for the Enriques Calabi-Yau. It is also convenient to introduce,

$$\Phi(S,t) = \eta^{24}(S) \Phi(t), \tag{4.35}$$

so that we can write

$$\mathcal{F}^{(1)}(S,t) = -\frac{1}{2}\log\,\Phi(S,t). \tag{4.36}$$

We presented above formulae for the holomorphic limit of $F_E^{(g)}(t,\bar{t})$, but heterotic-type II duality can be used as well to obtain the antiholomorphic dependence on $\bar{t}$. At genus one, one finds [28, 46]

$$F_E^{(1)}(t,\bar{t}) = -2\log Y - \log\Big|\Phi(t)\Big|. \tag{4.37}$$

The antiholomorphic dependence on $\bar{S}$ is the usual one for the torus [6] and one has

$$F^{(1)}(S,\bar{S},t,\bar{t}) = F_E^{(1)}(t,\bar{t}) - 6\log\Big((S+\bar{S})|\eta^2(S)|^2\Big). \tag{4.38}$$

Equivalently, we can write

$$F^{(1)}(S,\bar{S},t,\bar{t}) = -2\log\Big[(S+\bar{S})^3 Y\Big] - \log\Big|\Phi(S,t)\Big|. \tag{4.39}$$

As a consistency check one shows that this anti-holomorphic dependence can also be inferred from the holomorphic anomaly equation (2.11) for $F^{(1)}$.

The antiholomorphic dependence in the heterotic calculation at higher genus is much more complicated, but was written down for the STU model in [46]. As we show in appendix C, this computation can be considerably simplified and adapted to the Enriques case. We find that the non-holomorphic free energy $F_E^{(g)}(t,\bar{t})$ can be cast into the form

$$\begin{aligned} F_E^{(g)}(t,\bar{t}) \;=\; & \sum_{l=0}^{g-1}\sum_{C=0}^{\min(l,2g-3-l)} \begin{pmatrix} 2g-3-l \\ C \end{pmatrix} \frac{(t+\bar{t})^{a_1}\dots(t+\bar{t})^{a_{l-C}}\partial_{a_1}\dots\partial_{a_{l-C}}\mathcal{F}_E^{(g-l)}(t)}{(l-C)!2^l\;Y^l} \\ & -\frac{1}{2^{g-2}(g-1)Y^{g-1}}\,, \end{aligned} \tag{4.40}$$

where $\mathcal{F}_E^{(r)}(t)$ is the holomorphic fiber expression given in (4.25). It is easy to check that the $F_E^{(g)}(t,\bar{t})$ fulfill the holomorphic anomaly equation on the fiber.

So far we have discussed the heterotic results for the fiber limit by using the Kähler parameters (4.7) appropriate for the large radius limit. As shown in [41], one can also compute them in the coordinates t_D^a introduced in (4.19). This was called the BHM reduction in [41], and leads to the holomorphic couplings,

$$\mathcal{F}_E^{(g)}(t_D) = \sum_{r>0} d_g(r^2/2)(-1)^{n+m}\mathrm{Li}_{3-2g}(\mathrm{e}^{-r\cdot t_D}) \tag{4.41}$$

where the coefficients $d_g(n)$ are defined by

$$\sum_n d_g(n)q^n = \frac{2^{2+g}\mathcal{P}_g(q^4) - 2^{2-g}\mathcal{P}_g(q)}{\eta^{12}(2\tau)}, \tag{4.42}$$

and in (4.41) we regard r as a vector in $\Gamma^{1,1} \oplus E_8(-2)$. Note that in comparison to (4.10) we now need to include the lattice $E_8(-2)$ with inner product given by -2 times the Cartan matrix of E_8, such that $r^2 = 2nm - 2\vec{q}^{\,2}$. One has, in particular,

$$\mathcal{F}_E^{(1)}(t_D) = -\frac{1}{2}\log \Phi_B(t_D)\,, \tag{4.43}$$

where

$$\Phi_B(t_D) = \prod_{r>0}\left(1 - \mathrm{e}^{-r\cdot t_D}\right)^{(-1)^{n+m} c_B(r^2/2)} \tag{4.44}$$

with coefficients

$$\sum_n c_B(n) q^n = \frac{\eta(2\tau)^8}{\eta(\tau)^8 \eta(4\tau)^8}. \tag{4.45}$$

This is the modular form introduced by Borcherds in [9], and the above expression for F_1 agrees with that found by Harvey and Moore in [29] (up to a factor of 1/2 due to different choice of normalizations).

4.3 An all-genus product formula on the fiber

As we have already mentioned, the infinite product (4.34) was first considered by Borcherds in [10]. Borcherds also noticed that (4.34) is the denominator formula for a generalized Kac-Moody (or Borcherds) superalgebra (see [25, 28] for a review of Borcherds algebras). The root lattice of this superalgebra is $\Gamma^{1,1} \oplus E_8(-1)$ (i.e. the cohomology lattice of the Enriques surface), and the simple roots are the positive, norm 0 vectors. Each simple root appears also as a superroot, both with multiplicity 8, and this is why the product of (4.34) has a "supersymmetric" structure: the numerator is a trace over fermionic degrees of freedom, while the denominator traces over bosonic degrees of freedom. Both have the same multiplicity $2c_1(r^2)$. In addition, the fact that $c_1(-1) = 0$ is equivalent to the absence of tachyons in the spectrum.

We will now write down a formula for the total partition function of topological string theory, restricted to the fiber, and we will show that it preserves the structure found by Borcherds for (4.34). As a first step, we define a generating functional $\xi(q, g_s)$ closely related to (4.29),

$$\xi(q, g_s) = \prod_{n=1}^{\infty} \frac{(1-q^n)^2}{1 - 2q^n \cos g_s + q^{2n}}. \tag{4.46}$$

We have the identity

$$\sum_{g=0}^{\infty} \mathcal{P}_g(q) g_s^{2g-2} = \left(2\sin\frac{g_s}{2}\right)^{-2} \xi^2(q, g_s), \tag{4.47}$$

Let us now define the Enriques degeneracies $\Omega_E(r, \ell)$ as

$$\sum_{r,\ell} 8\Omega_E(r,\ell) q^{r^2} q_s^{\ell} = \frac{2}{(q_s^{\frac{1}{4}} - q_s^{-\frac{1}{4}})^2} \frac{1}{\eta^{12}(2\tau)} (\xi^2(q, g_s/2) - \xi^2(-q, g_s/2)), \tag{4.48}$$

where

$$q_s = \mathrm{e}^{\mathrm{i}g_s} \tag{4.49}$$

The r.h.s. of (4.48) only involves *integer* powers of $q_s^{\pm 1}$. We can collect the Enriques degeneracies in the generating polynomials

$$\Omega_n(z) = \sum_{r^2=2n, \ell\geq 0} \Omega_E(r,\ell) z^\ell, \tag{4.50}$$

which are of degree n in z. We have for the first few:

$$\begin{aligned}
\Omega_0(z) &= 1, \\
\Omega_1(z) &= 12 + 2z, \\
\Omega_2(z) &= 90 + 24z + 3z^2, \\
\Omega_3(z) &= 520 + 180z + 36z^2 + 4z^3, \\
\Omega_4(z) &= 2538 + 1040z + 270z^2 + 48z^3 + 5z^4, \\
\Omega_5(z) &= 10944 + 5070z + 1560z^2 + 360z^3 + 60z^4 + 6z^5.
\end{aligned} \tag{4.51}$$

Notice that the constant terms of $\Omega_n(z)$ are closely related to the Euler characteristics of the Hilbert schemes of the Enriques surface, but there are "deviations" which become more and more important as the degree increases. Finally, notice that

$$\sum_\ell \Omega_E(r,\ell) q^{r^2} q_s^\ell = \Omega_n(q_s) + \Omega_n(q_s^{-1}) - \Omega_n(0). \tag{4.52}$$

We now define

$$F_E = \sum_{g=1}^{\infty} g_s^{2g-2} \mathcal{F}_E^{(g)}(t), \qquad Z_E = \mathrm{e}^{-2F_E}. \tag{4.53}$$

Notice that, as $g_s \to 0$, Z_E is precisely the Borcherds product $\Phi(t)$. It is now an easy exercise to evaluate it for finite g_s from (4.25), and we find

$$Z_E(g_s,t) = \prod_{r,\ell} \left(\frac{1 - q_s^\ell \mathrm{e}^{-r\cdot t}}{1 + q_s^\ell \mathrm{e}^{-r\cdot t}} \right)^{8\Omega_E(r,\ell)}. \tag{4.54}$$

As in the $g = 1$ case, (4.54) has a supersymmetric structure, with the same degeneracies for fermionic and bosonic states. This formula in fact suggests the existence of a superalgebra structure for the all-genus result as well. By including g_s we have extended the lattice to

$$\Gamma^{1,1} \oplus E_8(-1) \to \Gamma^{1,1} \oplus E_8(-1) \oplus \mathbb{Z} \tag{4.55}$$

which is reminiscent of the growth of an eleven-dimensional direction associated to the string coupling constant. The fact that the all-genus heterotic results seem to lead to an extra direction in the heterotic lattice has been pointed out in [17, 38]. It would be very interesting to see if there is indeed a superalgebra associated to the all-genus result (4.54). If this was the case, the quantities $8\Omega_E(r,\ell)$ would correspond to root multiplicities.

Finally, we mention that according to the conjecture in [47] and the results of [48], (4.54) is essentially the generating functional of an infinite family of Donaldson-Thomas invariants on the Enriques surface (written already in the right variables). Such product formulas for Z exist generically if the latter is expressed in terms of of Gopakumar-Vafa invariants [40]. Our comments above indicate that the Donaldson-Thomas theory on this manifold has a highly nontrivial algebraic structure (see section 3.2.6 in [48] for a related observation).

4.4 Automorphic forms

The free energies $F_E^{(g)}(t,\bar{t})$ on the fiber turn out to be automorphic forms on the coset space $\mathcal{N}_8$. Here we will study in some detail automorphic forms on the space $\mathcal{N}_s$. We will say that a function on the moduli space $\mathcal{N}_s$ is *automorphic* if it has well-defined transformation properties under the discrete subgroup $O(s+2,2;\mathbb{Z})$.

The transformation properties are easier to understand if we consider explicit generators of the symmetry group. We consider the explicit parametrization of the coset space (4.5) induced by a reduction

$$\Gamma^{s+2,2} = \Gamma^{s+1,1} \oplus \Gamma^{1,1}, \tag{4.56}$$

and let $t \in \mathbb{C}^{s+1,1}$ be the vector of complex coordinates parametrizing the coset. Our conventions are such that t has *positive* real part. For an element $t^a \in \mathbb{C}^{s+1,1}$ we define the inner product

$$t^2 = \frac{1}{2} C_{ab} t^a t^b, \tag{4.57}$$

where C_{ab} is the intersection matrix.

The generators of the symmetry group are taken to be [28]:

- $t \mapsto t + 2\pi i \lambda$, $\lambda \in \Gamma^{s+1,1}$.
- $t \mapsto w(t)$, $w \in O(s+1,1;\mathbb{Z})$.
- The automorphic analog of an S-duality transformation

$$t^a \quad \mapsto \quad \tilde{t}^a = \frac{t^a}{t^2}. \tag{4.58}$$

We say that a function $\Psi(t)$ is an *automorphic function of weight k* if it is invariant under the first two transformations above, and if under (4.58), it behaves as follows:

$$\Psi_k(\tilde{t}) = t^{2k} \Psi_k(t). \tag{4.59}$$

We can also have automorphic forms of weight $(k,\bar{k})$ which transform as

$$\Psi_{k,\bar{k}}(\tilde{t}) = t^{2k}\, \bar{t}^{2\bar{k}} \Psi_{k,\bar{k}}(t). \tag{4.60}$$

Although we have not indicated it explicitly, these functions might have a non-holomorphic dependence on $\bar{t}$. Automorphic forms are in general non-holomorphic. Some automorphic forms are meromorphic (they have poles at divisors). If they do not have poles, they are called *holomorphic*.

Notice that (4.58) transforms the metric $Y = (t+\bar{t})^2$ on the "upper half plane" as follows:

$$Y \;\mapsto\; t^{-2}\,\bar{t}^{-2} Y \;. \tag{4.61}$$

Following the definition (4.60) this identifies Y as an automorphic form of weight $(-1,-1)$. Recalling the form of the Kähler potential for the classical moduli space (4.11) this is nothing but a Kähler transformation [43]

$$K \;\mapsto\; K + \log t^2 + \log \bar{t}^2 \;. \tag{4.62}$$

in special coordinates where $X^0 = 1$. Note that, if we keep X^0, this shift can be absorbed by the transformation of X^0

$$X^0 \;\mapsto\; t^2 X^0 \;. \tag{4.63}$$

This can be traced back to the fact that K as given in (A.8) is a scalar under the full symplectic group.

In order to understand how the automorphic properties mix with taking derivatives, it is useful to derive the Jacobian J_a^b of the change of coordinates (4.58). We immediately find,

$$\frac{\partial \tilde{t}^a}{\partial t^b} \equiv (J^{-1})^a_b = \frac{1}{t^4}\Big(\delta^a{}_b t^2 - t^a C_{be} t^e\Big)\,, \qquad\qquad \frac{\partial t^a}{\partial \tilde{t}^b} = J^a_b = \delta^a{}_b t^2 - t^a C_{be} t^e \;. \tag{4.64}$$

Notice that J_a^b obeys the following useful identities

$$J_a^b = t^4 (J^{-1})^b_a\,, \qquad C_{ab} = t^{-4} C_{cd} J_a^c J_b^d\,, \qquad C^{ab} J_a^c J_b^d = t^4 C^{cd} \;. \tag{4.65}$$

Let us now assume that Ψ is an automorphic form of weight $(k,0)$. We want to determine the transformation behavior of $D_a\Psi$ and $D_a D_b \Psi$ under the dualities (4.58). D_a are here the derivatives covariant both with respect to Christoffel connection and the canonical connection on the vacuum bundle $\mathcal{L}$, as introduced in section 4.1. Therefore,

$$D_a \Psi = (\partial_a - k K_a)\Psi \;. \tag{4.66}$$

Notice that, since K transforms as given in (4.62), its first derivative K_a shifts as

$$K_a \;\mapsto\; J_a^b \Big(K_b + t^{-2} C_{bc} t^c \Big) \;. \tag{4.67}$$

Combining this with the transformation of the automorphic form Ψ itself we conclude

$$D_a \Psi \;\mapsto\; t^{2k} J_a^b D_b \Psi \;. \tag{4.68}$$

Similarly, we show that the second derivative of Ψ transforms as

$$D_b D_a \Psi \;\mapsto\; t^{2k} J_d^b J_a^c D_b D_c \Psi\,, \tag{4.69}$$

where we have used that the Christoffel symbols in the second connection transform as

$$J_b^d \partial_d J_a^c - \widetilde{\Gamma}_{ba}^d J_a^c = \Gamma_{ba}^d J_a^c \ . \tag{4.70}$$

Hence, we have shown that the covariant derivatives D_a of Ψ transform with a factor t^{2k} but are also rotated by the Jacobian J_b^a containing another factor of t^2. Note however, that we can easily obtain automorphic forms containing the derivatives $D_a\Psi$. More precisely, if Ψ and Ψ' are automorphic forms of weight $(k,0)$ and $(k',0)$ we find by using (4.65) that

$$C^{ab} D_a D_b \Psi \, , \qquad C^{ab} D_a \Psi D_b \Psi' \tag{4.71}$$

are automorphic forms of weight $k+2$ and $k+k'+2$ respectively. Such automorphic combinations arise in the derivation of all $F^{(g)}(S,\bar{S},t,\bar{t})$, $g>1$. More precisely, we will argue in the next sections that as function of $t,\bar{t}$, $F^{(g)}(S,\bar{S},t,\bar{t})$ itself is an automorphic form of weight $(2g-2,0)$ such that

$$F^{(g)} \ \mapsto \ t^{4g-4} F^{(g)} \qquad \text{for} \quad g>1 \ . \tag{4.72}$$

An important example of an automorphic form is the heterotic integral (4.24). It is easy to show from the properties of the Narain-Siegel theta function that it has weight $(2g-2,0)$. Since this integral gives the fiber limit $F_E^{(g)}$, we obtain a check of the general property (4.72) from heterotic/type II duality. Note that it is straightforward to define amplitudes $F^{(g)}$ invariant under automorphic transformations by

$$(X^0)^{2-2g} \, F^{(g)} \ . \tag{4.73}$$

The invariance of this combination is readily checked by using (4.63) and (4.72). The expressions (4.73) are shown to be invariant under the full target space symmetry group $Sl(2,\mathbb{Z}) \times O(10,2)$. They are the direct analogs of the invariant free energies encountered in the Seiberg-Witten example in section 3.

A particularly important and simple example occurs at $g=1$. Since $F_E^{(1)}$ is invariant, one deduces from (4.37) and (4.61) that $\Phi(t)$ is an automorphic form of weight $(4,0)$ i.e.

$$\Phi(\tilde{t}) = t^8 \ \Phi(t) \, , \qquad \tilde{t}^a = \frac{t^a}{t^2} \ . \tag{4.74}$$

One can also show that $\Phi(t)$ is holomorphic. This is proved in [10], and it is in fact a consequence of the regularity of $\mathcal{F}_E^{(g)}(t)$ at the singular locus (4.18), which will be discussed in more detail in section 5.4. In addition, $\Phi(t)$ is what is called a singular automorphic form (see [8], section 3, for a definition). Singular automorphic forms are known to satisfy a wave equation

$$C^{ab} \frac{\partial^2}{\partial t^a \partial t^b} \Phi(t) = 0. \tag{4.75}$$

Equivalently, they have Fourier expansions involving only vectors of zero norm. It follows that $\mathcal{F}_E^{(1)}(t)$ satisfies

$$C^{ab} \partial_a \partial_b \mathcal{F}_E^{(1)} = 2 C^{ab} \partial_a \mathcal{F}_E^{(1)} \, \partial_b \mathcal{F}_E^{(1)} \ . \tag{4.76}$$

This is equivalent to the recursive relation found in [48] for genus one invariants on the fiber, and proves that the expression for $\mathcal{F}_E^{(1)}(t)$ obtained in [41] agrees with the Gromov-Witten calculation of [48].

5. Direct integration on the Enriques Calabi-Yau

In this section we illustrate the power of the method of direct integration by studying the topological string amplitudes $F^{(g)}$ on the Enriques Calabi-Yau. Our approach will follow and generalize the strategy developed for the Seiberg-Witten example in section 3. To begin with, we perform a direct integration along the $\mathbb{T}^2-$base in section 5.1. Using the fiber results obtained in the previous section as additional input, the first six free energies $F^{(g)}$ can be determined in a closed form. We then present a more general formalism combining direct integration in base and fiber directions. In section 5.2, we introduce the relevant holomorphic and non-holomorphic $O(10,2,\mathbb{Z})$ forms. A closed recursive expression for $F^{(g)}$ will be derived in section 5.3. It determines the $F^{(g)}$ up to a holomorphic ambiguity and we will briefly discuss possible boundary conditions in section 5.4. Finally, in section 5.5 we consider a reduced Enriques model with three parameters only, which was already studied in [41]. This model has the advantage that the mirror map can be determined explicitly. We also study in more detail the boundary conditions (such as the gap condition), which lead to valuable conclusions also applying to the full model.

5.1 A simple direct integration and $F^{(g)}$ to genus six

Let us now perform the direct integration along the $\mathbb{T}^2$ base and derive the first few amplitudes $F^{(g)}$. In order to do that we carefully keep track of their dependence of on the base direction $S, \bar{S}$. As in the case of Seiberg-Witten theory studied in section 3, it is easy to see from the structure of the holomorphic anomaly equations that the only antiholomorphic dependence of $F^{(g)}$ on $\bar{S}$ appears through $\widehat{E}_2(S,\bar{S})$. By taking derivatives with respect to S we will also generate in the holomorphic anomaly equations the modular forms $E_4(S), E_6(S)$, and by keeping track of the modular weight one immediately finds that $F^{(g)}$ is an element of weight $2g-2$ in the ring generated by

$$\widehat{E}_2(S,\bar{S}), \quad E_4(S), \quad E_6(S) \, . \tag{5.1}$$

Our only assumption here is that the holomorphic ambiguity for $F^{(g)}$ is also a modular form of weight $2g-2$ in this ring. This assumption (as well as the details of the direct integration) can be checked in a highly nontrivial way by comparing the resulting expressions to the field theory limit in the $N_f = 4$ locus of figure 1. This check will be performed in section 6.

To perform the direct integration let us first rewrite the holomorphic anomaly equation for the base direction $\bar{S}$. The general expression (2.10) reduces to

$$\partial_{\bar{S}} F^{(g)} = -\frac{1}{2}\frac{C^{ab}}{(S+\bar{S})^2}\left(D_a D_b F^{(g-1)} + \sum_{r=1}^{g-1} D_a F^{(r)} D_b F^{(g-r)}\right) \, . \tag{5.2}$$

We now convert the derivative $\partial_{\bar{S}}$ into a derivative with respect to $\widehat{E}_2$. The definition of $\widehat{E}_2$ was already given in (3.14). Since we now consider an expansion in $q_S = e^{-S}$ it takes the form

$$\widehat{E}_2(S,\bar{S}) = -\frac{12}{S+\bar{S}} + E_2(S) \, . \tag{5.3}$$

Using the above assumption that the dependence of $F^{(g)}$ on $\bar{S}$ is only through this quantity, we can rewrite the anomaly equation as

$$\frac{\partial F^{(g)}}{\partial \widehat{E}_2} = -\frac{1}{24} C^{ab} \left(D_a D_b F^{(g-1)} + \sum_{r=1}^{g-1} D_a F^{(r)} D_b F^{(g-r)} \right) . \tag{5.4}$$

Here the covariant derivatives D_a are only taken with respect to the fiber directions and do not depend on the base due to the simple special geometry of the Enriques Calabi-Yau. This implies that all dependence on $\widehat{E}_2$ arises directly through the $F^{(r)}$. We thus expand $F^{(g)}$ in powers of $\widehat{E}_2$ by writing

$$F^{(g)} = \sum_{k=0}^{g-1} \widehat{E}_2^k(S, \bar{S})\ c_k^{(g)} , \qquad g > 1 . \tag{5.5}$$

We see that (5.4) determines all the coefficients $c_k^{(g)}$ for $k = 1, \ldots, g-1$ in terms of quantities at lower genera. Explicitly, we have the solution

$$c_k^{(g)} = -\frac{1}{24k} C^{ab} \left(D_a D_b c_{k-1}^{(g-1)} + \sum_{r=1}^{g-1} \sum_{l+m=k-1} D_a c_l^{(r)} D_b c_m^{(g-r)} \right) , \tag{5.6}$$

where we have set

$$c_0^{(1)} = F^{(1)} , \qquad c_i^{(1)} = 0 , \qquad i \neq 0 . \tag{5.7}$$

The $\widehat{E}_2$-independent term $c_0^{(g)}$ arises as an integration constant and hence cannot be determined by the holomorphic anomaly equation. However, given our assumptions, we can fix it up to genus 6 as follows. Let us denote the coefficients in the fiber limit by

$$c_{E|\,k}^{(g)} = \lim_{S, \bar{S} \to \infty} c_k^{(g)} . \tag{5.8}$$

By also taking the fiber limit of (5.5) we find

$$\sum_{k=0}^{g-1} c_{E|\,k}^{(g)} = F_E^{(g)}(t, \bar{t}) . \tag{5.9}$$

The free energies $F_E^{(g)}(t, \bar{t})$ are known from the heterotic computation and given in (4.40). Together with the fact that all $c_{E|\,k}^{(g)}$ for $k \geq 1$ are uniquely determined by the direct integration we can use (5.9) to derive $c_{E|\,0}^{(g)}$ i.e. the fiber limit of the integration constant. But the condition that $c_0^{(g)}$ is a modular form in the ring generated by (5.1) and does not involve $\widehat{E}_2$ fixes it uniquely in terms of $c_{E|\,0}^{(g)}$ as

$$\begin{aligned} c_0^{(2)} &= 0 , & c_0^{(3)} &= c_{E|\,0}^{(3)}\ E_4 , & c_0^{(4)} &= c_{E|\,0}^{(4)}\ E_6 , \\ c_0^{(5)} &= c_{E|\,0}^{(5)}\ E_4^2 , & c_0^{(6)} &= c_{E|\,0}^{(6)}\ E_4\, E_6 , \end{aligned} \tag{5.10}$$

where $E_4(S)$ and $E_6(S)$ are the two holomorphic generators in (5.1). This can be checked by noting that the definition (B.9) of the Eisenstein series implies that

$$E_2 , \quad E_4 , \quad E_6 \quad \rightarrow \quad 1 , \tag{5.11}$$

in the fiber limit $S, \bar{S} \to \infty$. For $g \geq 7$, the number of possible modular forms is greater than one and $c^{(g)}_{E|0}$ is no longer uniquely determined in terms of its fiber limit. For example, at genus seven $c^{(7)}_0$ can contain terms proportional to E_4^3 as well as E_6^2.

