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Two-loop amplitudes and master integrals for the production of a Higgs boson via a massive quark and a scalar-quark loop

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Abstract: We compute all two-loop master integrals which are required for the evaluation of next-to-leading order QCD corrections in Higgs boson production via gluon fusion. Many two-loop amplitudes for $2 \to 1$ processes in the Standard Model and beyond can be expressed in terms of these integrals using automated reduction techniques. These integrals also form a subset of the master integrals for more complicated $2 \to 2$ amplitudes with massive propagators in the loops. As a first application, we evaluate the two-loop amplitude for Higgs boson production in gluon fusion via a massive quark. Our result is the first independent check of the calculation of Spira, Djouadi, Graudenz and Zerwas. We also present for the first time the two-loop amplitude for $gg \to h$ via a massive squark.

1. Introduction

Next-to-leading (NLO) QCD corrections are important for a wide range of processes at the LHC. Recently, there have been new breakthroughs in developing efficient techniques for one-loop amplitudes, e.g. [1]. It is realistic that these theoretical advances will improve the data analysis for many interesting observables. However, processes which cannot occur with tree-level interactions at leading order, require two-loop rather than one-loop amplitudes at NLO.

Loop induced processes should be rather sensitive to new physics (see for example [2]). It is possible that we revisit their NLO corrections several times in the future, in order to include new types of particle interactions in the loops. In this paper, we study a basic LHC process of this kind: the production of a Higgs boson in gluon fusion.

The two-loop QCD corrections for $gg \to H$ in the minimal Standard Model and its two-Higgs-doublet extensions have already been computed in [3]. In this calculation, the mass effects of the quark coupled to the Higgs boson were also fully accounted for. The impact of the NLO correction is striking; for example, the SM Higgs boson production cross-section increases by more than 70%.

This calculation has never been verified in the literature, except in well known limits such as within the heavy top-quark approximation [4]. It is thus important to perform an independent computation. Our main objective, however, is to automatize the evaluation of the two-loop amplitude in QCD for this and other processes with similar distribution of massive particles in the loops. This is essential for future applications in gluon fusion and other processes for a variety of extensions of the Standard Model.

In 2001 Laporta introduced a new algorithm for an automated reduction of multi-loop integrals to master integrals [5]. A parallel rapid development of the Mellin-Barnes method [6–10], the differential equation method [11], and sector decomposition [12–14], yielded robust technology for computing master integrals. As a result, two-loop calculations for three and four point functions with more than one scale are now tractable.

In this paper we study the two-loop integrals which are required in gluon fusion processes. We apply the algorithm of Laporta, using the package AIR [15] and an independent MATHEMATICA implementation [16], to perform a reduction to master integrals. Some master integrals were already known in the literature [17–23]. We have recomputed them using the method of differential equations [11]. We present here for the first time the remaining master integrals, including the most complicated scalar cross-triangle. The same master integrals enter the evaluation of two-loop amplitudes of more complicated $2 \to 2$ processes, such as heavy quark pair production.

Our results are first given as an expansion in the dimension parameter $\epsilon = (4-d)/2$ in terms of harmonic polylogarithms. We present the series coefficients through the order where harmonic polylogarithms with transcendentality four appear. The polylogarithms are real valued in a non-physical kinematic region corresponding to imaginary center of mass energy $s < 0$. Then we perform explicitly the analytic continuation in the physical regions below and above the threshold corresponding to two on-shell heavy particles in the loops: $s = 4m_t^2$. For $s > 4m_t^2$ the analytic continuation proceeds as in [24,29]. For $s < 4m_t^2$, we find the analytic continuation of harmonic polylogarithms using the procedure described in [19]. Interestingly, the result for the master integrals is in general simpler than the analytic continuation for individual harmonic polylogarithms.

As a first application, we compute the two-loop amplitude for $gg \to h$ in the Standard Model. We have compared our result with the expression in [25], in the non-physical region. The result of [25] was derived by series expanding the integral representation of the two-loop amplitude from [3] in a kinematic variable, and mapping the expansion to a carefully chosen ansatz. Our result agrees with [25]. This is the first independent check of the two-loop amplitude in [3].

We present here a new result for the two-loop amplitude $gg \to h$ via scalar-quarks. In the heavy squark limit, our result agrees with [27]. With a completed setup for the reduction procedure and the expressions for the master integrals, the evaluation of this new result is fully automated. Other amplitudes with different particle content in the loops can be obtained easily. We note that preliminary numerical results for the NLO K-factor for the squark case have been presented in [28].

2. Reduction of the amplitudes

We consider QCD virtual corrections to the process

$$g + g \to h \tag{2.1}$$

with just one massive particle running in the loops. Some typical Feynman diagrams are shown in Figure 1 for the cases of a heavy quark and a squark in the loop.

It is convenient to project the two-loop amplitudes onto scalar form factors. In this way we are left only with loop integrals involving scalar products in the numerator. The scalar integrals can be classified into topologies according to their denominators. In $gg \to h$ we find diagrams with at most six propagators, of which five can be massive. In the numerators, however, one can find seven independent scalar products. The irreducible scalar product can be dealt with by introducing an additional propagator, which is raised to negative powers in the expressions for the physical amplitudes. After the introduction of the auxiliary propagators, all scalar integrals belong to subtopologies of the three topologies, shown in Figure 2, with denominators:

$$\begin{array}{lll}
\mathrm{TP}_1 & \mathrm{TP}_2 & \mathrm{TP}_3 \\
D_{11} = k^2 & D_{21} = k^2 - m_t^2 & D_{31} = k^2 - m_t^2 \\
D_{12} = (k+p_1)^2 & D_{22} = (k+p_2)^2 - m_t^2 & D_{32} = (k-l-p_1)^2 \\
D_{13} = (k+p_{12})^2 & D_{23} = (k+p_{12})^2 - m_t^2 & D_{33} = (k+p_{12})^2 - m_t^2 \\
D_{14} = (l+p_{12})^2 - m_t^2 & D_{24} = (l+p_{12})^2 - m_t^2 & D_{34} = (l+p_{12})^2 - m_t^2 \\
D_{15} = (l+p_1)^2 - m_t^2 & D_{25} = (l+p_2)^2 - m_t^2 & D_{35} = (l+p_1)^2 - m_t^2 \\
D_{16} = l^2 - m_t^2 & D_{26} = l^2 - m_t^2 & D_{36} = (k+p_1)^2 - m_t^2 \\
D_{17} = (k-l)^2 - m_t^2 & D_{27} = (k-l)^2 & D_{37} = (k-l)^2\,,
\end{array} \tag{2.2}$$

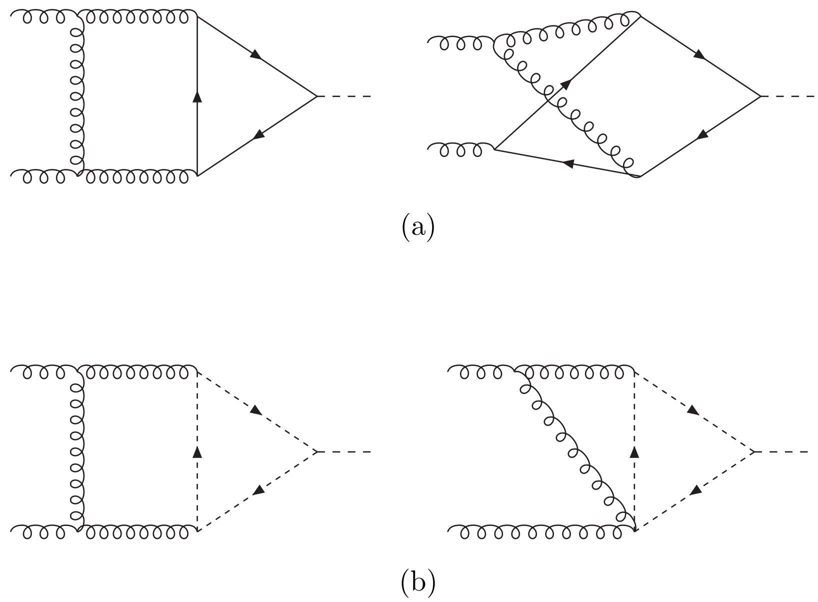


Figure 1: Typical Feynman diagrams in the two loop contributions to $gg \to H$ with (a) a heavy fermion in the loop, (b) a heavy scalar in the loop .

where $p_{12} = p_1 + p_2$, m_t is the mass of the particle running in the loops and k and l are the loop momenta.

The reduction to master integrals is done using integration by part identities [30, 31] combined with the Laporta algorithm [5] in [15, 16]. We found 17 master integrals, which are shown in Figure 3. It is possible to choose a different basis of master integrals; the basis we choose is particularly convenient for the method of differential equations.

The master integrals in the first two lines of Figure 3 are products of known one-loop integrals [17, 19]. The master integrals in the third, fourth and fifth line in Figure 3 are non-factorizable. Integrals in the third and fourth line were calculated already in [18][1] and [19, 21, 22]. respectively. The double triangle, last diagram in the third line was calculated in [21–23]. Also the six propagators triangle - third diagram in the last line of Figure 3 - has been calculated in [20].

3. Master integrals

We computed all master integrals using the differential equation method [11, 32–36]. The natural variable to express the results is

$$x = \frac{\sqrt{1-\tau}-1}{\sqrt{1-\tau}+1} + i\varepsilon \qquad \text{where} \qquad \tau = \frac{4{m_t}^2}{s}\,, \tag{3.1}$$

[1]Our results fully agree with the results quoted in this reference taken from the electronic file in `http://pheno.physik.uni-freiburg.de/bonciani/`. The printed version contains several typographical mistakes.

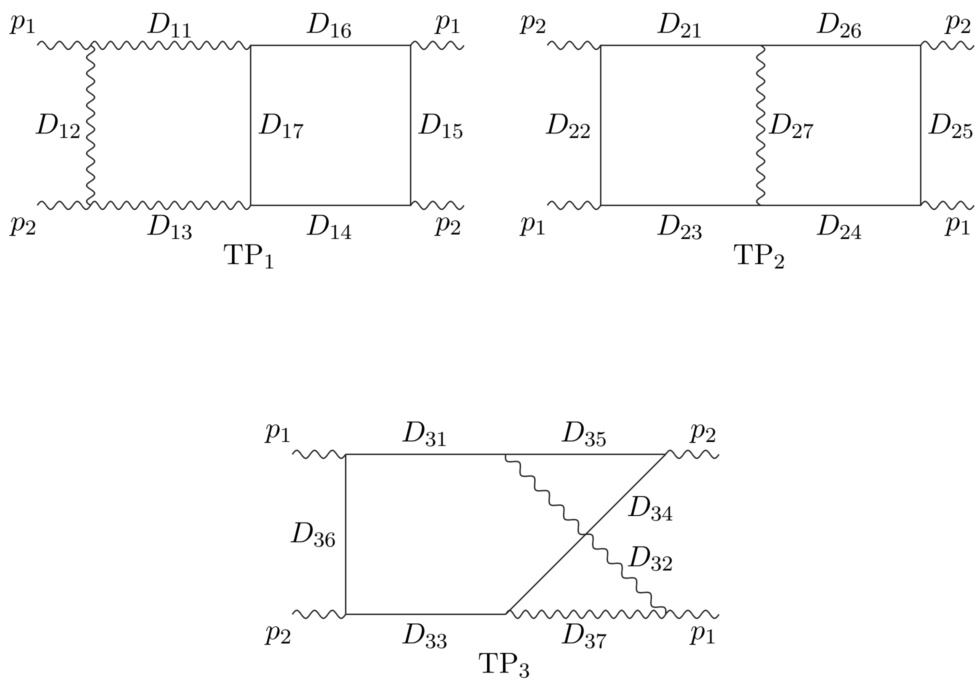


Figure 2: Master topologies. Wavy lines denote massless particles both external and internal. Internal massive lines are denoted by single straight lines whereas the double line denotes a massive external particle.

with $s = (p_1 + p_2)^2$. The variable x is real valued in the space-like region ($s < 0$) and in the physical region above threshold ($s > 4\,m_t^2$). Below threshold x lies on the unit circle in the complex plane. In that region, we introduce the variable θ such that $x = e^{i\theta}$. For quick reference, in Table 1 we list the domain of each variable in the different kinematical regions.

Region	s	τ	x
space-like	$-\infty < s < 0$	$0 > \tau > -\infty$	$0 < x < 1$
below threshold	$0 < s < 4{m_t}^2$	$\infty > \tau > 1$	$x = e^{i\theta}$ with $0 < \theta < \pi$
above threshold	$4{m_t}^2 < s < \infty$	$1 > \tau > 0$	$-1 < x < 0$

Table 1: Domain spanned by the variables in the different kinematical regions

The dependence on x of the master integrals is determined by solving the associated differential equations. These are obtained by taking the derivative with respect to x of the loop integrals and exchanging the order of the differential operator and the loop integration. In this way, the derivative of a given master integral is expressed in terms of scalar integrals which can again be reduced to masters. Applying this procedure to all integrals in our master basis, we derive a closed system of differential equations.

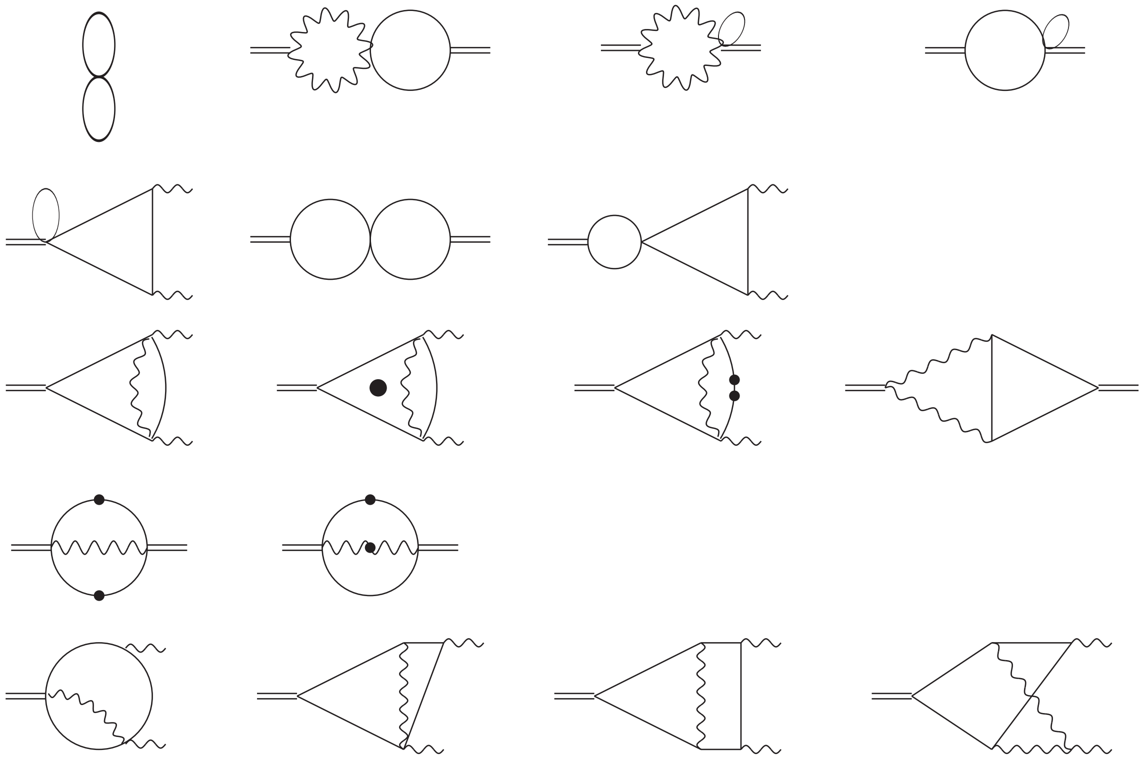
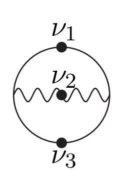


Figure 3: Set of master integrals. The conventions for the lines are as in Figure 2. Each dot on a propagator line denotes an additional power of the propagator in the denominator. The diagram with a big dot contains a numerator, it is defined in the following section.

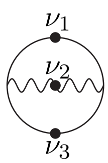
We solve the differential equations order by order in powers of ϵ. Only integrations with kernels $1/x$, $1/(1-x)$ and $1/(1+x)$ are required, and the solutions can be written exclusively in terms of harmonic polylogarithms [29].

To fully determine the solution of the differential equations, we require the value of the master integrals at a certain kinematic point. The value at $x = 1$ is very easy to obtain. This limit corresponds to setting the external momenta to zero ($s = 0$). All non-factorizable master integrals $\mathrm{MI}^{(\mathrm{NF})}$ collapse to a vacuum sunset diagram with extra powers of propagators:

$$\lim_{x\to 1} \mathrm{MI}^{(\mathrm{NF})} = \qquad (3.2)$$



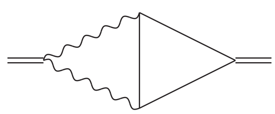
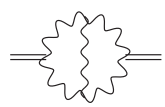
The exponent ν_2 corresponds to the number of massless propagators in the integral whereas $\nu_1 + \nu_3$ is the number of massive propagators. One can easily compute:



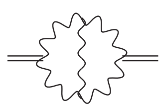
$$= (-1)^{\nu_{123}}\, {m_t}^{2\,(d-\nu_{123})} \frac{\Gamma\left(\nu_{123}-d\right)\Gamma\left(\nu_{12}-\frac{d}{2}\right)\Gamma\left(\nu_{23}-\frac{d}{2}\right)\Gamma\left(\frac{d}{2}-\nu_2\right)}{\Gamma(\nu_1)\Gamma(\nu_3)\Gamma\left(\frac{d}{2}\right)\Gamma\left(\nu_{13}+2\,\nu_2-d\right)}\ . \tag{3.3}$$

We have observed that one could fix the solution of the differential equations by requiring simply that the $x \to 1$ limit is finite, since the homogeneous solutions usually diverge at $x = 1$. The explicit formula for the limit $x = 1$ in Eq. 3.3 was then an additional consistency check of our calculation.

In one master integral, the $x = 1$ limit does not commute with the expansion around $\epsilon = 0$, due to a collinear singularity as s vanishes. For this master integral, we have used the massless limit $x \to 0$, which is well behaved:

$$\lim_{x\to 0}$$  $$=$$  $$, \tag{3.4}$$

with



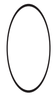
$$= (-s)^{-1-2\,\epsilon}\left(\frac{\Gamma(1+\epsilon)}{(1-\epsilon)}\right)^2\left(-6\,\zeta(3)-\epsilon\,\frac{\pi^4}{10}+\mathcal{O}\left(\epsilon^2\right)\right) \tag{3.5}$$

In the following sections we shall present our results in the space-like and $0 < s < 4m_t^2$ regions, and describe the analytic continuation procedure. We have performed several checks on our results for the master integrals. We have verified that our expressions satisfy the differential equations before and after analytic continuation. We have also computed all master integrals numerically from their Feynman parameterization, by using sector decomposition [12–14]. Our analytic expressions agree fully with the direct numerical evaluation.

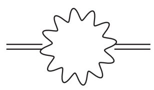
4. Results in the region $s < 0$

We now present the results for the master integrals in the space-like region. We give the definition of the master integrals in terms of the propagators listed in section 2, and their ϵ-expansion in terms of harmonic polylogarithms.

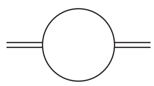
4.1 One loop integrals



$$= \int \frac{d^dk}{i\pi^{d/2}} \frac{1}{k^2 - m_t^2 + i\varepsilon} = \frac{\Gamma(1+\epsilon)}{1-\epsilon}\,(m_t^2)^{-\epsilon+1}\cdot\frac{1}{\epsilon} \tag{4.1}$$



$$= \int \frac{d^dk}{i\pi^{d/2}} \frac{1}{D_{11}D_{13}} = \frac{\Gamma(1+\epsilon)}{1-\epsilon}\,(m_t^2)^{-\epsilon} \cdot \frac{1}{\epsilon}\frac{\Gamma(2-\epsilon)\Gamma(1-\epsilon)}{\Gamma(2-2\epsilon)} \left(\frac{(1-x)^2}{x} - i\varepsilon\right)^{-\epsilon} \tag{4.2}$$



$$= \int \frac{d^dk}{i\pi^{d/2}} \frac{1}{D_{21}D_{23}} = \frac{\Gamma(1+\epsilon)}{1-\epsilon}\,(m_t^2)^{-\epsilon} \sum_{i=-1}^{3} \epsilon^i F^i_{\rm mbub}(x) + \mathcal{O}(\epsilon^4) \tag{4.3}$$

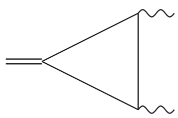
$$F^{-1}_{\rm mbub}(x) = 1 \tag{4.4}$$

$$F^{0}_{\rm mbub}(x) = \frac{1}{1-x}\Big\{ -x + (x+1)H(0;x) + 1\Big\} \tag{4.5}$$

$$\begin{aligned} F^{1}_{\rm mbub}(x) = {} & \frac{1}{1-x}\Big\{\frac{1}{6}\left(-\pi^2 x - 12x - \pi^2 + 12\right) + (x+1)H(0;x) - 2(x+1)H(-1,0;x) \\ & + (x+1)H(0,0;x)\Big\} \end{aligned} \tag{4.6}$$

$$\begin{aligned} F^{2}_{\rm mbub}(x) = {} & \frac{1}{1-x}\Big\{ -\frac{1}{6}\pi^2(x+1) - 2((2+\zeta(3))x + \zeta(3) - 2) \\ & + \frac{1}{3}\pi^2(x+1)H(-1;x) - \frac{1}{6}\left(-12+\pi^2\right)(x+1)H(0;x) \\ & - 2(x+1)H(0,-1,0;x) - 2(x+1)H(-1,0;x) + (x+1)H(0,0;x) \\ & + 4(x+1)H(-1,-1,0;x) - 2(x+1)H(-1,0,0;x) + (x+1)H(0,0,0;x)\Big\} \end{aligned} \tag{4.7}$$

$$\begin{aligned} F^{3}_{\rm mbub}(x) = {} & \frac{1}{1-x}\Big\{ -\frac{1}{40}\pi^4(x+1) - \frac{1}{3}\pi^2(x+1) - 2((4+\zeta(3))x + \zeta(3) - 4) \\ & + \frac{1}{3}\pi^2(x+1)H(0,-1;x) + \frac{1}{3}(x+1)H(-1;x)\left(\pi^2 + 12\zeta(3)\right) \\ & - \frac{1}{6}(x+1)\left(\pi^2 + 12(-2+\zeta(3))\right)H(0;x) - 2(x+1)H(0,0,-1,0;x) \\ & - 2(x+1)H(0,-1,0;x) - \frac{2}{3}\pi^2(x+1)H(-1,-1;x) \\ & + \frac{1}{3}\left(-12+\pi^2\right)(x+1)H(-1,0;x) - \frac{1}{6}\left(-12+\pi^2\right)(x+1)H(0,0;x) \\ & + 4(x+1)H(0,-1,-1,0;x) - 2(x+1)H(0,-1,0,0;x) + 4(x+1)H(-1,0,-1,0;x) \\ & + 4(x+1)H(-1,-1,0;x) - 2(x+1)H(-1,0,0;x) + (x+1)H(0,0,0;x) \\ & + 4(x+1)H(-1,-1,0,0;x) - 8(x+1)H(-1,-1,-1,0;x) \\ & + (x+1)H(0,0,0,0;x) - 2(x+1)H(-1,0,0,0;x)\Big\} \end{aligned} \tag{4.8}$$



$$= \int \frac{d^dk}{i\pi^{d/2}} \frac{1}{D_{21}D_{22}D_{23}} = \frac{\Gamma(1+\epsilon)}{1-\epsilon} \, (m_t^2)^{-\epsilon-1} \sum_{i=0}^{2} \epsilon^i F^i_{\rm mtri}(x) + \mathcal{O}(\epsilon^3) \tag{4.9}$$

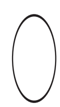
$$F^0_{\rm mtri}(x) = -\frac{xH(0,0;x)}{(x-1)^2} \tag{4.10}$$

$$F^1_{\rm mtri}(x) = \frac{x}{(1-x)^2}\Big\{\frac{1}{6}\pi^2 H(0;x) + 2H(0,-1,0;x) + H(0,0;x) + 3\zeta(3) - H(0,0,0;x)\Big\} \tag{4.11}$$

$$F^2_{\rm mtri}(x) = \frac{x}{(1-x)^2}\Big\{ -\frac{1}{3}\pi^2 H(0,-1;x) + H(0;x)\left(-\frac{\pi^2}{6} + 2\zeta(3)\right) - 3\zeta(3) + \frac{\pi^4}{72} + 2H(0,0,-1,0;x) - 2H(0,-1,0;x) + \frac{1}{6}\pi^2 H(0,0;x) - 4H(0,-1,-1,0;x) + 2H(0,-1,0,0;x) + H(0,0,0;x) - H(0,0,0,0;x)\Big\} \tag{4.12}$$

4.2 Factorizable integrals

$$= \int \frac{d^dk}{i\pi^{d/2}} \int \frac{d^dl}{i\pi^{d/2}} \frac{1}{D_{15}D_{17}} =$$



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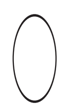

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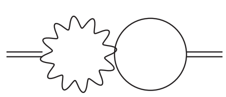
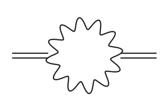


$$= \int \frac{d^dk}{i\pi^{d/2}} \int \frac{d^dl}{i\pi^{d/2}} \frac{1}{D_{11}D_{13}D_{14}D_{16}} =$$



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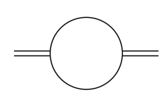

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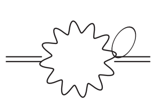
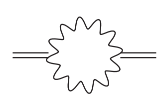


$$= \int \frac{d^dk}{i\pi^{d/2}} \int \frac{d^dl}{i\pi^{d/2}} \frac{1}{D_{11}D_{13}D_{16}} =$$



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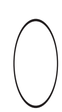

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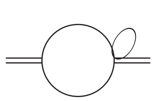
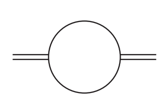


$$= \int \frac{d^dk}{i\pi^{d/2}} \int \frac{d^dl}{i\pi^{d/2}} \frac{1}{D_{14}D_{16}D_{17}} =$$



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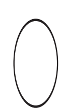
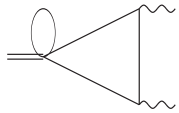
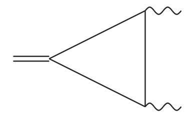
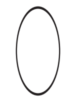
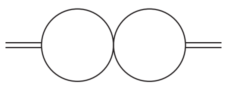


(4.16)

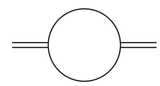
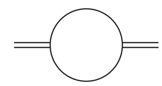


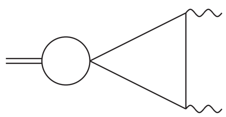
$$= \int \frac{d^dk}{i\pi^{d/2}} \int \frac{d^dl}{i\pi^{d/2}} \frac{1}{D_{14}D_{15}D_{16}D_{17}} = $$

 $\times$  (4.17)

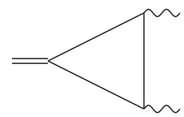
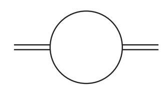


$$= \int \frac{d^dk}{i\pi^{d/2}} \int \frac{d^dl}{i\pi^{d/2}} \frac{1}{D_{21}D_{23}D_{24}D_{26}} = $$

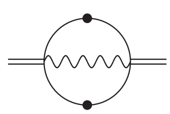
 $\times$  (4.18)



$$= \int \frac{d^dk}{i\pi^{d/2}} \int \frac{d^dl}{i\pi^{d/2}} \frac{1}{D_{21}D_{23}D_{24}D_{25}D_{26}} = $$

 $\times$  (4.19)

4.3 Three propagator integrals



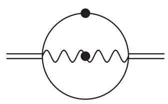
$$\begin{aligned} &= \int \frac{d^dk}{i\pi^{d/2}} \int \frac{d^dl}{i\pi^{d/2}} \frac{1}{D_{11}D_{14}^2D_{17}^2} \\ &= \left(\frac{\Gamma(1+\epsilon)}{1-\epsilon}\right)^2 (m_t^2)^{-2\epsilon-1} \sum_{i=0}^{2} \epsilon^i F_1^i(x) + \mathcal{O}(\epsilon^3) \end{aligned} \tag{4.20}$$

$$F_1^0(x) = -\frac{2xH(0,0;x)}{(x-1)^2} \tag{4.21}$$

$$\begin{aligned} F_1^1(x) = \frac{x}{(1-x)^2}\Big\{ &\frac{1}{3}\pi^2 H(0;x) + 12H(0,-1,0;x) + 4H(0,0;x) + 6\zeta(3) \\ &- 4H(0,1,0;x) - 6H(0,0,0;x) + 4H(1,0,0;x)\Big\} \end{aligned} \tag{4.22}$$

$$\begin{aligned} F_1^2(x) = \frac{x}{(1-x)^2}\Big\{ &-2\pi^2 H(0,-1;x) - 12\zeta(3) + \frac{13\pi^4}{180} \\ &+ \left(-\frac{2\pi^2}{3} + 16\zeta(3)\right) H(0;x) + \frac{2}{3}\pi^2 H(0,1;x) - 12H(1;x)\zeta(3) \\ &+ 36H(0,0,-1,0;x) - 24H(0,-1,0;x) + \left(-2+\pi^2\right) H(0,0;x) - \frac{2}{3}\pi^2 H(1,0;x) \\ &+ 8H(0,1,0;x) - 12H(0,0,1,0;x) - 72H(0,-1,-1,0;x) + 48H(0,-1,0,0;x) \end{aligned}$$

$$+ 24H(0,-1,1,0;x) + 12H(0,0,0;x) - 24H(1,0,-1,0;x) - 8H(1,0,0;x)$$
$$+ 8H(1,0,1,0;x) + 24H(0,1,-1,0;x) - 20H(0,1,0,0;x) - 8H(0,1,1,0;x)$$
$$- 14H(0,0,0,0;x) + 12H(1,0,0,0;x) - 8H(1,1,0,0;x)\Big\} \quad (4.23)$$



$$= \int \frac{d^dk}{i\pi^{d/2}} \int \frac{d^dl}{i\pi^{d/2}} \frac{1}{D_{11}^2 D_{14}^2 D_{17}}$$
$$= \left(\frac{\Gamma(1+\epsilon)}{1-\epsilon}\right)^2 (m_t^2)^{-2\epsilon-1} \sum_{i=-1}^{2} \epsilon^i F_2^i(x) + \mathcal{O}(\epsilon^3) \quad (4.24)$$

$$F_2^{-1}(x) = \frac{xH(0;x)}{x^2-1} \quad (4.25)$$

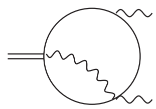
$$F_2^0(x) = \frac{x}{(x-1)^3(x+1)}\Big\{ -2H(0;x)(x-1)^2 - \frac{1}{6}\pi^2(x-1)^2$$
$$+ (x-1)(5x-3)H(0,0;x) - 6(x-1)^2 H(-1,0;x)$$
$$+ 2(x-1)^2 H(1,0;x)\Big\} \quad (4.26)$$

$$F_2^1(x) = \frac{x}{(x-1)^3(x+1)}\Big\{\frac{1}{3}(x-1)\left(6\zeta(3)(4-7x) + \pi^2(x-1)\right)$$
$$+ \pi^2(x-1)^2 H(-1;x) - \frac{1}{6}(x-1)\left(-6x + \pi^2(5x-3) + 6\right) H(0;x)$$
$$- \frac{1}{3}\pi^2 H(1;x)(x-1)^2 - 6(5x-3)H(0,-1,0;x)(x-1)$$
$$+ 12(x-1)^2 H(-1,0;x) - 2(x-1)(5x-3)H(0,0;x)$$
$$+ 2(x-1)(5x-3)H(0,1,0;x) - 4(x-1)^2 H(1,0;x)$$
$$+ 36(x-1)^2 H(-1,-1,0;x) - 24(x-1)^2 H(-1,0,0;x)$$
$$+ (x-1)(13x-7)H(0,0,0;x) - 12(x-1)^2 H(-1,1,0;x)$$
$$+ 2(x-1)(3x-5)H(1,0,0;x) - 12(x-1)^2 H(1,-1,0;x)$$
$$+ 4(x-1)^2 H(1,1,0;x)\Big\} \quad (4.27)$$

$$F_2^2(x) = \frac{x}{(x-1)^3(x+1)}\Big\{ -\frac{1}{360}(x-1)\left(60\pi^2(x-1) + \pi^4(61x-35) - 1440(7x-4)\zeta(3)\right)$$
$$+ \pi^2(x-1)(5x-3)H(0,-1;x) - 2(x-1)^2 H(-1;x)\left(\pi^2 - 33\zeta(3)\right)$$
$$+ \frac{1}{3}(x-1)H(0;x)\left(6\zeta(3)(9-17x) + \pi^2(5x-3)\right)$$

$$
\begin{aligned}
&+\frac{2}{3}(x-1)H(1;x)\left(6\zeta(3)(7-4x)+\pi^2(x-1)\right)\\
&-\frac{1}{3}\pi^2(x-1)(5x-3)H(0,1;x)-6(x-1)(13x-7)H(0,0,-1,0;x)\\
&+12(x-1)(5x-3)H(0,-1,0;x)-6\pi^2(x-1)^2H(-1,-1;x)\\
&+2\left(-3+2\pi^2\right)H(-1,0;x)(x-1)^2+2\pi^2H(-1,1;x)(x-1)^2\\
&-\frac{1}{6}(x-1)\left(-30x+\pi^2(13x-7)+18\right)H(0,0;x)\\
&+2\pi^2(x-1)^2H(1,-1;x)-\frac{1}{3}(x-1)\left(-6x+\pi^2(3x-5)+6\right)H(1,0;x)\\
&-\frac{2}{3}\pi^2H(1,1;x)(x-1)^2-4(5x-3)H(0,1,0;x)(x-1)\\
&+2(x-1)(13x-7)H(0,0,1,0;x)+36(x-1)(5x-3)H(0,-1,-1,0;x)\\
&-24(x-1)(5x-3)H(0,-1,0,0;x)-12(x-1)(5x-3)H(0,-1,1,0;x)\\
&+144(x-1)^2H(-1,0,-1,0;x)-72(x-1)^2H(-1,-1,0;x)\\
&+48H(-1,0,0;x)(x-1)^2+24H(-1,1,0;x)(x-1)^2\\
&-48H(-1,0,1,0;x)(x-1)^2-2(13x-7)H(0,0,0;x)(x-1)\\
&+24(x-1)^2H(1,-1,0;x)-12(x-1)(3x-5)H(1,0,-1,0;x)\\
&-8H(1,1,0;x)(x-1)^2-4(3x-5)H(1,0,0;x)(x-1)\\
&+4(x-1)(3x-5)H(1,0,1,0;x)-12(x-1)(5x-3)H(0,1,-1,0;x)\\
&+2(x-1)(27x-17)H(0,1,0,0;x)+4(x-1)(5x-3)H(0,1,1,0;x)\\
&+144(x-1)^2H(-1,-1,0,0;x)-216(x-1)^2H(-1,-1,-1,0;x)\\
&+72(x-1)^2H(-1,-1,1,0;x)-60(x-1)^2H(-1,0,0,0;x)\\
&+72(x-1)^2H(-1,1,-1,0;x)-48(x-1)^2H(-1,1,0,0;x)\\
&+(x-1)(29x-15)H(0,0,0,0;x)-24(x-1)^2H(-1,1,1,0;x)\\
&+72(x-1)^2H(1,-1,-1,0;x)-48(x-1)^2H(1,-1,0,0;x)\\
&+2(x-1)(7x-13)H(1,0,0,0;x)-24(x-1)^2H(1,-1,1,0;x)\\
&+4(x-1)(5x-3)H(1,1,0,0;x)-24(x-1)^2H(1,1,-1,0;x)\\
&+8(x-1)^2H(1,1,1,0;x)\Big\}
\end{aligned}
\tag{4.28}
$$