Let us now write down some explicit formula for lower genera. For $g = 2$ we find,

$$F^{(2)}(S, \bar{S}, t, \bar{t}) = \widehat{E}_2(S, \bar{S})\; c^{(2)}_1 \,, \tag{5.12}$$

where we use (5.10) and apply (5.6) to derive

$$c^{(2)}_1 = -\frac{1}{24} C^{ab} \left(D_a D_b F^{(1)}_E + D_a F^{(1)}_E D_b F^{(1)}_E \right). \tag{5.13}$$

Consistency of the fiber limit requires that $c^{(2)}_1 = F^{(2)}_E(t, \bar{t})$. This can be checked by using the heterotic expression (4.40) for $F^{(2)}_E(t, \bar{t})$, the property (4.76), and the identity [41]

$$\mathcal{F}^{(2)}_E = -\frac{1}{16} C^{ab} \partial_a \partial_b \mathcal{F}^{(1)}_E \,, \tag{5.14}$$

which follows directly from (4.24). In the holomorphic limit we find,

$$\mathcal{F}^{(2)}(S, t) = E_2(S) \mathcal{F}^{(2)}_E(t) \,, \tag{5.15}$$

in agreement with the results of [41, 48]. In the following sections we will also need a slightly different form of $F^{(2)}$. Namely, it is straightforward to apply (4.76) to write

$$F^{(2)} = -\frac{1}{8} C^{ab} \partial_a F^{(1)} \partial_b F^{(1)} \,. \tag{5.16}$$

Let us now consider the $g = 3$ case. The amplitude $F^{(3)}$ can be expanded by using (5.5) and (5.10) as

$$F^{(3)} = \widehat{E}^2_2(S, \bar{S})\, c^{(3)}_2 + E_4(S)\, c^{(3)}_{E|0} \,. \tag{5.17}$$

Using the result of the direct integration (5.6) we obtain

$$c^{(3)}_2 = -\frac{1}{48} C^{ab} \left(D_a D_b F^{(2)}_E + 2 D_a F^{(2)}_E D_b F^{(1)}_E \right) \,. \tag{5.18}$$

To determine $c^{(3)}_{E|0}$ we use (5.9), which gives

$$c^{(3)}_2 + c^{(3)}_{E|0} = F^{(3)}_E(t, \bar{t}) \,. \tag{5.19}$$

On the other hand, one finds that

$$F^{(3)}_E(t, \bar{t}) = -\frac{1}{24} C^{ab} D_a D_b F^{(2)}_E. \tag{5.20}$$

This can be derived in the holomorphic limit by using (4.24), and it is similar to (5.14). The antiholomorphic part can be checked with (4.40). Using all this, we finally obtain the following simple expression for $F^{(3)}(S, \bar{S}, t, \bar{t})$,

$$F^{(3)} = -\frac{1}{24} E_4\, C^{ab} D_a D_b F^{(2)}_E - \frac{1}{48} (\widehat{E}^2_2 - E_4) C^{ab} \left(D_a D_b F^{(2)}_E + 2 D_a F^{(2)}_E D_b F^{(1)}_E \right), \tag{5.21}$$

with the holomorphic limit

$$\mathcal{F}^{(3)}(S,t) = -\frac{1}{24}E_4\, C^{ab}\partial_a\partial_b\mathcal{F}_E^{(2)} - \frac{1}{48}(E_2^2 - E_4)C^{ab}\Big(\partial_a\partial_b\mathcal{F}_E^{(2)} + 2\partial_a\mathcal{F}_E^{(2)}\partial_b\mathcal{F}_E^{(1)}\Big)\,. \quad (5.22)$$

Note that the second term in these expressions vanishes identically in the fiber limit where $E_2, E_4 \to 1$. As we will discuss in more detail in section 5.4, this is the first $F^{(g)}$ where the inclusion of the base yields a behavior near the singular loci that differs significantly from the fiber limit.

Explicit calculations at genus 4 proceed in the same way. Modular invariance with respect to S gives

$$F^{(4)}(S,\bar{S},t,\bar{t}) = \widehat{E}_2^3\; c^{(4)}_{E|\,3} + \widehat{E}_2 E_4\; c^{(4)}_{E|\,1} + E_6\; c^{(4)}_{E|\,0}. \quad (5.23)$$

Once again, the general equation (5.48) allows us to determine the coefficients as

$$\begin{aligned} c^{(4)}_{E|\,3} &= -\frac{1}{72}C^{ab}\Big(D_aD_bc^{(3)}_{E|\,2} + 2D_aF_E^{(1)}D_bc^{(3)}_{E|\,2} + D_aF_E^{(2)}D_bF_E^{(2)}\Big)\,,\\ c^{(4)}_{E|\,1} &= -\frac{1}{24}C^{ab}\Big(D_aD_bc^{(3)}_{E|\,0} + 2D_aF_E^{(1)}D_bc^{(3)}_{E|\,0}\Big)\,. \end{aligned} \quad (5.24)$$

The ambiguity $c^{(4)}_{E|\,0}$ is again determined by the heterotic computation in the fiber limit. More precisely, one specializes (5.9) to

$$c^{(4)}_{E|\,0} + c^{(4)}_{E|\,1} + c^{(4)}_{E|\,3} = F_E^{(4)}(t,\bar{t})\,, \quad (5.25)$$

and solves for $c^{(4)}_{E|\,0}$ by inserting the fiber result (4.40). This determines the free energy $F^{(4)}$. A similar analysis also applies to $g = 5, 6$. As already discussed above, the main obstacle that has to be overcome in order to proceed to higher genus is the difficulty to fix the ambiguities $c_0^{(g)}$. We will discuss possible additional boundary conditions in sections 5.4, 5.5 and 6.

5.2 Propagators and homolomorphic automorphic forms

In the previous section we calculated the first free energies $F^{(g)}$ by a direct integration along the base direction. The results were expressed in terms of the holomorphic fiber energies $\mathcal{F}_E^{(g)}$, which are known from heterotic-type II duality. Even though the results were rather compact and transparent, the information we have extracted is somewhat partial, since we have not used the holomorphic anomaly equations for the fiber moduli. In order to exploit the information they contain, we will construct building blocks for the automorphic forms in the fiber which enable us to perform the direct integration of the remaining holomorphic anomaly equations. Recall that we argued in the previous sections that the almost holomorphic modular form

$$\widehat{E}_2(S,\bar{S}) = -\frac{12}{S+\bar{S}} + E_2(S)\,, \qquad E_2(S) = \partial_S \log\Phi\,, \quad (5.26)$$

contains all non-holomorphic dependence of $F^{(g)}$ along the base direction S. It will be the task of this section to introduce the analog of $\widehat{E}_2$ for the fiber directions t^a. Furthermore we will define the fiber analogs of the holomorphic modular forms $E_4(S)$ and $E_6(S)$. This will lead us to the definition of a new class of holomorphic automorphic forms of $O(10,2,\mathbb{Z})$. Eventually, in section 5.3 we will argue that a direct integration along the fiber direction allows us to express all $F^{(g)}$ in terms of these almost holomorphic and holomorphic forms of $O(10,2,\mathbb{Z})$.

Let us now introduce the fiber analog of the almost holomorphic modular form $\widehat{E}_2(S,\bar{S})$. This can be done by recalling that the genus one free energy $F^{(1)}$ is an invariant of the full symmetry group $Sl(2,\mathbb{Z}) \times O(10,2,\mathbb{Z})$ and hence its first derivatives transform in a particularly simple way. For the derivative with respect to S one finds $\partial_S F^{(1)} = \frac{1}{2}\widehat{E}_2$. The derivative with respect to t^a we denote by $\Delta^a = -\frac{1}{2}C^{ab}\partial_b F^{(1)}$ and evaluate

$$\Delta^a = \frac{t^a + \bar{t}^a}{Y} + \epsilon^a(t) = \epsilon^a(t) - K_b(t)C^{ba}\,, \qquad \epsilon^a(t) = \frac{1}{4}C^{ab}\partial_{t^b}\log\Phi\,, \tag{5.27}$$

where $Y = \frac{1}{2}C_{ab}(t+\bar{t})^b(t+\bar{t})^b$ and Φ is given in (4.34). The function $\epsilon^a(t)$ is holomorphic in the coordinates t^a and is the fiber analog of $E_2(S)$, while Δ^a plays the role of $\widehat{E}_2$. To see this note that ϵ^a transforms with a shift under the duality $t^a \mapsto t^a/t^2$:

$$\epsilon^a \quad \mapsto \quad t^4(J^{-1})^a_b(\epsilon^b + t^{-2}t^b)\ . \tag{5.28}$$

This shift is precisely canceled by the shift of the non-holomorphic term in (5.27) such that Δ^a simply transforms as

$$\Delta^a \quad \mapsto \quad t^4(J^{-1})^a_b\Delta^b(t)\ . \tag{5.29}$$

Note that $\widehat{E}_2$ and Δ^a are sufficient to parametrize all propagators $\hat{\Delta}^{ij}, \hat{\Delta}^i, \hat{\Delta}$ introduced in (2.17). Indeed, one has

$$\begin{aligned} \hat{\Delta}^{ab} &= -\frac{1}{12}C^{ab}\widehat{E}_2\,, &\quad \hat{\Delta}^{aS} &= \Delta^a\,, & & \\ \hat{\Delta}^{S} &= -\frac{1}{2}C_{ab}\Delta^a\Delta^b\,, &\quad \hat{\Delta}^{a} &= \frac{1}{12}\widehat{E}_2\Delta^a\,, &\quad \hat{\Delta} &= -\frac{1}{12}\widehat{E}_2 C_{ab}\Delta^a\Delta^b\ . \end{aligned} \tag{5.30}$$

Using the explicit form of $\widehat{E}_2$ and Δ^a it is straightforward to check that these propagators fulfill the defining conditions (2.17). The fact that all $\hat{\Delta}-$propagators can be expressed as polynomials in $\widehat{E}_2$ and Δ^a will be used in the next section to argue that all non-holomorphic dependence of $F^{(g)}$ only arises through $\widehat{E}_2, \Delta^a$. However, we also have to extract the non-holomorphic dependence in the covariant derivatives D_a defined in (A.3). Following the logic of section 3 we will show that each derivative can be split into a holomorphic covariant derivative $\hat{D}_a$ plus holomorphic terms times the propagators Δ^a. As an important byproduct, the definition of $\hat{D}_a$ will also allow us to find an interesting construction of holomorphic automorphic forms.

Let us now construct a holomorphic covariant derivative $\hat{D}_a$, which has the same properties as D_a under automorphic transformations (4.58). More precisely, given an automorphic form Ψ of weight k we define its first derivative as

$$\hat{D}_a\Psi \equiv \Big(\partial_a - kC_{ab}\epsilon^b\Big)\Psi\,, \tag{5.31}$$

where ϵ^a is defined in (5.27), and note that $\hat{D}_a = D_a - kC_{ab}\Delta^b$. $\hat{D}_a$ can be viewed as the analog of the Serre derivative (3.19) for modular forms of subgroups of $Sl(2,\mathbb{Z})$. It is not hard to check that it transforms under (4.58) exactly as D_a. This transformation property was given in (4.68). Note however, that $\hat{D}_a$ maps holomorphic forms into holomorphic forms, while D_a contains an anti-holomorphic contribution. Moreover, by definition of ϵ^a one has

$$\hat{D}_a \Phi(t) = 0\,, \tag{5.32}$$

for the automorphic form $\Phi(t)$ given in (4.34). In order to evaluate second derivatives we need to introduce the holomorphic analog of the Christoffel symbol in the definition (A.3) of D_k. To do that, let us consider a section Ψ_a which transforms as $\Psi_a \mapsto t^{2k} J_a^b \Psi_b$ under the action (4.58). The covariant derivative is then defined to act as

$$\hat{D}_a \Psi_b = \Big(\partial_a - kC_{ac}\epsilon^c\Big)\Psi_b - \hat{\Gamma}^c_{ab}\Psi_c\,. \tag{5.33}$$

Here we have included the holomorphic Christoffel symbol

$$\hat{\Gamma}^c_{ab} = \hat{\Gamma}^c_{ab|d}\epsilon^d = \frac{1}{2}\hat{C}^{cd}\Big(\partial_b \hat{C}_{da} + \partial_a \hat{C}_{db} - \partial_d \hat{C}_{ab}\Big)\,, \tag{5.34}$$

where $\hat{\Gamma}^b_{cd|a}$ is defined in (4.14) and related to Γ^b_{cd} by $\Gamma^b_{cd} = \hat{\Gamma}^b_{cd|a} C^{ae} K_e$. We also have introduced the holomorphic 'metric' $\hat{C}_{ab}$. Explicitly, $\hat{C}_{ab}$ is defined as

$$\hat{C}_{ab} = \Phi^{1/2} C_{ab}\,, \qquad \hat{C}_{ab} \;\mapsto\; J_a^c J_b^d \hat{C}_{cd}\,, \tag{5.35}$$

where Φ is given in (4.34) and we have also displayed the transformation behavior of $\hat{C}_{ab}$ under (4.58) as inferred from (4.74) and (4.65). Once again we evaluate the transformation behavior of $\hat{D}_a\Psi_b$ under (4.58) and finds the holomorphic analog of (4.69). It is now easy to show that every non-holomorphic derivative D_a can be split as

$$D_a\Psi_b = \hat{D}_a\Psi_b + kC_{ac}\Delta^c\,\Psi_b + \hat{\Gamma}^c_{ab|d}\Delta^d\,\Psi_c\,. \tag{5.36}$$

In other words, whenever Ψ_b is holomorphic the non-holomorphic dependence in $D_a\Psi_b$ arises through the propagators Δ^a only.

Let us now discuss a second interesting application of the holomorphic covariant derivative $\hat{D}_a$. Namely, we will now show how it can be used to construct new holomorphic automorphic forms. To start with let us note that $\epsilon_a = C_{ab}\epsilon^b$ transforms in (5.28) similarly to a vector field. We can use this analogy and define a field strength

$$\epsilon^4_{ab} = \partial_a\epsilon_b - \frac{1}{2}\hat{\Gamma}^c_{ab}\epsilon_c = \partial_a\epsilon_b - \epsilon_a\epsilon_b + C_{ab}\epsilon^2\,, \qquad\qquad \epsilon_a = C_{ab}\epsilon^b\,, \tag{5.37}$$

which transforms covariantly, $\epsilon^4_{ab} \mapsto J_a^c J_b^d \epsilon^4_{cd}$, under automorphic transformations (4.58). Note that by using the wave-equation (4.76) one shows that $\partial_a\epsilon^a = -4C_{ab}\epsilon^a\epsilon^b$ such that

$$C^{ab}\epsilon^4_{ab} = 0\,. \tag{5.38}$$

Nevertheless, we can use ϵ^4_{ab} to construct holomorphic automorphic forms. To do that, we define

$$\epsilon^{2k}_{a_1\ldots a_k} = \hat{D}_{a_k}\ldots\hat{D}_{a_3}\epsilon^4_{a_2a_1}\,, \tag{5.39}$$

which is shown to be totally symmetric in the indices. Holomorphic automorphic forms are now constructed by contraction with C^{ab}. For example, forms of weight 4 and 6 are given by

$$\begin{aligned} &\text{weight } 4: \qquad C^{ab}C^{cd}\epsilon^4_{ac}\epsilon^4_{bd}\,, \\ &\text{weight } 6: \qquad C^{ac}C^{be}C^{df}\epsilon^4_{ab}\epsilon^4_{cd}\epsilon^4_{ef}\,, \qquad C^{ac}C^{be}C^{df}\epsilon^6_{abd}\epsilon^6_{cef}\,. \end{aligned} \tag{5.40}$$

It is tempting to conjecture that holomorphic automorphic forms of this type are sufficient to parametrize the holomorphic ambiguity of $F^{(g)}$. The fact that there is no holomorphic weight 2 automorphic form of this type due to (5.38) matches nicely the fact that there is no holomorphic ambiguity for $F^{(2)}$. Also the forms in (5.40) can be shown to be sufficient to parametrize the ambiguities of $F^{(3)}$ and $F^{(4)}$. This will be analyzed in further work.

5.3 Direct integration of the holomorphic anomaly

We will now use the material developed in the previous section to perform the direct integration in both fiber and base directions. This will allow us to give closed expressions which determine the $F^{(g)}$ up to a holomorphic ambiguity. To begin with, we show that each $F^{(g)}$ can be written as

$$F^{(g)} = \sum_{k=0}^{g-1}\sum_{n=0}^{2g-2} \widehat{E}_2^k\Delta^{a_1}\ldots\Delta^{a_n}c^{(g)}_{k\,|\,a_1\ldots a_n}\,, \qquad g>1 \tag{5.41}$$

where $c^{(g)}_{k\,|\,a_1\ldots a_n}$ are holomorphic functions of S, t^a and all anti-holomorphic dependence arises through the propagators Δ^a and $\widehat{E}_2$ introduced in (5.26) and (5.27). Note that by using the transformation properties of $F^{(g)}$ and Δ^a given in (4.72) and (5.29) one infers that

$$c^{(g)}_{k\,|\,a_1\ldots a_n} \;\mapsto\; t^{4g-4-4n}J^{b_1}_{a_1}\ldots J^{b_n}_{a_n}\,c^{(g)}_{k\,|\,b_1\ldots b_n} \tag{5.42}$$

under automorphic transformations (4.58).

Let us now show that each $F^{(g)}$ for $g>1$ can indeed be written as (5.41) by using induction. We first note that $F^{(2)}$ is of the form (5.41),

$$F^{(2)} = -\frac{1}{2}\widehat{E}_2C_{ab}\Delta^a\Delta^b\,, \tag{5.43}$$

as is immediately inferred from (5.16) and (5.27). So let us assume that (5.41) is true for all $r<g$ and show that this implies that (5.41) is true for g. In order to do that we use the Feynman graph expansion (2.14) of $F^{(g)}$ [7], which states that each $F^{(g)}$ can be written as an expansion with propagators $\hat{\Delta}^{ij}, \hat{\Delta}^i, \hat{\Delta}$ and vertices $C^{(r)}_{i_1\ldots i_n}$ with $r<g$. We have already shown that the $\hat{\Delta}$-propagators are polynomials in $\widehat{E}_2$ and Δ^a in (5.30). Hence, it remains to show that also the vertices $C^{(r)}_{i_1\ldots i_n}$ are polynomials in $\widehat{E}_2$ and Δ^a. By definition (2.7) and our assertion, the vertices are defined as the covariant derivatives of amplitudes $F^{(r)}$

of the form (5.41). Using (5.36) each of these covariant derivatives D_a can be split into a holomorphic covariant derivative $\hat{D}_a$ and an expansion in Δ^a. So we only have to show that $\hat{D}_a\Delta^b$ admits again an expansion into Δ's. A straightforward computation shows that

$$\hat{D}_a\Delta^b = C^{bd}\epsilon^4_{da} - \frac{1}{2}\hat{\Gamma}^b_{cd|a}\Delta^c\Delta^d\,, \tag{5.44}$$

where ϵ^4_{ab} and $\hat{\Gamma}^b_{cd|a}$ are defined in (5.37) and (4.14). Altogether one infers that all vertices and $\hat{\Delta}$-propagators are polynomial in Δ^a and hence that $F^{(g)}$ is of the form (5.41).

Having shown that every $F^{(g)}$ is of the form (5.41) we will now derive a closed expression for $F^{(g)}$ by direct integration of the holomorphic anomaly equation (2.10). Applying the definition (2.17) of the propagators we can write the holomorphic anomaly equation as

$$\partial_{\bar{\imath}}F^{(g)} = \frac{1}{2}\partial_{\bar{\imath}}\hat{\Delta}^{ik}\left(D_jD_kF^{(g-1)} + \sum_{r=1}^{g-1}D_jF^{(r)}D_kF^{(g-r)}\right)\,. \tag{5.45}$$

This equation captures the anti-holomorphic derivatives $\partial_{\bar{S}}F^{(g)}$ along the base as well as the derivative $\partial_{\bar{a}}F^{(g)}$ along the fiber of the Enriques Calabi-Yau. Recall that the only non-vanishing propagators are $\hat{\Delta}^{ab} = -\frac{1}{12}C^{ab}\widehat{E}_2$ and $\Delta^a = \hat{\Delta}^{aS}$. As we have shown, they contain all anti-holomorphic dependence such that we can rewrite (5.45) as

$$\frac{\partial F^{(g)}}{\partial \widehat{E}_2} = -\frac{1}{24}C^{ab}\left(D_aD_bF^{(g-1)} + \sum_{r=1}^{g-1}D_aF^{(r)}D_bF^{(g-r)}\right)\,, \tag{5.46}$$

$$\frac{\partial F^{(g)}}{\partial \Delta^a} = D_aD_SF^{(g-1)} + \sum_{r=1}^{g-1}D_aF^{(r)}D_SF^{(g-r)}\,. \tag{5.47}$$

As we have seen above, the first equation is already very powerful and can be integrated easily. We can write the solution (5.6) as

$$F^{(g)} = -\frac{1}{24}\sum_{k=1}^{\infty}\frac{1}{k}\widehat{E}_2^kC^{ab}\left(D_aD_bc^{(g-1)}_{k-1} + \sum_{r=1}^{g-1}\sum_{l+m=k-1}D_ac^{(r)}_lD_bc^{(g-r)}_m\right) + c^{(g)}_0\,, \tag{5.48}$$

where $c^{(1)}_m$ is defined in (5.7). Note that $c^{(g)}_0(\Delta,S,t)$ arises an integration constant of the $\widehat{E}_2$ integration and hence can be a function of the propagators Δ^a but not $\widehat{E}_2$.

Let us now determine a second closed expression for $F^{(g)}$ by integrating the second anomaly equation (5.47). Since $F^{(1)}$ is not of the form (5.41) we first split off terms involving $F^{(1)}$. Inserting the definitions of the propagators Δ^a and $\widehat{E}_2$ we find for $g > 2$ that

$$\frac{\partial F^{(g)}}{\partial \Delta^a} = \left(D_S + \frac{1}{2}\widehat{E}_2\right)D_aF^{(g-1)} - 2C_{ac}\Delta^cD_SF^{(g-1)} + \sum_{r=2}^{g-2}D_aF^{(r)}D_SF^{(g-r)}. \tag{5.49}$$

To make the dependence on the propagators Δ^a explicit we expand the covariant derivative $D_aF^{(g)}$. The covariant derivative D_a can be split into a holomorphic derivative $\hat{D}_a$ defined in (5.33) plus a propagator expansion using (5.36). Moreover, using the chain rule one rewrites

$$\hat{D}_a = \hat{d}_a + (\hat{D}_a \Delta^b)\partial_{\Delta^b}\,, \tag{5.50}$$

where $\hat{d}_a$ is the covariant holomorphic derivative not acting on the propagators, i.e. we set

$$\hat{d}_a\Big(\Delta^{a_1}\dots\Delta^{a_n}c_{a_1\dots a_n}\Big) = \Delta^{a_1}\dots\Delta^{a_n}\hat{D}_a c_{a_1\dots a_n}\,. \tag{5.51}$$

Combining (5.36), (5.50) and (5.44) we immediately derive

$$D_a F^{(g)} = \Big[\hat{d}_a + \epsilon^4_{ac}C^{cb}\partial_{\Delta^b} + (2g-2)C_{ad}\Delta^d - \frac{1}{2}\hat{\Gamma}^b_{cd|a}\Delta^c\Delta^d\partial_{\Delta^b}\Big]F^{(g)}\,. \tag{5.52}$$

This expansion makes the dependence of D_a on the propagators Δ^a explicit. We note that the $\hat{d}_a$ term on the right-hand side of this expansion does not change the number of propagators. The second term lowers the number of propagators by one, while the two last terms raise the number of propagators by one. Inspecting the holomorphic anomaly equation we note that only the first derivative along the fiber direction appears on the right-hand side of (5.49). Hence, at least for the integration of (5.49) it will not be necessary to evaluate the second derivative $D_a D_b F^{(g)}$ as a propagator expansion.