4.4 Four propagator integrals



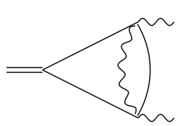
$$= \int \frac{d^dk}{i\pi^{d/2}} \int \frac{d^dl}{i\pi^{d/2}} \frac{1}{D_{11}D_{14}D_{15}D_{17}}$$
$$= \left(\frac{\Gamma(1+\epsilon)}{1-\epsilon}\right)^2 (m_t^2)^{-2\epsilon} \sum_{i=-2}^{1} \epsilon^i F_3^i(x) + \mathcal{O}(\epsilon^2) \tag{4.29}$$

$$F_3^{-2}(x) = \frac{1}{2} \tag{4.30}$$

$$F_3^{-1}(x) = -\frac{1}{2} \tag{4.31}$$

$$\begin{aligned} F_3^0(x) = \; & \frac{1}{(1-x)^2}\Big\{ -3x^2 + (6-4\zeta(3))x + 2\left(x^2-1\right) H(0;x) - 3 \\ & - H(0,0;x)(x-1)^2 + 2xH(0,0,0;x) + 4xH(1,0,0;x)\Big\} \end{aligned} \tag{4.32}$$

$$\begin{aligned} F_3^1(x) = \; & \frac{1}{(1-x)^2}\Big\{3(-4+\zeta(3))x^2 - 2(-12+\zeta(3))x + \frac{2\pi^4 x}{45} - \frac{1}{3}\pi^2\left(x^2-1\right) + 3(-4+\zeta(3)) \\ & + \frac{1}{6}H(0;x)\left(\pi^2(x-1)^2 + 12\left(4x^2 - 3\zeta(3)x - 4\right)\right) - 12xH(1;x)\zeta(3) \\ & + 6H(0,-1,0;x)(x-1)^2 - 12xH(0,0,-1,0;x) - 12\left(x^2-1\right)H(-1,0;x) \\ & + \left(11x^2 - \frac{1}{3}\left(6+\pi^2\right)x - 5\right)H(0,0;x) + \left(4x^2 - \frac{2\pi^2 x}{3} - 4\right)H(1,0;x) \\ & - 2H(0,1,0;x)(x-1)^2 + 4xH(0,0,1,0;x) + \left(-3x^2+4x-3\right)H(0,0,0;x) \\ & - 24xH(1,0,-1,0;x) + 2\left(x^2-4x+1\right)H(1,0,0;x) + 8xH(1,0,1,0;x) \\ & - 4xH(0,1,0,0;x) + 6xH(0,0,0,0;x) + 12xH(1,0,0,0;x) \\ & - 8xH(1,1,0,0;x)\Big\} \end{aligned} \tag{4.33}$$



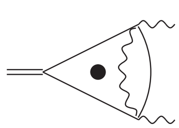
$$= \int \frac{d^dk}{i\pi^{d/2}} \int \frac{d^dl}{i\pi^{d/2}} \frac{1}{D_{12}D_{14}D_{16}D_{17}}$$
$$= \left(\frac{\Gamma(1+\epsilon)}{1-\epsilon}\right)^2 (m_t^2)^{-2\epsilon} \sum_{i=-2}^{1} \epsilon^i F_4^i(x) + \mathcal{O}(\epsilon^2) \tag{4.34}$$

$$F_4^{-2}(x) = \frac{1}{2} \tag{4.35}$$

$$F_4^{-1}(x) = \frac{1}{(1-x)^2}\Big\{\frac{3}{2}(x-1)^2 + \left(1-x^2\right) H(0;x)\Big\} \tag{4.36}$$

$$\begin{aligned} F_4^{0}(x) = \; & \frac{1}{(1-x)^2}\Big\{5x^2 + 2(-5+2\zeta(3))x + \left(-3x^2 + \frac{\pi^2 x}{3} + 3\right) H(0;x) + 5 \\ & + 2\left(x^2-1\right) H(-1,0;x) - (x-1)(x+2)H(0,0;x) + \left(1-x^2\right) H(1,0;x) \\ & + 2xH(0,1,0;x) + xH(0,0,0;x)\Big\} \end{aligned} \tag{4.37}$$

$$\begin{aligned} F_4^{1}(x) = \; & \frac{1}{(1-x)^2}\Big\{ -(-16+\zeta(3))x^2 + \left(-32 - \frac{11\pi^4}{90} + \zeta(3)\right) x - 4(-4+\zeta(3)) \\ & + H(0;x)\left(-10x^2 - \frac{1}{6}\pi^2(x+1)x - 3\zeta(3)x + 10\right) - \frac{2}{3}\pi^2 x H(0,-1;x) \\ & + \frac{1}{3}\pi^2 x H(0,1;x) + \frac{1}{6}H(1;x)\left(-48\zeta(3)x - \pi^2\left(x^2-1\right)\right) \\ & - 2xH(0,0,-1,0;x) + 2(x-1)(x+2)H(0,-1,0;x) + 6\left(x^2-1\right) H(-1,0;x) \\ & + \left(-3x^2 + \frac{1}{6}\left(-18+\pi^2\right)x + 6\right) H(0,0;x) + \left(-3x^2 - \frac{2\pi^2 x}{3} + 3\right) H(1,0;x) \\ & - 2\left(x^2+x-1\right) H(0,1,0;x) + 2xH(0,0,1,0;x) - 2xH(0,-1,0,0;x) \\ & - 4xH(0,-1,1,0;x) + \left(4-4x^2\right) H(-1,-1,0;x) + 3\left(x^2-1\right) H(-1,0,0;x) \\ & + 2\left(x^2-1\right) H(-1,1,0;x) + \left(-2x^2-3x+4\right) H(0,0,0;x) \\ & + 2\left(x^2-1\right) H(1,-1,0;x) + \left(-x^2-2x+3\right) H(1,0,0;x) \\ & + \left(2-2x^2\right) H(1,1,0;x) - 4xH(1,0,1,0;x) - 4xH(0,1,-1,0;x) \\ & + 4xH(0,1,0,0;x) + 4xH(0,1,1,0;x) + 3xH(0,0,0,0;x) \\ & - 2xH(1,0,0,0;x)\Big\} \end{aligned} \tag{4.38}$$



$$\begin{aligned} & = \int \frac{d^d k}{i\pi^{d/2}} \int \frac{d^d l}{i\pi^{d/2}} \frac{(k+p_1)\cdot(l-k)}{D_{12}D_{14}D_{16}D_{17}} \\ & = \left(\frac{\Gamma(1+\epsilon)}{1-\epsilon}\right)^2 (m_t^2)^{1-2\epsilon} \sum_{i=-2}^{1} \epsilon^i F_5^i(x) + \mathcal{O}(\epsilon^2) \end{aligned} \tag{4.39}$$

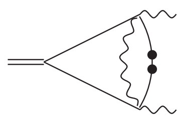
$$F_5^{-2}(x) = \frac{(x-1)^2}{8x} \tag{4.40}$$

$$F_5^{-1}(x) = \frac{1}{(1-x)^2 x}\Big\{\frac{5}{16}(x-1)^4 + \left(-\frac{x^4}{4}+x^3-x+\frac{1}{4}\right)H(0;x)\Big\} \tag{4.41}$$

$$\begin{aligned}
F_5^{0}(x) = \; & \frac{1}{(1-x)^2 x}\Big\{\frac{1}{32}\left(31x^4-140x^3+(218-64\zeta(3))x^2-140x+31\right) \\
& + \frac{1}{24}\left(-15x^4+54x^3-4\pi^2x^2-54x+15\right)H(0;x) \\
& + \frac{1}{2}\left(x^4-4x^3+4x-1\right)H(-1,0;x) \\
& + \frac{1}{4}\left(-x^4+4x^3+3x^2-8x+2\right)H(0,0;x) + \left(-\frac{x^4}{4}+x^3-x+\frac{1}{4}\right)H(1,0;x) \\
& - H(0,1,0;x)x^2 - \frac{1}{2}H(0,0,0;x)x^2\Big\}
\end{aligned} \tag{4.42}$$

$$\begin{aligned}
F_5^{1}(x) = \; & \frac{1}{(1-x)^2 x}\Big\{\left(\frac{189}{64}-\frac{\zeta(3)}{4}\right)x^4 + \left(-\frac{233}{16}+\frac{\pi^2}{24}+\zeta(3)\right)x^3 + \left(\frac{743}{32}+\frac{11\pi^4}{180}-\frac{3\zeta(3)}{4}\right)x^2 \\
& - \frac{1}{48}\left(699+2\pi^2-192\zeta(3)\right)x - \zeta(3) + \frac{189}{64} + \frac{1}{3}\pi^2x^2H(0,-1;x) \\
& + \frac{1}{48}H(0;x)\left(-\left(93+2\pi^2\right)x^4+\left(342+8\pi^2\right)x^3+6\left(\pi^2+12\zeta(3)\right)x^2-342x+93\right) \\
& + \frac{1}{24}H(1;x)\left(96x^2\zeta(3)-\pi^2\left(x^4-4x^3+4x-1\right)\right) \\
& + x^2H(0,0,-1,0;x) - \frac{1}{6}\pi^2x^2H(0,1;x) \\
& + \frac{1}{2}\left(x^4-4x^3-3x^2+8x-2\right)H(0,-1,0;x) \\
& + \frac{1}{4}\left(5x^4-18x^3+18x-5\right)H(-1,0;x) \\
& + \left(-\frac{5x^4}{8}+2x^3-\frac{1}{24}\left(-57+2\pi^2\right)x^2-5x+\frac{5}{4}\right)H(0,0;x) \\
& + \frac{1}{24}\left(-15x^4+60x^3+8\pi^2x^2-60x+15\right)H(1,0;x) \\
& + \frac{1}{2}\left(-x^4+4x^3+3x^2-4x+1\right)H(0,1,0;x) - x^2H(0,0,1,0;x) \\
& + H(0,-1,0,0;x)x^2 + 2H(0,-1,1,0;x)x^2 + \left(-x^4+4x^3-4x+1\right)H(-1,-1,0;x) \\
& + \frac{3}{4}\left(x^4-4x^3+4x-1\right)H(-1,0,0;x) \\
& + \frac{1}{2}\left(x^4-4x^3+4x-1\right)H(-1,1,0;x) \\
& + \left(-\frac{x^4}{2}+2x^3+\frac{9x^2}{4}-4x+1\right)H(0,0,0;x) \\
& + \frac{1}{2}\left(x^4-4x^3+4x-1\right)H(1,-1,0;x)
\end{aligned}$$

$$+\left(-\frac{x^4}{4}+x^3+\frac{3x^2}{2}-3x+\frac{3}{4}\right)H(1,0,0;x)$$
$$+2H(1,0,1,0;x)x^2+\frac{1}{2}\left(-x^4+4x^3-4x+1\right)H(1,1,0;x)$$
$$+2H(0,1,-1,0;x)x^2-2H(0,1,0,0;x)x^2-2H(0,1,1,0;x)x^2$$
$$+x^2H(1,0,0,0;x)-\frac{3}{2}x^2H(0,0,0,0;x)\Big\} \tag{4.43}$$



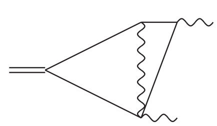
$$= \int\frac{d^dk}{i\pi^{d/2}}\int\frac{d^dl}{i\pi^{d/2}}\frac{1}{D_{12}D_{14}D_{16}D_{17}^3}$$
$$= \left(\frac{\Gamma(1+\epsilon)}{1-\epsilon}\right)^2 (m_t^2)^{-2\epsilon-2}\sum_{i=0}^{2}\epsilon^i F_6^i(x)+\mathcal{O}(\epsilon^3) \tag{4.44}$$

$$F_6^0(x) = \frac{xH(0,0;x)}{2(x-1)^2} \tag{4.45}$$

$$F_6^1(x) = \frac{x}{(1-x)^2}\Big\{-\frac{1}{4}\pi^2H(0;x)-H(0,-1,0;x)-H(0,0;x)-\frac{9\zeta(3)}{2}$$
$$-H(0,1,0;x)+\frac{1}{2}H(0,0,0;x)+H(1,0,0;x)\Big\} \tag{4.46}$$

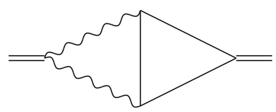
$$F_6^2(x) = \frac{x}{(1-x)^2}\Big\{\frac{1}{2}\pi^2H(0,-1;x)+\frac{1}{2}\pi^2H(0;x)+9\zeta(3)+\frac{11\pi^4}{144}$$
$$+3\zeta(3)H(1;x)-\frac{1}{6}\pi^2H(0,1;x)-H(0,0,-1,0;x)+2H(0,-1,0;x)$$
$$+\frac{1}{12}\left(6-\pi^2\right)H(0,0;x)+\frac{1}{2}\pi^2H(1,0;x)+2H(0,1,0;x)$$
$$+2H(0,-1,-1,0;x)+2H(0,-1,1,0;x)-H(0,0,0;x)-2H(1,0,-1,0;x)$$
$$-2H(1,0,0;x)+4H(1,0,1,0;x)+2H(0,1,-1,0;x)-H(0,1,0,0;x)$$
$$-2H(0,1,1,0;x)+\frac{1}{2}H(0,0,0,0;x)+4H(1,0,0,0;x)+2H(1,1,0,0;x)\Big\} \tag{4.47}$$

4.5 Five propagator integrals



$$= \int\frac{d^dk}{i\pi^{d/2}}\int\frac{d^dl}{i\pi^{d/2}}\frac{1}{D_{22}D_{23}D_{24}D_{26}D_{27}}$$
$$= \left(\frac{\Gamma(1+\epsilon)}{1-\epsilon}\right)^2 (m_t^2)^{-2\epsilon-1}F_7^0(x)+\mathcal{O}(\epsilon^1) \tag{4.48}$$

$$F_7^0(x) = \frac{1}{(1-x)^2}\Big\{ -\frac{1}{6}\pi^2 H(0,0;x)x - \frac{1}{3}\pi^2 H(1,0;x)x - \frac{\pi^4 x}{36} - xH(0,0,1,0;x) - 2xH(1,0,1,0;x) - 2xH(0,1,0,0;x) - 3xH(1,0,0,0;x) - 4xH(1,1,0,0;x)\Big\} \quad (4.49)$$

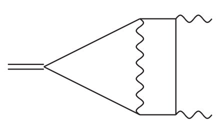


$$= \int \frac{d^dk}{i\pi^{d/2}} \int \frac{d^dl}{i\pi^{d/2}} \frac{1}{D_{11}D_{13}D_{14}D_{16}D_{17}} = \left(\frac{\Gamma(1+\epsilon)}{1-\epsilon}\right)^2 (m_t^2)^{-2\epsilon-1} \sum_{i=0}^{1} \epsilon^i F_8^i(x) + \mathcal{O}(\epsilon^2) \quad (4.50)$$

$$F_8^0(x) = \frac{x}{(1-x)^2}\Big\{ -2H(0,0,1;x) - 2H(0,1,0;x) + 4H(1,0,0;x) - 6\zeta(3)\Big\} \quad (4.51)$$

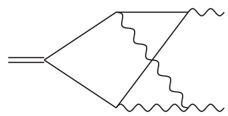
$$F_8^1(x) = \frac{x}{(1-x)^2}\Big\{ -12\zeta(3)H(0;x) + \frac{1}{3}\pi^2 H(0,1;x) - 24H(1;x)\zeta(3) - \frac{\pi^4}{10} - 8H(0,0,0,1;x) - 10H(0,0,-1,0;x) + 4H(0,-1,0,1;x) - \frac{2}{3}\pi^2 H(1,0;x) - 4H(1,0,0,1;x) - 4H(0,1,0,1;x) - 4H(0,0,1,0;x) - 4H(0,0,1,1;x) + 4H(0,-1,0,0;x) + 4H(0,-1,1,0;x) - 24H(1,0,-1,0;x) + 4H(1,0,1,0;x) + 4H(0,1,-1,0;x) - 6H(0,1,0,0;x) - 4H(0,1,1,0;x) + 12H(1,0,0,0;x)\Big\} \quad (4.52)$$

4.6 Six propagator integrals



$$= \int \frac{d^dk}{i\pi^{d/2}} \int \frac{d^dl}{i\pi^{d/2}} \frac{1}{D_{21}D_{23}D_{24}D_{25}D_{26}D_{27}} = \left(\frac{\Gamma(1+\epsilon)}{1-\epsilon}\right)^2 (m_t^2)^{-2\epsilon-2} F_9^0(x) + \mathcal{O}(\epsilon^1) \quad (4.53)$$

$$F_9^0(x) = \frac{x^2}{(1-x)^3(x+1)}\Big\{ 8\zeta(3)H(0;x) + 16H(0,0,-1,0;x) + \frac{\pi^4}{10} + \frac{2}{3}\pi^2 H(0,0;x) - 4H(0,0,1,0;x) - 8H(0,-1,0,0;x) + 14H(0,1,0,0;x) + H(0,0,0,0;x)\Big\} \quad (4.54)$$



$$
\begin{aligned}
&= \int \frac{d^dk}{i\pi^{d/2}} \int \frac{d^dl}{i\pi^{d/2}} \frac{1}{D_{31}D_{32}D_{33}D_{34}D_{35}D_{37}} \\
&= \left(\frac{\Gamma(1+\epsilon)}{1-\epsilon}\right)^2 (m_t^2)^{-2\epsilon-2} \sum_{i=-1}^{0} \epsilon^i F_{10}^i(x) + \mathcal{O}(\epsilon^1) \qquad (4.55)
\end{aligned}
$$

$$
F_{10}^{-1}(x) = \frac{x^2}{(1-x)^4}\Big\{ -\frac{2}{3}\pi^2 H(0;x) - 8H(0,-1,0;x) + 4H(0,0,0;x) - 12\zeta(3)\Big\} \qquad (4.56)
$$

$$
\begin{aligned}
F_{10}^{0}(x) = \frac{x^2}{(1-x)^4}\Big\{ & \frac{8}{3}\pi^2 H(0,-1;x) + 24\zeta(3) - \frac{16\pi^4}{45} \\
& + \frac{4}{3}\left(\pi^2 - 33\zeta(3)\right) H(0;x) - \frac{4}{3}\pi^2 H(0,1;x) - 48H(1;x)\zeta(3) \\
& - 56H(0,0,-1,0;x) + 16H(0,-1,0;x) - \frac{10}{3}\pi^2 H(0,0;x) - \frac{8}{3}\pi^2 H(1,0;x) \\
& + 8H(0,0,1,0;x) + 64H(0,-1,-1,0;x) - 40H(0,-1,0,0;x) - 16H(0,-1,1,0;x) \\
& - 8H(0,0,0;x) - 32H(1,0,-1,0;x) - 16H(0,1,-1,0;x) + 8H(0,1,0,0;x) \\
& + 12H(0,0,0,0;x) + 16H(1,0,0,0;x)\Big\} \qquad (4.57)
\end{aligned}
$$

5. Analytic continuation

The expressions in section 4 correspond to the unphysical, space-like region. The results must be analytically continued towards the time-like region. Due to the threshold in $s = 4m_t^2$, the physical region splits into two subregions, namely above and below threshold.

5.1 Analytic continuation above threshold

The region above threshold corresponds to the range $-1 < x < 0$. The analytic continuation to this region is straightforward, using the properties of harmonic polylogarithms under the transformation $x \to -x$ [24,29]. For harmonic polylogarithms $H(a_n,\ldots,a_1;x+i\varepsilon)$ with $a_1 \neq 0$ the analytic continuation is obtained trivially

$$
H(a_n,\ldots,\pm1;x+i\varepsilon) = (-1)^{\pm1+\cdots+a_n} H(-a_n,\ldots,\mp1;-x)\ . \qquad (5.1)
$$

where $a_k = -1,0,1$ for $k \neq 1$. We can eliminate higher rank harmonic polylogarithms with $a_1 = 0$ by applying integration by parts and product identities [29]. For instance:

$$
H(1,0;x) = H(0;x)\,H(1;x) - H(0,1;x)\ . \qquad (5.2)
$$

Recursively, one can write similar identities for the harmonic polylogarithms of higher rank. At the end of this procedure we only require the analytic continuation of simple logarithms:

$$\begin{aligned} H(1;x+i\varepsilon) &= -H(-1;-x+i\varepsilon) \\ H(0;x+i\varepsilon) &= H(0;-x+i\varepsilon)+i\pi \\ H(-1;x+i\varepsilon) &= -H(1;-x+i\varepsilon)\ . \end{aligned} \tag{5.3}$$

The analytic continuation described here is incorporated in the Mathematica package [37]. The expressions for the master integrals above threshold can be easily obtained from the ones in the space-like region using the routines implemented in this package.

5.2 Analytic continuation below threshold

Below threshold, the variable x lies on the unit circle of the complex plane. For this analytic continuation we follow the procedure in [19], which we summarize here.

We first express our results in terms of the variable θ given by $x = \exp(i\,\theta)$, and introduce the following notation for the harmonic polylogarithms as functions of θ:

$$H_c(a_n,\ldots,a_1;\theta) \stackrel{\text{def}}{=} H(a_n,\ldots,a_1;e^{i\theta})\ . \tag{5.4}$$

We now eliminate polylogarithms $H_c(a_n,\ldots,a_1;\theta)$ with $a_n = 1$, using integration by parts and product identities [24]. Then we use the analytic continuation of harmonic polylogarithms of weight one as kernels. For $0 < \theta < \pi$ we have

$$H_c(1;\theta) = -\ln 2\left|\sin\frac{\theta}{2}\right| \;+\; i\left(\frac{\pi}{2}-\frac{\theta}{2}\right) \tag{5.5}$$

$$H_c(0;\theta) = i\theta \tag{5.6}$$

$$H_c(-1;\theta) = \ln 2\left|\cos\frac{\theta}{2}\right| + i\frac{\theta}{2} \tag{5.7}$$

We can find the analytic continuation for the harmonic polylogarithms of higher weights recursively, using

$$H_c(a_n,a_{n-1},\ldots,a_1;\theta) = H(a_n,a_{n-1},\ldots,a_1;1) + i\int_0^\theta \mathrm{d}\theta' g(a_n;\theta')H_c(a_{n-1},\ldots,a_1;\theta') \tag{5.8}$$

where

$$g(1;\theta) = \frac{e^{i\theta}}{1-e^{i\theta}} = -\frac{1}{2} \;+\; i\frac{1}{2}\cot\frac{\theta}{2}\ , \tag{5.9}$$

$$g(0;\theta) = \frac{e^{i\theta}}{e^{i\theta}} = 1\ , \tag{5.10}$$

$$g(-1;\theta) = \frac{e^{i\theta}}{1+e^{i\theta}} = \frac{1}{2} \;+\; i\frac{1}{2}\tan\frac{\theta}{2}\ . \tag{5.11}$$

In this way, we have obtained expressions for the analytically continued harmonic polylogarithms through weight 4. The analytically continued master integrals can be expressed in terms of the following functions [19, 38, 39]:

$$\mathrm{Cl}_1(\theta) = -\ln 2\left|\sin\frac{\theta}{2}\right| , \tag{5.12}$$

$$\mathrm{Ls}_j^{(k)}(\theta) = -\int_0^\theta \mathrm{d}\theta'\, \theta'^k \ln^{j-k-1} 2\left|\sin\frac{\theta'}{2}\right| , \tag{5.13}$$

$$\mathrm{Lsc}_{i,j}(\theta) = -\int_0^\theta \mathrm{d}\theta'\, \ln^{i-1} 2\left|\sin\frac{\theta'}{2}\right| \ln^{j-1} 2\left|\cos\frac{\theta'}{2}\right| , \tag{5.14}$$

$$\mathrm{LsLsc}_{n,i,j}(\theta) = \int_0^\theta \mathrm{d}\theta'\, \mathrm{Ls}_{n+1}^{(0)}(\theta') \ln^{i-1} 2\left|\sin\frac{\theta'}{2}\right| \ln^{j-1} 2\left|\cos\frac{\theta'}{2}\right| . \tag{5.15}$$

Interestingly, these functions are a smaller set than the functions that appear in the analytic continuation of individual harmonic polylogarithms.

6. Results for the master integrals in the region below threshold

We now present the master integrals in the kinematic region $0 < s < 4m_t^2$. The expressions $\tilde{F}_j^i(\theta)$ correspond to the analytic continuation of the coefficients $F_j^i(x)$ in section 4.

6.1 One loop integrals

$$\tilde{F}^{-1}_{\mathrm{mbub}}(\theta) = 1 \tag{6.1}$$

$$\tilde{F}^{0}_{\mathrm{mbub}}(\theta) = -\theta\cot\left(\frac{\theta}{2}\right) + 1 \tag{6.2}$$

$$\tilde{F}^{1}_{\mathrm{mbub}}(\theta) = \cot\left(\frac{\theta}{2}\right)\Big(-\theta - 2\theta\mathrm{Cl}_1(\theta-\pi) + 2\mathrm{Ls}_2^{(0)}(\theta-\pi)\Big) + 2 \tag{6.3}$$

$$\begin{aligned}\tilde{F}^{2}_{\mathrm{mbub}}(\theta) = {}& \frac{1}{6}\cot\left(\frac{\theta}{2}\right)\Big(6\Big(-2\theta-\frac{\pi^3}{6}\Big) - 12\theta\mathrm{Cl}_1(\theta-\pi) - 12\theta\mathrm{Cl}_1(\theta-\pi)^2 \\ & + (24\mathrm{Cl}_1(\theta-\pi)+12)\mathrm{Ls}_2^{(0)}(\theta-\pi) + 12\mathrm{Ls}_3^{(0)}(\theta-\pi)\Big) + 4\end{aligned} \tag{6.4}$$

$$\begin{aligned}\tilde{F}^{3}_{\mathrm{mbub}}(\theta) = {}& \frac{1}{6}\cot\left(\frac{\theta}{2}\right)\Big(-\pi^3 - 24\theta + 6\Big(-4\theta-\frac{\pi^3}{3}\Big)\mathrm{Cl}_1(\theta-\pi) - 12\theta\mathrm{Cl}_1(\theta-\pi)^2 \\ & - 8\theta\mathrm{Cl}_1(\theta-\pi)^3 + \big(24\mathrm{Cl}_1(\theta-\pi)^2 + 24\mathrm{Cl}_1(\theta-\pi) + 24\big)\,\mathrm{Ls}_2^{(0)}(\theta-\pi) \\ & + (24\mathrm{Cl}_1(\theta-\pi)+12)\mathrm{Ls}_3^{(0)}(\theta-\pi) + 8\mathrm{Ls}_4^{(0)}(\theta-\pi) + 12\pi\zeta(3)\Big) + 8\end{aligned} \tag{6.5}$$

$$\tilde{F}^0_{\rm mtri}(\theta) = -\frac{1}{8}\theta^2\csc^2\left(\frac{\theta}{2}\right) \tag{6.6}$$

$$\tilde{F}^1_{\rm mtri}(\theta) = \frac{1}{8}\csc^2\left(\frac{\theta}{2}\right)\Big(\theta^2+(4\theta-8\pi){\rm Ls}_2^{(0)}(\theta-\pi)-8{\rm Ls}_3^{(1)}(\theta-\pi)-14\zeta(3)\Big) \tag{6.7}$$

$$\begin{aligned}\tilde{F}^2_{\rm mtri}(\theta) = {}& \frac{1}{24}\csc^2\left(\frac{\theta}{2}\right)\Big(-\pi^3\theta+12(2\pi-\theta){\rm Ls}_2^{(0)}(\theta-\pi)+12{\rm Ls}_2^{(0)}(\theta-\pi)^2\\ &+12\theta{\rm Ls}_3^{(0)}(\theta-\pi)+24{\rm Ls}_3^{(1)}(\theta-\pi)+42\zeta(3)\Big)\end{aligned} \tag{6.8}$$

6.2 Three propagator integrals

$$\tilde{F}^0_1(\theta) = -\frac{1}{4}\theta^2\csc^2\left(\frac{\theta}{2}\right) \tag{6.9}$$

$$\begin{aligned}\tilde{F}^1_1(\theta) = {}& \frac{1}{2}\csc^2\left(\frac{\theta}{2}\right)\Big(\theta^2+\theta^2{\rm Cl}_1(\theta)+2\theta{\rm Ls}_2^{(0)}(\theta)+6{\rm Ls}_2^{(0)}(\theta-\pi)(-2\pi+\theta)\\ &-6{\rm Ls}_3^{(1)}(\theta)-12{\rm Ls}_3^{(1)}(\theta-\pi)-21\zeta(3)\Big)\end{aligned} \tag{6.10}$$

$$\begin{aligned}\tilde{F}^2_1(\theta) = {}& \frac{1}{240}\csc^2\left(\frac{\theta}{2}\right)\Big(79\pi^4-180\pi^3\theta-60\theta^2-30\theta^4-120\theta^2{\rm Cl}_1(\theta)^2+120\pi^2\log^2(2)-120\log^4(2)\\ &-480\theta{\rm Ls}_2^{(0)}(\theta)+480{\rm Ls}_2^{(0)}(\theta)^2+{\rm Ls}_2^{(0)}(\theta-\pi)\big(2880\pi-1440\theta+2880{\rm Ls}_2^{(0)}(\theta)\big)\\ &+2160{\rm Ls}_2^{(0)}(\theta-\pi)^2+240\theta{\rm Ls}_3^{(0)}(\theta)+720{\rm Ls}_3^{(0)}(\theta-\pi)\big(-\pi+3\theta\big)+1440{\rm Ls}_3^{(1)}(\theta)\\ &+2880{\rm Ls}_3^{(1)}(\theta-\pi)+180{\rm Ls}_4^{(1)}(2\theta)-720{\rm Ls}_4^{(1)}(\theta-\pi)+1440\theta{\rm Lsc}_{2,2}(\theta)+1440{\rm LsLsc}_{1,1,2}(\theta)\\ &-2880{\rm Li}_4\left(\frac{1}{2}\right)+5040\zeta(3)-2520\log(2)\zeta(3)+{\rm Cl}_1(\theta)\Big(-240\theta^2-480\theta{\rm Ls}_2^{(0)}(\theta)\\ &-1440(\theta-2\pi){\rm Ls}_2^{(0)}(\theta-\pi)+1440{\rm Ls}_3^{(1)}(\theta)+2880{\rm Ls}_3^{(1)}(\theta-\pi)+5040\zeta(3)\Big)\Big)\end{aligned} \tag{6.11}$$

$$\tilde{F}^{-1}_2(\theta) = \frac{1}{2}\theta\csc(\theta) \tag{6.12}$$

$$\tilde{F}^0_2(\theta) = \frac{1}{4}\theta^2\cot\left(\frac{\theta}{2}\right)\csc(\theta)+\csc(\theta)\Big(-\theta+\theta{\rm Cl}_1(\theta)+3\theta{\rm Cl}_1(\theta-\pi)-{\rm Ls}_2^{(0)}(\theta)-3{\rm Ls}_2^{(0)}(\theta-\pi)\Big) \tag{6.13}$$

$$\begin{aligned}\tilde{F}^1_2(\theta) = {}& +\frac{1}{2}\cot\left(\frac{\theta}{2}\right)\csc(\theta)\Big(-\theta^2-\theta^2{\rm Cl}_1(\theta)-2\theta{\rm Ls}_2^{(0)}(\theta)+(12\pi-6\theta){\rm Ls}_2^{(0)}(\theta-\pi)+6{\rm Ls}_3^{(1)}(\theta)\\ &+12{\rm Ls}_3^{(1)}(\theta-\pi)+21\zeta(3)\Big)+\frac{\csc(\theta)}{4}\Big(+3\pi^3+2\theta+2\theta^3+4\theta{\rm Cl}_1(\theta)^2-24\theta{\rm Cl}_1(\theta-\pi)\\ &+36\theta{\rm Cl}_1(\theta-\pi)^2+{\rm Cl}_1(\theta)(24\theta{\rm Cl}_1(\theta-\pi)-8\theta)+(-8{\rm Cl}_1(\theta)-24{\rm Cl}_1(\theta-\pi)+8){\rm Ls}_2^{(0)}(\theta)\end{aligned}$$

$$+ (-24\mathrm{Cl}_1(\theta) - 72\mathrm{Cl}_1(\theta-\pi) + 24)\mathrm{Ls}_2^{(0)}(\theta-\pi) - 4\mathrm{Ls}_3^{(0)}(\theta) - 36\mathrm{Ls}_3^{(0)}(\theta-\pi)$$

$$- 24\mathrm{Lsc}_{2,2}(\theta)\Big) \tag{6.14}$$

$$\tilde{F}_2^2(\theta) = \frac{1}{240}\cot\left(\frac{\theta}{2}\right)\csc(\theta)\Big(- 79\pi^4 + 180\pi^3\theta + 60\theta^2 + 30\theta^4 + 120\theta^2\mathrm{Cl}_1(\theta)^2$$

$$- 120\pi^2\log^2(2) + 120\log^4(2) - 480\mathrm{Ls}_2^{(0)}(\theta)^2 + \mathrm{Ls}_2^{(0)}(\theta)(480\mathrm{Cl}_1(\theta)\theta + 480\theta$$

$$- 2880\mathrm{Ls}_2^{(0)}(\theta-\pi)) + (240(6\theta-12\pi) + 1440(\theta-2\pi)\mathrm{Cl}_1(\theta))\mathrm{Ls}_2^{(0)}(\theta-\pi) - 2160\mathrm{Ls}_2^{(0)}(\theta-\pi)^2$$

$$- 240\theta\mathrm{Ls}_3^{(0)}(\theta) + 240(3\pi-9\theta)\mathrm{Ls}_3^{(0)}(\theta-\pi) + (-1440\mathrm{Cl}_1(\theta) - 1440)\mathrm{Ls}_3^{(1)}(\theta)$$

$$+ (-2880\mathrm{Cl}_1(\theta) - 2880)\mathrm{Ls}_3^{(1)}(\theta-\pi) - 180\mathrm{Ls}_4^{(1)}(2\theta) + 720\mathrm{Ls}_4^{(1)}(\theta-\pi) - 1440\theta\mathrm{Lsc}_{2,2}(\theta)$$

$$- 1440\mathrm{LsLsc}_{1,1,2}(\theta) + 2880\mathrm{Li}_4\left(\frac{1}{2}\right) + 240\mathrm{Cl}_1(\theta)\left(\theta^2 - 21\zeta(3)\right)$$

$$- 5040\zeta(3) + 2520\log(2)\zeta(3)\Big) + \frac{\csc(\theta)}{6}\Big(4\theta\mathrm{Cl}_1(\theta)^3 + \left(18\theta^3 + 18\theta + 27\pi^3\right)\mathrm{Cl}_1(\theta-\pi)$$

$$- 108\theta\mathrm{Cl}_1(\theta-\pi)^2 + 108\theta\mathrm{Cl}_1(\theta-\pi)^3 + \mathrm{Cl}_1(\theta)^2(36\theta\mathrm{Cl}_1(\theta-\pi) - 12\theta)$$

$$+ \mathrm{Cl}_1(\theta)\left(6\theta^3 + 108\mathrm{Cl}_1(\theta-\pi)^2\theta - 72\mathrm{Cl}_1(\theta-\pi)\theta + 6\theta + 9\pi^3\right) + \Big(- 18\theta^2 - 12\mathrm{Cl}_1(\theta)^2$$

$$- 108\mathrm{Cl}_1(\theta-\pi)^2 + \mathrm{Cl}_1(\theta)(24 - 72\mathrm{Cl}_1(\theta-\pi)) + 72\mathrm{Cl}_1(\theta-\pi) - 6\Big)\mathrm{Ls}_2^{(0)}(\theta)$$

$$+ \Big(- 54\theta^2 + 216\pi\theta - 36\mathrm{Cl}_1(\theta)^2 - 324\mathrm{Cl}_1(\theta-\pi)^2 + \mathrm{Cl}_1(\theta)(72 - 216\mathrm{Cl}_1(\theta-\pi))$$

$$+ 216\mathrm{Cl}_1(\theta-\pi) - 216\pi^2 - 18\Big)\mathrm{Ls}_2^{(0)}(\theta-\pi) + (-12\mathrm{Cl}_1(\theta) - 36\mathrm{Cl}_1(\theta-\pi)$$

$$+ 12)\mathrm{Ls}_3^{(0)}(\theta) + (-108\mathrm{Cl}_1(\theta) - 324\mathrm{Cl}_1(\theta-\pi) + 108)\mathrm{Ls}_3^{(0)}(\theta-\pi)$$

$$+ 108\theta\mathrm{Ls}_3^{(1)}(\theta) + (216\theta - 432\pi)\mathrm{Ls}_3^{(1)}(\theta-\pi) - 4\mathrm{Ls}_4^{(0)}(\theta) - 108\mathrm{Ls}_4^{(0)}(\theta-\pi)$$

$$- 126\mathrm{Ls}_4^{(2)}(\theta) - 216\mathrm{Ls}_4^{(2)}(\theta-\pi) + (-72\mathrm{Cl}_1(\theta) - 216\mathrm{Cl}_1(\theta-\pi) + 72)\mathrm{Lsc}_{2,2}(\theta)$$

$$- 108\mathrm{Lsc}_{2,3}(\theta) - 36\mathrm{Lsc}_{3,2}(\theta) - 3\left(2\theta^3 - 126\zeta(3)\theta + 198\pi\zeta(3) + 3\pi^3\right)\Big) \tag{6.15}$$