To integrate expressions such as (5.52) for $D_a F^{(g)}$ we also need to keep track of the number of propagators in the expansion of $F^{(g)}$. Therefore, we introduce the following short-hand notation

$$F^{(g)} = \sum_n c^{(g)}_{(n)}\,, \qquad c^{(g)}_{(n)} = \sum_{k=0}^{g-1}\widehat{E}_2^k\Delta^{a_1}\dots\Delta^{a_n}c^{(g)}_{k\,|\,a_1\dots a_n}\,, \tag{5.53}$$

where each $c^{(g)}_{(n)}$ contains n propagators Δ^a. By counting the number of propagators one finds

$$\int D_a F^{(g)} d\Delta^a = \sum_n \Big\{\frac{1}{n+1}\Delta^a\hat{d}_a + \frac{1}{n}\Delta^a\epsilon^4_{ac}C^{cb}\partial_{\Delta^b} + \frac{4g-4-n}{n+2}\Delta^2\Big\}\,c^{(g)}_{(n)}\,, \tag{5.54}$$

where as defined above $\Delta^2 = \frac{1}{2}C_{ab}\Delta^a\Delta^b$. This integral together with similar ones for the remaining terms in (5.49) yields a closed expression for $F^{(g)}$ of the form

$$\begin{aligned} F^{(g)} = {} & \Big(D_S + \frac{1}{2}\widehat{E}_2\Big)\sum_n\Big\{\frac{1}{n+1}\Delta^a\hat{d}_a + \frac{1}{n}\Delta^a\epsilon^4_{ac}C^{cb}\partial_{\Delta^b} + \frac{4g-8-n}{n+2}\Delta^2\Big\}c^{(g-1)}_{(n)} \\ & -\sum_n\frac{4}{n+2}\Delta^2 D_S c^{(g-1)}_{(n)} + \sum_{r=2}^{g-2}\sum_n\sum_{k+l=n} D_S c^{(g-r)}_{(l)}\Big\{\frac{1}{n+1}\Delta^a\hat{d}_a \\ & +\frac{1}{n}\Delta^a\epsilon^4_{ac}C^{cb}\partial_{\Delta^b} + \frac{4r-4-n}{n+2}\Delta^2\Big\}c^{(r)}_{(k)} + c^{(g)}_{(0)}\,. \end{aligned} \tag{5.55}$$

Here $c^{(g)}_{(0)}(\widehat{E}_2, S, t)$ is the integration constant of the Δ^a integration and hence can depend on $\widehat{E}_2$ but not on Δ^a.

Before turning to the discussion of an explicit example, let us consider the fiber limit of (5.55). We therefore apply (B.13) and (5.11) to show that

$$\lim_{S,\bar{S}\to\infty} D_S F^{(g)} = 0 \,. \tag{5.56}$$

We also denote by $c^{(g)}_{E\,(k)}$ the fiber limit of the coefficients $c^{(g)}_{(k)}$ in (5.53). Inserting (5.56) into the formula (5.55) for direct integration along the fiber direction one finds

$$F^{(g)}_E = \frac{1}{2}\sum_n \left(\frac{1}{n+1}\Delta^a \hat{d}_a + \frac{1}{n}\Delta^a \epsilon^4_{ac} C^{cb}\partial_{\Delta^b} + \frac{4g-8-n}{n+2}\Delta^2\right) c^{(g-1)}_{E\,(n)} + c^{(g)}_{E\,(0)}\,, \tag{5.57}$$

where $c^{(g)}_{E\,(0)}(t)$ is a holomorphic ambiguity in the fiber. Recall that the full expression (4.40) for $F^{(g)}_E(t,\bar{t})$ is known from heterotic-type II duality. Therefore, verifying that this closed expression fulfills the differential equation (5.57) provides a non-trivial check of our derivations.

Let us end this section by presenting the first non-trivial solution to the closed expressions (5.48) and (5.55) for $F^{(g)}$. More precisely, one derives that the free energy $F^{(3)}$ admits the following propagator expansion

$$\begin{aligned} F^{(3)} &= -\frac{1}{48}\hat{E}_2^2\left(14\Delta^4 + 10\epsilon^4_{ab}\Delta^a\Delta^b - \epsilon^4_{ac}\epsilon^4_{bd}C^{ab}C^{cd}\right) \\ &\quad -\frac{1}{48}E_4\left(-2\Delta^4 + 2\epsilon^4_{ab}\Delta^a\Delta^b - \epsilon^4_{ac}\epsilon^4_{bd}C^{ab}C^{cd}\right), \end{aligned} \tag{5.58}$$

where ϵ^4_{ab} is defined in (5.37). Note that the last term in the first line has to be determined by the direct integration with respect to $\widehat{E}_2$ by using (5.48). Moreover, the purely holomorphic term

$$f^{(3)}(S,t) = \frac{1}{48}E_4\epsilon^4_{ac}\epsilon^4_{bd}C^{ab}C^{cd} \tag{5.59}$$

is the holomorphic ambiguity at genus 3, determined by the fiber limit. In other words, applying (5.11) one easily derives

$$F^{(3)}_E = -\frac{1}{4}\epsilon^4_{ab}\Delta^a\Delta^b - \frac{1}{4}\Delta^4 + \frac{1}{24}\epsilon^4_{ac}\epsilon^4_{bd}C^{ab}C^{cd}\,, \tag{5.60}$$

which is readily compared with the general expression (4.40) for the fiber free energies. It is straightforward to derive all $F^{(g)}$ for $g<7$ by evaluating (5.48) and (5.55) and fixing the ambiguity by comparison with the fiber result (4.40). Clearly, at genus greater than 6 we will encounter the same difficulties as in section 5.1. Only additional boundary conditions can help to fix the ambiguities in these cases. In the next section we will summarize possible additional conditions.

5.4 Boundary conditions

One important feature of the formalism of direct integration is that modular and holomorphic properties of the $F^{(g)}$ are manifest. In particular the ambiguity is holomorphic, modular invariant and for given genus expressible in terms of a modular form of finite weight. This implies that a finite number of data will fix it. The latter must be provided from additional information at the boundaries of the moduli space of the Calabi-Yau manifold. Let us give a short overview over the the nature of these boundary conditions.

In the large radius limit the holomorphic limit of the $F^{(g)}$ has an expansion in terms of Gromov-Witten invariants $N_\beta^{(g)}$. Since the an-holomorphic part is fixed, the $F^{(g)}$ can be completely determined by calculating a finite number of Gromov-Witten invariants. The reorganisation of the expansion in terms of Gopakumar-Vafa invariants $n_\beta^{(g)}$ is useful here, because the latter vanish if the degree is higher then the maximal degree for which a smooth curve exists in a given class.

For K3-fibered Calabi-Yau threefolds, the limit of large base volume corresponds generically to a perturbative heterotic string theory on K3$\times\mathbb{T}^2$. If the heterotic theory is known one can calculate the dependence of the $F^{(g)}$ on the fiber moduli by calculating a BPS saturated one loop amplitude in the heterotic string [46, 41]. In the Enriques CY case this yields most of the information and is the reason that one can tackle an 11 parameter model at all. Even if the heterotic dual is not known, one may get all the holomorphic $F^{(g)}$ in the fiber from the modular properties of the B-model on the K3 and the formula for the cohomology of the Hilbert scheme of points on the fiber [40].

If the Calabi-Yau admits controllable local limits, e.g. to toric Fano varieties with anti-canonical bundle, then the $F^{(g)}$ can be unambiguously calculated using the topological vertex [2].

One can also find boundary conditions by looking at the behavior of the topological string amplitudes near the conifold point, as we discussed in section 3.3. When there is only one hypermultiplet becoming massless at the conifold point, the amplitudes behave like (3.32), where t_D is a suitable coordinate transverse to the conifold divisor. This yields $2g-2$ independent conditions on the holomorphic ambiguity.

In contrast to generic $\mathcal{N}=2$ compactifications, the four dimensional massless spectrum at singularities of the Enriques Calabi-Yau is conformal, which requires hyper- and vector multiplets to become simultaneously massless. The leading behavior of the corresponding effective action is less characteristic. We will find a partial gap in the reduced model considered in section 5.5, which is similar to the partial gap structures that were found in [34] at a point where likewise several RR states become massless. The determination of the subleading behavior is possible in the field theory limit and yields conditions on the anomaly. We will consider here only the complex codimension singularities that we discussed in section 4. The nontrivial information about the $F^{(g)}$ comes from the $N_f=4$ locus: as we will show in section 6, the residue of the leading singularity near (4.18) can be computed using instanton counting in field theory.

Let us now analyze the leading singularity of $\mathcal{F}^{(g)}$ near the singular loci in the fiber limit. This can be done with the heterotic computations of [41] reviewed in section 4. These computations give us expansions around two special regions in moduli space, the large radius limit (where t^a are large) and the region appropriate to the BHM reduction (where t^a_D are large). As in [3, 46], we can use the computation at large radius to obtain the leading behavior of the fiber amplitudes near (4.18), and the computation in the BHM reduction to obtain the behavior near (4.20).

Let us first look at the behavior near (4.18). A possible singular behavior there must come from the vector $r = (1, -1)$ in (4.25), since this leads to a polylogarithm which, when expanded at the singular locus (4.18),

$$\mathrm{Li}_{3-2g}(\mathrm{e}^{-z}) = \frac{(2g-3)!}{z^{2g-2}} + \mathcal{O}(z^0), \quad g \geq 2, \tag{5.61}$$

exhibits a pole. Here, $z = t^1 - t^2$. However, since $c_g(-2) = 0$, the coefficient of this polylogarithm vanishes and we conclude that the amplitudes are regular at (4.18). This is indeed consistent with the fact that the field theory limit of this model at (4.18) is massless SU(2), $\mathcal{N} = 4$ super Yang-Mills theory, which has $\mathcal{F}^{(g)} = 0$ for all $g \geq 2$ [52, 53, 12].

Let us now look at the behavior near (4.20). To understand this, we look at the heterotic result for the holomorphic couplings in the BHM reduction (4.41). Again, the singular behavior comes from the vector $r = (1, -1)$. Since the coefficients are defined now by (4.42), we find

$$d_g(-1) = \frac{4^g - 1}{2^{g-2}} \frac{|B_{2g}|}{2g(2g-2)!}. \tag{5.62}$$

If we set

$$\mu = t_D^1 - t_D^2, \tag{5.63}$$

and we take into account the behavior of the polylogarithm (5.61), we find that the singular behavior of $\mathcal{F}_E^{(g)}(t_D)$ near (4.20) is given by

$$\mathcal{F}_E^{(g)}(t_D) \to \frac{4^g - 1}{2^{g-2}} \frac{|B_{2g}|}{2g(2g-2)} \frac{1}{\mu^{2g-2}} + \mathcal{O}(\mu^0) \tag{5.64}$$

for $g \geq 2$, while for $g = 1$ we have a logarithm singularity

$$-\frac{1}{2} \log \mu. \tag{5.65}$$

Since the full $\mathcal{F}^{(g)}(S, t_D)$ can be written for $g \leq 6$ in terms of (4.41), as we showed in section 5.1, we can compute its leading singular behavior at the locus (4.20). This will be useful in section 6 to compare to the field theory limit. The above computation shows that along the fiber direction the topological string amplitudes $\mathcal{F}_E^{(g)}$ show the gap behavior discovered in [33, 34]. In order to see if the gap also holds in the mixed directions, it is clear from the formulae above that we need a precise knowledge of the regular terms in μ in the expansion of $\mathcal{F}_E^{(g)}$. Unfortunately, this is something we cannot extract from the heterotic expressions. We will however be able to do this in the reduced model introduced in [41] and studied in more detail below. We will see that indeed the strong gap condition obtained for the fiber direction in (5.64) does not hold for the mixed directions.

5.5 The reduced Enriques model

In this section we discuss a reduced model for the Enriques Calabi-Yau introduced in [41]. The main advantage of this model is that the target symmetry group becomes much simpler, and one can easily parametrize the holomorphic functions which appear in the expansion of $F^{(g)}$ in the propagators $\Delta^a(t, \bar{t})$ and $\widehat{E}_2(S, \bar{S})$. In particular, the holomorphic ambiguity can be parametrized in terms of a finite number of coefficients at each genus. Also the mirror map is known explicitly and can be used to translate the $F^{(g)}$ into a simple polynomial form. In these aspects, the reduced model is very closely related to the Seiberg-Witten theory studied in section 3.

5.5.1 Special geometry and the mirror map

We begin with a brief discussion of the reduced special geometry and recall the mirror map derived in [41]. Out of the eleven special coordinates S, t^a the reduced model is only parametrized by three parameters. More precisely, it is obtained by shrinking 8 out of the 10 cycles in the Enriques fiber as

$$(S, t^a) = (S, t^i, t^\alpha) \quad \rightarrow \quad (S, t^i, 0)\,, \qquad i = 1, 2\,, \quad \alpha = 3, \ldots, 10\,. \tag{5.66}$$

We denote the reduced moduli space spanned by the remaining coordinates S, t^1, t^2 by $\mathcal{M}_{\rm r}$. Explicitly, the full coset (4.5) reduces in this limit to

$$\mathcal{M}_{\rm r} = \frac{Sl(2,\mathbb{R})}{\mathrm{SO}(2)} \times \left(\frac{Sl(2,\mathbb{R})}{\mathrm{SO}(2)}\right)^2\,, \tag{5.67}$$

inducing a split of the full target space symmetry group as

$$Sl(2,\mathbb{Z}) \times O(10,2,\mathbb{Z}) \quad \rightarrow \quad Sl(2,\mathbb{Z}) \times \Gamma(2) \times \Gamma(2)\,. \tag{5.68}$$

The generators of $Sl(2,\mathbb{Z})$ are precisely the Eisenstein series $\widehat{E}_2(S,\bar{S})$, $E_4(S)$, $E_6(S)$ as already introduced for the full model in (5.1). The generators for $\Gamma(2)$ have been introduced in the Seiberg-Witten section 3. More precisely, we will generate the ring of almost holomorphic modular forms of $\Gamma(2)$ by $\widehat{E}_2(t,\bar{t})$, $K_2(t)$ and $K_4(t)$ explicitly defined in (5.3) and (3.5). In the following we will simplify expressions by abbreviating

$$\begin{aligned} E_2 &= E_2(t^1)\,, \qquad & K_2 &= K_2(t^1)\,, \qquad & K_4 &= K_4(t^1)\,, \\ \tilde{E}_2 &= E_2(t^2)\,, \qquad & \tilde{K}_2 &= K_2(t^2)\,, \qquad & \tilde{K}_4 &= K_4(t^2)\,. \end{aligned} \tag{5.69}$$

Whenever not stated otherwise, we will keep the S-dependence explicit. Let us also note that the matrix C_{ab} splits as

$$C_{ab} = \begin{pmatrix} C_{ij} & 0 \\ 0 & C_{\alpha\beta} \end{pmatrix}\,, \qquad C_{ij} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\,, \tag{5.70}$$

as already given in (4.10). Hence, the holomorphic prepotential (4.8) and the fiber Kähler potential $Y = (t+\bar{t})^2$ reduce to

$$\mathcal{F}_{\rm r} = iSt^1t^2\,, \qquad\qquad Y_{\rm r} = (t^1 + \bar{t}^1)(t^2 + \bar{t}^2)\,. \tag{5.71}$$

As we have already noted in section 4.1 this prepotential and fiber potential are exact and receive no instanton corrections.

Let us now turn to a discussion of the mirror map for the reduced Enriques model. In order to determine this duality map we first note that the reduced Enriques has an algebraic realization. Applying standard techniques, one can thus derive the three Picard-Fuchs equations for the holomorphic three-form $\Omega(z)$ as

$$\mathcal{L}_1\Omega(z) = 0\,, \qquad \mathcal{L}_2\Omega(z) = 0\,, \qquad \mathcal{L}_3\Omega(z) = 0\,, \tag{5.72}$$

where $z^i(t), z^3(S)$ with $i = 1, 2$ are the mirror coordinates of t^i, S respectively. The Picard-Fuchs operators are found to be

$$\mathcal{L}_i = \theta_i^2 - 4(4\theta_i + 4\theta_j - 3)(4\theta_i + 4\theta_j - 1)z_i\,, \qquad i, j = 1, 2\,, \quad i \neq j\,, \tag{5.73}$$

$$\mathcal{L}_3 = 36(z^3 - 1)^2(z^3 - 2)\theta_3^2 + 36z^3(z^3 - 1)\theta_3 + z^3\Big(8z^3 - 4(z^3)^2 - 31\Big)\,, \tag{5.74}$$

where $\theta_i = z^i \frac{\partial}{\partial z^i}$. The Picard-Fuchs equations (5.72) can be solved to determine the mirror maps $z^i(t), z^3(S)$. This was done in [41] and we will only quote the result here. We first abbreviate

$$z(q_i) = \frac{K_4(t^i)}{K_2^2(t^i)}\ . \tag{5.75}$$

Using this shorthand notation the fiber mirror map reads

$$z^1(t) = z(q_1)\Big(1 - z(q_2)\Big)\,, \qquad z^2(t) = z(q_2)\Big(1 - z(q_1)\Big)\ . \tag{5.76}$$

These coordinates are related by a factor of 64 to z_1, z_2 used in ref. [41]. In the base one evaluates

$$z^3(S) = 1 - E_4^{-3/2}E_6\ . \tag{5.77}$$

Compared to [41] we have rescaled z^3 by a factor 864. Using these explicit expressions for z^1, z^2 and z^3, one immediately verifies their invariance under the target space symmetry group $Sl(2,\mathbb{Z}) \times \Gamma(2) \times \Gamma(2)$. Also the fundamental period X^0 can be obtained from the Picard-Fuchs system (5.72) and reads

$$X^0 = x^0\hat{X}^0\,, \qquad (\hat{X}^0)^2 = \frac{1}{4}K_2\tilde{K}_2\,, \qquad (x^0)^4 = E_4\ . \tag{5.78}$$

We immediately verify that X^0 is not invariant under the symmetry group $Sl(2,\mathbb{Z}) \times \Gamma(2) \times \Gamma(2)$. The S-duality transformation (4.58) reads for the reduced model $t^1 \mapsto 1/t^2$ and $t^2 \mapsto 1/t^1$. Applied to X^0 this yields precisely the transformation behavior given in (4.63). Before turning to the higher genus amplitudes in the next section let us also note that the discriminant of the reduced model is given by

$$\Delta(z^1, z^2)\ D(z^3)\,, \tag{5.79}$$

where $\Delta(z^1, z^2)$ is the discriminant along the fiber and $D(z^3)$ is the discriminant along the base. Explicitly, we find in the coordinates (5.75) and (5.76) that

$$\Delta(z^1, z^2) = \Big(1 - z(q_1) - z(q_2)\Big)^2 \tag{5.80}$$

$$= 1 - 2(z^1 + z^2 + z^1z^2) + (z^1)^2 + (z^2)^2\ . \tag{5.81}$$

The second discriminant $D(z^3)$ is given by

$$D(z^3) = \frac{1}{2^6 3^3}\Big((z^3)^2 - z^3\Big) . \tag{5.82}$$

In the next section we will use the mirror coordinates z^1, z^2 to express the reduced free energies $F_r^{(g)}$. Since along the base direction all equations are expressed in terms of simple Eisenstein series $E_{2n}(S)$ we choose to keep this S-parametrization also in the following discussions.

5.5.2 Reduced free energies and direct integration

Let us now discuss the free energies $F_r^{(g)}$ and their holomorphic limits $\mathcal{F}_r^{(g)}$ for the reduced model. In the limit (5.66) they are simply defined as

$$F_r^{(g)}(S, t^1, t^2) = F^{(g)}(S, t^1, t^2, t^\alpha = 0) . \tag{5.83}$$

The reduced form of $F^{(1)}$ can be derived by direct computation as was already discussed in [41]. Explicitly one finds

$$F_{\rm r}^{(1)} = -2\log\Big[(S+\bar{S})^3(t^1+\bar{t}^1)(t^2+\bar{t}^2)\Big] - \log|\Phi_{\rm r}(S,t)| , \tag{5.84}$$

where

$$\Phi_{\rm r}(S, t^1, t^2) = \eta^{24}(S)\prod_{m,n}\left(\frac{1-q^n q^m}{1+q^n q^m}\right)^{c_1^{\rm r}(2mn)} . \tag{5.85}$$

The coefficients $c_1^r(n)$ are given through the modular form

$$\sum_n c_1^{\rm r}(n) q^n = -\frac{64}{3\eta^6(q)\vartheta_2^6(q)} E_2(q) E_4(q^2) . \tag{5.86}$$

Note that in comparison with the expression (4.28) for the full Enriques model the Eisenstein series $E_4(q^2)$ appears in (5.86). This extra factor arises due to the summation over the E_8 vectors in (4.34) and precisely counts their degeneracy. It was further shown in [41] that the following denominator formula holds

$$\Phi_{\rm r}(S, t^1, t^2) = \frac{1}{16}\eta^{24}(S)\,\delta = \eta^{24}(S)(\hat{X}^0)^4\Delta^{1/2} \tag{5.87}$$

where

$$\delta(t^1, t^2) = K_2^2\tilde{K}_2^2 - K_4\tilde{K}_2^2 - K_2^2\tilde{K}_4 . \tag{5.88}$$

Here the $\Gamma(2)$ generators $K_2, \tilde{K}_2$ as well as $K_4, \tilde{K}_4$ are defined in (5.69), while the fundamental period $\hat{X}^0$ and the discriminant Δ were given in (5.78) and (5.80).

The holomorphic reduced amplitudes restricted to the Enriques fiber can also be computed directly by reducing the heterotic expressions (4.25) and (4.41). The result reads [41]

$$\begin{aligned}\mathcal{F}_{{\rm r},E}^{(g)}(t) &= \sum_{r>0} c_g^{\rm r}(r^2)\Big[2^{3-2g}{\rm Li}_{3-2g}(e^{-r\cdot t}) - {\rm Li}_{3-2g}(e^{-2r\cdot t})\Big] ,\\ \mathcal{F}_{{\rm r},E}^{(g)}(t_D) &= \sum_{r>0} d_g^{\rm r}(r^2/2)(-1)^{n+m}{\rm Li}_{3-2g}({\rm e}^{-r\cdot t_D}) ,\end{aligned} \tag{5.89}$$

where the coefficients $c^{\mathrm{r}}(n)$, $d^{\mathrm{r}}_g(n)$ are defined by

$$\begin{aligned} \sum_n c^{\mathrm{r}}_g(n) q^n &= -2E_4(q^2)\frac{\mathcal{P}_g(q)}{\eta^{12}(2\tau)}\,, \\ \sum_n d^{\mathrm{r}}_g(n) q^n &= E_4(q^2)\frac{2^{2+g}\mathcal{P}_g(q^4) - 2^{2-g}\mathcal{P}_g(q)}{\eta^{12}(2\tau)}\,. \end{aligned} \tag{5.90}$$

Once again we recognize the additional factor $E_4(q^2)$ counting the degeneracies of the E_8 lattice. Clearly, also the expressions $\mathcal{F}^{(g)}_{\mathrm{r},E}(t)$ and $\mathcal{F}^{(g)}_{\mathrm{r},E}(t_D)$ can be expressed in terms of the holomorphic generators (5.69) depending on t^i and t^i_D respectively.

Let us now turn to the discussion of the complete reduced amplitudes including the base and the non-holomorphic dependence. In order to do that we describe the direct integration for the reduced model focusing on the essential differences to the considerations presented in section 5.3. To begin with, note that the propagators of the full model reduce as

$$\Delta^i \quad \to \quad \Box^i\,, \qquad\qquad \Delta^\alpha \quad \to \quad 0\,, \tag{5.91}$$

where $\Box^i$ is obtained from (5.27) by setting $t^\alpha = 0$ and using (5.85). That Δ^α reduces to zero arises from the fact that in summation over the E_8 lattice the vectors cancel pairwise. In order to perform the direct integration we first have to find recursive relations which are valid for the reduced free energies $F^{(g)}_{\mathrm{r}}$. Recall that in the full Enriques model we found two sorts of recursive relations (5.46) and (5.47) capturing the properties $F^{(g)}$ in the base and in the fiber of the Enriques. It turns out that only the second anomaly equation (5.47) admits a simple reduction. More precisely, it can be rewritten for the reduced model as

$$\frac{\partial F^{(g)}_{\mathrm{r}}}{\partial \Box^i} = D_S D_i F^{(g-1)}_{\mathrm{r}} + \sum_{r=1}^{g-1} D_i F^{(r)}_{\mathrm{r}} D_S F^{(g-r)}_{\mathrm{r}}\,, \tag{5.92}$$

since performing the reduction $t^\alpha = 0$ interchanges with the differentiation with respect to t^1, t^2. Note that this is no longer true for derivatives with respect to t^α. In particular, the first equation (5.46) involves a summation over the α indices and one shows that the resulting terms do not vanish under the reduction $t^\alpha = 0$. Nevertheless, one can directly integrate (5.92) for the reduced free energies

$$F^{(g)}_{\mathrm{r}} = \sum_{n=1}^{2g-2} \Box^{i_1} \dots \Box^{i_n}\, \hat{c}^{(g)}_{i_1 \dots i_n} + \hat{c}^{(g)}\,, \qquad\qquad g > 1\,. \tag{5.93}$$

The function $\hat{c}^{(g)}$ is holomorphic in t^i and generally depends on $\widehat{E}_2(S,\bar{S}), E_4(S), E_6(S)$. Note that due to (5.91) the coefficients of the full and reduced model are related by $\hat{c}^{(g)}_{i_1 \dots i_n} = c^{(g)}_{i_1 \dots i_n}(t^\alpha = 0)$. The direct integration is performed in analogy to the integration in the full model and results in a closed expression similar to (5.55). The important difference is that the $\epsilon^4_{\mathrm{r}\,ij}$ as well as the covariant derivatives $\hat{D}^{\mathrm{r}}_a$ are not obtained from the full ϵ^4_{ab} and $\hat{D}_a$ by simply restricting to the i, j indices and setting $t^\alpha = 0$. Both $\epsilon^4_{\mathrm{r}\,ij}$ as well as $\hat{D}^{\mathrm{r}}_a$ have to be defined with respect to a new holomorphic metric $\hat{C}^{\mathrm{r}}_{ij} = \Phi^{1/2}_{\mathrm{r}} C_{ij}$ but otherwise analog to (5.37) and (5.33). If one had been using the old connection, an additional summation over the α indices would arise and yield extra contributions. Applied to the specific free energy $F^{(3)}$ one finds the reduction of the holomorphic ambiguity (5.59)

$$f_{\rm r}^{(3)}(S,t) = \frac{1}{48}E_4\left(\epsilon^4_{{\rm r}\,ik}\epsilon^4_{{\rm r}\,jl}C^{ij}C^{kl} + \frac{1}{8}(\epsilon^4_{{\rm r}\,ij}C^{ij})^2\right) \tag{5.94}$$

After these considerations it is not surprising that the contraction of the new $\epsilon^4_{{\rm r}\,ij}$ with C^{ij} does not vanish as it is the case in the full model (5.38).

5.5.3 The free energies $F^{(g)}$ on the mirror

So far the reduced free energies $F_{\rm r}^{(g)}$ were expressed as functions of the variables t^i, S or t^i_D, S. In the reduced model we know the mirror map explicitly and thus will be able to translate the expansion (5.93) of $F_{\rm r}^{(g)}$ into a function of the complex coordinates z^i. We will show that the holomorphic coefficients become polynomials in z^i divided by an appropriate power of the discriminant. Since the dependence of $F_{\rm r}^{(g)}$ is rather transparent we chose to keep this variable and do not replace it by its mirror counterpart z^3.

The $F^{(g)}$ transform non-trivially under the reduced automorphic transformations. We already discussed the actually invariant combination in (4.73). In the coordinates z^i, S we thus set

$$F^{(g)}(z,\bar z,S,\bar S) = (\hat X^0)^{2-2g}\, F^{(g)}(t,\bar t,S,\bar S)\ . \tag{5.95}$$

This definition is consistent with the fact that the $z^i(t)$ are invariant under the target space group (5.68), while $(\hat X^0)^{2g-2}$ transforms exactly as $F^{(g)}(t,S)$. To rewrite the expansion (5.93) we first note that the propagator $\Box^i$ can be written in the z^i coordinates as

$$\Box^i = (\hat X^0)^2\frac{\partial t^i}{\partial z^j}\Box^{z^j}\ , \qquad\qquad \Box^{z^i} = -C^{z^iz^j}\left(\hat K_{z^j} - \frac{1}{8}\partial_{z^j}\log\Delta\right)\ , \tag{5.96}$$

where $\Box^{z^i}$ is a function of $z^i, \bar z^i$ and we have used

$$C_{ij} = (\hat X^0)^{-2}C_{z^kz^l}\frac{\partial z^k}{\partial t^i}\frac{\partial z^l}{\partial t^j}\ , \qquad\qquad \hat K(z,\bar z) \equiv -\log\Big[|\hat X^0|^2 Y_{\rm r}(z,\bar z)\Big]. \tag{5.97}$$

It is not hard to use the expressions (5.76) for z^1 and z^2 to evaluate $C_{z^iz^j}$ explicitly as

$$C_{z^1z^2} = \frac{1}{z^1z^2\Delta}(1-z^1-z^2)\,, \qquad C_{z^1z^1} = \frac{1}{z^1z^2\Delta}2z^2\,, \qquad C_{z^2z^2} = \frac{1}{z^1z^2\Delta}2z^1\ . \tag{5.98}$$

Once again (5.96) and (5.97) are in accordance with the transformation behavior of the $\Box^i$ and C_{ij} given in (5.29) and (4.65). Similarly, we transform the coefficients $\hat c^{(g)}_{i_1\ldots i_n}$ in (5.93) and set

$$\hat c^{(g)}_{i_1\ldots i_n} = (\hat X^0)^{2g-2-2n}\,\frac{\partial z^{j_1}}{\partial t^{i_1}}\cdots\frac{\partial z^{j_n}}{\partial t^{i_n}}\ \hat c^{(g)}_{z^{j_1}\ldots z^{j_n}}(z)\,, \tag{5.99}$$

which is consistent with (5.43). It is also straightforward to rewrite the direct integration expression for $F_{\rm r}^{(g)}$ by using the z^i coordinates. Let us once again only discuss the appearing building blocks. We begin by noting that the holomorphic covariant derivative transforms as

$$\hat{D}_i V_j = (\hat{X}^0)^{2k} \frac{\partial z^l}{\partial t^i} \frac{\partial z^m}{\partial t^j} \hat{D}_{z^l} V_{z^m} \, , \tag{5.100}$$

where the covariant derivative $\hat{D}_{z^i}$ is given by

$$\hat{D}_{z^i} V_{z^j} = \partial_{z^i} V_{z^j} - \frac{k}{8} (\partial_{z^i} \log \Delta) V_{z^j} + \hat{\Gamma}^{z^l}_{z^i z^j} V_{z^l} \ . \tag{5.101}$$

The holomorphic Christoffel symbol in this expression is defined by

$$\hat{\Gamma}^{z^l}_{z^i z^j} = \frac{1}{2} \hat{C}^{z^l z^m} \left(\partial_{z^i} \hat{C}_{z^m z^j} + \partial_{z^j} \hat{C}_{z^i z^m} - \partial_{z^m} \hat{C}_{z^i z^j} \right), \qquad \hat{C}_{z^i z^j} = \Delta^{1/4} C_{z^i z^j} \ . \tag{5.102}$$

The second important object in the general equation for the direct integration is the automorphic form $\epsilon^4_{\mathrm{r}\, ij}$. One shows that it can be decomposed as

$$\epsilon^4_{\mathrm{r}\, ij} = \frac{1}{z^1 z^2 \Delta^2} \frac{\partial z^l}{\partial t^i} \frac{\partial z^m}{\partial t^j} \ \epsilon^4_{z^l z^m} \ . \tag{5.103}$$

where for $i = 1, 2$, $i \neq j$ one finds that

$$\begin{aligned}
\epsilon^4_{z^i z^i} &= -\frac{1}{16} z^j \left((z^i)^2 \ \left(1 + 3\, z^j\right) + \left(-1 + z^j\right)^2 \ \left(1 + 3\, z^j\right) - 2\, z^i \ \left(1 - 5\, z^j + 3\, (z^j)^2\right) \right), \\
\epsilon^4_{z^i z^j} &= \frac{3}{16}\, z^i\, z^j \ \left(-2 + z^i + (z^i)^2 + z^j + (z^j)^2 - 2\, z^i\, z^j\right) \ .
\end{aligned} \tag{5.104}$$

Note that $\epsilon^4_{z^i z^i}$ is polynomial due to the fact that we extracted the denominator $z^1 z^2 \Delta^2$ in (5.103). This turns out to be possible for all the coefficients $\hat{c}^{(g)}_{z^{i_1} \ldots z^{i_n}}$ appearing in (5.99). We thus define

$$P^{(g)}_{i_1 \ldots i_n}(z, \widehat{E}_2, E_4, E_6) = (z^1 z^2 \Delta)^{g-1} \ \hat{c}^{(g)}_{z^{i_1} \ldots z^{i_n}} \, , \tag{5.105}$$

where $P^{(g)}$ are polynomials in z^i as well as $\widehat{E}_2, E_4, E_6$. The reduced free energies are thus of the form

$$F^{(g)}_{\mathrm{r}}(z, \bar{z}, S, \bar{S}) = \frac{1}{(z^1 z^2 \Delta)^{g-1}} \sum_n \square^{z^{i_1}} \ldots \square^{z^{i_n}} \ P^{(g)}_{i_1 \ldots i_n} \, , \qquad g > 1 \ . \tag{5.106}$$

In particular, this implies that at each genus the holomorphic ambiguity is parametrized by a polynomial $P^{(g)}(z, E_4, E_6)$ holomorphic in z^i and S. As it was the case before the coefficients in $P^{(g)}$ have to be determined by boundary conditions. For the lower genera this can be done explicitly by using the fiber limes. At higher genus additional information are needed and we will discuss in the next section the possible input from a small gap condition. We believe that essentially all results on the mirror rewriting can be generalized to the full model in case one is able to determine the full mirror map. For the ten parameters along the fiber this is however a technically challenging task.

5.5.4 Boundary conditions and the small gap

As we have seen in (5.68), the automorphism group acting on the fiber variables is simply

$$\Gamma(2)\times\Gamma(2)\,, \tag{5.107}$$

where these groups act on $t^{1,2}$, respectively, plus the exchange $t^1 \leftrightarrow t^2$. Moreover, we see from (4.19) that the $\{t^i = 0 : i = 3,\cdots,10\}$ locus maps to the $\{t^i_D = 0 : i = 3,\cdots,10\}$ locus. If we now define

$$2\pi\mathrm{i}\tau^{1,2} = -t^{1,2}, \quad 2\pi\mathrm{i}\tau^{1,2}_D = -t^{1,2}_D. \tag{5.108}$$

we see that the transformation (4.19) relating the geometric and the BHM expressions reduces to

$$\tau^1_D = \tau^1, \quad \tau^2_D = -\frac{1}{2}\frac{1}{\tau^2}. \tag{5.109}$$

By using the explicit expressions for $\mathcal{F}^{(g)}_{\mathrm{r},E}(t)$ in terms of modular forms (which can be obtained for example by direct integration), one finds that under (5.109)

$$\mathcal{F}^{(g)}_{\mathrm{r},E}(t) \to 2^{1-g}\mathcal{F}^{(g)}_{\mathrm{r},E}(t_D), \tag{5.110}$$

where the factor of 2 is inherited from the factor of 2 in (5.109) and $\mathcal{F}^{(g)}_{\mathrm{r},E}(t_D)$ are also given in (5.89). Therefore, one can obtain expressions for the amplitudes in the BHM reduction in terms of modular forms by simply applying the transformation (5.109) to the results of the direct integration in the reduced model (which are valid for the geometric reduction).

These expressions for the BHM amplitudes can also be used to study in detail the behavior near the singularity (4.20), and in particular to calculate the subleading terms. One can verify that the discriminant (5.80) transforms under (5.109) as

$$\Delta(t^1,t^2) \quad \mapsto \quad \Delta(t^1_D,t^2_D) = (z(q^1_D)-z(q^2_D))^2, \tag{5.111}$$

which vanishes at the locus (4.20). This leads to the singular behavior of $\mathcal{F}^{(g)}_{\mathrm{r}}(t_D)$, and one can now verify the behavior (5.64) by expanding the expressions in terms of modular forms. One finds,

$$\begin{aligned}
\mathcal{F}^{(1)}_{\mathrm{r},E}(t_D) &= -\frac{1}{2}\log(\mu) - \frac{1}{2}\log\left[\frac{1}{128}K_2K_4(K_2^2-K_4)(q^2_D)\right] + \mathcal{O}(\mu),\\
\mathcal{F}^{(2)}_{\mathrm{r},E}(t_D) &= \frac{1}{16\mu^2} - \frac{80E_2^2K_2^2-16K_2^4+3K_2^2K_4+9K_4^2+16E_2(K_2^2+3K_2K_4)}{9216K_2^2}(q^2_D) + \mathcal{O}(\mu),\\
\mathcal{F}^{(3)}_{\mathrm{r},E}(t_D) &= \frac{1}{32\mu^4} + \frac{1}{53084160K_2^4}\big(-800E_2^4K_2^4+214K_2^8-726K_2^6K_4+1431K_2^4K_4^2\\
&\quad +405K_4^4-320E_2^3(K_2^5+3K_2^3K_4)+120E_2^2(10K_2^6-15K_2^4K_4+9K_2^2K_4^2)\\
&\quad -540K_2^2K_4-40E_2(14K_2^7-54K_2^5K_4+27K_2^3K_4^2-27K-2K_4^3)\big)(q^2_D) + O(\mu).
\end{aligned}$$

However, if one includes the base directions, the gap is "partially filled" starting at genus three (for $\mathcal{F}^{(2)}(S,t_D)$, the gap property away from the fiber limit is trivially satisfied).

Indeed, one finds that the term $C^{ab}\partial_a\mathcal{F}_E^{(1)}(t_D)\partial_b\mathcal{F}_E^{(2)}(t_D)$, leads, in the reduced model, to the expansion

$$\partial_1\mathcal{F}^{(1)}_{\mathrm{r},E}(t_D)\partial_2\mathcal{F}^{(2)}_{\mathrm{r},E}(t_D)+\partial_2\mathcal{F}^{(1)}_{\mathrm{r},E}(t_D)\partial_1\mathcal{F}^{(2)}_{\mathrm{r},E}(t_D)= \\ =-\frac{1}{8\mu^4}-\frac{20E_2^2K_2+17K_2^3+3K_2K_4+4E_2(K_2^2+3K_4)}{9216K_2}(q_D^2)\frac{1}{\mu^2}+\cdots \tag{5.112}$$

Although there are some nontrivial cancellations (for example, there is no term in μ^{-3}), generically one finds, for finite S, singular terms in μ beyond the leading one.

6. The field theory limit

As we reviewed in section 4, there is a line of enhanced symmetry in the moduli space of the Enriques Calabi-Yau which leads in the field theory limit to SU(2), $\mathcal{N}=2$ QCD with four massless hypermultiplets. This occurs at the locus (4.20). Similarly to what happens for other K3 fibrations [35], we expect that near this locus the leading singularities of the topological string partition functions become field theory amplitudes of the $N_f=4$ theory. At genus zero one should recover the prepotential, and at higher genus the gravitational amplitudes introduced by Nekrasov in [52] by using instanton counting techniques. In this section we will explain this in some detail, and as spin-off we will obtain some new results on the modularity properties of the $N_f=4$ theory and its gravitational corrections.

We first note that the behavior of the amplitudes near (4.18), in the fiber limit, has been already determined with heterotic techniques in (5.64). The results of section 5 including the base were obtained in principle in the large radius limit, in terms of the "electric" coordinates t. However, the calculations of $F^{(g)}$ performed there are also valid in the t_D coordinates, due to general covariance. In particular, the holomorphic limit $\mathcal{F}^{(g)}(S,t_D)$ can be expanded in polynomials in $E_2(S),E_4(S),E_6(S)$ as explained before (5.1), and we can write

$$\mathcal{F}^{(g)}(S,t_D)=\sum_k p_k^g(S)f_k^g(t_D). \tag{6.1}$$

Near the locus (4.20) the f_k^g should show display a singular behavior of the form

$$f_k^g(t_D)=\frac{b_k^g}{\mu^{2g-2}}+\cdots, \tag{6.2}$$

as we checked in the fiber limit in (5.64). How does this compare to the field theory?

The prepotential and gravitational corrections of the massless $N_f=4$, SU(2) $\mathcal{N}=2$ Yang-Mills theory depend on the vector multiplet variable a and on the microscopic coupling τ_0. They can be put together into a generating functional

$$\mathcal{F}^{\mathrm{YM}}(a,\tau_0,\hbar)=\sum_{g=0}^{\infty}\hbar^{2g}\mathcal{F}_g^{\mathrm{YM}}(a,\tau_0), \tag{6.3}$$

where $\mathcal{F}_0^{\mathrm{YM}}(a,\tau_0)$ is the $\mathcal{N}=2$ prepotential and the higher g amplitudes are the gravitational corrections. The statement that the type II theory on the Enriques Calabi-Yau has this gauge theory as its field theory limit near the locus (4.20) implies that the leading singularity of the topological string amplitudes is given by

$$\mathcal{F}^{(g)}(S, t_D) \to \frac{1}{\mu^{2g-2}} \sum_k b_k^g p_k^g(S) = \mathcal{F}_g^{\mathrm{YM}}(a, \tau), \tag{6.4}$$

where S is related to the coupling constant of the theory τ_0, and μ is related to the a variable of Seiberg and Witten in a way that we will make precise in a moment. Let us first look at the prepotential. While it has been originally assumed [55] that the prepotential of the self-dual theories with $\mathcal{N} = 2$, gauge group $\mathrm{SU}(N)$ and $2N$ flavors is classically exact, it was found in [20] that it does get instanton corrections. Those can however be absorbed in the following redefinition of the coupling [21],

$$\tau_0 \to \tau = \frac{1}{2}\frac{\partial^2}{\partial a^2}\mathcal{F}_0^{\mathrm{YM}}(a, \tau_0) = \tau_0 + \sum_k c_k q_0^k, \tag{6.5}$$

where

$$q_0 = \exp(2\pi i \tau_0). \tag{6.6}$$

We then have for the instanton-corrected prepotential

$$\mathcal{F}_0^{\mathrm{YM}}(a, \tau_0) = \frac{1}{2}\tau a^2, \tag{6.7}$$

in terms of the renormalized coupling τ. This is needed in order to match the type II prepotential (4.8), which does not exhibit instanton corrections. We will then express the $\mathcal{F}_g^{\mathrm{YM}}$ obtained by instanton computations not as functions of q_0, but of $q = \mathrm{e}^{2\pi i \tau}$.

The computation of the field theory amplitudes proceeds as follows. The functional (6.3) has the structure

$$\mathcal{F}^{\mathrm{YM}}(a, \tau_0, \hbar) = \mathcal{F}_{\mathrm{pert}}^{\mathrm{YM}}(a, \hbar) - \hbar^2 \log Z(a, \tau_0, \hbar), \tag{6.8}$$

where

$$\mathcal{F}_{\mathrm{pert},g}^{\mathrm{YM}}(a, \hbar) = -\frac{2B_{2g}}{4^{(g-1)}2g(2g-2)}(1-4^g)\frac{1}{a^{2g-2}} \tag{6.9}$$

is the perturbative piece computed in [53], and

$$Z(a, \tau_0, \hbar) = \sum_k Z_k(a, \hbar) q_0^k \tag{6.10}$$

is an instanton sum. Nekrasov's formula for the k-instanton contribution to the partition sum $Z_k(a, \hbar)$ can be written as [12]

$$Z_k(a, \hbar) = \sum_{\{Y_\lambda\}} \prod_{\lambda}^{N} \prod_{s \in Y_\lambda} \frac{\varphi_\lambda(s)^4}{\prod_{\tilde{\lambda}} E(s)^2}. \tag{6.11}$$

The sum runs over sets $\{Y_\lambda\}$ of Young diagrams labeled in the $\mathrm{SU}(2)$ case by $\lambda = 1, 2$. For massless flavors,

$$\varphi_\lambda(s) = a_\lambda - (s_j - s_i)\hbar, \tag{6.12}$$

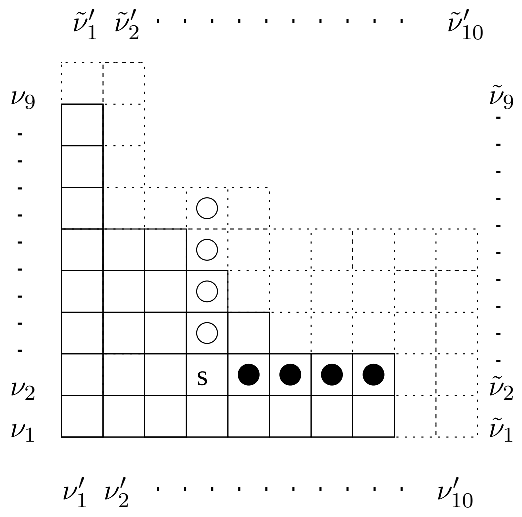

Figure 2: A sample pair of Young diagrams $Y_\lambda, Y_{\tilde{\lambda}}$ contributing to (6.11).

where s_i, s_j are the coordinates of the cell s inside the Young diagram Y_λ. We also have

$$E(s) = a_{\lambda\tilde{\lambda}} - \hbar(h(s) - v(s) - 1), \qquad h(s) = \nu_{s_i} - s_j, \qquad v(s) = \tilde{\nu}'_{s_j} - s_i, \tag{6.13}$$

where ν_{s_i} is the length of row s_i in Y_λ, $\tilde{\nu}'_{s_j}$ the length of column s_j in $Y_{\tilde{\lambda}}$ and $h(s), v(s)$ are the number of boxes to the right of s inside Y_λ respectively above s inside $Y_{\tilde{\lambda}}$, see figure 2. The constants $a_\lambda = (a_1, a_2)$ are set to $(-a, a)$.

The relative normalizations between the results in [52] and the Calabi-Yau case can be obtained from the limit $q \to 0$, which is the limit $S \to \infty$. The only remaining singularity on the Enriques is then (5.64), while in the Yang-Mills case we are left with the perturbative piece (6.9). Comparing this to (5.64) and taking into account the relative sign in (6.4) we find

$$(-2)^{g-1} \frac{a^{2g-2}}{\mu^{2g-2}} = 1, \tag{6.14}$$

and one can immediately read off the normalization of a with respect to $\mu = t_D^1 - t_D^2$:

$$a = \frac{\mu}{i\sqrt{2}}. \tag{6.15}$$

We notice the following factorization,

$$\mathcal{F}_g^{\mathrm{YM}}(q_0, a) = \frac{1}{a^{2g-2}} \Xi_g(q_0), \tag{6.16}$$

where $\Xi_g(q_0)$ is a power series in q_0. The relation between q_0 and q is defined by

$$q = q_0 \exp[\Xi_0(q_0)], \tag{6.17}$$

which can be inverted to obtain the relation between q_0 and q. The explicit power series one finds is

$$q_0 = q - \frac{q^2}{2} + \frac{11q^3}{64} - \frac{3q^4}{64} + \frac{359q^5}{32768} - \frac{75q^6}{32768} + \frac{919q^7}{2097152} - \frac{41q^8}{524288} + \mathcal{O}(q^9). \tag{6.18}$$

If we now plug this series into $\mathcal{F}_g^{\rm YM}(a, q_0)$ we find that all gravitational couplings are functions of q^2, that is to say, there are no odd instanton contributions, as it should be since those are forbidden by a $\mathbb{Z}_2$-symmetry of the theory [55]. The power series (6.18) should be given by a mirror map, corresponding to some algebraic realization of an elliptic curve. Indeed, when expressed in terms of

$$q = 2^4 q_S^{\frac{1}{2}}, \quad q_S = {\rm e}^{-S}, \tag{6.19}$$

we find

$$q_0 = 16\, q_S^{\frac{1}{2}} - 128\, q_S + 704\, q_S^{\frac{3}{2}} - 3072\, q_S^2 + \cdots = \frac{\vartheta_2^4(q_S)}{\vartheta_3^4(q_S)}, \tag{6.20}$$

which is (up to an overall factor 16) the Hauptmodul of $\Gamma_0(4)$. This equality between q_0 and the Hauptmodul has only be checked for the first few terms of the instanton expansion, and we don't have a general proof.

We can now express the couplings $\mathcal{F}_g^{\rm YM}(a, q_0)$, computed from (6.11), in terms of q_S, μ. Due to the connection to the Enriques results and the field theory limit (6.4), we expect them to be (up to an overall factor μ^{2-2g}) quasi-modular forms in q_S of weight $2g-2$, and belonging to the ring generated by $E_2(S)$, $E_4(S)$ and $E_6(S)$. The results obtained with the instanton expansion are in perfect agreement with this. We find at $g = 2$

$$\begin{aligned} \mu^2 \mathcal{F}_2^{\rm YM} &= \frac{1}{16} - \frac{3q_S}{2} - \frac{9q_S^2}{2} - 6q_S^3 - \frac{21q_S^4}{2} - 9q_S^5 + 18q_S^6 + \mathcal{O}\left(q_S^7\right) \\ &= \frac{1}{2^4} E_2(q_S)\ . \end{aligned} \tag{6.21}$$

Proceeding in the same way we find,

$$\begin{aligned}
\mu^4 \mathcal{F}_3^{\rm YM} &= \frac{1}{2^5}\left(\frac{2}{3}E_2^2 + \frac{1}{3}E_4\right), \\
\mu^6 \mathcal{F}_4^{\rm YM} &= \frac{1}{2^6}\left(\frac{11}{12}E_2^3 + \frac{4}{3}E_2E_4 + \frac{7}{12}E_6\right), \\
\mu^8 \mathcal{F}_5^{\rm YM} &= \frac{1}{2^7}\left(\frac{17}{9}E_2^4 + \frac{97}{18}E_2^2E_4 + \frac{32}{9}E_4^2 + \frac{14}{3}E_2E_6\right), \\
\mu^{10} \mathcal{F}_6^{\rm YM} &= \frac{1}{2^8}\left(\frac{619}{120}E_2^5 + \frac{218}{9}E_2^3E_4 + \frac{427}{9}E_2E_4^2 + \frac{4501}{144}E_2^2E_6 + \frac{4337}{144}E_4E_6\right), \\
\mu^{12} \mathcal{F}_7^{\rm YM} &= \frac{1}{2^9}\left(\frac{1418}{81}E_2^6 + \frac{52837}{432}E_2^4E_4 + \frac{12848}{27}E_2^2E_4^2 + \frac{22631}{108}E_2^3E_6\right. \\
&\qquad \left. + \frac{5423}{9}E_2E_4E_6 + \frac{6529}{54}E_6^2 + \frac{352069}{1296}E_4^3\right),
\end{aligned} \tag{6.22}$$

We point out that we have not proved these equalities, but rather verified them by using the instanton expansion up to high order. It is however highly non-trivial that this expansion can be matched to a quasimodular form of the required weight. In addition, one can verify that the coefficients of the above combinations agree with the Enriques results. For example, if we look at the singular behavior of (5.22) by using (5.64), one finds,

$$\mathcal{F}^{(3)}(S,t_D) \to \frac{1}{32\mu^4}E_4(S) + \frac{1}{48\mu^4}(E_2^2(S) - E_4(S)) = \frac{1}{96}(2E_2^2(S) + E_4(S)), \qquad (6.23)$$

in agreement with the result above. We have checked that the above polynomials are in accordance with the field theory limit of the Enriques model also for $g = 4, 5, 6$. For higher genus the instanton results for the $N_f = 4$ theory provide a boundary condition for the holomorphic anomaly equation, since they determine the coefficient of the leading singularity near (4.20) as a function of S, and generalize the heterotic result (5.64) away from the fiber.