6.3 Four propagator integrals

$$\tilde{F}_3^{-2}(\theta) = \frac{1}{2} \tag{6.16}$$

$$\tilde{F}_3^{-1}(\theta) = -\frac{1}{2} \tag{6.17}$$

$$\tilde{F}_3^{0}(\theta) = \frac{1}{4}\csc^2\left(\frac{\theta}{2}\right)\Big(- 6 + \theta^2 + 2\theta^2\mathrm{Cl}_1(\theta) + 6\cos(\theta) - \theta^2\cos(\theta) - 4\mathrm{Ls}_3^{(1)}(\theta) + 4\theta\sin(\theta)\Big) \tag{6.18}$$

$$\tilde{F}_3^{1}(\theta) = \frac{1}{240}\csc^2\left(\frac{\theta}{2}\right)\Big(- 1440 + 79\pi^4 - 60\theta^2 - 120\theta^2\mathrm{Cl}_1(\theta)^2 + 1440\cos(\theta) + 180\theta^2\cos(\theta)$$

$$+ 120\pi^2\log^2(2) - 120\log^4(2) + 240\mathrm{Ls}_2^{(0)}(\theta)^2 - 720\pi\mathrm{Ls}_3^{(0)}(\theta-\pi) + (1440\mathrm{Cl}_1(\theta)$$

$$- 240(3\cos(\theta)-4))\mathrm{Ls}_3^{(1)}(\theta) + (2880\mathrm{Cl}_1(\theta) - 1440(\cos(\theta)-1))\mathrm{Ls}_3^{(1)}(\theta-\pi) + 180\mathrm{Ls}_4^{(1)}(2\theta)$$
$$- 720\mathrm{Ls}_4^{(1)}(\theta-\pi) + 1440\mathrm{LsLsc}_{1,1,2}(\theta) - 2880\mathrm{Li}_4\left(\frac{1}{2}\right) + \mathrm{Ls}_2^{(0)}(\theta)(-480\theta\mathrm{Cl}_1(\theta)$$
$$+ 1440\mathrm{Ls}_2^{(0)}(\theta-\pi) + 240(\cos(\theta)\theta - \theta - 2\sin(\theta))) + 960\theta\sin(\theta) + 1440\theta\mathrm{Cl}_1(\theta-\pi)\sin(\theta)$$
$$+ \mathrm{Ls}_2^{(0)}(\theta-\pi)(1440(2\pi-\theta)\mathrm{Cl}_1(\theta) - 720(-\cos(\theta)\theta + \theta + 2\pi\cos(\theta) + 2\sin(\theta) - 2\pi))$$
$$+ 2520\zeta(3) - 2520\cos(\theta)\zeta(3) - 2520\log(2)\zeta(3)$$
$$+ 120\mathrm{Cl}_1(\theta)\left(\cos(\theta)\theta^2 - 2\theta^2 + 4\sin(\theta)\theta + 42\zeta(3)\right)\Big) \tag{6.19}$$

$$\tilde{F}_4^{-2}(\theta) = \frac{1}{2} \tag{6.20}$$

$$\tilde{F}_4^{-1}(\theta) = -\theta\cot\left(\frac{\theta}{2}\right) + \frac{3}{2} \tag{6.21}$$

$$\tilde{F}_4^{0}(\theta) = \frac{1}{8}\csc^2\left(\frac{\theta}{2}\right)\Big(20 - \theta^2 - 20\cos(\theta) + \theta^2\cos(\theta) + 8\mathrm{Ls}_3^{(1)}(\theta) - 4\mathrm{Ls}_2^{(0)}(\theta)(\theta - \sin(\theta))$$
$$- 12\theta\sin(\theta) - 4\theta\mathrm{Cl}_1(\theta)\sin(\theta) - 8\theta\mathrm{Cl}_1(\theta-\pi)\sin(\theta) + 8\mathrm{Ls}_2^{(0)}(\theta-\pi)\sin(\theta)\Big) \tag{6.22}$$

$$\tilde{F}_4^{1}(\theta) = \frac{1}{96}\csc^2\left(\frac{\theta}{2}\right)\Big(768 - 36\theta^2 - \theta^4 - 768\cos(\theta) + 36\theta^2\cos(\theta) - 96\mathrm{Ls}_2^{(0)}(\theta)^2 + (-192\mathrm{Cl}_1(\theta)$$
$$- 48(\cos(\theta)+1))\mathrm{Ls}_3^{(1)}(\theta) + 96(\cos(\theta)-1)\mathrm{Ls}_3^{(1)}(\theta-\pi) - 96\mathrm{Ls}_4^{(1)}(\theta) - 48\mathrm{Ls}_3^{(0)}(\theta)(\theta - \sin(\theta))$$
$$- 96\mathrm{Lsc}_{2,2}(\theta)(\theta - \sin(\theta)) - 8\pi^3\sin(\theta) - 480\theta\sin(\theta) + 4\theta^3\sin(\theta) - 48\theta\mathrm{Cl}_1(\theta)^2\sin(\theta)$$
$$- 288\theta\mathrm{Cl}_1(\theta-\pi)\sin(\theta) - 96\theta\mathrm{Cl}_1(\theta-\pi)^2\sin(\theta) + 96\mathrm{Ls}_3^{(0)}(\theta-\pi)\sin(\theta)$$
$$+ \mathrm{Ls}_2^{(0)}(\theta)(-96\mathrm{Ls}_2^{(0)}(\theta-\pi) + 96\mathrm{Cl}_1(\theta-\pi)\sin(\theta) + 96\mathrm{Cl}_1(\theta)(\theta + \sin(\theta)) + 48(\theta + 3\sin(\theta)))$$
$$+ \mathrm{Ls}_2^{(0)}(\theta-\pi)(96\mathrm{Cl}_1(\theta)\sin(\theta) + 192\mathrm{Cl}_1(\theta-\pi)\sin(\theta) + 48(-\cos(\theta)\theta + \theta + 2\pi\cos(\theta)$$
$$+ 6\sin(\theta) - 2\pi)) + \mathrm{Cl}_1(\theta)\left(24\left(\cos(\theta)\theta^2 - \theta^2 - 6\sin(\theta)\theta\right) - 96\theta\mathrm{Cl}_1(\theta-\pi)\sin(\theta)\right) - 168\zeta(3)$$
$$+ 168\cos(\theta)\zeta(3)\Big) \tag{6.23}$$

$$\tilde{F}_5^{-2}(\theta) = -\frac{1}{2}\sin^2\left(\frac{\theta}{2}\right) \tag{6.24}$$

$$\tilde{F}_5^{-1}(\theta) = \frac{1}{32}\csc^2\left(\frac{\theta}{2}\right)(20\cos(\theta) - 5\cos(2\theta) + 16\theta\sin(\theta) - 4\theta\sin(2\theta) - 15) \tag{6.25}$$

$$\tilde{F}_5^{0}(\theta) = \frac{1}{64}\csc^2\left(\frac{\theta}{2}\right)\Big(-109 + 6\theta^2 + 140\cos(\theta) - 8\theta^2\cos(\theta) - 31\cos(2\theta) + 2\theta^2\cos(2\theta)$$

$$
\begin{aligned}
&- 32\mathrm{Ls}_3^{(1)}(\theta) + 72\theta\sin(\theta) - 16\mathrm{Ls}_2^{(0)}(\theta-\pi)(4\sin(\theta)-\sin(2\theta)) - 20\theta\sin(2\theta) \\
&+ 8\mathrm{Ls}_2^{(0)}(\theta)(2\theta - 4\sin(\theta)+\sin(2\theta)) + 8\mathrm{Cl}_1(\theta)(4\theta\sin(\theta)-\theta\sin(2\theta)) \\
&+ 16\mathrm{Cl}_1(\theta-\pi)(4\theta\sin(\theta)-\theta\sin(2\theta))\Big)
\end{aligned} \tag{6.26}
$$

$$
\begin{aligned}
\tilde{F}_5^1(\theta) \quad = \quad & \frac{1}{384}\csc^2\left(\frac{\theta}{2}\right)\Big(- 2229 + 114\theta^2 + 2\theta^4 + 2796\cos(\theta) - 144\theta^2\cos(\theta) - 567\cos(2\theta) \\
&+ 30\theta^2\cos(2\theta) + 192\mathrm{Ls}_2^{(0)}(\theta)^2 + (384\mathrm{Cl}_1(\theta) + 48(4\cos(\theta)-\cos(2\theta)+3))\mathrm{Ls}_3^{(1)}(\theta) \\
&- 96(4\cos(\theta)-\cos(2\theta)-3)\mathrm{Ls}_3^{(1)}(\theta-\pi) + 192\mathrm{Ls}_4^{(1)}(\theta) + 32\pi^3\sin(\theta) + 1368\theta\sin(\theta) \\
&- 16\theta^3\sin(\theta) + \mathrm{Ls}_2^{(0)}(\theta-\pi)(-48(-4\cos(\theta)\theta + \cos(2\theta)\theta + 3\theta + 8\pi\cos(\theta) - 2\pi\cos(2\theta) \\
&+ 18\sin(\theta) - 5\sin(2\theta) - 6\pi) - 96\mathrm{Cl}_1(\theta)(4\sin(\theta)-\sin(2\theta)) - 192\mathrm{Cl}_1(\theta-\pi)(4\sin(\theta) \\
&- \sin(2\theta))) + \mathrm{Ls}_2^{(0)}(\theta)(192\mathrm{Ls}_2^{(0)}(\theta-\pi) - 24(6\theta + 20\sin(\theta) - 5\sin(2\theta)) \\
&- 96\mathrm{Cl}_1(\theta-\pi)(4\sin(\theta)-\sin(2\theta)) - 96\mathrm{Cl}_1(\theta)(2\theta + 4\sin(\theta) - \sin(2\theta))) \\
&- 96\mathrm{Ls}_3^{(0)}(\theta-\pi)(4\sin(\theta)-\sin(2\theta)) - 8\pi^3\sin(2\theta) - 372\theta\sin(2\theta) + 4\theta^3\sin(2\theta) \\
&+ 48\mathrm{Ls}_3^{(0)}(\theta)(2\theta - 4\sin(\theta)+\sin(2\theta)) + 96\mathrm{Lsc}_{2,2}(\theta)(2\theta - 4\sin(\theta)+\sin(2\theta)) \\
&+ 48\mathrm{Cl}_1(\theta-\pi)(18\theta\sin(\theta) - 5\theta\sin(2\theta)) + 48\mathrm{Cl}_1(\theta)^2(4\theta\sin(\theta)-\theta\sin(2\theta)) \\
&+ 96\mathrm{Cl}_1(\theta-\pi)^2(4\theta\sin(\theta)-\theta\sin(2\theta)) + \mathrm{Cl}_1(\theta)\left(96\mathrm{Cl}_1(\theta-\pi)(4\theta\sin(\theta)-\theta\sin(2\theta))\right. \\
&\left.-24\left(4\cos(\theta)\theta^2 - \cos(2\theta)\theta^2 - 3\theta^2 - 20\sin(\theta)\theta + 5\sin(2\theta)\theta\right)\right) \\
&+ 504\zeta(3) - 672\cos(\theta)\zeta(3) + 168\cos(2\theta)\zeta(3)\Big)
\end{aligned} \tag{6.27}
$$

$$
\tilde{F}_6^0(\theta) \quad = \quad \frac{1}{16}\theta^2\csc^2\left(\frac{\theta}{2}\right) \tag{6.28}
$$

$$
\begin{aligned}
\tilde{F}_6^1(\theta) \quad = \quad & \frac{1}{8}\csc^2\left(\frac{\theta}{2}\right)\Big(- \theta^2 + \theta^2\mathrm{Cl}_1(\theta) + 2\theta\mathrm{Ls}_2^{(0)}(\theta) + 2(2\pi-\theta)\mathrm{Ls}_2^{(0)}(\theta-\pi) \\
&- 6\mathrm{Ls}_3^{(1)}(\theta) + 4\mathrm{Ls}_3^{(1)}(\theta-\pi) + 7\zeta(3)\Big)
\end{aligned} \tag{6.29}
$$

$$
\begin{aligned}
\tilde{F}_6^2(\theta) \quad = \quad & \frac{1}{2880}\csc^2\left(\frac{\theta}{2}\right)\Big(79\pi^4 + 60\pi^3\theta + 180\theta^2 + 15\theta^4 + 360\theta^2\mathrm{Cl}_1(\theta)^2 + 120\pi^2\log^2(2) - 120\log^4(2) \\
&+ 2160\mathrm{Ls}_2^{(0)}(\theta)^2 + (1440\theta - 720(2\theta - 4\pi)\mathrm{Cl}_1(\theta) - 2880\pi)\mathrm{Ls}_2^{(0)}(\theta-\pi) \\
&- 720\mathrm{Ls}_2^{(0)}(\theta-\pi)^2 + \mathrm{Ls}_2^{(0)}(\theta)(-2880\mathrm{Cl}_1(\theta)\theta - 1440\theta + 2880\mathrm{Ls}_2^{(0)}(\theta-\pi)) + 720\theta\mathrm{Ls}_3^{(0)}(\theta) \\
&- 720(\theta+\pi)\mathrm{Ls}_3^{(0)}(\theta-\pi) + (4320\mathrm{Cl}_1(\theta) + 4320)\mathrm{Ls}_3^{(1)}(\theta) + (2880\mathrm{Cl}_1(\theta) - 2880)\mathrm{Ls}_3^{(1)}(\theta-\pi) \\
&+ 1440\mathrm{Ls}_4^{(1)}(\theta) + 180\mathrm{Ls}_4^{(1)}(2\theta) - 720\mathrm{Ls}_4^{(1)}(\theta-\pi) + 1440\theta\mathrm{Lsc}_{2,2}(\theta) + 1440\mathrm{LsLsc}_{1,1,2}(\theta)
\end{aligned}
$$

$$- 2880\mathrm{Li}_4\left(\frac{1}{2}\right) - 720\mathrm{Cl}_1(\theta)\left(\theta^2 - 7\zeta(3)\right) - 5040\zeta(3) - 2520\log(2)\zeta(3)\Big) \tag{6.30}$$

6.4 Five propagator integrals

$$\tilde{F}_7^0(\theta) = \frac{1}{64}\csc^2\left(\frac{\theta}{2}\right)\Big(- \theta^4 - 16\theta^2\mathrm{Cl}_1(\theta)^2 + 32\theta\mathrm{Cl}_1(\theta)\mathrm{Ls}_2^{(0)}(\theta) - 16\mathrm{Ls}_2^{(0)}(\theta)^2\Big) \tag{6.31}$$

$$\tilde{F}_8^0(\theta) = \frac{1}{8}\csc^2\left(\frac{\theta}{2}\right)\Big(- i\pi\theta^2 + 4\theta^2\mathrm{Cl}_1(\theta) - 12\mathrm{Ls}_3^{(1)}(\theta)\Big) \tag{6.32}$$

$$\begin{aligned}\tilde{F}_8^1(\theta) = \; & \frac{1}{1440}\csc^2\left(\frac{\theta}{2}\right)\Big(553\pi^4 + 120\pi^2\theta^2 + 840\pi^2\log^2(2) - 840\log^4(2) + 1440\mathrm{Ls}_2^{(0)}(\theta)^2 \\ & + \big(1440\big(\frac{i\pi\theta}{2} - i\pi^2\big) + 8640(2\pi - \theta)\mathrm{Cl}_1(\theta)\big)\mathrm{Ls}_2^{(0)}(\theta - \pi) \\ & + \mathrm{Ls}_2^{(0)}(\theta)(10080\mathrm{Ls}_2^{(0)}(\theta - \pi) - 2880\theta\mathrm{Cl}_1(\theta)) - 5040\pi\mathrm{Ls}_3^{(0)}(\theta - \pi) + 4320\mathrm{Cl}_1(\theta)\mathrm{Ls}_3^{(1)}(\theta) \\ & + (17280\mathrm{Cl}_1(\theta) - 1440i\pi)\mathrm{Ls}_3^{(1)}(\theta - \pi) - 2880\mathrm{Ls}_4^{(1)}(\theta) + 1260\mathrm{Ls}_4^{(1)}(2\theta) - 5040\mathrm{Ls}_4^{(1)}(\theta - \pi) \\ & + 10080\mathrm{LsLsc}_{1,1,2}(\theta) - 20160\mathrm{Li}_4\left(\frac{1}{2}\right) - 360i\mathrm{Cl}_1(\theta)\left(\pi\theta^2 + 84i\zeta(3)\right) \\ & - 2520i\pi\zeta(3) - 17640\log(2)\zeta(3)\Big)\end{aligned} \tag{6.33}$$