In summary, we have verified with the instanton computations of [52] our general results about the structure of the topological string amplitudes in the Enriques Calabi-Yau (in particular our assumption after (5.1) about the modular properties of the holomorphic ambiguity). Conversely, the results on the Enriques side have been instrumental in clarifying the modularity structure of the massless $N_f = 4$ theory.

7. Direct integration on generic Calabi-Yau manifolds

In this section we present a general formalism which allows for direct integration of the holomorphic anomaly equation (2.10) for a generic Calabi-Yau manifold. In order to do that we will first have to rewrite these equations by using new coordinates and introduce the so-called big moduli space $\widehat{\mathcal{M}}$ in section 7.1. The holomorphic anomaly equations on the big moduli space have been also discussed in [18, 59]. The target space symmetry group acts naturally on the coordinates of this extended moduli space and we will briefly discuss modular forms on $\widehat{\mathcal{M}}$ in section 7.2. This has been also studied in [1], see also [27]. Alternatively to the direct integration, the higher genus amplitudes can be derived using a Feynman graph expansion in generalization of [7, 59, 1]. We introduce the appropriate propagators and vertices in section 7.3. Finally, in section 7.4 we derive a closed expression for the $F^{(g)}$ using direct integration. This can be viewed as the generalization of the discussion of the Seiberg-Witten example in section 3 for compact Calabi-Yau manifolds with an arbitrary number of moduli.

7.1 The recursive anomaly for $F^{(g)}$

In this section we rewrite the holomorphic anomaly equations (2.10) for an enlarged moduli space in which the $2h^{(2,1)}$ coordinates $t^i, \bar{t}^{\bar{i}}$ on $\mathcal{M}$ are promoted to $2h^{(2,1)} + 2$ coordinates $Y^K, \bar{Y}^K$. From a geometric point of view, this amounts to working on the moduli space of complex normalized $(3,0)$-forms Ω on the Calabi-Yau manifold under consideration. We denote this moduli space by $\widehat{\mathcal{M}}$ and call it the big moduli space. The coordinates Y^K on $\widehat{\mathcal{M}}$ are defined as functions of t^k by using the homogeneous coordinates $X^K(t)$ arising in the expansion (A.6) of the holomorphic three-form Ω. Explicitly, we define

$$Y^I = \lambda^{-1} X^I(t)\,, \qquad I = 0, \ldots, h^{(2,1)}\,, \qquad (7.1)$$

where λ is the complex string coupling. The big moduli space $\widehat{\mathcal{M}}$ is shown to be a rigid special Kähler manifold with Kähler potential $\widehat{K}$ and Kähler metric $\widehat{K}_{I\bar{J}}$ given by

$$\widehat{K} = \frac{i}{2}\left(Y^K \bar{\mathcal{F}}_K(\bar{Y}) - \bar{Y}^K \mathcal{F}_K(Y)\right), \qquad \widehat{K}_{I\bar{J}} = \partial_I \partial_{\bar{J}} \widehat{K} = \mathrm{Im}\tau_{IJ}\,, \tag{7.2}$$

where $\partial_I \equiv \partial_{Y^I}$ and $\partial_{\bar{I}} \equiv \partial_{\bar{Y}^I}$ are the derivatives with respect to the coordinates on $\widehat{\mathcal{M}}$ and $\tau_{IJ} = \partial_I \partial_J \mathcal{F}$ is the second derivative of the prepotential. Note that the Kähler metric $\mathrm{Im}\tau_{IJ}$ is not positive definite, but rather has complex signature $(h^{2,1}, 1)$, i.e. has one complex negative direction. The metric connection is shown to be

$$\Gamma^I_{JK} = \widehat{K}^{I\bar{M}} \partial_J \widehat{K}_{K\bar{M}} = -\frac{i}{2} C_{JK}{}^{I}\,, \tag{7.3}$$

where $C_{IJK}(Y) = \partial_I \partial_J \partial_K \mathcal{F}$ is the third derivative of the prepotential $\mathcal{F}$. This implies that the covariant derivative of a tensor V_K on $\widehat{M}$ is given by

$$D_I V_K \equiv \partial_I V_K - \Gamma^J_{IK} V_J = \partial_I V_K + \frac{i}{2} C_{IK}{}^{J} V_J\ . \tag{7.4}$$

Here and in the following, we will raise and lower indices using the metric $\widehat{K}_{I\bar{J}} = \mathrm{Im}\tau_{IJ}$. For a more exhaustive discussion of rigid special geometry we refer to the existing literature [14, 15, 24].

Let us now lift the holomorphic anomaly equations (2.10) for the free energies $F^{(g)}$ to the big moduli space $\widehat{\mathcal{M}}$. In order to do that, we evaluate $F^{(g)}(t, \bar{t})$ as functions of the homogeneous coordinates X^K. As reviewed in section 2, they transform as sections of $\mathcal{L}^{2-2g}$ such that

$$F^{(g)}(Y, \bar{Y}) = \lambda^{2g-2} F^{(g)}(X, \bar{X})\,, \qquad\qquad Y^K \partial_K F^{(g)} = (2-2g) F^{(g)}\ . \tag{7.5}$$

Rewriting the holomorphic anomaly equations (2.10) using the Y^K coordinates and the functions $F^{(g)}(Y, \bar{Y})$ we find

$$\partial_{\bar{I}} F^{(g)} = -\frac{i}{8} \bar{C}_{\bar{I}}{}^{JK} \left(D_J \partial_K F^{(g-1)} + \sum_{r=1}^{g-1} \partial_J F^{(r)} \partial_K F^{(g-r)} \right)\ . \tag{7.6}$$

A detailed derivation of (7.6) can be found in appendix D.1. We can also lift the equation (2.11) for $F^{(1)}$ to the big moduli space $\widehat{\mathcal{M}}$ in a way similar to the lift of the holomorphic anomaly equations for $g > 1$. First recall that $F^{(1)}$ is a section of $\mathcal{L}^0$ and hence as a function of the homogeneous coordinates $X^K(t)$ homogeneous of degree 0 as seen in (7.5). This implies that

$$Y^K \partial_{Y^K} \partial_{\bar{Y}^M} F^{(1)} = \bar{Y}^K \partial_{\bar{Y}^K} \partial_{Y^M} F^{(1)} = 0\ . \tag{7.7}$$

Using this property and the special geometry identities summarized in appendix A, one derives the holomorphic anomaly for $F^{(1)}(Y, Y)$ on the big moduli space (see appendix D.2)

$$\partial_I \partial_{\bar{J}} F^{(1)} = \frac{1}{8} C_{ILM} \bar{C}_{\bar{J}}{}^{LM} - \left(\frac{\chi}{24} - 1\right) K_{I\bar{J}}(Y, \bar{Y})\,, \tag{7.8}$$

where the second derivative of the Kähler potential (2.5) is shown to be

$$K_{I\bar{J}} \equiv \partial_{Y^I}\partial_{\bar{Y}^J} K(Y,\bar{Y}) = 2e^K \widehat{K}_{IJ} + 4e^{2K}\bar{Y}_I Y_J\,, \tag{7.9}$$

and indices were lowered by contraction with the metric (7.2). Note that the last term in the expression for $K_{I\bar{J}}$ ensures that the holomorphic anomaly (7.8) also implies (7.7). In this big moduli space formulation, it is straightforward to integrate the holomorphic anomaly equation (7.8) for $F^{(1)}$. One thus shows that the genus one free energy is locally of the form

$$F^{(1)}(Y,\bar{Y}) = -\frac{1}{2}\log\det(\mathrm{Im}\tau_{IJ}) - \left(\frac{\chi}{24}-1\right)K(Y,\bar{Y}) - \ln|\Phi| + f^{(1)} + \bar{f}^{(1)}\,, \tag{7.10}$$

where $\Phi(Y)$ and $f^{(1)}(Y)$ are holomorphic functions arising as integration constants. For reasons which will become clear later, we introduced the seemingly artificial split of the holomorphic ambiguity into Φ and $f^{(1)}$. The expression (7.10) provides the direct generalization for $F^{(1)}$ in Seiberg-Witten theory (3.11) and also reduces to the Enriques result (4.39). Clearly, the holomorphic anomaly does not determine Φ, $f^{(1)}$ which were derived in the Seiberg-Witten and Enriques example by using additional information due to modularity and string dualities. In the next section we will briefly discuss modularity from the point of view of the big moduli space $\widehat{\mathcal{M}}$.

7.2 Monodromy, symplectic group and modular forms

In this section we discuss the action of the target space symmetry group on the coordinates of the big moduli space $\widehat{\mathcal{M}}$ and introduce some basic modular forms and modular derivatives. To begin with, let us note that there is a natural symplectic action on the periods $(\mathcal{F}_J, Y^I)$ of the holomorphic three-form given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \mathcal{F} \\ Y \end{pmatrix} = \begin{pmatrix} \mathcal{F}' \\ Y' \end{pmatrix}, \tag{7.11}$$

where a, b, c and d are real integer-valued matrices obeying

$$a^T c = c^T a\,, \qquad b^T d = d^T b\,, \qquad a^T d - c^T b = 1\,. \tag{7.12}$$

These transformations change the basis of the third cohomology of the Calabi-Yau manifold and form the symplectic group $\mathrm{Sp}(H^3,\mathbb{Z})$. Note that in general only a subgroup Γ_M of $\mathrm{Sp}(H^3,\mathbb{Z})$ provides a true symmetry of the topological string theory. Γ_M is the monodromy group. We encountered specific examples for Γ_M in the sections on Seiberg-Witten theory and the Enriques Calabi-Yau: $\Gamma_M(\mathrm{SW}) = \Gamma(2)$ and $\Gamma_M(\mathrm{E}) = Sl(2,\mathbb{Z}) \times O(10,2,\mathbb{Z})$. The monodromy group Γ_M is a symmetry of all higher genus amplitudes $F^{(g)}(Y,\bar{Y})$.

Given the action of $\mathrm{Sp}(H^3,\mathbb{Z})$ on the periods, we can also investigate its induced action on the geometrical objects on $\widehat{\mathcal{M}}$. First note that both Kähler potentials K and $\widehat{K}$ are invariant under (7.11) since they contain the symplectic scalars $Y^K\bar{\mathcal{F}}_K - \bar{Y}^K\mathcal{F}_K$. The second derivative $\tau_{IJ} = \partial_{Y^I}\mathcal{F}_J$ of the prepotential transforms as

$$\tau \quad \mapsto \quad (a\tau + b)(c\tau + d)^{-1}\,. \tag{7.13}$$

This implies that τ_{IJ} transforms as a modular parameter and is the higher-dimensional analog of (3.4). Once again one easily shows that the inverse of $\mathrm{Im}\tau_{IJ}$ transforms with a shift

$$\mathrm{Im}\tau^{IJ} \quad \mapsto \quad (c\tau+d)^{I}{}_{K}(c\tau+d)^{J}{}_{L}\mathrm{Im}\tau^{KL} - 2ic^{IK}(c\tau+d)^{J}_{K}\;. \tag{7.14}$$

On the other hand, the third derivative C_{IJK} of the prepotential $\mathcal{F}$ transforms without such a shift

$$C_{IJK} \quad \mapsto \quad (c\tau+d)^{-1\,M}_{I}(c\tau+d)^{-1\,N}_{J}(c\tau+d)^{-1\,P}_{K}C_{MNP} \tag{7.15}$$

This is precisely the transformation property of a modular form of weight -3. In general, we say that a modular form is of weight $-n$ if it transforms as

$$M_{I_1\ldots I_n} \quad \mapsto \quad (c\tau+d)^{-1\,J_1}_{I_1}\ldots(c\tau+d)^{-1\,J_n}_{I_n}M_{J_1\ldots J_n}\;. \tag{7.16}$$

The holomorphic form Φ appearing in (7.10) has no indices, but nevertheless transforms under the modular group Γ_M. It is chosen such that $F^{(1)}$ as well as $f^{(1)}$ are invariant. This implies that it has to transform as

$$\Phi \quad \mapsto \quad \det(c\tau+d)\;\Phi \tag{7.17}$$

to compensate the transformation of $\det(\mathrm{Im}\tau_{IJ})$ in (7.10). A major challenge is to find the appropriate Φ for a given Calabi-Yau manifold and show that it can be expressed as a function of τ_{IJ} only. In order to do that one can change $f^{(1)}$ by holomorphic modular invariant combinations. $\Phi(\tau_{IJ})$ can be explicitly derived for the Enriques Calabi-Yau. It is desirable to explore further examples such as the quintic Calabi-Yau.

As we have seen before, the derivative $\partial_{I_0}M_{I_1\ldots I_n}$ of a modular form M is no longer a modular form, since the derivative also acts on the matrices $(C\tau+D)^{-1\,J_i}_{I_i}$. However, this action can be compensated by using covariant derivatives on $\widehat{\mathcal{M}}$. One easily shows that the Christoffel symbols (7.3) shift under (7.11) such that the covariant derivative $D_{I_0}M_{I_1\ldots I_n}$ of a modular form is again a modular form of weight reduced by one. If we express M_K as a function of τ_{IJ}, we can also take derivatives

$$D^{IJ}M_K \equiv \partial_{\tau_{IJ}}M_K - \frac{i}{2}\delta^{\{J}_{K}\mathrm{Im}\tau^{I\}L}M_L\,, \qquad D_I = C_{IJK}D^{JK}\,, \tag{7.18}$$

where $\{IJ\}$ indicates symmetrization in the indices I and J with symmetry factor $\frac{1}{2}$. In order to relate D_I and D^{JK} we have used that $\partial_I\tau_{KL} = C_{IKL}$. Since C_{IKL} has weight -3 this also implies that D^{IJ} raises the weight of the modular form by 2. Let us note that D_I and D^{IJ} are the higher-dimensional analogs of the derivatives D_t and D_τ displayed in (3.18).

7.3 Feynman rules for $F^{(g)}$: vertices and the propagators

Let us now come back to the discussion of the holomorphic anomaly equations (7.6). As argued in [7] and briefly recalled in section 2, the traditional way of finding a solution to equations of the form (7.6) is via a Feynman graph expansion. In this section we will derive the vertices and propagators to describe such an expansion in the large moduli space.

This can be done by first directly solving (7.6) for the smallest possible genus $g = 2$. The resulting section $F^{(2)}$ can be identified as a sum over Feynman graphs counting degeneracies of Riemann surfaces. This example allows us to identify the vertices and propagators, which can be used to systematically construct every solution $F^{(g)}$. The generating functional encoding these Feynman rules is then derived and can be shown to be equivalent to the generating functional of Bershadsky, Cecotti, Ooguri and Vafa [7].

In order to extract the solutions for the free energies $F^{(g)}$ we first define complex tensors

$$C^{(g)}_{I_1\dots I_n}(Y,\bar{Y}) \equiv \begin{cases} D_{I_1}\dots D_{I_n}F^{(g)}(Y,\bar{Y}) & \text{for } g \geq 1 \\ iD_{I_1}\dots D_{I_n}C_{I_{n-2}I_{n-1}I_n} & \text{for } g = 0 \end{cases} . \tag{7.19}$$

and demand

$$C^{(g)}_{I_1\dots I_n} = 0 \quad \text{for} \quad 2g-2+n \leq 0 \, . \tag{7.20}$$

These two equations are the big moduli space equivalents of (2.7) and (2.8). They imply that $C^{(g)}_{I_1\dots I_n}$ is a section of $\mathcal{L}^{2-2g-n}$ such that we can infer the homogeneity relation

$$Y^K C^{(g)}_{KI_1\dots I_n} = (2-2g-n)C^{(g)}_{I_1\dots I_n} \, . \tag{7.21}$$

In equation (7.10) we already displayed the general local form of solutions for the free energy $F^{(1)}$. The next function to determine is $F^{(2)}(Y,\bar{Y})$. Evaluating (7.6) for $g = 2$ one obtains

$$\partial_{\bar{Y}^I}F^{(2)} = -\frac{i}{8}\bar{C}_I{}^{JK}\left(D_J\partial_K F^{(1)} + \partial_J F^{(1)}\partial_K F^{(1)}\right), \tag{7.22}$$

As we discuss in appendix D.1 such an equation can be solved by an integration by parts method. This amounts to writing the right-hand side of (7.22) as an anti-holomorphic derivative of some expression $\Gamma^{(2)}(Y,\bar{Y})$. The solution to (7.22) is then given by $F^{(2)} = \Gamma^{(2)}(Y,\bar{Y}) + f^{(2)}(Y)$, where $f^{(2)}$ is the holomorphic ambiguity at genus two. This method of solving (7.22) is equivalent to the one used in ref. [7] to solve the holomorphic anomaly equations (2.10) for $F^{(g)}(t,\bar{t})$. However, in contrast to [7] it will be sufficient to introduce one type of propagator denoted by Δ^{IJ}. The propagator Δ^{IJ} has to be chosen such that

$$\partial_{\bar{K}}\Delta^{IJ} = \frac{i}{4}\bar{C}_{\bar{K}}{}^{IJ} \, . \tag{7.23}$$

Clearly, this fixes the form of Δ^{IJ} only up to an holomorphic function. As for the examples discussed in the previous sections this ambiguity can be fixed by modular invariance and compatibility with the solution $F^{(1)}$.

In order to derive an explicit expression for the propagator Δ^{IJ} we note that a solution to (7.23) is always of the form

$$\Delta^{IJ} = -\frac{1}{2}\mathrm{Im}\tau^{IJ} + \mathcal{E}^{IJ}(Y) \, , \tag{7.24}$$

where $\mathcal{E}^{IJ}(Y)$ is a holomorphic function, which compensates the shift transformation (7.14) of $\mathrm{Im}\tau^{IJ}$. As in the Seiberg-Witten and Enriques example we want to express $\mathcal{E}^{IJ}$ as a derivative of the holomorphic part of $F^{(1)}$ given in (7.10). In order to do that, let us assume that we can express $\Phi(Y)$ as a function of τ_{IJ} itself. To achieve this, it might be necessary to appropriately split the holomorphic ambiguity of $F^{(1)}$ into $\Phi(\tau)$ and an additional function $f^{(1)}(Y)$. $f^{(1)}$ is a modular invariant function which might not be expressible as a function of τ_{IJ}. We then identify the holomorphic part in (7.24) to be

$$\mathcal{E}^{IJ} = -\frac{i}{\Phi}\frac{\partial\Phi(\tau)}{\partial\tau_{IJ}} \; . \tag{7.25}$$

From this definition one can immediately conclude that Δ^{IJ} is a modular form of weight 2 under the target space symmetry group Γ_M. To see this, note that since $F^{(1)}$ and K are invariant under Γ_M also the section

$$\tilde{F}^{(1)} = -\frac{1}{2}\log\det(\mathrm{Im}\tau_{IJ}) - \ln|\Phi(\tau)| + f^{(1)} + \bar{f}^{(1)} \, , \tag{7.26}$$

is trivially invariant under Γ_M. But evaluating the first derivative on the weight zero forms $\tilde{F}^{(1)}$ and $f^{(1)}$ one finds

$$\partial_I \tilde{F}^{(1)} = -\frac{1}{2}C^{(0)}_{IJK}\Delta^{JK} + \partial_I f^{(1)} \, , \tag{7.27}$$

and infers from the discussion of section 7.2 that Δ^{IJ} is of weight 2 and does not shift under Γ_M.

Now that we have discussed the propagator Δ^{IJ}, let us turn to the definition of the vertices. We do that by continuing the evaluation of the $F^{(2)}$ example. In appendix D.1 we determine by the partial integration method of [7] that $F^{(2)}(Y,\bar{Y})$ to be

$$\begin{aligned} F^{(2)}(Y,\bar{Y}) \;=\; & f^{(2)} - \Delta^{JK}\left(\frac{1}{2}\tilde{C}^{(1)}_{JK} + \frac{1}{2}\tilde{C}^{(1)}_{J}\tilde{C}^{(1)}_{K}\right) - \Delta^{JK}\Delta^{LM}\left(\frac{1}{8}C^{(0)}_{KLMJ} + \frac{1}{2}C^{(0)}_{JLM}\tilde{C}^{(1)}_{K}\right) \\ & -\Delta^{JK}\Delta^{LM}\Delta^{QP}\left(\frac{1}{12}C^{(0)}_{KMQ}C^{(0)}_{PLJ} + \frac{1}{8}C^{(0)}_{KJQ}C^{(0)}_{PML}\right) , \end{aligned} \tag{7.28}$$

where $f^{(2)}(Y)$ is the holomorphic ambiguity. Note that in this expansion we introduced the shifted $F^{(1)}$ vertices

$$\tilde{C}^{(1)}_{JK} = C^{(1)}_{JK} + \left(\frac{\chi}{24} - 1\right)K_J K_K \, , \qquad\qquad \tilde{C}^{(1)}_{K} = C^{(1)}_{K} + \left(\frac{\chi}{24} - 1\right)K_K \; . \tag{7.29}$$

It is not hard to interpret the resulting $F^{(2)}$ as being obtained from a Feynman graph expansion. Each term in (7.28) corresponds to one Feynman diagram representing a degeneration of a genus 2 Riemann surface. The vertices are $C^{(0)}_{IJK}$, $C^{(0)}_{IJKL}$ and $\tilde{C}^{(1)}_{I}$, $\tilde{C}^{(1)}_{IJ}$ which are connected by propagators Δ^{IJ}. The whole Feynman sum is shown in figure 3.

From this example we can also infer the general Feynman rules which generate the solutions $F^{(g)}(Y,\bar{Y})$ to the holomorphic anomaly equation (7.6). The propagator is defined in (7.24) and (7.27) as the derivative of $\tilde{F}^{(1)}$. The vertices take the form

$$\tilde{C}^{(g)}_{I_1\ldots I_n} = C^{(g)}_{I_1\ldots I_n} \qquad \text{for} \quad g \neq 1 \, , \qquad\qquad \tilde{C}^{(1)} = 0 \; . \tag{7.30}$$

$$\tilde{C}^{(1)}_{I_1\ldots I_n} = C^{(1)}_{I_1\ldots I_n} + (n-1)!\left(\frac{\chi}{24} - 1\right)K_{I_1}\ldots K_{I_n} \qquad \text{for} \quad n > 0 \, , \tag{7.31}$$

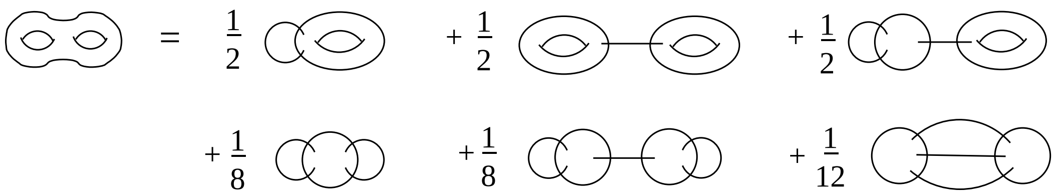

Figure 3: The Feynman graph expansion for $F^{(2)}$.

where $K_I = \partial_{Y^I} K(Y, \bar{Y})$ are the first derivatives of the Kähler potential (2.5). It is straightforward to check that by using $K_I = -\widehat{K}_I/\widehat{K}$ and $D_I \widehat{K}_J = 0$ one finds

$$\tilde{C}^{(1)}_{I_1 \dots I_n} = D_{I_1} \dots D_{I_n} \tilde{F}^{(1)} \,, \tag{7.32}$$

where $\tilde{F}^{(1)}$ is defined in (7.26). This Feynman graph expansion can be obtained as a saddle point expansion of the formal integral

$$\hat{Z}\Big[Y\Big] = \int dZ \, \sqrt{\det \Delta} \, \exp\Big(-\frac{1}{2} g_s^{-2} \, \Delta^{-1}_{IJ} \, Z^I Z^J + W[Z;\, Y, \bar{Y}] \Big) \,, \tag{7.33}$$

where g_s is the expansion constant playing the role of $\hbar$. Here $W[Z;\, Y, \bar{Y}]$ contains the vertices (7.30) and reads

$$\begin{aligned} W[Z;\, Y, \bar{Y}] &= \sum_{g=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{n!} \, g_s^{2g-2} \, \tilde{C}^{(g)}_{I_1 \dots I_n} Z^{I_1} \dots Z^{I_n} \\ &= \sum_{g=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{n!} \, g_s^{2g-2} \, C^{(g)}_{I_1 \dots I_n} Z^{I_1} \dots Z^{I_n} - \Big(\frac{\chi}{24} - 1 \Big) \ln \Big(1 - Z^I K_I \Big) \,. \end{aligned} \tag{7.34}$$

Note that the holomorphic anomaly equations on the big moduli space together with (7.19) and (7.20) imply that the integrand of (7.33) transforms as a wavefunction [61, 18, 59, 1]. Moreover, following [7, 59] one shows that $\hat{Z}[Y]$ is actually holomorphic in Y^I. One thus concludes that each coefficient of g_s^{2g-2} in the saddle point expansion of $\log \hat{Z}$ is a holomorphic ambiguity $f^{(g)}(Y)$. On the other hand, each coefficient is of the form $F^{(g)}(Y, \bar{Y}) - \Gamma^{(g)}(Y, \bar{Y})$, where $\Gamma^{(g)}$ are the Feynman graphs described above. We thus solve for $F^{(g)} = \Gamma^{(g)}(Y, \bar{Y}) + f^{(g)}(Y)$ and find the desired result. In the remainder of this section, we will argue that the big moduli space formulation is indeed completely equivalent to the results obtained in [7].

Let us now turn to the comparison of the big moduli space formalism with the standard results of [7] reviewed in section 2. Firstly, note that we only needed one type of propagator Δ^{IJ}. This propagator is related to the propagators $\hat{\Delta}^{ij}, \hat{\Delta}^i$ and $\hat{\Delta}$ introduced in (2.17) by

$$\begin{aligned} \Delta^{IJ} &= \chi^I_i \hat{\Delta}^{ij} \chi^J_j - \chi^I_i \hat{\Delta}^i X^J - X^I \hat{\Delta}^i \chi^J_i + X^I \hat{\Delta} X^J \,, \\ &= \begin{pmatrix} X^I & \chi^I_i \end{pmatrix} \begin{pmatrix} \hat{\Delta} & -\hat{\Delta}^j \\ -\hat{\Delta}^i & \hat{\Delta}^{ij} \end{pmatrix} \begin{pmatrix} X^J \\ \chi^J_j \end{pmatrix} \,, \end{aligned} \tag{7.35}$$

where χ_i^I is defined in (A.6). To check this identity, we can evaluate the $\bar{t}^{\bar{i}}$-derivative of Δ^{IJ}. Clearly, from the form (7.24) we find

$$\partial_{\bar{t}^{\bar{i}}} \Delta^{IJ} = \frac{i}{4} \lambda^{-1} \bar{\chi}_{\bar{i}}^K \bar{C}_K{}^{IJ} \ . \tag{7.36}$$

Precisely the same equation is obtained by using the identification (7.35) the special geometry identities (A.13) and the derivatives (2.17) of the small propagators $\hat{\Delta}^{ij}, \hat{\Delta}^i$ and $\hat{\Delta}$. In other words, we found a non-holomorphic lift of the small propagators $\hat{\Delta}$ to $\widehat{\mathcal{M}}$ such that Δ^{IJ} takes the simple form (7.24). Even though we did not completely specify the holomorphic dependence of Δ^{IJ} we already notice that all non-holomorphic dependence entirely arises through the inverse of $\mathrm{Im}\tau_{IJ}$. This already hints to the fact that in the formulation on $\widehat{\mathcal{M}}$ we have much better control over the $\bar{Y}^I$ dependence of each $F^{(g)}(Y, \bar{Y})$. In section 7.4 we will show that this fact can be used to directly integrate the holomorphic anomaly equations, which provides an efficient and direct method to find $F^{(g)}$. In order to show that expressions such as (7.28) are completely equivalent to the ones of [7], we also need the projection of the vertices $\tilde{C}^{(g)}_{I_1 \ldots I_n}$. These vertices are related to the correlation functions $C^{(g)}_{i_1 \ldots i_n}$ defined in (2.7) by

$$C^{(g)}_{i_1 \ldots i_n}(t, \bar{t}) = \lambda^{2-2g-n} \chi_{i_1}^{I_1} \ldots \chi_{i_n}^{I_n} \tilde{C}^{(g)}_{I_1 \ldots I_n}(Y, \bar{Y}) \ . \tag{7.37}$$

In order to derive this equation we have used the special geometry relations (A.13) as well as the scaling behavior of $C^{(g)}_{I_1 \ldots I_n}$ when inserting $Y^K = \lambda^{-1} X^K(t)$. This equation also holds for $\tilde{C}^{(1)}_{I_1 \ldots I_n}$ since due to (D.11) the additional terms are zero under the contraction with χ_i^I. They are however of importance once one contracts $\tilde{C}^{(g)}_{I_1 \ldots I_n}$ by Y^K yielding

$$Y^K \tilde{C}^{(g)}_{K I_1 \ldots I_n} = (2 - 2g - n)\, \tilde{C}^{(g)}_{I_1 \ldots I_n} \ . \tag{7.38}$$

By using these identities and the identification (7.35) of the propagators the expansion (7.28) on $\widehat{\mathcal{M}}$ gets transformed into the known result of [7]. Moreover, also the generating function (7.34) reduces to the one found in [7] if we identify

$$Z^I = -\varphi Y^I + x^i \chi_i^I(Y, \bar{Y}) \ . \tag{7.39}$$

This proves that the solutions for $F^{(g)}$ are actually identical in the formulation of section 2 and the big moduli space formalism presented here. As already mentioned above, the advantage of this new formulation is that all non-holomorphic dependence arises entirely through $\mathrm{Im}\tau^{IJ}$ in the unified propagator Δ^{IJ} and the covariant derivatives D_I. We will use this fact in the next section to perform a direct integration and to derive a closed expression for $F^{(g)}$.

7.4 Direct integration of the holomorphic anomaly

In this section we make use of the special properties of the big moduli space formulation to directly integrate the holomorphic anomaly equations (7.6). To begin with, we will argue that every $F^{(g)}$ for $g > 1$ can be expressed as a finite power series in the propagators Δ^{IJ} as

$$F^{(g)}(Y, \bar{Y}) = \sum_{k=0}^{3g-3} \Delta^{I_1 J_1} \ldots \Delta^{I_k J_k} \; c^{(g)}_{I_1 J_1 \ldots I_k J_k} \,, \tag{7.40}$$

where $c^{(g)}$ without indices is the holomorphic ambiguity at genus g. Due to modular invariance of $F^{(g)}$ the coefficients $c^{(g)}_{I_1 J_1 \ldots I_k J_k}(Y)$ are shown to be holomorphic modular forms of weight $-2k$ on the big moduli space $\widehat{\mathcal{M}}$. All non-holomorphic dependence of $F^{(g)}$ arises entirely through $\mathrm{Im}\tau^{IJ}$ appearing in the propagators Δ^{IJ} defined in (7.24), It is this fact which will allow us to directly integrate the holomorphic anomaly equations (7.6).

First of all, we have to show that indeed each $F^{(g)}$ for $g > 1$ can be written as a power series in the propagators Δ^{IJ} with holomorphic coefficients. We check this for $F^{(2)}$ first. $F^{(2)}$ was expressed in (7.28) as a power series in Δ^{IJ} with coefficients containing $\tilde{C}^{(0)}_{IJKL}$, $\tilde{C}^{(1)}_{I}$ and $\tilde{C}^{(1)}_{IJ}$. From their definitions (7.30), it is clear that these three quantities are not holomorphic. Hence, in order to establish that (7.40) is true for $F^{(2)}$, they have to be written as power series in Δ^{IJ}. For $\tilde{C}^{(0)}_{IJKL} \equiv D_I C^{(0)}_{JKL}$ this requires that we have to expand the connection D_I. Note that D_I contains the Christoffel symbol $\Gamma^K_{IJ} = -\frac{i}{2} C_{IJL} \mathrm{Im}\tau^{LK}$ and is only non-holomorphic due to the appearance of $\mathrm{Im}\tau^{IJ}$. However, by using (7.24) one can replace $\mathrm{Im}\tau^{LK}$ and split the connection as

$$D_I V_J = \check{D}_I V_J - C^{(0)}_{IJK} \Delta^{KL} V_L \,, \qquad \Gamma^K_{IJ} = \check{\Gamma}^K_{IJ} + C^{(0)}_{IJM} \Delta^{MK} \,, \tag{7.41}$$

where we introduced the holomorphic connection

$$\check{D}_I V_J = \partial_I V_J - \check{\Gamma}^M_{IJ} V_M = \partial_I V_J + i C_{IJK} \mathcal{E}^{KM} V_M \,. \tag{7.42}$$

The holomorphic connection $\check{D}_I$ maps holomorphic sections $V_K(Y)$ into holomorphic sections $\check{D}_K V_L(Y)$. Moreover, it maps modular forms into modular forms, decreasing the weight of the modular form by one. $\check{D}_I$ are the generalizations of the holomorphic covariant derivatives (3.19) and (5.33) for the Seiberg-Witten example and the Enriques Calabi-Yau. We can now split $\tilde{C}^{(0)}_{IJKL}$ into a holomorphic part and a term linear in the propagator

$$\tilde{C}^{(0)}_{IJKL} = \check{D}_I C^{(0)}_{JKL} - \Delta^{MN} \Big(C^{(0)}_{IJM} C^{(0)}_{NKL} + C^{(0)}_{IKM} C^{(0)}_{NJL} + C^{(0)}_{ILM} C^{(0)}_{NJK} \Big) \,. \tag{7.43}$$

Clearly, due to the holomorphicity of $C^{(0)}_{IJK}$ both $\check{D}_I C^{(0)}_{JKL}$ and the coefficient of Δ^{MN} are holomorphic functions.

Let us now evaluate $\tilde{C}^{(1)}_{I}$ and $\tilde{C}^{(1)}_{IJ}$. The first derivative $\tilde{C}^{(1)}_{I} = \partial_I \tilde{F}^{(1)}$ was already given in (7.27) and shown to have an expansion in the propagator Δ^{IJ} with holomorphic coefficients. In order to also evaluate the remaining vertices, we will need to take derivatives of $\mathcal{E}^{IJ}$. As in the examples, note that the first derivative $\partial_K \mathcal{E}^{IJ}$ is not a modular form, but rather transforms with a shift. These shift transformations can be compensated by adding another term quadratic in $\mathcal{E}^{IJ}$. Indeed, we find that the linear combination

$$\mathcal{E}^{KL}{}_{I} \equiv \partial_I \mathcal{E}^{KL} - \mathcal{E}^{KM} \mathcal{E}^{LN} C^{(0)}_{IMN} \,, \tag{7.44}$$

transforms as a modular form without an additional shift. Not surprisingly, $\mathcal{E}^{KL}_{\ \ I}$ is not the same as $\check{D}_I \mathcal{E}^{KL}$ but rather the field strength of $\mathcal{E}^{IJ}$. However, there is another important representation of $\mathcal{E}^{KL}_{\ \ I}$ in terms of derivatives of $\Phi(\tau)$. Using (7.44) one finds

$$\mathcal{E}^{KL}_{\ \ I} = \mathcal{E}_4^{KLMN} C^{(0)}_{MNI}\,, \tag{7.45}$$

where

$$\mathcal{E}_{2k}^{I_1 J_1 \ldots I_k J_k} = (-i)^k \frac{1}{\Phi} \frac{\partial \Phi(\tau)}{\partial \tau_{I_1 J_1} \ldots \partial \tau_{I_k J_k}}\ . \tag{7.46}$$

These holomorphic modular forms of weight $2k$ are the direct generalizations of the forms $\epsilon^{2k}_{a_1 \ldots a_k}$ introduced in (5.39). A direct calculation shows that we can also express the holomorphic modular derivative of Δ^{IJ} as a propagator expansion,

$$\check{D}_I \Delta^{KL} = -\Delta^{KM} \Delta^{LN} C^{(0)}_{MNI} + \mathcal{E}_4^{KLMN} C^{(0)}_{MNI}\ . \tag{7.47}$$

We are now in the position to evaluate the vertex $\tilde{C}^{(1)}_{IJ} \equiv D_J \partial_I \tilde{\mathcal{F}}^{(1)}$. Using the derivatives (7.47) of the propagators together with (7.41), one easily derives

$$\begin{aligned}\tilde{C}^{(1)}_{IJ} \ =\ & -\frac{1}{2}\mathcal{E}_4^{KLMN} C^{(0)}_{MNJ} C^{(0)}_{IKL} + \check{D}_J \partial_I f^{(1)} - \frac{1}{2}\Delta^{KL}\Big(\check{D}_J C^{(0)}_{IKL} + 2 C^{(0)}_{IJK}\partial_L f^{(1)}\Big) \\ & + \frac{1}{2}\Delta^{KL}\Delta^{MN}\Big(C^{(0)}_{JIM} C^{(0)}_{NKL} + C^{(0)}_{JKM} C^{(0)}_{NIL}\Big)\ .\end{aligned} \tag{7.48}$$

Inserting (7.27), (7.43) and (7.48) into the expansion (7.28) for $F^{(2)}$ one finds

$$\begin{aligned}F^{(2)} \ =\ & \Delta^{I_1J_1}\Delta^{I_2J_2}\Delta^{I_3J_3}\Big(\frac{1}{24} C^{(0)}_{I_1I_2I_3} C^{(0)}_{J_1J_2J_3} + \frac{1}{16} C^{(0)}_{I_1J_1I_2} C^{(0)}_{J_2I_3J_3}\Big) \\ & -\frac{1}{8}\Delta^{I_1J_1}\Delta^{I_2J_2}\Big(\check{D}_{I_1} C^{(0)}_{J_1I_2J_2} + 4 C^{(0)}_{I_1J_1I_2}\partial_{J_2} f^{(1)}\Big) \\ & -\frac{1}{4}\Delta^{I_1J_1}\mathcal{E}_4^{KLMN} C^{(0)}_{I_1MN} C^{(0)}_{J_1KL} + \frac{1}{2}\check{D}_{I_1}\partial_{J_1} f^{(1)} + \frac{1}{2}\partial_{I_1} f^{(1)}\partial_{J_1} f^{(1)} + c^{(2)}\ .\end{aligned} \tag{7.49}$$

This shows that the calculation of $F^{(2)}$ using the partial integration and the expansion of the non-holomorphic coefficients yields the desired expansion (7.40) of $F^{(2)}$. We will now use an inductive argument to show that every $F^{(g)}$ can be written in the form (7.40) and derive a recursive expression by direct integration.

Let us now go one step further and show that if all $F^{(r)}$ for $1 < r < g$ can be written in the form (7.40) also $F^{(g)}$ itself admits this expansion. To do that, we use the Feynman graph expansion introduced in section 7.3. It was shown there that each $F^{(g)}(Y, \bar{Y})$ can be obtained from vertices $\tilde{C}^{(r)}_{I_1 \ldots I_k}$, $r < g$ connected with propagators Δ^{IJ}. But it is not hard to see that $\tilde{C}^{(r)}_{I_1 \ldots I_k}$ is actually an expansion in Δ^{IJ} with holomorphic coefficients. More precisely, note that the vertices are obtained by taking covariant derivatives D_I of $F^{(r)}$ and we can apply (7.41) to rewrite these into holomorphic covariant derivatives $\check{D}_I$ and a propagator contribution. But since by our induction assumption $F^{(r)}$ is of the form (7.40) for $r < g$ we can apply (7.47) to show that this is equally true for $F^{(g)}$ itself. This proves that (7.40) is true for all $g > 1$. It is also straightforward to count the number of propagators arising in the expansion (7.40). One simply notes that the term in the Feynman graph expansion with coefficients $C^{(0)}_{IJK}$ only is already an expansion in Δ^{IJ} with holomorphic coefficients. It has the maximal number of propagators, namely $3g-3$.

Having shown that $F^{(g)}$ can be always brought to the form (7.40), let us now determine a closed expression by direct integration. Since all non-holomorphic dependence arises through Δ^{IJ}, the holomorphic anomaly equation can be rewritten as

$$\frac{\partial F^{(g)}}{\partial \Delta^{IJ}} = \frac{1}{2} D_I \partial_J F^{(g-1)} + \frac{1}{2}\sum_{r=1}^{g-1} \partial_I F^{(r)} \partial_J F^{(g-r)} \ . \tag{7.50}$$

To integrate this expression we introduce the following shorthand notation

$$F^{(g)}(Y,\bar{Y}) = \sum_{k=0}^{3g-3} c^{(g)}_{(k)}\,, \qquad\qquad c^{(r)}_{(k)} = \Delta^{I_1J_1}\ldots\Delta^{I_kJ_k}\, c^{(r)}_{I_1J_1\ldots I_kJ_k}\,, \tag{7.51}$$

where $c^{(g)}_{(k)}$ is the term containing k propagators Δ^{IJ}. We also rewrite the right-hand side of the holomorphic anomaly equation as

$$D_I\partial_J F^{(g-1)} + \partial_I \tilde{F}^{(1)}\partial_J F^{(g-1)} + \partial_I F^{(g-1)}\partial_J \tilde{F}^{(1)} + \sum_{r=2}^{g-2} \partial_I F^{(r)} \partial_J F^{(g-r)} \ . \tag{7.52}$$

Here the first three terms can be rewritten as

$$\begin{aligned} &D_I\partial_J F^{(g-1)} + \partial_I \tilde{F}^{(1)}\partial_J F^{(g-1)} + \partial_I F^{(g-1)}\partial_J \tilde{F}^{(1)} \\ &\quad = \check{D}_I \partial_J F^{(g-1)} - \Delta^{KL} C^{(0)}_{IJK}\partial_L F^{(g-1)} - \Delta^{KL} C^{(0)}_{KL\{I}\partial_{J\}} F^{(g-1)} + 2\partial_{\{I} f^{(1)} \partial_{J\}} F^{(g-1)}\,, \end{aligned} \tag{7.53}$$

where we have applied (7.41) and inserted (7.27). We also introduce the derivative $\check{d}$, which acts on the coefficients of the Δ-expansion as the holomorphic covariant derivative $\check{D}_I$ but leaving Δ^{IJ} invariant. For $c^{(g)}_{(k)}$ given in (7.51) we thus set

$$\check{d}_I c^{(g)}_{(k)} = \Delta^{I_1J_1}\ldots\Delta^{I_kJ_k}\, \check{D}_I c^{(g)}_{I_1J_1\ldots I_kJ_k}(Y) \tag{7.54}$$

Using this definition, we calculate

$$\partial_I c^{(g)}_{(k)} = \check{d}_I c^{(g)}_{(k)} + \Big(\check{D}_I \Delta^{MN}\Big)\frac{\partial}{\partial\Delta^{MN}} c^{(g)}_{(k)} \tag{7.55}$$

$$\qquad = \check{d}_I c^{(g)}_{(k)} + C^{(0)}_{IPQ}\Big(\mathcal{E}_4^{PQMN} - \Delta^{PM}\Delta^{QN}\Big)\frac{\partial}{\partial\Delta^{MN}} c^{(g)}_{(k)} \ . \tag{7.56}$$

Note that the first term is homogeneous of degree k in Δ, the second is homogeneous of degree $k-1$, while the last is homogeneous of degree $k+1$. We also evaluate the second derivative

$$\begin{aligned} \check{D}_I\partial_J c^{(g-1)}_{(k)} &= \Big[\mathcal{E}_4^{MNQP}\mathcal{E}_4^{TURS} C^{(0)}_{IMN} C^{(0)}_{JTU}\Big]\frac{\partial^2}{\partial\Delta^{QP}\partial\Delta^{RS}} c^{(g-1)}_{(k)} \\ &\quad + \Big[\mathcal{E}_6^{CDKLMN} C^{(0)}_{KLI} C^{(0)}_{MNJ} + \mathcal{E}_4^{CDFG}\check{C}^{(0)}_{JFGI} + 2\mathcal{E}_4^{CDFG} C^{(0)}_{FG\{I}\check{d}_{J\}}\Big]\frac{\partial}{\partial\Delta^{CD}} c^{(g-1)}_{(k)} \end{aligned} \tag{7.57}$$

$$
\begin{aligned}
&+\Bigg[\check{d}_I\check{d}_J - 2\mathcal{E}_4^{QCFG}C^{(0)}_{FG\{I}C^{(0)}_{J\}QB}\Delta^{BD}\frac{\partial}{\partial\Delta^{CD}}\\
&\qquad\qquad -2\mathcal{E}_4^{CDEF}C^{(0)}_{MN\{I}C^{(0)}_{J\}EF}\Delta^{MQ}\Delta^{NP}\frac{\partial^2}{\partial\Delta^{QP}\partial\Delta^{CD}}\Bigg]c^{(g-1)}_{(k)}\\
&-\Big[\check{C}^{(0)}_{IJAB}+2C^{(0)}_{AB\{I}\check{d}_{J\}}\Big]\Delta^{AC}\Delta^{BD}\frac{\partial}{\partial\Delta^{CD}}c^{(g-1)}_{(k)}\\
&+\Bigg[2C^{(0)}_{IMN}C^{(0)}_{JQB}\Delta^{QM}\Delta^{NC}\Delta^{BD}\frac{\partial}{\partial\Delta^{CD}}\\
&\qquad\qquad +C^{(0)}_{IMN}C^{(0)}_{JAB}\Delta^{QM}\Delta^{NP}\Delta^{AC}\Delta^{BD}\frac{\partial^2}{\partial\Delta^{QP}\partial\Delta^{CD}}\Bigg]c^{(g-1)}_{(k)}\,,
\end{aligned}
$$

where $\{IJ\}$ indicates the symmetrization of the indices and we abbreviated

$$
\check{C}^{(0)}_{IJKL}=\check{D}_I C^{(0)}_{JKL}\,. \tag{7.58}
$$

Once again, we can specify the Δ-homogeneity of the terms: first line $k-2$, second line $k-1$, third and fourth line k, fifth line $k+1$, sixth and seventh line $k+2$. In performing the direct integration of the holomorphic anomaly equation we keep track of the number of propagators on the right-hand side of (7.50). We can do this explicitly by inserting (7.53) together with (7.55) and (7.57) into (7.50). Due to the vast number of indices the result looks rather complicated and will be presented in the following.