6.5 Six propagator integrals

$$\begin{aligned}\tilde{F}_9^0(\theta) = \; & -\frac{1}{16}\csc^3\left(\frac{\theta}{2}\right)\sec\left(\frac{\theta}{2}\right)\Big(2\mathrm{Ls}_2^{(0)}(\theta)\theta^2 - 22\mathrm{Ls}_3^{(1)}(\theta)\theta - 40\mathrm{Ls}_3^{(1)}(\theta - \pi)\theta - 70\zeta(3)\theta + 27\mathrm{Ls}_4^{(2)}(\theta) \\ & + 4\left(2\theta^2 - 10\pi\theta + 9\pi^2\right)\mathrm{Ls}_2^{(0)}(\theta - \pi) + 72\pi\mathrm{Ls}_3^{(1)}(\theta - \pi) + 36\mathrm{Ls}_4^{(2)}(\theta - \pi) + 72\pi\zeta(3)\Big)\end{aligned} \tag{6.34}$$

$$\tilde{F}_{10}^{-1}(\theta) = \frac{1}{4}\csc^4\left(\frac{\theta}{2}\right)\Big((2\theta - 4\pi)\mathrm{Ls}_2^{(0)}(\theta - \pi) - 4\mathrm{Ls}_3^{(1)}(\theta - \pi) - 7\zeta(3)\Big) \tag{6.35}$$

$$\begin{aligned}\tilde{F}_{10}^0(\theta) = \; & \frac{1}{1440}\csc^4\left(\frac{\theta}{2}\right)\Big(- 158\pi^4 - 240\pi^3\theta - 75\theta^4 - 240\pi^2\log^2(2) + 240\log^4(2) \\ & + (-1440\theta - 1440(4\pi - 2\theta)\mathrm{Cl}_1(\theta) - 1440\mathrm{Ls}_2^{(0)}(\theta) + 2880\pi)\mathrm{Ls}_2^{(0)}(\theta - \pi) \\ & + 2880\mathrm{Ls}_2^{(0)}(\theta - \pi)^2 + 1440(2\theta + \pi)\mathrm{Ls}_3^{(0)}(\theta - \pi) + (2880 - 5760\mathrm{Cl}_1(\theta))\mathrm{Ls}_3^{(1)}(\theta - \pi) \\ & + 1440\mathrm{Ls}_4^{(1)}(\theta) - 360\mathrm{Ls}_4^{(1)}(2\theta) + 1440\mathrm{Ls}_4^{(1)}(\theta - \pi) + 1440\theta\mathrm{Lsc}_{2,2}(\theta) \\ & - 2880\mathrm{LsLsc}_{1,1,2}(\theta) + 5760\mathrm{Li}_4\left(\frac{1}{2}\right) + 5040\zeta(3) - 10080\mathrm{Cl}_1(\theta)\zeta(3) + 5040\log(2)\zeta(3)\Big)\end{aligned} \tag{6.36}$$

7. Virtual corrections to $gg \to h$

In this section we will present the results for the two loop corrections to the Higgs boson production process via gluon fusion, with either quarks or scalar quarks running in the loops, in the region below threshold.

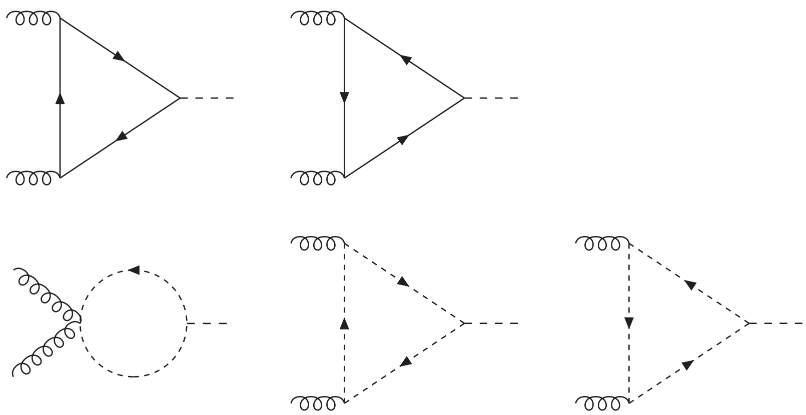


Figure 4: Contributions to $gg \to h$ at the lowest order

At order α_s^2, the unrenormalized amplitude for the process $gg \to H$, is given by

$$\mathcal{M} = \frac{\alpha_s}{4\pi} \sum_{i=s,f} \left(\mathcal{M}_i^{(0)} + \frac{\alpha_s}{4\pi} \mathcal{M}_i^{(1)} \right) + \mathcal{O}(\alpha_s^3) \,, \tag{7.1}$$

where s and f denote the contributions of scalars and fermions in the loop. The Born amplitudes are given by the diagrams in Figure 4. They are given by

$$\mathcal{M}_i^{(0)} = \delta_{ab} \, K_{ab} \, \Lambda_i \left(\frac{m_i^2}{4\pi\mu^2} \right)^{-\epsilon} e^{-\epsilon \, \gamma_E} \, M_i^{(0)} \,. \tag{7.2}$$

Explicit expressions for the form factors $M_i^{(0)}$ are given in sections 7.1 and 7.2 for fermions and scalars respectively. Indices a and b denote the colors of the gluons, $\epsilon_i(p_i)$ the corresponding polarization vectors and m_i the mass of the particle running in the loop. The couplings of fermions and scalars to the Higgs boson have been written as $g_{ffH} = \Lambda_f \, m_f$ and $g_{ssH} = \Lambda_s \, m_s^2$ respectively. The constants Λ_i have inverse mass dimensions, in the case of the SM, $\Lambda_f = 1/v$, where v is the VEV of the Higgs Boson. Finally, the helicity projector K_{ab} is given by

$$K_{ab} = \epsilon_a(p_1) \cdot \epsilon_b(p_2) \, p_1 \cdot p_2 - p_1 \cdot \epsilon_b(p_2) \, p_2 \cdot \epsilon_a(p_1) \,. \tag{7.3}$$

The two loop contributions to the amplitudes can be written as

$$\left(\frac{\alpha_s}{4\pi} \right)^2 \mathcal{M}_i^{(1)} = \mathcal{M}_{i,\mathrm{ir}}^{(1)} + \mathcal{M}_{i,\mathrm{uv}}^{(1)} + \delta_{ab} \, K_{ab} \, \Lambda_i \left(\frac{\alpha_s}{4\pi} \right)^2 \left(M_{i,\mathrm{fin}}^{(1)} + 2 \, i \, \pi \beta_0 \right) + \mathcal{O}(\epsilon) \,, \tag{7.4}$$

where the infrared and ultraviolet poles have been extracted into $\mathcal{M}^{(1)}_{i,\mathrm{ir}}$ and $\mathcal{M}^{(1)}_{i,\mathrm{uv}}$ respectively. The form factors $M^{(1)}_{i,\mathrm{fin}}$ are finite in the limit $\epsilon \to 0$, their explicit expressions in the region below threshold are given in sections 7.1 and 7.2. We have omitted contributions from diagrams with gluon self-energies in external lines. These drop out if one renormalizes in a heavy quark and squark decoupling scheme; the running of the renormalized strong coupling is then determined from the light flavors below the decoupling scale [40, 41].

The singular contributions can be written in terms of the Born amplitudes. For the infrared contributions we have

$$\mathcal{M}^{(1)}_{i,\mathrm{ir}} = -\frac{\alpha_s}{4\pi}\left(\frac{-s}{4\pi\mu^2}\right)^{-\epsilon} e^{-\epsilon\,\gamma_E}\left[N\left(\frac{2}{\epsilon^2}-\frac{\pi^2}{6}\right)+\frac{2\,\beta_0}{\epsilon}\right]\mathcal{M}^{(0)}_i + \mathcal{O}(\epsilon)\,. \tag{7.5}$$

The ultraviolet pieces, in turn, are given by

$$\mathcal{M}^{(1)}_{i,\mathrm{uv}} = -\left(\frac{s}{\mu^2}\right)^{-\epsilon}\left(2\,\delta Z_g\,\mathcal{M}^{(0)}_i + \delta Z_{m_i,\mathrm{gluon}}\,\frac{\partial}{\partial m_i}\left(\mathcal{M}^{(0)}_i\right)\right)\,, \tag{7.6}$$

where δZ_g is the strong coupling renormalization constant at one loop in the $\overline{\mathrm{MS}}$ scheme, given by

$$\delta Z_g = -\frac{\alpha_s}{4\pi}\,(4\pi)^\epsilon\, e^{-\epsilon\,\gamma_E}\,\frac{\beta_0}{\epsilon}\,, \tag{7.7}$$

and $\delta Z_{m_i,\mathrm{gluon}}$ are the gluonic contributions to the mass renormalization to fermions and scalars, also in the $\overline{\mathrm{MS}}$ scheme:

$$\delta Z_{m_i,\mathrm{gluon}} = -\frac{\alpha_s}{4\pi}\,(4\pi)^\epsilon\, e^{-\epsilon\,\gamma_E}\, C_F\,\frac{3}{\epsilon}\,. \tag{7.8}$$

As discussed in the introduction, the fermionic contributions have been computed in [3] and expressed in terms of 8 one-dimensional integrals. In turn, these integrals were computed in terms of harmonic polylogarithms in [25], giving the analytical result for the two loop corrections with fermions in the loop. Our results for these pieces fully agree with the ones quoted in [25].

In the case of scalars mediating the gluon-Higgs boson interaction, there are additional contributions originated in quartic couplings between the scalars. In supersymmetric theories, these interactions have a component proportional to g_s^2. Contributions containing the quartic interactions involve, additionally, the mixing between different scalar quarks. Therefore, the NLO corrections associated to them contain more than one massive particle running in the loops. However, as the gluon couplings to the scalars are diagonal, the corrections due to mixing only give contributions in the form of products of one loop integrals. In what follows we will only consider the gluonic corrections and postpone the treatment of contributions due to self interactions of the scalars to a forthcoming publication.

If the mass of the Higgs particle is significantly smaller than the mass of the particles circulating in the loops, the amplitudes can be approximated by their limit when $m_i \to \infty$. These results have been obtained in the context of effective field theories, both for fermions and scalar loops in references [4, 42] and [27] respectively. Choosing the $\overline{\mathrm{MS}}$ scheme -notice that in the infinite mass limit, the scheme dependence due to mass renormalization cancels out- we obtain

$$\mathcal{M}_{\rm f,\infty} = \delta_{ab}\, K_{ab}\, \Lambda_f\, \frac{\alpha_s}{3\,\pi}\, \left(1 + \frac{\alpha_s}{\pi}\,\frac{11}{4}\right) + \mathcal{M}^{(1)}_{\rm f,ir,\infty} + \mathcal{O}(\epsilon)\,, \tag{7.9}$$

for fermion loops, whereas for scalar quarks, the amplitude is given by

$$\mathcal{M}_{\rm s,\infty} = \delta_{ab}\, K_{ab}\, \Lambda_s\, \frac{\alpha_s}{24\,\pi}\, \left(1 + \frac{\alpha_s}{\pi}\,\frac{9}{2}\right) + \mathcal{M}^{(1)}_{\rm s,ir,\infty} + \mathcal{O}(\epsilon)\,. \tag{7.10}$$

As mentioned above, this last result, which agrees with [28], does not include the self interactions of the scalars, thus it differs from the one quoted in [27]. Including a four scalar vertex with coupling given by $\sum_a g_s^2\, \left(T^a_{ij}\, T^a_{kl} + T^a_{il} T^a_{jk}\right)$, and modifying accordingly the mass renormalization of the scalars, we find

$$\mathcal{M}_{\rm s,\infty,4} = \delta_{ab}\, K_{ab}\, \Lambda_s\, \frac{\alpha_s}{24\,\pi}\, \left(1 + \frac{\alpha_s}{\pi}\,\frac{25}{6}\right) + \mathcal{M}^{(1)}_{\rm s,ir,\infty} + \mathcal{O}(\epsilon)\,, \tag{7.11}$$

in agreement with [27].

Notice that in the expressions quoted above, we have taken the limit $\epsilon \to 0$, except in the infrared singular pieces, before evaluating the limit $m_i \to \infty$. The infrared contributions contain a prefactor $(m_i^2)^{-\epsilon}$ that can be expanded only after combining with the real radiation pieces.

The following two subsections contain the explicit results for the form factors $M_i^{(0)}$ and $M_{i,\rm fin}^{(1)}$ in the region below threshold, $s < 4\, m_i^2$. As discussed above, in this region the natural variable to use is θ defined by $x = \exp(i\theta)$, with $0 \le \theta < \pi$. In Appendix A we also give the results for the form factors as linear combinations of the master integrals introduced in the previous sections.

7.1 Amplitudes for quarks

At the one loop level, the contributions from fermion loops to the process $gg \to H$ when $s < 4\, m_f^2$ are given by

$$\begin{aligned}
M_{\rm f}^{(0)} =\ & \frac{1}{4}\,(4 - \theta^2 - (4+\theta^2)\, \cos(\theta))\, \csc^4\left(\frac{\theta}{2}\right) \\
& +\epsilon\Bigg[-2\,(2\,\pi - \theta)\, \cot^2\left(\frac{\theta}{2}\right)\, \csc^2\left(\frac{\theta}{2}\right)\, {\rm Ls}_2^{(0)}(\theta-\pi) - 4\, \cot^2\left(\frac{\theta}{2}\right)\, \csc^2\left(\frac{\theta}{2}\right)\, {\rm Ls}_3^{(1)}(\theta-\pi) \\
& \quad +\frac{1}{2}\csc^4\left(\frac{\theta}{2}\right)\, \left(6 - \theta^2 - 2\,\theta\,\sin(\theta) - 7\,\zeta(3)(1+\cos(\theta)) - 6\,\cos(\theta)\right) \Bigg] \\
& +\epsilon^2\Bigg[-4\,\theta\,\cot\left(\frac{\theta}{2}\right)\, \csc^2\left(\frac{\theta}{2}\right)\, {\rm Cl}_1(\theta-\pi) + 2\,\cot^2\left(\frac{\theta}{2}\right)\, \csc^2\left(\frac{\theta}{2}\right)\, {\rm Ls}_2^{(0)}(\theta-\pi)^2 \\
& \quad +2\,\theta\,\cot^2\left(\frac{\theta}{2}\right)\, \csc^2\left(\frac{\theta}{2}\right)\, {\rm Ls}_3^{(0)}(\theta-\pi) - 4\,\csc^4\left(\frac{\theta}{2}\right)\, {\rm Ls}_3^{(1)}(\theta-\pi) \\
& \quad -2\,\csc^4\left(\frac{\theta}{2}\right)\, (2\,\pi - \theta - \sin(\theta))\, {\rm Ls}_2^{(0)}(\theta-\pi) + \frac{1}{48}\csc^4\left(\frac{\theta}{2}\right)\, \big(336 + 4\,\pi^2 - 4\,\pi^3\,\theta - 24\,\theta^2
\end{aligned}$$

$$
-\pi^2\,\theta^2 - (336 + 4\,\pi^3\,\theta + \pi^2\,(4+\theta^2))\,\cos(\theta) - 144\,\theta\,\sin(\theta) - 336\,\zeta(3)\Big)\Bigg] + \mathcal{O}(e^3)\,. \qquad (7.12)
$$

The form factor at two loops, in turn, is given by:

$$
\begin{aligned}
M^{(1)}_{\text{f,fin}} = \frac{1}{N}\Bigg[\ & \frac{27}{2}\,\cos\left(\frac{\theta}{2}\right)\,\cos(\theta)\,\text{Ls}_4^{(2)}(\theta) + 18\,\cos\left(\frac{\theta}{2}\right)\,\cos(\theta)\,\text{Ls}_4^{(2)}(\theta-\pi) \\
&+\theta\,\cos\left(\frac{\theta}{2}\right)\,(\theta\,\cos(\theta)-\sin(\theta))\,\text{Ls}_2^{(0)}(\theta) \\
&+4\,\cos\left(\frac{\theta}{2}\right)\,((9\,\pi-5\,\theta)\,\cos(\theta)+2\,\sin(\theta))\,\text{Ls}_3^{(1)}(\theta-\pi) \\
&+2\,\cos\left(\frac{\theta}{2}\right)\,((9\,\pi^2-10\,\pi\,\theta+2\,\theta^2)\,\cos(\theta)+2\,(2\,\pi-\theta)\,\sin(\theta))\,\text{Ls}_2^{(0)}(\theta-\pi) \\
&-\sin\left(\frac{\theta}{2}\right)\,((6+4\,\theta^2)\,\cos(\theta)+3\,(-2+\theta^2-\theta\,\sin(\theta)))\,\text{Cl}_1(\theta) \\
&-\frac{1}{2}\left(11\,\theta\,\cos\left(\frac{\theta}{2}\right)+11\,\theta\,\cos\left(\frac{3\theta}{2}\right)+9\,\sin\left(\frac{\theta}{2}\right)-7\,\sin\left(\frac{3\theta}{2}\right)\right)\text{Ls}_3^{(1)}(\theta) \\
&+\frac{1}{8}\Bigg(\cos\left(\frac{\theta}{2}\right)\,\big(-3\,\theta^3+\theta\,(4-140\,\zeta(3))+144\,\pi\,\zeta(3)\big) \\
&\qquad -4\,\sin\left(\frac{\theta}{2}\right)\,(18-9\,\theta^2-28\,\zeta(3)-2\,\cos(\theta)\,(9+\theta^2+14\,\zeta(3))) \\
&\qquad -\cos\left(\frac{3\theta}{2}\right)\,\big(\theta^3-144\,\pi\,\zeta(3)+4\,\theta\,(1+35\,\zeta(3))\big)\Bigg)\Bigg]\,\csc^5\left(\frac{\theta}{2}\right) \\
-N\Bigg[\ & -16\,\cos^2\left(\frac{\theta}{2}\right)\,\sin\left(\frac{\theta}{2}\right)\,\text{LsLsc}_{1,1,2}(\theta) \\
&+\frac{27}{2}\,\cos\left(\frac{\theta}{2}\right)\,\cos(\theta)\,\text{Ls}_4^{(2)}(\theta)+18\,\cos\left(\frac{\theta}{2}\right)\,\cos(\theta)\,\text{Ls}_4^{(2)}(\theta-\pi) \\
&+8\,\pi\,\cos^2\left(\frac{\theta}{2}\right)\,\sin\left(\frac{\theta}{2}\right)\,\text{Ls}_3^{(0)}(\theta-\pi)+8\,\cos^2\left(\frac{\theta}{2}\right)\,\sin\left(\frac{\theta}{2}\right)\,\text{Ls}_4^{(1)}(\theta) \\
&-2\,\cos^2\left(\frac{\theta}{2}\right)\,\sin\left(\frac{\theta}{2}\right)\,\text{Ls}_4^{(1)}(2\,\theta)+8\,\cos^2\left(\frac{\theta}{2}\right)\,\sin\left(\frac{\theta}{2}\right)\,\text{Ls}_4^{(1)}(\theta-\pi) \\
&-\left(16\,\cos^2\left(\frac{\theta}{2}\right)\,\sin\left(\frac{\theta}{2}\right)\,\text{Ls}_2^{(0)}(\theta-\pi)-\theta\,\cos\left(\frac{\theta}{2}\right)\,(\theta\,\cos(\theta)-\sin(\theta))\right)\text{Ls}_2^{(0)}(\theta) \\
&-4\Bigg(8\,\text{Cl}_1(\theta)\,\cos^2\left(\frac{\theta}{2}\right)\,\sin\left(\frac{\theta}{2}\right) \\
&\qquad -\cos\left(\frac{\theta}{2}\right)\,((9\,\pi-5\,\theta)\,\cos(\theta)+2\,\sin(\theta))\Bigg)\,\text{Ls}_3^{(1)}(\theta-\pi)
\end{aligned}
$$