Performing the direct integration one finds

$$
\begin{aligned}
F^{(g)} = \frac{1}{2}\sum_{k=0}^{3g-6}\Bigg[&\frac{1}{k-1}\mathcal{E}_4^{MNQP}\mathcal{E}_4^{TURS}C^{(0)}_{IMN}C^{(0)}_{JTU}\Delta^{IJ}\frac{\partial^2}{\partial\Delta^{QP}\partial\Delta^{RS}}\\
&+\frac{1}{k}\Bigg(\mathcal{E}_6^{CDKLMN}C^{(0)}_{KLI}C^{(0)}_{MNJ}+\mathcal{E}_4^{CDFG}\check{C}^{(0)}_{JFGI}\\
&\qquad\qquad +2\mathcal{E}_4^{CDFG}C^{(0)}_{FGI}\Big(\check{d}_J+\partial_J f^{(1)}\Big)\Bigg)\Delta^{IJ}\frac{\partial}{\partial\Delta^{CD}}\\
&+\frac{1}{k+1}\Bigg(\Big(\check{d}_I+\partial_I f^{(1)}\Big)\Big(\check{d}_J+\partial_J f^{(1)}\Big)\Delta^{IJ}-\partial_I f^{(1)}\partial_J f^{(1)}\Delta^{IJ}\\
&-2\Big(\mathcal{E}_4^{QCFG}C^{(0)}_{FGI}C^{(0)}_{JQB}\Delta^{IJ}\Delta^{BD}+\mathcal{E}_4^{PQCD}C^{(0)}_{IJK}C^{(0)}_{LPQ}\Delta^{IJ}\Delta^{KL}\Big)\frac{\partial}{\partial\Delta^{CD}}\\
&\qquad\qquad -2\mathcal{E}_4^{CDEF}C^{(0)}_{MNI}C^{(0)}_{JEF}\Delta^{IJ}\Delta^{MQ}\Delta^{NP}\frac{\partial^2}{\partial\Delta^{QP}\partial\Delta^{CD}}\Bigg)\\
&-\frac{1}{k+2}\Bigg(2C^{(0)}_{IJK}\check{d}_L\Delta^{IJ}\Delta^{KL}+\Big(\check{C}^{(0)}_{IJAB}+2C^{(0)}_{ABI}(\check{d}_J+\partial_J f^{(1)})\Big)\times\\
&\qquad\qquad\times\Delta^{IJ}\Delta^{AC}\Delta^{BD}\frac{\partial}{\partial\Delta^{CD}}\Bigg)\\
&+\frac{1}{k+3}\Bigg(2\Big(C^{(0)}_{IJK}C^{(0)}_{LPQ}\Delta^{IJ}\Delta^{KL}\Delta^{PC}\Delta^{QD}\\
&\qquad\qquad +C^{(0)}_{IMN}C^{(0)}_{JQB}\Delta^{IJ}\Delta^{QM}\Delta^{NC}\Delta^{BD}\Big)\frac{\partial}{\partial\Delta^{CD}}
\end{aligned}
$$

$$
\begin{aligned}
&\qquad\qquad +C^{(0)}_{IMN}C^{(0)}_{JAB}\Delta^{IJ}\Delta^{QM}\Delta^{NP}\Delta^{AC}\Delta^{BD}\frac{\partial^2}{\partial\Delta^{QP}\partial\Delta^{CD}}\Big)\Big]c^{(g-1)}_{(k)}\\
&+\frac{1}{2}\sum_{r=2}^{g-2}\sum_{k=0}^{3g-6}\sum_{m+n=k}\left[\frac{1}{k+1}\Delta^{IJ}\left(\check{d}_I c^{(g-r)}_{(m)}\right)\left(\check{d}_J c^{(r)}_{(n)}\right)\right.\\
&+\frac{1}{k-1}\mathcal{E}_4^{PQMN}\mathcal{E}_4^{RSTU}C^{(0)}_{IPQ}C^{(0)}_{JRS}\Delta^{IJ}\left(\frac{\partial}{\partial\Delta^{MN}}c^{(g-r)}_{(m)}\right)\left(\frac{\partial}{\partial\Delta^{TU}}c^{(r)}_{(n)}\right)\\
&+\frac{1}{k+3}C^{(0)}_{IPQ}C^{(0)}_{JRS}\Delta^{IJ}\Delta^{PM}\Delta^{QN}\Delta^{PT}\Delta^{SU}\left(\frac{\partial}{\partial\Delta^{MN}}c^{(g-r)}_{(m)}\right)\left(\frac{\partial}{\partial\Delta^{TU}}c^{(r)}_{(n)}\right)\\
&+\frac{1}{k}\mathcal{E}_4^{RSTU}C^{(0)}_{JRS}\Delta^{IJ}\left\{\left(\check{d}_I c^{(g-r)}_{(m)}\right)\left(\frac{\partial}{\partial\Delta^{TU}}c^{(r)}_{(n)}\right)+\left(\frac{\partial}{\partial\Delta^{TU}}c^{(g-r)}_{(m)}\right)\left(\check{d}_I c^{(r)}_{(n)}\right)\right\}\\
&-\frac{1}{k+2}C^{(0)}_{JRS}\Delta^{IJ}\Delta^{PT}\Delta^{SU}\left\{\left(\check{d}_I c^{(g-r)}_{(m)}\right)\left(\frac{\partial}{\partial\Delta^{TU}}c^{(r)}_{(n)}\right)\right.\\
&\qquad\qquad\qquad\left.\left.+\left(\frac{\partial}{\partial\Delta^{TU}}c^{(g-r)}_{(m)}\right)\left(\check{d}_I c^{(r)}_{(n)}\right)\right\}\right]\\
&-\frac{1}{k+1}\mathcal{E}_4^{PQMN}C^{(0)}_{IPQ}C^{(0)}_{JRS}\Delta^{IJ}\Delta^{PT}\Delta^{SU}\left\{\left(\frac{\partial}{\partial\Delta^{MN}}c^{(g-r)}_{(m)}\right)\left(\frac{\partial}{\partial\Delta^{TU}}c^{(r)}_{(n)}\right)\right.\\
&\left.\left.+\left(\frac{\partial}{\partial\Delta^{TU}}c^{(g-r)}_{(m)}\right)\left(\frac{\partial}{\partial\Delta^{MN}}c^{(r)}_{(n)}\right)\right\}\right]+c^{(g)}_{(0)}\,. \qquad (7.59)
\end{aligned}
$$

Let us end with some brief remarks about the properties of the direct integration in the big moduli space. Firstly, we note that the building blocks of $F^{(g)}$ are the propagators Δ^{IJ} as well as the holomorphic modular forms

$$
\mathcal{E}_{2k}^{I_1\ldots J_k}\,,\qquad \check{D}_{I_1}\ldots\check{D}_{I_k}f^{(1)}\,,\qquad \check{D}_{I_1}\ldots\check{D}_{I_k}C^{(0)}_{KLM}\,, \qquad (7.60)
$$

induced by $F^{(1)}$ and $F^{(0)}$. It seems likely that also the holomorphic ambiguity can be parametrized by (7.60). To determine these forms it is essential to find $\Phi(\tau)$, which will be harder for examples other than the Enriques Calabi-Yau. Moreover, in order to efficiently derive all $F^{(g)}$ one also needs to show that the forms (7.60) are generated by a finite number of holomorphic modular forms of Γ_M. Clearly, the most challenging task is then to fix the ambiguity by appropriate boundary conditions. To explore these issues for other interesting examples will be left for further work.

8. Conclusion and outlook

In this paper we have developed a new approach to solving the holomorphic anomaly equations of [7], based on the interplay between modularity and non-holomorphicity, which makes possible to perform a direct integration of the equations at each genus. This approach is more efficient than the diagram expansion of [7] and leads to closed expressions for the topological string amplitudes, once the ambiguities are fixed by appropriate boundary conditions. The amplitudes obtained with this procedure can be written as polynomials in a finite set of generators that transform in a particularly simple way under the space-time symmetry group, making the modularity properties manifest. There are many open questions and possible avenues for future work. We list here some of them.

Although we have been able to improve the results of [41] for the Enriques Calabi-Yau manifold, it would be interesting to push further the formalism developed in this paper. In section 5 we have introduced a set of holomorphic automorphic forms on the Enriques moduli space which might be enough to parametrize the holomorphic ambiguity. Using these forms, the boundary conditions obtained from the field theory and the fiber limits, and some extra information coming for example from Gromov-Witten theory, one might be able to obtain the topological string amplitudes at higher genus.

As explained in [37], Gopakumar-Vafa invariants should provide a microscopic counting of degrees of freedom for 5d spinning black holes, although in order to make contact with the macroscopic Bekenstein-Hawking entropy one typically needs a knowledge of these invariants (therefore of the topological string amplitudes $F^{(g)}$) at arbitrary high genus. Some of the results of this paper might be useful in studies of these black holes. For example, the all-genus fiber result for the Enriques Calabi-Yau manifold should give a detailed microscopic counting for small 5d black holes obtained by wrapping M2 branes in the Enriques fiber.

Vast progress has been made in the understanding of compactifications which allow to stabilize many or all moduli in $\mathcal{N} = 1$ supersymmetric vacua [19]. These vacua often rely on the inclusion of background fluxes and D brane instanton effects. Orientifolds of the Enriques Calabi-Yau might serve as very controllable examples in which certain corrections to the $\mathcal{N} = 1$ low energy effective theory can be derived. In particular, it is an interesting task to identify and compute corrections to the four-dimensional super- and Kähler potentials encoded by the higher genus amplitudes.

As shown in [33, 22] the free energies of matrix models satisfy the holomorphic anomaly conditions. Hence, the techniques of this paper could lead to a useful method to analyze matrix models. Using matrix model technology one can also write down holomorphic anomaly equations for open string amplitudes in local Calabi-Yau manifolds [22], and it would be interesting to study them using the methods of this paper. In view of the results of [45], this could lead to a powerful approach to compute open string amplitudes on toric Calabi-Yau manifolds.

Acknowledgments

We would like to thank Mina Aganagic, Richard Borcherds, Vincent Bouchard, Robbert Dijkgraaf, Ian Ellwood, Eberhard Freitag, Min-xin Huang, Peter Mayr, Nikita Nekrasov and Don Zagier, for very valuable discussions and correspondences. M.M. would like to thank Greg Moore for conversations on the field theory limit of the FHSV model in 1998. A.K. and T.G. are supported by DOE Grant DE-FG-02-95ER40896. M.W. is supported by the Marie Curie EST program. This work has been initiated during a stay of A.K., M.M. and T.G. at the MSRI in Berkeley.

A. $\mathcal{N} = 2$ special geometry

In this appendix we summarize some basics about $\mathcal{N} = 2$ special geometry [13 – 15, 24]. Let

Y be a Calabi-Yau threefold, i.e. a complex three-dimensional Kähler manifold with SU(3) or SU(2) $\times \mathbb{Z}_2$, but no smaller, holonomy group. In particular Y has a no-where vanishing holomorphic three-form Ω, which is unique up to complex rescaling. Ω depends on the complex structure of Y and hence varies over the space of complex structure deformations $\mathcal{M}$. Local coordinates on $\mathcal{M}$ are denoted by $t^i, \bar{t}^{\bar{\imath}}$. $\Omega(t)$ can be used to define a Kähler potential

$$K(t,\bar{t}) = -\log\left[i\int_Y \Omega\wedge\bar{\Omega}\right] \tag{A.1}$$

K induces the following Kähler metric structures on $\mathcal{M}$

$$\begin{aligned} G_{i\bar{\jmath}} &= \partial_i\partial_{\bar{\jmath}}K, & \Gamma^k_{ij} &= G^{k\bar{l}}\partial_i G_{j\bar{l}}, \qquad \Gamma^{\bar{k}}_{\bar{\imath}\bar{\jmath}} = G^{l\bar{k}}\bar{\partial}_{\bar{\imath}}G_{l\bar{\jmath}} \\ R_{i\bar{\jmath}k\bar{l}} &= -\partial_i\bar{\partial}_{\bar{\jmath}}G_{k\bar{l}} + G^{m\bar{n}}(\partial_i G_{k\bar{n}})(\bar{\partial}_{\bar{\jmath}}G_{m\bar{l}}), & R_{i\bar{\jmath}k}{}^{l} &= -\bar{\partial}_{\bar{\jmath}}\Gamma^l_{ik} \\ R_{i\bar{\jmath}} &\equiv G^{k\bar{l}}R_{i\bar{\jmath}k\bar{l}} = -\partial_i\bar{\partial}_{\bar{\jmath}}\log\det(G_{i\bar{\jmath}})\ . \end{aligned} \tag{A.2}$$

Ω and $\bar{\Omega}$ are sections of holomorphic and anti-holomorphic lines bundles $\mathcal{L}$ and $\overline{\mathcal{L}}$ over $\mathcal{M}$ respectively and holomorphic gauge transformations $\Omega \to e^f\Omega$ in $\mathcal{L}$ correspond to Kähler transformations, i.e. $e^{-K} \in \mathcal{L}\otimes\overline{\mathcal{L}}$. The derivatives ∂_i are with respect to coordinates t_i of $\mathcal{M}$, and sections like $V_{j\bar{\jmath}}$ in $T\mathcal{M}^*_{(1,0)}\otimes T\mathcal{M}^*_{(0,1)}\otimes\mathcal{L}^m\otimes\overline{\mathcal{L}}^n$ have a natural connection with respect to the Weil-Petersson metric $G_{i\bar{\jmath}}$ and the line bundle $K_i = \partial_i K$, $K_{\bar{\imath}} = \partial_{\bar{\imath}}K$

$$D_i V_{j\bar{\jmath}} = \partial_i - \Gamma^l_{ij}V_{l\bar{\jmath}} + mK_iV_{j\bar{\jmath}}, \quad D_{\bar{\imath}}V_{j\bar{\jmath}} = \partial_{\bar{\imath}} - \Gamma^{\bar{l}}_{\bar{\imath}\bar{\jmath}}V_{j\bar{l}} + nK_{\bar{\imath}}V_{j\bar{\jmath}}\ . \tag{A.3}$$

For a given complex structure Ω defines a Hodge decomposition

$$H^3(Y,\mathbb{C}) = H^{(3,0)}\oplus H^{(2,1)}\oplus H^{(1,2)}\oplus H^{(0,3)}\ . \tag{A.4}$$

The forms Ω, $\chi_i \equiv D_i\Omega$, $\overline{\chi}_{\bar{\imath}} \equiv D_{\bar{\imath}}\bar{\Omega}$ and $\bar{\Omega}$ provide a basis which spans the above cohomology groups over $\mathbb{C}$. Since it depends on the complex structure we call it the moving basis. By Kodaira theory, infinitesimal deformations of the complex structure are elements of $H^1(Y,TY)$. Ω induces an isomorphism $H^1(Y,TY)\sim H^{(2,1)}(Y)$. Hence the harmonic $(2,1)$-forms χ_i, $i = 1,\ldots,h^{21}$ can be identified as (co)tangent vectors to $\mathcal{M}$ and these deformations are unobstructed on a CY manifold [57].

We also introduce a fixed integer symplectic basis (α_K,β^L) of $H^3(Y,\mathbb{Z})$ with

$$\int_Y \alpha_K\wedge\beta^L = -\int_Y\beta^L\wedge\alpha_K = \delta^L_K\,, \qquad \int_Y\alpha_K\wedge\alpha_L = \int_Y\beta^K\wedge\beta^L = 0\,, \tag{A.5}$$

which is independent of the complex structure. We can expand the moving basis in terms of the fixed basis

$$\Omega = X^I\alpha_I - \mathcal{F}_I\beta^I\,, \qquad \chi_i = \chi^I_i\alpha_I - \chi_{Ii}\beta^I\,, \qquad \text{etc}\ . \tag{A.6}$$

The expansion coefficients are the period integrals

$$X^I = \int_{A^I}\Omega, \qquad F_I = \int_{B_I}\Omega, \qquad \chi^I_i = \int_{A^I}\chi_i, \qquad \chi_{I\,i} = \int_{B_I}\chi_i\,, \tag{A.7}$$

where (A^K, B_I) is a basis of $H_3(Y,\mathbb{Z})$ dual to (α_K, β^L). Using (A.5) and (A.7) we can express (A.1) in terms of the periods

$$K = -\log i\Big[\bar{X}^K \mathcal{F}_K - X^K \bar{\mathcal{F}}_K\Big] \ . \tag{A.8}$$

Note that $X^I \in \mathcal{L}$, $\overline{\mathcal{F}}_I \in \overline{\mathcal{L}}$, $\chi_i^I \in T^*_{(1,0)}\mathcal{M}\otimes\mathcal{L}$ etc. Obviously the periods carry the information about the complex structure deformations. The X^I, $I = 0, \ldots h^{21}$ can serve locally as homogeneous coordinates on $\mathcal{M}$. Local special coordinates on $\mathcal{M}$ are defined by $t^i = X^i/X^0$, $i = 1, \ldots h^{(2,1)}$. The $\mathcal{F}_I$ on the other hand are not independent. It follows rather from

$$\int_Y \Omega \wedge \frac{\partial}{\partial X^I}\Omega = 0 \tag{A.9}$$

that there is a holomorphic section $\mathcal{F}$ of $\mathcal{L}^2$ called prepotential obeying

$$\mathcal{F} = \frac{1}{2} X^I F_I \,, \qquad \mathcal{F}_I = \partial_{X^I}\mathcal{F} \ . \tag{A.10}$$

This also implies that $\mathcal{F}(X)$ is homogeneous of degree two in X^I. In special coordinates t^i one also writes $\mathcal{F}(t) = (X^0)^{-2}\mathcal{F}(X)$. It turns out to be useful to introduce the second and third derivative of the prepotential as

$$\tau_{IJ} = \partial_I\partial_J\mathcal{F} \,, \qquad C_{IJK} = \partial_I\partial_J\partial_K\mathcal{F} \,, \tag{A.11}$$

which are homogeneous of degree zero and minus one respectively.

Special Kähler geometry describes the relation between the metric structure and the Yukawa coupling

$$C^{(0)}_{ijk} \equiv iC_{ijk} \equiv -\int_Y \Omega\wedge\partial_i\partial_j\partial_k\Omega = -\int_Y \Omega\wedge D_iD_jD_k\Omega \,, \tag{A.12}$$

a section of $C_{ijk} \in \mathrm{Sym}^3(T^*_{(1,0)})\otimes\mathcal{L}^2$. Using $\langle\chi_i, \bar{\chi}_{\bar{\imath}}\rangle = G_{i\bar{\jmath}}e^{-K}$ and transversality of $\langle , \rangle$ under the the decomposition (A.4), i.e. $\langle\gamma_{(k,l)}, \gamma_{(m,n)}\rangle = 0$ unless $k+m = l+n = 3$ one gets the special geometry identities [13]

$$D_iX^I \equiv \chi_i^I \,, \qquad D_i\chi_j^I = iC_{ijk}G^{k\bar{k}}\bar{\chi}^I_{\bar{k}}e^K \,, \qquad D_i\bar{\chi}^I_{\bar{\jmath}} = G_{i\bar{\jmath}}\bar{X}^I \ . \tag{A.13}$$

From (A.3) and (A.2) follows $[D_i, D_{\bar{\jmath}}]\chi_k = -G_{i\bar{\jmath}} + R_{i\bar{\jmath}k}{}^l\chi_l$ and using (A.13) one gets

$$[D_i, D_{\bar{k}}]_j{}^l = R_{i\bar{k}j}{}^l = G_{i\bar{k}}\delta^l_j + G_{j\bar{k}}\delta^l_i - C_{ijm}\bar{C}^{ml}_{\bar{k}} \,, \tag{A.14}$$

where we abbreviated

$$\bar{C}^{(0)ml}_{\bar{k}} = e^{2K}\bar{C}^{(0)}_{\bar{k}\bar{\imath}\bar{\jmath}}G^{m\bar{\imath}}G^{l\bar{\jmath}} \,, \qquad \bar{C}^{ml}_{\bar{k}} = i\bar{C}^{(0)ml}_{\bar{k}} \ . \tag{A.15}$$

Let us also summarize some relations obeyed by τ_{IJ} and C_{IJK}. One first notes that by homogeneity and (A.12) and (A.13) one has

$$C_{IJK}X^K = 0 \,, \qquad C_{ijk} = C_{IJK}\chi^I_i\chi^J_j\chi^K_k \ . \tag{A.16}$$

Using the above definitions and the degree two homogeneity of $\mathcal{F}$ one also shows that

$$2e^K X^I \mathrm{Im}\tau_{IJ}\bar{X}^J = 1\,, \qquad \bar{X}^I \mathrm{Im}\tau_{IJ}\chi_i^J = 0\,, \qquad 2e^K \chi_i^I \mathrm{Im}\tau_{IJ}\bar{\chi}_{\bar{j}}^J = G_{i\bar{j}}\,. \tag{A.17}$$

Denoting by $\mathrm{Im}\tau^{IJ}$ the inverse of $\mathrm{Im}\tau_{IJ}$ it follows from these conditions that

$$\chi_i^I G^{i\bar{j}}\bar{\chi}_{\bar{j}}^J e^K = \frac{1}{2}\mathrm{Im}\tau^{IJ} + X^I\bar{X}^J e^K\,. \tag{A.18}$$

B. Theta functions and modular forms

Our conventions for the Jacobi theta functions are:

$$\begin{aligned}
\vartheta_1(\nu|\tau) &= \vartheta[{}^1_1](\nu|\tau) = i\sum_{n\in\mathbf{Z}}(-1)^n q^{\frac{1}{2}(n+1/2)^2}e^{i\pi(2n+1)\nu},\\
\vartheta_2(\nu|\tau) &= \vartheta[{}^1_0](\nu|\tau) = \sum_{n\in\mathbf{Z}} q^{\frac{1}{2}(n+1/2)^2}e^{i\pi(2n+1)\nu},\\
\vartheta_3(\nu|\tau) &= \vartheta[{}^0_0](\nu|\tau) = \sum_{n\in\mathbf{Z}} q^{\frac{1}{2}n^2}e^{i\pi 2n\nu},\\
\vartheta_4(\nu|\tau) &= \vartheta[{}^0_1](\nu|\tau) = \sum_{n\in\mathbf{Z}}(-1)^n q^{\frac{1}{2}n^2}e^{i\pi 2n\nu},
\end{aligned} \tag{B.1}$$

where $q = e^{2\pi i\tau}$. When $\nu = 0$ we will simply denote $\vartheta_2(\tau) = \vartheta_2(0|\tau)$ (notice that $\vartheta_1(0|\tau) = 0$). The theta functions $\vartheta_2(\tau)$, $\vartheta_3(\tau)$ and $\vartheta_4(\tau)$ have the following product representation:

$$\begin{aligned}
\vartheta_2(\tau) &= 2q^{1/8}\prod_{n=1}^{\infty}(1-q^n)(1+q^n)^2,\\
\vartheta_3(\tau) &= \prod_{n=1}^{\infty}(1-q^n)(1+q^{n-\frac{1}{2}})^2,\\
\vartheta_4(\tau) &= \prod_{n=1}^{\infty}(1-q^n)(1-q^{n-\frac{1}{2}})^2
\end{aligned} \tag{B.2}$$

and under modular transformations they behave as:

$$\begin{aligned}
\vartheta_2(-1/\tau) &= \sqrt{\frac{\tau}{i}}\vartheta_4(\tau), & \vartheta_2(\tau+1) &= \mathrm{e}^{i\pi/4}\vartheta_2(\tau),\\
\vartheta_3(-1/\tau) &= \sqrt{\frac{\tau}{i}}\vartheta_3(\tau), & \vartheta_3(\tau+1) &= \vartheta_4(\tau),\\
\vartheta_4(-1/\tau) &= \sqrt{\frac{\tau}{i}}\vartheta_2(\tau), & \vartheta_4(\tau+1) &= \vartheta_3(\tau).
\end{aligned} \tag{B.3}$$

The theta function $\vartheta_1(\nu|\tau)$ has the product representation

$$\vartheta_1(\nu|\tau) = -2q^{\frac{1}{8}}\sin(\pi\nu)\prod_{n=1}^{\infty}(1-q^n)(1-2\cos(2\pi\nu)q^n + q^{2n}). \tag{B.4}$$

We also have the following useful identities:

$$\vartheta_3^4(\tau) = \vartheta_2^4(\tau) + \vartheta_4^4(\tau), \tag{B.5}$$

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and

$$\vartheta_2(\tau)\vartheta_3(\tau)\vartheta_4(\tau) = 2\,\eta^3(\tau), \tag{B.6}$$

where

$$\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n) \tag{B.7}$$

is the Dedekind eta function. One has the following doubling formulae,

$$\begin{aligned} \eta(2\tau) =& \sqrt{\frac{\eta(\tau)\vartheta_2(\tau)}{2}}, \qquad \vartheta_2(2\tau) = \sqrt{\frac{\vartheta_3^2(\tau)-\vartheta_4^2(\tau)}{2}}, \\ \vartheta_3(2\tau) =& \sqrt{\frac{\vartheta_3^2(\tau)+\vartheta_4^2(\tau)}{2}}, \qquad \vartheta_4(2\tau) = \sqrt{\vartheta_3(\tau)\vartheta_4(\tau)}, \\ \eta(\tau/2) =& \sqrt{\eta(\tau)\vartheta_4(\tau)}. \end{aligned} \tag{B.8}$$

The Eisenstein series are defined by

$$E_{2n}(q) = 1 - \frac{4n}{B_{2n}}\sum_{k=1}^{\infty}\frac{k^{2n-1}q^k}{1-q^k}, \tag{B.9}$$

where B_m are the Bernoulli numbers. The covariant version of E_2 is

$$\widehat{E}_2(\tau,\bar\tau) = E_2(\tau) - \frac{3}{\pi\mathrm{Im}\,\tau} = E_2(\tau) - \frac{6\mathrm{i}}{\pi(\tau-\bar\tau)}. \tag{B.10}$$

The formulae for the derivatives of the theta functions are also useful:

$$\begin{aligned} q\frac{d}{dq}\log\,\vartheta_4 =& \frac{1}{24}\left(E_2 - \vartheta_2^4 - \vartheta_3^4\right), \\ q\frac{d}{dq}\log\,\vartheta_3 =& \frac{1}{24}\left(E_2 + \vartheta_2^4 - \vartheta_3^4\right), \\ q\frac{d}{dq}\log\,\vartheta_2 =& \frac{1}{24}\left(E_2 + \vartheta_3^4 + \vartheta_4^4\right), \end{aligned} \tag{B.11}$$

and from these one finds

$$q\frac{d}{dq}\log\,\eta = \frac{1}{24}E_2(\tau) \tag{B.12}$$

and the Ramanujan identities

$$\begin{aligned} q\frac{d}{dq}\;E_2(q) =& \frac{1}{12}(E_2^2(q) - E_4(q)), \\ q\frac{d}{dq}\;E_4(q) =& \frac{1}{3}(E_2(q)E_4(q) - E_6(q)), \\ q\frac{d}{dq}\;E_6(q) =& \frac{1}{2}(E_2(q)E_6(q) - E_4^2(q)). \end{aligned} \tag{B.13}$$

These can be used to compute the q-derivatives of the generators K_2, K_4 introduced in (3.5):

$$\begin{aligned} q\partial_q K_2 =& \frac{1}{6}E_2(q)K_2(q) + \frac{1}{4}K_4(q) - \frac{1}{12}K_2^2(q), \\ q\partial_q K_4 =& \frac{1}{3}K_4(q)(E_2(q)+K_2(q)). \end{aligned} \tag{B.14}$$

The doubling formulae for $E_2(\tau), E_4(\tau)$ are

$$\begin{aligned} E_2(2\tau) =& \frac{1}{2}E_2(\tau) + \frac{1}{4}(\vartheta_3^4(\tau) + \vartheta_4^4(\tau)), \\ E_4(2\tau) =& \frac{1}{16}E_4(2\tau) + \frac{15}{16}\vartheta_3^4(\tau)\vartheta_4^4(\tau). \end{aligned} \tag{B.15}$$

C. The antiholomorphic dependence of the heterotic $F^{(g)}$

In this appendix we find the antiholomorphic dependence of $F^{(g)}(t, \bar{t})$ in the heterotic theory. In section C.1, we show how the complicated result of the heterotic computation of the $F^{(g)}$ in the STU-model given in [46] can be simplified, along the lines of [10]. In section C.2 we write down the result for $F_E^{(g)}$ in the Enriques Calabi-Yau and derive (4.40).