$$
\begin{aligned}
&-2\left(8\,(2\,\pi-\theta)\,\mathrm{Cl}_1(\theta)\,\cos^2\left(\frac{\theta}{2}\right)\,\sin\left(\frac{\theta}{2}\right)\right.\\
&\left.-\cos\left(\frac{\theta}{2}\right)\,\left((9\,\pi^2-10\,\pi\,\theta+2\,\theta^2)\,\cos(\theta)+2\,(2\,\pi-\theta)\,\sin(\theta)\right)\right)\mathrm{Ls}_2^{(0)}(\theta-\pi)\\
&-\frac{1}{2}\left(11\,\theta\,\cos\left(\frac{\theta}{2}\right)+11\,\theta\,\cos\left(\frac{3\theta}{2}\right)-5\,\sin\left(\frac{\theta}{2}\right)-13\,\sin\left(\frac{3\theta}{2}\right)\right)\mathrm{Ls}_3^{(1)}(\theta)\\
&-\sin\left(\frac{\theta}{2}\right)\,\left(-6+8\,\theta^2-3\,\theta\,\sin(\theta)+28\,\zeta(3)+\cos(\theta)\,(6+7\,\theta^2+28\,\zeta(3))\right)\,\mathrm{Cl}_1(\theta)\\
&-\frac{1}{8}\cos\left(\frac{\theta}{2}\right)\,(9\,\theta^3-4\,\theta\,(1-35\,\zeta(3))-144\,\pi\,\zeta(3))\\
&-\frac{1}{8}\,\cos\left(\frac{3\theta}{2}\right)\,(3\,\theta^3+4\,\theta\,(1+35\,\zeta(3))-144\,\pi\,\zeta(3))\\
&-\sin\left(\frac{\theta}{2}\right)\,\Big(-\frac{7}{2}\,\theta^2+30+\cos^2\left(\frac{\theta}{2}\right)\,\big(\frac{79}{90}\,\pi^4+\frac{4}{3}\,\pi^2\,\log(2)^2-3\,\theta^2+\frac{1}{4}\,\theta^4\\
&\left.-\frac{2}{3}\,(45+2\,\log(2)^4+48\,\mathrm{Li}_4(1/2)+42\,\zeta(3)+42\,\log(2)\,\zeta(3))\big)\Big)\right]\,\csc^5\left(\frac{\theta}{2}\right)+\mathcal{O}(\epsilon)
\end{aligned}
\tag{7.13}
$$

7.2 Amplitudes for scalar quarks

The one loop form factor for scalar quarks, $s<4\,m_f^2$ is given by

$$
\begin{aligned}
M_{\mathrm{s}}^{(0)}=&-\frac{1}{8}(2-\theta^2-2\,\cos(\theta))\,\csc^4\left(\frac{\theta}{2}\right)\\
&+\frac{\epsilon}{8}\Big[4(2\,\pi-\theta)\,\csc^4\left(\frac{\theta}{2}\right)\,\mathrm{Ls}_2^{(0)}(\theta-\pi)+8\,\csc^4\left(\frac{\theta}{2}\right)\,\mathrm{Ls}_3^{(1)}(\theta-\pi)\\
&\qquad+\csc^4\left(\frac{\theta}{2}\right)\,(-6+\theta^2+6\,\cos(\theta)+2\,\theta\,\sin(\theta)+14\,\zeta(3))\Big]\\
&+\frac{\epsilon^2}{96}\Big[96\,\theta\,\cot\left(\frac{\theta}{2}\right)\,\csc^2\left(\frac{\theta}{2}\right)\,\mathrm{Cl}_1(\theta-\pi)-48\,\csc^4\left(\frac{\theta}{2}\right)\,\mathrm{Ls}_2^{(0)}(\theta-\pi)^2\\
&\qquad-48\,\theta\,\csc^4\left(\frac{\theta}{2}\right)\,\mathrm{Ls}_3^{(0)}(\theta-\pi)+96\,\csc^4\left(\frac{\theta}{2}\right)\,\mathrm{Ls}_3^{(1)}(\theta-\pi)\\
&\qquad-48\,\csc^4\left(\frac{\theta}{2}\right)\,(-2\,\pi+\theta+\sin(\theta))\,\mathrm{Ls}_2^{(0)}(\theta-\pi)-\csc^4\left(\frac{\theta}{2}\right)\,(168+2\,\pi^2-4\,\pi^3\,\theta-12\,\theta^2\\
&\qquad-\pi^2\,\theta^2-2\,(84+\pi^2)\,\cos(\theta)-72\,\theta\,\sin(\theta)-168\,\zeta(3))\Big]+\mathcal{O}(\epsilon^3)\,.
\end{aligned}
\tag{7.14}
$$

The form factor at two loops, in turn is given by:

$$
M_{\mathrm{s,fin}}^{(1)}=-\frac{1}{N}\Big[\;\frac{27}{4}\,\cos(\theta)\,\mathrm{Ls}_4^{(2)}(\theta)+9\,\cos(\theta)\,\mathrm{Ls}_4^{(2)}(\theta-\pi)
$$

$$
\begin{aligned}
&-\frac{1}{4}\sin\left(\frac{\theta}{2}\right)\left((-3+8\,\theta^2)\cos\left(\frac{\theta}{2}\right)+3\left(\cos\left(\frac{3\theta}{2}\right)-2\,\theta\sin\left(\frac{\theta}{2}\right)\right)\right)\mathrm{Cl}_1(\theta)\\
&-\frac{1}{2}\left(11\,\theta\cos(\theta)-3\sin(\theta)\right)\mathrm{Ls}_3^{(1)}(\theta)+\frac{1}{2}\,\theta\left(\theta\cos(\theta)-\sin(\theta)\right)\mathrm{Ls}_2^{(0)}(\theta)\\
&+2\left((9\,\pi-5\,\theta)\cos(\theta)+2\sin(\theta)\right)\mathrm{Ls}_3^{(1)}(\theta-\pi)\\
&+\left((9\,\pi^2-10\,\pi\,\theta+2\,\theta^2)\cos(\theta)+2\,(2\,\pi-\theta)\sin(\theta)\right)\mathrm{Ls}_2^{(0)}(\theta-\pi)\\
&+\frac{1}{16}\left(-11\,\theta+\theta^3-3\,\theta\cos(2\,\theta)-28\sin(\theta)+17\,\theta^2\sin(\theta)+14\sin(2\,\theta)\right.\\
&\qquad\left.+112\sin(\theta)\,\zeta(3)+\cos(\theta)\left(\theta^3+\theta\,(14-280\,\zeta(3))+288\,\pi\,\zeta(3)\right)\right)\Bigg]\frac{\csc^5\left(\frac{\theta}{2}\right)}{2\cos\left(\frac{\theta}{2}\right)}\\
-N\Bigg[&\;8\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)\mathrm{LsLsc}_{1,1,2}(\theta)\\
&-\frac{27}{4}\cos(\theta)\,\mathrm{Ls}_4^{(2)}(\theta)-9\cos(\theta)\,\mathrm{Ls}_4^{(2)}(\theta-\pi)\\
&-4\,\pi\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)\mathrm{Ls}_3^{(0)}(\theta-\pi)-4\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)\mathrm{Ls}_4^{(1)}(\theta)\\
&+\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)\mathrm{Ls}_4^{(1)}(2\,\theta)-4\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)\mathrm{Ls}_4^{(1)}(\theta-\pi)\\
&-\frac{1}{2}\,\theta\left(\theta\cos(\theta)-\sin(\theta)\right)\mathrm{Ls}_2^{(0)}(\theta)+\frac{11}{2}\left(\theta\cos(\theta)-\sin(\theta)\right)\mathrm{Ls}_3^{(1)}(\theta)\\
&+2\Big(4\,\mathrm{Cl}_1(\theta)\sin(\theta)-\left((9\,\pi-5\,\theta)\cos(\theta)+2\sin(\theta)\right)\Big)\mathrm{Ls}_3^{(1)}(\theta-\pi)\\
&+\Big(4\,(2\,\pi-\theta)\,\mathrm{Cl}_1(\theta)\sin(\theta)+4\sin(\theta)\,\mathrm{Ls}_2^{(0)}(\theta)\\
&-\left((9\,\pi^2-10\,\pi\,\theta+2\,\theta^2)\cos(\theta)+2\,(2\,\pi-\theta)\sin(\theta)\right)\Big)\mathrm{Ls}_2^{(0)}(\theta-\pi)\\
&+\frac{1}{4}\sin\left(\frac{\theta}{2}\right)\left(3\cos\left(\frac{3\theta}{2}\right)-6\,\theta\sin\left(\frac{\theta}{2}\right)-\cos\left(\frac{\theta}{2}\right)\left(3-24\,\theta^2-112\,\zeta(3)\right)\right)\mathrm{Cl}_1(\theta)\\
&+\frac{1}{16}\Big(11\,\theta+3\,\theta^3+3\,\theta\cos(2\,\theta)+52\sin(\theta)+\frac{158}{45}\,\pi^4\sin(\theta)\\
&-21\,\theta^2\sin(\theta)+\theta^4\sin(\theta)+\frac{16}{3}\,\pi^2\log^2(2)\sin(\theta)-\frac{16}{3}\log^4(2)\sin(\theta)\\
&-128\,\mathrm{Li}_4\left(\frac{1}{2}\right)\sin(\theta)-26\sin(2\,\theta)-112\sin(\theta)\,\zeta(3)-112\log(2)\sin(\theta)\,\zeta(3)\\
&+\cos(\theta)\left(3\,\theta^3-14\,\theta\,(1-20\,\zeta(3))-288\,\pi\,\zeta(3)\right)\Big)\Bigg]\frac{\csc^5\left(\frac{\theta}{2}\right)}{2\cos\left(\frac{\theta}{2}\right)}+\mathcal{O}(\epsilon)\,.
\end{aligned}
\tag{7.15}
$$

8. Summary

In this paper we have computed the two-loop master integrals which are needed for the evaluation of the two-loop QCD amplitude in the gluon fusion process $gg \to h$. This is a loop induced process and generally requires a new calculation when modifying the particle content in the loops.

We have automatized the evaluation of the two-loop amplitude using modern reduction methods and providing analytic expressions for the master integrals. We computed the master integrals using the differential equation method. Our results agree with the literature when a comparison is available and with a direct numerical evaluation which is performed with an independent method.

In this paper we have evaluated analytically the two-loop amplitudes for $gg \to h$ via a quark and a scalar quark. The first result agrees with the result of Spira et al., in the analytic form written by Harlander and Kant. The amplitude for the scalar quark is a new result, and agrees with the result derived within the heavy squark approximation.

The master integrals we have presented here, are relevant for other $2 \to 1$ processes in the Standard Model and its extensions and more complicated $2 \to 2$ processes with heavy particles in the loops.

Acknowledgments

We gratefully thank Uli Haisch for very enlightening and useful discussions and suggestions. C.A. is grateful to ETH Zürich for their hospitality and providing access to MAPLE v10. The work of A.D. was partially supported by the Swiss National Science Foundation (SNF) under contract number 200020-109162 and by the Forschungskredit der Universität Zürich.

A. Amplitudes in terms of master integrals

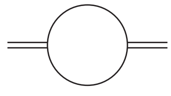
We present the results for the amplitudes in eq. (7.1) in terms of the master integrals in Section 3. We write the amplitudes as

$$\mathcal{M}_i^{(n)} = \delta_{ab}\, K_{ab}\, \Lambda_i\, (4\,\pi\,\mu^2)^{(n+1)\,\epsilon}\, \bar{\mathcal{M}}_i^{(n)}\,. \tag{A.1}$$

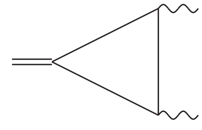
A.1 Fermionic amplitudes

For the fermionic contribution at one loop, we have

$$\bar{\mathcal{M}}_{\rm f}^{(0)} = \Bigg\{ \frac{8\,\epsilon\,(1+\epsilon)\,x}{(1-x)^2}$$



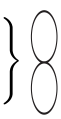
$$- \frac{4\,s\,x\,(1+(2+4\,\epsilon+4\,\epsilon^2)\,x+x^2)}{(1-x)^4}$$



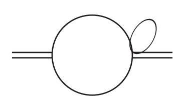
$$\Bigg\}\,. \tag{A.2}$$

At two loops:

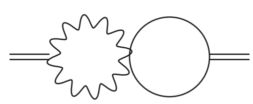
$$
\begin{aligned}
\bar{\mathcal{M}}_{\rm f}^{(1)} =& \Bigg\{ \frac{N}{\epsilon\, s^2\, x^2} \Big[-24\, x\, (1+x)^2 - \epsilon\, (1 + 56\, x + 238\, x^2 + 56\, x^3 + x^4) \\
&+ \epsilon^2\, (9 - 360\, x - 994\, x^2 - 360\, x^3 + 9\, x^4) + 2\, \epsilon^3\, (19 - 900\, x - 2014\, x^2 - 900\, x^3 + 19\, x^4) \Big] \\
&+ \frac{C_F}{s^2\, x^2\, (1+x)^2} \Big[4\, (1 - 12\, x - 25\, x^2 + 8\, x^3 - 25\, x^4 - 12\, x^5 + x^6) \\
&- 8\, \epsilon\, (1 + 10\, x + 51\, x^2 + 36\, x^3 + 51\, x^4 + 10\, x^5 + x^6) \\
&- 8\, \epsilon^2\, (5 + 37\, x + 203\, x^2 + 214\, x^3 + 203\, x^4 + 37\, x^5 + 5\, x^6) \Big] \Bigg\}
\end{aligned}
$$



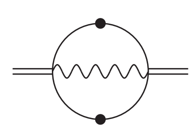
$$
\begin{aligned}
&+ \Bigg\{ \frac{N}{s\, (1-x)^2} \Big[24\, (1+x)^2 + 20\, \epsilon^2\, (17 + 46\, x + 17\, x^2) + 4\, \epsilon\, (23 + 42\, x + 23\, x^2) \Big] \\
&+ \frac{C_F}{s\, (1-x)^2\, (1+x)^2} \Big[8\, (1-x)^2\, (1 - 6\, x + x^2) + 8\, \epsilon\, (9 - 8\, x + 30\, x^2 - 8\, x^3 + 9\, x^4) \\
&+ 8\, \epsilon^2\, (21 + 8\, x + 102\, x^2 + 8\, x^3 + 21\, x^4) \Big] \Bigg\}
\end{aligned}
$$



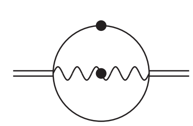
$$
+ \frac{N}{(1-x)^2} \Big[16\, x - 16\, \epsilon\, x - 16\, \epsilon^2\, x \Big]
$$



$$
\begin{aligned}
&+ \Bigg\{ \frac{N}{\epsilon^2\, (1-x)^4} \Big[4\, s\, x\, (1+x)^2 + 2\, \epsilon\, s\, x\, (3+x)\, (1 + 3\, x) + \epsilon^2\, s\, (1 + 2\, x + 122\, x^2 + 2\, x^3 + x^4) \Big] \\
&+ \frac{C_F}{\epsilon\, (1-x)^4} \Big[8\, s\, x\, (1+x)^2 - 4\, \epsilon\, s\, (1 - 8\, x - 10\, x^2 - 8\, x^3 + x^4) \Big] \Bigg\}
\end{aligned}
$$



$$
+ \Bigg\{ \frac{N}{(1-x)^4} \Big[4\, \epsilon\, s\, (1+x)^2\, (1 - 26\, x + x^2) \Big] + \frac{C_F}{(1-x)^2} \Big[-16\, \epsilon\, s\, (1+x)^2 \Big] \Bigg\}
$$

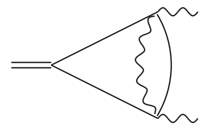


$$
+ \Bigg\{ \frac{N}{\epsilon\, (1-x)^4} \Big[-24\, (1+x)^2\, (1 - 4\, x + x^2) - 4\, \epsilon\, (1 - 4\, x + x^2)\, (7 + 26\, x + 7\, x^2)
$$

$$- 4\,\epsilon^2\,(43 - 46\,x - 442\,x^2 - 46\,x^3 + 43\,x^4) - 4\,\epsilon^3\,(1 - 4\,x + x^2)\,(201 + 550\,x + 201\,x^2)\Big]$$