C.1 A simple form for $F^{(g)}$ in the STU-model

In [46], an explicit expression for the holomorphic and antiholomorphic dependence of the topological amplitudes in the fiber limit of the STU-model was found. This expression is obtained from a one-loop computation in the dual heterotic theory, given by the integral (4.24), which is then performed by using the technique of lattice reduction [10]. One finds that $F^{(g)} = F^{(g)}_{\rm deg} + F^{(g)}_{\rm ndeg}$, where [46]

$$F^{(g)}_{\rm deg} = 4\pi^2 U_1 \delta_{g,1} + \frac{2^{2g-1}\pi^{4g-3}}{T_1^{2g-3}} \sum_{l=0}^{g} c_g(0,l) \frac{l!}{\pi^{l+3}} \left(\frac{T_1}{U_1}\right)^l \zeta(2(2+l-g)), \tag{C.1}$$

$$\begin{aligned} F^{(g>1)}_{\rm ndeg} =& 4\pi^{2g-2}(-1)^{g-1} \sum_{r\neq 0} \sum_{l=0}^{g} \sum_{h=0}^{2g-2} \sum_{j=0}^{[g-1-h/2]} \sum_{a=0}^{s} c_g(r^2/2, l) \frac{(2\pi)^l (2g-2)!}{j!h!(2g-h-2j-2)!} \\ &\times \frac{(-1)^{j+h}}{2^{j+a}} \frac{(s+a)!}{a!(s-a)!} (\mathrm{sgn}\,(\mathrm{Re}(r\cdot y))^h \frac{1}{(T_1U_1)^l} \left(\mathrm{Re}(r\hat{\cdot} y)\right)^{l-j-a} \mathrm{Li}_{3+a+j+l-2g}(\mathrm{e}^{-r\hat{\cdot} y}) \\ &+ \frac{2\pi^{3g-3} c_g(0,g-1)}{(T_1U_1)^{g-1}} \sum_{s=0}^{g-1} (-1)^s \frac{(2g-2)!}{s!(g-1-s)!} \psi\left(\frac{1}{2}+s\right) \\ &+ \sum_{\substack{l=0 \\ l\neq g-1}}^{g} 4^{l+g}\pi^{2g+l-5/2} c_g(0,l) \frac{\zeta(3+2(l-g))}{(T_1U_1)^l} \\ &\times \sum_{s=0}^{g-1} (-1)^s 2^{2(s-2g)+5} \frac{(2g-2)!}{(2s)!(g-1-s)!} \Gamma\left(\frac{3}{2}+s+l-g\right). \end{aligned} \tag{C.2}$$

We refer to $F^{(g)}_{\rm deg}$, $F^{(g)}_{\rm ndeg}$ as the degenerate and nondegenerate contributions, respectively. Also, $s := |2g-2-h-j-l-1/2| - 1/2$; $\quad y = (T, U)$, $\quad$ the complex norm is defined as $r^2 = 2r_1r_2$, and

$$r\hat{\cdot} y \equiv |\mathrm{Re}(r\cdot y)| + \mathrm{i}\mathrm{Im}(r\cdot y).$$

The coefficients $c_g(m,l)$ can be obtained from the expansion

$$\frac{E_4E_6}{\eta^{24}} \widehat{\mathcal{P}}_g = \sum_{m\in\mathbb{Q}} \sum_{l\geq 0} c_g(m,l) q^m \tau_2^{-l}, \tag{C.3}$$

where $\widehat{\mathcal{P}}_g$ are defined by

$$\left(\frac{2\pi\eta^3\lambda}{\vartheta_1(\lambda|\tau)}\right)^2 \mathrm{e}^{-\frac{\pi\lambda^2}{\tau_2}} = \sum_{g=0}^{\infty}(2\pi\lambda)^{2g}\widehat{\mathcal{P}}_g(\tau,\bar{\tau}). \tag{C.4}$$

Note that these $\widehat{\mathcal{P}}_g(\tau,\bar{\tau})$ are the modular, almost holomorphic extensions of the $\mathcal{P}_g(\tau)$ defined in (4.29), that is, $\widehat{\mathcal{P}}_g$ is obtained from $\mathcal{P}_g$ by replacing $E_2 \to \widehat{E}_2$. The only anti-holomorphic dependence in $\widehat{\mathcal{P}}_g$ thus lies in the $\widehat{E}_2(\tau,\bar{\tau})$. Using the explicit expressions for $\widehat{\mathcal{P}}_g$ given in [41], one can show that independently of the specific model,

$$c_g(m,l) = \frac{(-1)^l}{l!(4\pi)^l}c_{g-l}(m), \tag{C.5}$$

where $c_g(m)$ are defined analogously to (4.28), that is

$$\sum_n c_g(n)q^n = \mathcal{P}_g(q)\frac{E_4E_6}{\eta^{24}}. \tag{C.6}$$

In what follows, we will systematically express everything in terms of the coefficients $c_g(m)$.

It turns out that (C.2) can be dramatically simplified. We will need the identity:

$$\sum_j(-1)^j\binom{C}{j}\binom{A-2j+C-B-1}{A-2j} = \sum_j(-1)^j\binom{C}{A-j}\binom{B}{j} \tag{C.7}$$

This is valid for any $A,B,C\in\mathbb{Z}$, see [10] for a proof. A special case of the above formula is the following. Let C, l and m^+-h^+ be integers such that $0\le l<C<m^+-h^+-l$. Then,

$$\sum_j\frac{(-1)^j(m^+-h^++C-2j-1-l)!}{j!(m^+-h^+-2j)!(C-j)!} = 0.$$

The proof of this statement is easy. Set $B=l, A=m^+-h^+$ in (C.7) to obtain

$$\sum_j(-1)^j\frac{(m^+-h^++C-2j-1-l)!C!}{j!(m^+-h^+-2j)!(C-j)!(C-1-l)!} = \sum_j\binom{C}{m^+-h^+-j}\binom{l}{j}. \tag{C.8}$$

Since $C>l\ge 0$, any non-vanishing term on the right-hand side must fulfill $m^+-h^+-C\le j\le l$, in contradiction with the assumption $C<m^+-h^+-l$.

We also have the following three additional nontrivial identities. First of all, let $s:=|2g-2-h-j-l-1/2|-1/2$. Then,

$$\sum_{h=0}^{2g-2}\sum_{j=0}^{C}\frac{(2g-2)!}{2^{2g-2}}\frac{(s+C-j)!(-1)^{C-j}}{l!h!j!(2g-2-h-2j)!(s-C+j)!(C-j)!}$$
$$= \begin{cases}\binom{2g-3-l}{C}\frac{1}{(l-C)!} & C\le\min(l,2g-3-l)\\ 0 & \text{otherwise.}\end{cases}$$

This is valid for any pair of positive integers g,l. The second identity reads,

$$\sum_{s=0}^{g-1}(-1)^s\frac{(2g-2)!}{s!(g-1-s)!}\psi\left(s+\frac{1}{2}\right) = -2^{(2g-2)}(g-2)!. \tag{C.9}$$

The final identity we will need is

$$\sum_{s=0}^{g-1}(-1)^{l+s}2^{2(s-2g)+5}\frac{(2g-2)!}{(2s)!(g-1-s)!}\Gamma\left(\frac{3}{2}+s+l-g\right)=\frac{(-1)^{g-1}(2g-3-l)!}{(2g-3-2l)!}\sqrt{\pi}4^{-l}. \tag{C.10}$$

which is valid for any $l \in \mathbb{N}, l < g-1$. Making use of (C.8) and (C.9), we can convert the sums over h, j, a in (C.2) into a single one over $C = j + a = \{0, \cdots, l\}$. Then, (C.9) and (C.10) can be used to simplify the second respectively third term in (C.2). The sum over r can be restricted for all $g \geq 3$ to a sum over r for which $\mathrm{Re}(r \cdot y) < 0$, or equivalently to a sum over positive r and a finite number of boundary cases. At genus 2, however, there is a contribution from $\mathrm{Re}(r \cdot y) > 0$, it reads [46]

$$\frac{c_0(r^2/2)}{16T_1U_1}\mathrm{Li}_3(\mathrm{e}^{-r\cdot y}). \tag{C.11}$$

We can then write down a simplified expression for the nondegenerate part of $F^{(g)}$ in the STU model:

$$\begin{aligned}F^{(g>1)}_{\mathrm{nd,STU}} = &\sum_{l=0}^{g-1}\sum_{C=0}^{\min(l,2g-3-l)}\sum_{r>0}\frac{\binom{2g-l-3}{C}}{(l-C)!2^C}\frac{(-\mathrm{Re}(r\cdot y))^{l-C}}{(2T_1U_1)^l}c_{g-l}(\frac{r^2}{2})\mathrm{Li}_{3-2g+l+C}(\mathrm{e}^{-r\cdot y})\\ &+\frac{22}{2^g(g-1)}\frac{1}{(2T_1U_1)^{g-1}}+\sum_{l=0}^{g-2}\frac{c_{g-l}(0)}{l!(4T_1U_1)^l}\zeta(3+2(l-g))\frac{(2g-3-l)!}{(2g-3-2l)!},\end{aligned} \tag{C.12}$$

where we also have used the fact that in the STU model, $c_1(0) = -22$, and we have removed an overall prefactor of $4(2\pi\mathrm{i})^{2g-2}$ to agree with the normalization of the topological string amplitudes.

C.2 Application to the Enriques Calabi-Yau

The above expressions have to be adapted slightly for the Enriques Calabi-Yau. We only consider here the geometric reduction suited to the large radius limit. As shown in [41], the polylogarithm is replaced by $\mathrm{Li}_m(x) \to 2^m\mathrm{Li}_m(x^{\frac{1}{2}}) - \mathrm{Li}_m(x)$, and the norm of the reduced lattice is doubled. We also replace the quantity $2T_1U_1$ appearing in the STU-model by $Y = \mathrm{e}^{-K}$ as in (4.11), and the coefficients $c_g(m)$ are now defined by (4.28). There is a new important simplification: $c_0(r^2)$ and $c_{g>1}(0)$ vanish, and thus there is no contribution from negative r at any genus $g > 1$, since (C.11) becomes

$$\frac{c_0(r^2)}{8Y}\left(8\mathrm{Li}_3(\mathrm{e}^{-r\cdot y}) - \mathrm{Li}_3(\mathrm{e}^{-2r\cdot y})\right) = 0. \tag{C.13}$$

Furthermore, the degenerate contribution (C.1) and the last term in (C.2) vanish for all $g > 1$, while $c_1(0) = 4$, and the full $F^{(g)}_E(t,\bar{t})$ for the Enriques reads

$$\begin{aligned}F^{(g>1)}_E(t,\bar{t}) = &\sum_{l=0}^{g-1}\sum_{C=0}^{\min(l,2g-3-l)}\sum_{r>0}\frac{\binom{2g-l-3}{C}}{(l-C)!2^C}\frac{(-2\mathrm{Re}(r\cdot t))^{l-C}}{Y^l}c_{g-l}(r^2)\\ &\cdot\left(2^{3-2g+l+C}\mathrm{Li}_{3-2g+l+C}(\mathrm{e}^{-r\cdot t}) - \mathrm{Li}_{3-2g+l+C}(\mathrm{e}^{-2r\cdot t})\right) - \frac{1}{2^{g-2}(g-1)}\frac{1}{Y^{g-1}}.\end{aligned} \tag{C.14}$$

Using

$$\mathrm{Re}(t^a)\partial_{t^a}\mathrm{Li}_n(\mathrm{e}^{-r\cdot t}) = -\mathrm{Re}(r\cdot t)\mathrm{Li}_{n-1}(\mathrm{e}^{-r\cdot t}), \tag{C.15}$$

this can be cast into the following recursive form:

$$\begin{aligned} F_E^{(g)}(t,\bar t) = \sum_{l=0}^{g-1}\sum_{C=0}^{\min(l,2g-3-l)} \frac{(2g-3-l)!}{(2g-3-l-C)!(l-C)!C!2^l}\times \\ \times\frac{(t^{a_1}+\bar t^{a_1})\cdots(t^{a_{l-C}}+\bar t^{a_{l-C}})\partial_{a_1}\cdots\partial_{a_{l-C}}\mathcal{F}^{(g-l)}(t)}{Y^l} - \frac{1}{2^{g-2}(g-1)Y^{g-1}}. \end{aligned} \tag{C.16}$$

Notice that this exhibits the structure of the antiholomorphic amplitudes written down in [1].

D. Anomaly equations for $F^{(g)}$ on the big moduli space

D.1 Anomaly equation for $F^{(g)}$ $(g>1)$

Here we provide some details on the calculation of the recursive anomaly equations on the big moduli space. We like to rewrite the equation (2.10) in terms of the variables $Y^K = \lambda^{-1}X^K(t)$ and $\bar Y^K$. First note that

$$\frac{\partial}{\partial t^i} - K_i\lambda\frac{\partial}{\partial\lambda} = \lambda^{-1}\chi_i^I\frac{\partial}{\partial Y^K}\,, \tag{D.1}$$

where χ_i^I is defined in (A.6). This implies that the first derivative of $F^{(g)}$ can be written as

$$D_iF^{(g)} = \lambda^{-2g+1}\chi_i^I\,\partial_{Y^I}F^{(g)}(Y)\ . \tag{D.2}$$

where we have used the fact that $\lambda\partial_\lambda F^{(g)}(Y) = (2g-2)F^{(g)}(Y)$ due to (7.5). Moreover, one derives that the second derivative reads

$$\begin{aligned} D_iD_jF^{(g-1)} &= \lambda^{-2g+3}(D_i\chi_j^I)\,\partial_{Y^I}F^{(g-1)}(Y) + \lambda^{-2g+3}\chi_j^ID_i\,\partial_{Y^I}F^{(g-1)}(Y) \\ &= i\lambda^{-2g+3}C_{ijk}G^{k\bar k}\bar\chi_{\bar k}^I\,\partial_{Y^I}F^{(g-1)}(Y) + \lambda^{-2g+2}\chi_i^I\chi_j^J\,\partial_{Y^I}\partial_{Y^J}F^{(g-1)}(Y) \\ &= \lambda^{-2g+2}\chi_i^I\chi_j^J\left[\frac{i}{2}C^{(Y)\,K}_{IJ}\partial_{Y^K}F^{(g-1)}(Y) + \partial_{Y^I}\partial_{Y^J}F^{(g-1)}(Y)\right]\ . \end{aligned} \tag{D.3}$$

In order to evaluate the second identity we have used the special geometry relation (A.13) and (D.2) while for the third identity we have used (A.18). Also notice that from (D.2) one infers that

$$\sum_{r=1}^{g-1}D_iF^{(r)}D_jF^{(g-r)} = \lambda^{-2g+2}\chi_i^I\chi_j^J\sum_{r=1}^{g-1}\partial_{Y^I}F^{(r)}(Y)\,\partial_{Y^J}F^{(g-r)}(Y) \tag{D.4}$$

Finally, we need the identity

$$\frac{i}{2}e^{2K}\bar C_{\bar i\bar j\bar k}G^{\bar jj}G^{\bar kk}\chi_j^I\chi_k^J = \frac{i}{8}\lambda^{-1}\bar\chi_{\bar i}^K\bar C^{(Y)\,IJ}_{K}\ . \tag{D.5}$$

Hence, we conclude that

$$\partial_{\bar{Y}^I} F^{(g)} = \frac{i}{8} \bar{C}_K^{IJ} \Big[\partial_{Y^I} \partial_{Y^J} F^{(g-1)} + \sum_{r=1}^{g-1} \partial_{Y^I} F^{(r)} \, \partial_{Y^J} F^{(g-r)} \Big] - \frac{1}{16} \bar{C}_K^{IJ} C_{IJ}{}^K \partial_{Y^K} F^{(g-1)} \,, \tag{D.6}$$

where C_{IJK} and $F^{(r)}$ are functions of $Y^K, \bar{Y}^K$. This equation is precisely the recursive anomaly equation given in (7.6).

Let us also present the derivation of the simplest solution to (7.6). In other words, we calculate $F^{(2)}$ by using the integration by parts method of [7]. To do that we use the definition (7.23) of the propagator to replace $\bar{C}_K^{IJ}$ in (7.22). We pull the derivative $\partial_{\bar{I}}$ in front of all the terms and evaluate

$$\begin{aligned}\partial_{\bar{I}} \Big[F^{(2)} + \frac{1}{2} \Delta^{JK} \Big(D_J \partial_K F^{(1)} + \partial_J F^{(1)} \partial_K F^{(1)} \Big) \Big] &= - \Big(\frac{\chi}{24} - 1 \Big) \partial_{\bar{I}} \Big[\Delta^{JK} K_J \Big] \partial_K F^{(1)} \\ - \frac{1}{8} \partial_{\bar{I}} \Big[\Delta^{JK} \Delta^{LM} \Big] \Big(C^{(0)}_{KLMJ} + 4 C^{(0)}_{JLM} \partial_K F^{(1)} \Big) &- \frac{1}{2} \Big(\frac{\chi}{24} - 1 \Big) \partial_{\bar{I}} \Big[\Delta^{JK} K_J K_K \Big] \,,\end{aligned}$$

where $C^{(0)}_{IJK} = i C_{IJK}$ as defined in (7.19). In performing the derivative we used the equation (7.22) to eliminate the terms arising when $\partial_{\bar{I}}$ hits the propagator. Furthermore we commuted $\partial_{\bar{I}}$ with the covariant derivative D_J by using the identity

$$\Big[\partial_{\bar{I}}, D_J \Big] V_K = \frac{1}{4} C_{JK}{}^P \bar{C}_{IP}{}^M V_M \,. \tag{D.7}$$

One can then eliminate the second derivative $\partial_{\bar{I}} \partial_K F^{(1)}$ by inserting the equation (7.8) and applying the useful identities

$$D_I \widehat{K}_{I\bar{J}} = D_I \widehat{K}_J = 0 \,, \qquad \Delta^{IJ} D_I K_{J\bar{K}} = 2 \Delta^{IJ} K_{I\bar{K}} K_J \,, \qquad K_J \partial_{\bar{I}} \Delta^{JK} = 0 \,. \tag{D.8}$$

In the next step we once again pull the derivative $\partial_{\bar{I}}$ in front of all terms and evaluate

$$\begin{aligned}\partial_{\bar{I}} \Big[F^{(2)} &+ \frac{1}{2} \Delta^{JK} \Big(D_J \partial_K F^{(1)} + \partial_J F^{(1)} \partial_K F^{(1)} \Big) + \frac{1}{2} \Big(\frac{\chi}{24} - 1 \Big) \Delta^{JK} K_J K_K \\ &+ \frac{1}{8} \Delta^{JK} \Delta^{LM} \Big(C^{(0)}_{KLMJ} + 4 C^{(0)}_{JLM} \partial_K F^{(1)} \Big) + \Big(\frac{\chi}{24} - 1 \Big) \Delta^{JK} K_J \partial_K F^{(1)} \Big] \\ &= - \partial_{\bar{I}} \Big[\frac{1}{2} \Big(\frac{\chi}{24} - 1 \Big) C^{(0)}_{JLM} \Delta^{JK} \Delta^{LM} K_J + \frac{1}{2} \Big(\frac{\chi}{24} - 1 \Big)^2 \Delta^{JK} K_J K_K \\ &\quad + \Delta^{JK} \Delta^{LM} \Delta^{QP} \Big(\frac{1}{12} C^{(0)}_{KMQ} C^{(0)}_{PLJ} + \frac{1}{8} C^{(0)}_{KJQ} C^{(0)}_{PML} \Big) \Big] \,.\end{aligned} \tag{D.9}$$

We are now in the position to read off $F^{(2)}(Y, \bar{Y})$ up to a holomorphic ambiguity $f^{(2)}(Y)$. The corresponding solution can be found in (7.28).

D.2 Anomaly equation for $F^{(1)}$ on big phase space

In this appendix we discuss the lift of the holomorphic anomaly equation 2.11 for $F^{(1)}$ to the big moduli space $\widehat{\mathcal{M}}$. To begin with let us first note that

$$|\lambda|^{-2} \chi^I_i \bar{\chi}^J_{\bar{j}} \partial_{Y^I} \partial_{\bar{Y}^J} K = G_{i\bar{j}} \,, \qquad Y^I \partial_{Y^I} \partial_{\bar{Y}^J} K = 0 \,, \qquad \bar{Y}^J \partial_{Y^I} \partial_{\bar{Y}^J} K = 0 \,. \tag{D.10}$$

where K is the Kähler potential (A.8) and $G_{i\bar{j}}$ is the Weil-Petersson metric. We also evaluate the first derivative K_I of K and show that it satisfies

$$K_I \chi_i^I = 0\,, \qquad\qquad K_I Y^I = -1\ . \tag{D.11}$$

With these identities at hand we now lift the holomorphic anomaly equation (2.11). Using the homogeneity condition (7.7) one derives

$$\partial_i \partial_{\bar{j}} F^{(1)} = |\lambda|^{-2} \chi_i^I \bar{\chi}_{\bar{j}}^J \partial_{Y^I} \partial_{\bar{Y}^J} F^{(1)}(Y) \tag{D.12}$$

Moreover, one shows that

$$\frac{1}{2} e^{2K} G^{k\bar{k}} G^{l\bar{l}} C_{ikl} \bar{C}_{\bar{j}\bar{k}\bar{l}} = |\lambda|^{-2} \chi_i^I \bar{\chi}_{\bar{j}}^J \frac{1}{8} C_{ILM} \bar{C}_J{}^{LM}\,, \tag{D.13}$$

as well as

$$\left(\frac{\chi}{24} - 1\right) G_{i\bar{j}} = |\lambda|^{-2} \chi_i^I \bar{\chi}_{\bar{j}}^J \left(\frac{\chi}{12} - 2\right) e^{K(Y,\bar{Y})} \mathrm{Im}\tau_{IJ}\ . \tag{D.14}$$

Inserting (D.12)-(D.14) into the anomaly equation (2.11) we verify its big moduli space counterpart (7.8). It is straightforward to integrate (7.8) to find the local solution (7.10) for $F^{(1)}$ by applying the identity

$$R_{IJ} = \partial_{Y^I} \partial_{\bar{Y}^J} \log\det \mathrm{Im}\tau = -\frac{1}{4} C_{IKL} \bar{C}_J{}^{KL}\ . \tag{D.15}$$

It is however instructive to also recall a second alternative approach which integrates (2.11) rather then (7.8).

Let us end this appendix by recalling the direct integration of (2.11). First note that the Riemann tensor on a special Kähler manifold is given by

$$R_{i\bar{j}l\bar{m}} = G_{i\bar{j}} G_{l\bar{m}} + G_{i\bar{m}} G_{l\bar{j}} - e^{2K} C_{ilp} \bar{C}_{\bar{j}\bar{m}\bar{p}} G^{p\bar{p}}\ . \tag{D.16}$$

The Ricci tensor takes the form

$$R_{i\bar{j}} = \partial_i \partial_{\bar{j}} \log\det G = G_{i\bar{j}}(h^{2,1}+1) - e^{2K} C_{ilp} \bar{C}_{\bar{j}\bar{m}\bar{p}} G^{l\bar{m}} G^{p\bar{p}}\ ., \tag{D.17}$$

such that

$$\frac{1}{2} e^{2K} G^{k\bar{k}} G^{l\bar{l}} C_{ikl} \bar{C}_{\bar{j}\bar{k}\bar{l}} = G_{i\bar{j}}(h^{2,1}+1) - \partial_i \partial_{\bar{j}} \log\det G\ . \tag{D.18}$$

Using this equation we solve (2.11) as[4]

$$F^{(1)} = -\frac{1}{2} \log\det G + \left(\frac{1}{2}(h^{2,1}+1) - \frac{\chi}{24} + 1\right) K + h(t) + \bar{h}(\bar{t}) \tag{D.19}$$

where $h(t)$ is a holomorphic function arising as integration constant. Now note that it follows from (A.18) that [14]

$$\det(2\mathrm{Im}\tau_{IJ}) = -\det(G_{i\bar{j}}) e^{-(h^{(2,1)}+1)K} |\det(\chi_i^I, X^I)|^{-2}\ . \tag{D.20}$$

[4]The derivative of the determinate of the matrix A is given by $\partial_x \det A = \det A \cdot A^{-1\,IJ} \partial_x A_{IJ}$

This equation can be used to rewrite $F^{(1)}$ as

$$F^{(1)} = -\frac{1}{2}\log\det(2\mathrm{Im}\tau_{IJ}) + \left(1-\frac{\chi}{24}\right)K + \frac{1}{2}\log\left(-|\det(\chi_i^I, X^I)|^{-2}\right) + h(t) + \bar{h}(\bar{t}) \quad \text{(D.21)}$$

One can evaluate the determinate of the coordinate change and shows [14]

$$|\det(\chi_i^I, X^I)|^{-2} = |X^0|^{-2(h^{2,1}+1)}|\det e_j^i|^{-2}\ . \quad \text{(D.22)}$$

where $e_j^i = \partial_{t^i}(X^i/X^0)$. But since X^0 and e_j^i are holomorphic in the coordinates t^i they can be absorbed into h such that (D.21) becomes (7.10).

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