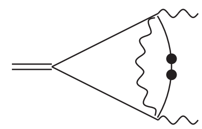
$$+ \frac{C_F}{(1-x)^4}\Big[- 16\,(1+x)^2\,(1 - 4\,x + x^2) - 8\,\epsilon\,(7 - 16\,x - 30\,x^2 - 16\,x^3 + 7\,x^4)$$

$$- 16\,\epsilon^2\,(10 - 19\,x - 70\,x^2 - 19\,x^3 + 10\,x^4)\Big]\Bigg\}$$

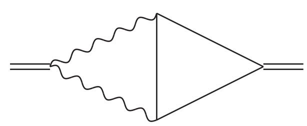


$$+ \Bigg\{ \frac{N}{\epsilon^2\,(1-x)^6}\Big[- 32\,s^2\,x^2\,(1+x)^2 - 16\,\epsilon\,s^2\,x^2\,(5 + 14\,x + 5\,x^2) - 8\,\epsilon^2\,s^2\,x^2\,(41 + 118\,x + 41\,x^2)\Big]$$

$$+ \frac{C_F}{\epsilon\,(1-x)^6}\Big[- 16\,s^2\,x^2\,(1+x)^2 - 32\,\epsilon\,s^2\,x^2\,(3 + 4\,x + 3\,x^2)\Big]\Bigg\}$$

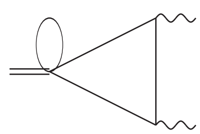


$$+ \Bigg\{ \frac{N}{\epsilon\,(1-x)^4}\Big[- 8\,s\,x\,(1+x)^2 + 16\,\epsilon\,s\,x\,(1+x^2)\Big]\Bigg\}$$



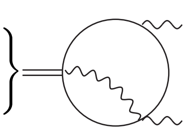
$$+ \Bigg\{ \frac{N}{(1-x)^2}\Big[32\,\epsilon\,(1+x)^2 + 4\,(1 + 6\,x + x^2)\Big]$$

$$- \frac{C_F}{(1-x)^2}\Big[8\,(1 + 14\,x + x^2) + 96\,\epsilon\,x\Big]\Bigg\}$$

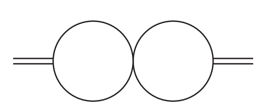


$$+ \Bigg\{ \frac{N}{(1-x)^2}\Big[- 80\,\epsilon^2\,(1+x)^2 + 2\,(1 - 18\,x + x^2) - 16\,\epsilon\,(1 + 4\,x + x^2)\Big]$$

$$+ \frac{C_F}{(1-x)^2}\Big[- 8\,(1 - 6\,x + x^2) + 8\,\epsilon\,(1 + 6\,x + x^2) + 8\,\epsilon^2\,(5 + 6\,x + 5\,x^2)\Big]\Bigg\}$$

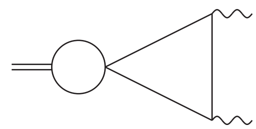


$$+ \quad \frac{C_F}{(1-x)^2\,(1+x)^2}\Big[- 32\,\epsilon^2\,x^2 + 16\,x\,(1+x^2) - 16\,\epsilon\,x\,(1 + 4\,x + x^2)\Big]$$

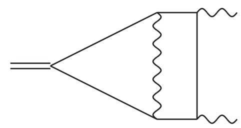


$$+\left\{\frac{N}{(1-x)^4}\left[8\,s\,x\,(1+x)^2+24\,\epsilon\,s\,x\,(1+x)^2\right]\right.$$

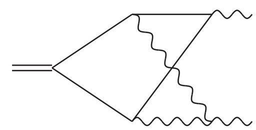
$$\left.+\frac{C_F}{(1-x)^4}\left[-16\,\epsilon\,s\,(1-x)^2\,x+8\,s\,x\,(1+x)^2\right]\right\}$$



$$+\frac{C_F}{(1-x)^6}\left[-8\,s^2\,x\,(1+x)^2\,(1+x^2)\right]$$



$$+\left\{\frac{N}{(1-x)^4}\left[-4\,s^2\,x\,(1+x)^2-2\,\epsilon\,s^2\,x\,(3+x)\,(1+3\,x)\right]\right\}$$

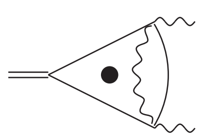


$$+\left\{\frac{N}{\epsilon\,s\,(1-x)^2}\left[-96\,(1+x)^2-32\,\epsilon\,(5+16\,x+5\,x^2)\right.\right.$$

$$\left.-8\,\epsilon^2\,(105+326\,x+105\,x^2)-24\,\epsilon^3\,(161+446\,x+161\,x^2)\right]$$

$$-\frac{16\,C_F}{s\,(1-x)^2}\left[4\,(1+x)^2+8\,\epsilon\,(2+3\,x+2\,x^2)\right.$$

$$\left.\left.+3\,\epsilon^2\,(17+38\,x+17\,x^2)\right]\right\}$$

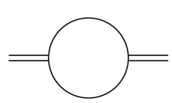


$$+\mathcal{O}(\epsilon)\,. \qquad \text{(A.3)}$$

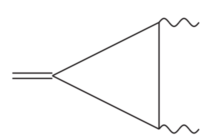
A.2 Scalar amplitudes

The scalar contributions are given by

$$\bar{\mathcal{M}}_{\rm s}^{(0)}=-\frac{2\,\epsilon\,(1+\epsilon)\,x}{(1-x)^2}$$



$$+\frac{4\,(1+\epsilon+\epsilon^2)\,s\,x^2}{(1-x)^4}$$



$$, \qquad \text{(A.4)}$$

at one loop, and

$$\bar{\mathcal{M}}_{\rm s}^{(1)}=+\left\{\frac{N}{\epsilon\,s^2\,x}\left[24\,x-4\,\epsilon^2\,(1-108\,x+x^2)-4\,\epsilon\,(1-24\,x+x^2)+4\,\epsilon^3\,(5+462\,x+5\,x^2)\right]\right.$$

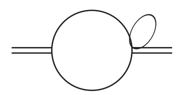
$$+\frac{C_F}{s^2\,x\,(1+x)^2}\left[4\,x\,(9-2\,x+9\,x^2)+4\,\epsilon\,(1+23\,x+32\,x^2+23\,x^3+x^4)\right.$$

$$+ 8\,\epsilon^2\,(1 + 48\,x + 78\,x^2 + 48\,x^3 + x^4)\Big]\Bigg\}$$

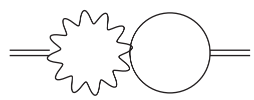
$$+ \Bigg\{ \frac{N}{s\,(1-x)^2}\Big[- 24\,x - 88\,\epsilon\,x - 400\,\epsilon^2\,x \Big]$$

$$+ \frac{C_F}{s\,(1-x)^2\,(1+x)^2}\Big[- 4\,(1-x)^2\,(2 - 3\,x + 2\,x^2) - 8\,\epsilon\,(1 + 3\,x + 3\,x^3 + x^4)$$

$$- 16\,\epsilon^2\,(1+x)^2\,(1 + 3\,x + x^2)\Big]\Bigg\}$$

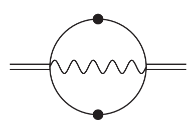


$$+ \frac{N}{(1-x)^2}\Big[- 4\,x + 4\,\epsilon\,x + 4\,\epsilon^2\,x \Big]$$

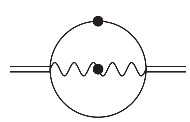


$$+ \Bigg\{ \frac{N}{\epsilon^2\,(1-x)^4}\Big[- 4\,s\,x^2 - 8\,\epsilon\,s\,x^2 + 4\,\epsilon^2\,s\,x\,(1 - 10\,x + x^2)\Big]$$

$$+ \frac{C_F}{\epsilon\,(1-x)^4}\Big[- 8\,s\,x^2 - 24\,\epsilon\,s\,x^2 \Big]\Bigg\}$$



$$+ \frac{N}{(1-x)^4}\Big[24\,\epsilon\,s\,x\,(1+x)^2 \Big]$$

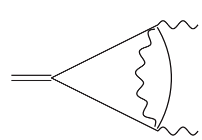


$$+ \Bigg\{ \frac{N}{\epsilon\,(1-x)^4}\Big[24\,x\,(1 - 4\,x + x^2) + 40\,\epsilon\,x\,(1 - 4\,x + x^2) + 8\,\epsilon^2\,x\,(27 - 110\,x + 27\,x^2)$$

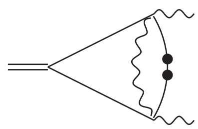
$$+ 952\,\epsilon^3\,x\,(1 - 4\,x + x^2)\Big]$$

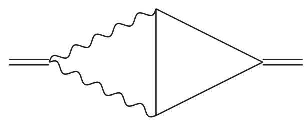
$$+ \frac{C_F}{(1-x)^4}\Big[16\,x\,(1 - 4\,x + x^2) + 8\,\epsilon\,x\,(5 - 22\,x + 5\,x^2)$$

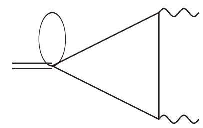
$$+ 8\,\epsilon^2\,x\,(21 - 86\,x + 21\,x^2)\Big]\Bigg\}$$

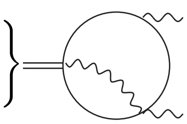


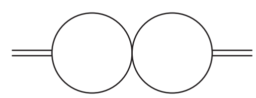
$$+\left\{\frac{N}{\epsilon^2\,(1-x)^6}\left[32\,s^2\,x^3+96\,\epsilon\,s^2\,x^3+400\,\epsilon^2\,s^2\,x^3\right]\right.$$

$$\left.+\frac{C_F}{\epsilon\,(1-x)^6}\left[16\,s^2\,x^3+80\,\epsilon\,s^2\,x^3\right]\right\}$$ 

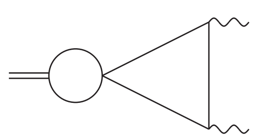
$$+\frac{N}{\epsilon\,(1-x)^4}\left[8\,s\,x^2-8\,\epsilon\,s\,x^2\right]$$ 

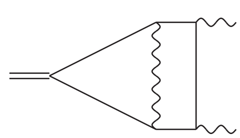
$$+\left\{\frac{N}{(1-x)^2}\left[-8\,x-32\,\epsilon\,x\right]+\frac{C_F}{(1-x)^2}\left[20\,x+32\,\epsilon\,x\right]\right\}$$ 

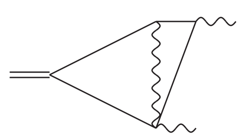
$$+\left\{\frac{N}{(1-x)^2}\left[8\,x+24\,\epsilon\,x+80\,\epsilon^2\,x\right]+\frac{C_F}{(1-x)^2}\left[-8\,x-16\,\epsilon\,x-32\,\epsilon^2\,x\right]\right\}$$ 

$$+\frac{C_F}{(1-x)^2\,(1+x)^2}\left[2\,\epsilon\,(1-x)^2\,x+2\,\epsilon^2\,(1-x)^2\,x-4\,x\,(1+x^2)\right]$$ 

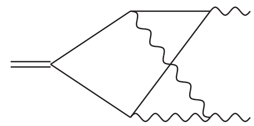
$$+\left\{\frac{N}{(1-x)^4}\left[-8\,s\,x^2-24\,\epsilon\,s\,x^2\right]\right.$$

$$\left.+\frac{C_F}{(1-x)^4}\left[4\,s\,x^2+4\,\epsilon\,s\,x^2\right]\right\}$$ 

$$+\frac{C_F}{(1-x)^6}\left[8\,s^2\,x^2\,(1+x^2)\right]$$ 

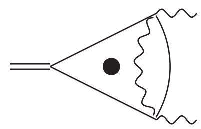
$$+\left\{\frac{N}{(1-x)^4}\left[s\,x^2\right]+\frac{C_F}{(1-x)^4}\left[-4\,s\,x^2\right]\right\}$$ 

$$+\frac{N}{(1-x)^4}\left[4\,s^2\,x^2+8\,\epsilon\,s^2\,x^2\right]$$



$$+\left\{\frac{N}{\epsilon\,s\,(1-x)^2}\left[96\,x+208\,\epsilon\,x+1072\,\epsilon^2\,x+4608\,\epsilon^3\,x\right]\right.$$

$$\left.+\frac{C_F}{s\,(1-x)^2}\left[64\,x+224\,\epsilon\,x+864\,\epsilon^2\,x\right]\right\}$$



$$+\,\mathcal{O}(\epsilon)\,, \tag{A.5}$$

at two loops.

References

[1] G. Ossola, C. G. Papadopoulos and R. Pittau, arXiv:hep-ph/0609007.

[2] A. V. Manohar and M. B. Wise, Phys. Lett. B **636**, 107 (2006) [arXiv:hep-ph/0601212].

[3] M. Spira, A. Djouadi, D. Graudenz and P. M. Zerwas, Phys. Lett. B **318**, 347 (1993); M. Spira, A. Djouadi, D. Graudenz and P. M. Zerwas, Nucl. Phys. B **453**, 17 (1995) [arXiv:hep-ph/9504378].

[4] S. Dawson, Nucl. Phys. B **359**, 283 (1991).

[5] S. Laporta, Int. J. Mod. Phys. A **15**, 5087 (2000) [arXiv:hep-ph/0102033].

[6] V. A. Smirnov and O. L. Veretin, Nucl. Phys. B **566**, 469 (2000) [arXiv:hep-ph/9907385].

[7] J. B. Tausk, Phys. Lett. B **469**, 225 (1999) [arXiv:hep-ph/9909506].

[8] C. Anastasiou, J. B. Tausk and M. E. Tejeda-Yeomans, Nucl. Phys. Proc. Suppl. **89**, 262 (2000) [arXiv:hep-ph/0005328].

[9] C. Anastasiou and A. Daleo, JHEP **0610**, 031 (2006) [arXiv:hep-ph/0511176].

[10] M. Czakon, Comput. Phys. Commun. **175**, 559 (2006) [arXiv:hep-ph/0511200].

[11] T. Gehrmann and E. Remiddi, Nucl. Phys. B **580**, 485 (2000) [arXiv:hep-ph/9912329].

[12] T. Binoth and G. Heinrich, Nucl. Phys. B **585**, 741 (2000) [arXiv:hep-ph/0004013].

[13] T. Binoth and G. Heinrich, Nucl. Phys. B **680**, 375 (2004) [arXiv:hep-ph/0305234].

[14] C. Anastasiou, K. Melnikov and F. Petriello, arXiv:hep-ph/0505069.

[15] C. Anastasiou and A. Lazopoulos, JHEP **0407**, 046 (2004) [arXiv:hep-ph/0404258].

[16] A. Daleo, unpublished.

[17] R. Bonciani, P. Mastrolia and E. Remiddi, Nucl. Phys. B **661**, 289 (2003) [Erratum-ibid. B **702**, 359 (2004)] [arXiv:hep-ph/0301170].

[18] R. Bonciani, P. Mastrolia and E. Remiddi, Nucl. Phys. B **690**, 138 (2004) [arXiv:hep-ph/0311145].

[19] A. I. Davydychev and M. Y. Kalmykov, Nucl. Phys. B **699**, 3 (2004) [arXiv:hep-th/0303162].

[20] J. Fleischer, O. V. Tarasov and V. O. Tarasov, Phys. Lett. B **584**, 294 (2004) [arXiv:hep-ph/0401090].

[21] A. I. Davydychev and M. Y. Kalmykov, Nucl. Phys. B **605** (2001) 266 [arXiv:hep-th/0012189].

[22] D. J. Broadhurst, J. Fleischer and O. V. Tarasov, Z. Phys. C **60**, 287 (1993) [arXiv:hep-ph/9304303].

[23] M. Y. Kalmykov and O. Veretin, Phys. Lett. B **483**, 315 (2000) [arXiv:hep-th/0004010].

[24] T. Gehrmann and E. Remiddi, Comput. Phys. Commun. **141**, 296 (2001) [arXiv:hep-ph/0107173].

[25] R. Harlander and P. Kant, JHEP **0512**, 015 (2005) [arXiv:hep-ph/0509189].

[26] M. Spira, A. Djouadi, D. Graudenz and P. M. Zerwas, Phys. Lett. B **318**, 347 (1993).

[27] S. Dawson, A. Djouadi and M. Spira, Phys. Rev. Lett. **77**, 16 (1996) [arXiv:hep-ph/9603423].

[28] Muhlleitner, M. Spira, presented during the HP^2 workshop, ETH Zürich, Sept. 2006

[29] E. Remiddi and J. A. M. Vermaseren, Int. J. Mod. Phys. A **15** (2000) 725 [arXiv:hep-ph/9905237].

[30] F. V. Tkachov, Phys. Lett. B **100**, 65 (1981).

[31] K. G. Chetyrkin and F. V. Tkachov, Nucl. Phys. B **192**, 159 (1981).

[32] A. V. Kotikov, Phys. Lett. B **254**, 158 (1991).

[33] A. V. Kotikov, Phys. Lett. B **259**, 314 (1991).

[34] A. V. Kotikov, Phys. Lett. B **267**, 123 (1991).

[35] E. Remiddi, Nuovo Cim. A **110**, 1435 (1997) [arXiv:hep-th/9711188].

[36] M. Caffo, H. Czyz, S. Laporta and E. Remiddi, Acta Phys. Polon. B **29**, 2627 (1998) [arXiv:hep-th/9807119]; M. Caffo, H. Czyz, S. Laporta and E. Remiddi, Nuovo Cim. A **111**, 365 (1998) [arXiv:hep-th/9805118].

[37] D. Maitre, Comput. Phys. Commun. **174**, 222 (2006) [arXiv:hep-ph/0507152].

[38] M. Y. Kalmykov, Nucl. Phys. B **718** (2005) 276 [arXiv:hep-ph/0503070].

[39] M. Y. Kalmykov and A. Sheplyakov, Comput. Phys. Commun. **172** (2005) 45 [arXiv:hep-ph/0411100].

[40] P. Nason, S. Dawson and R. K. Ellis, Nucl. Phys. B **327**, 49 (1989) [Erratum-ibid. B **335**, 260 (1990)].

[41] G. Rodrigo and A. Santamaria, Phys. Lett. B **313**, 441 (1993) [arXiv:hep-ph/9305305].

[42] A. Djouadi, M. Spira and P. M. Zerwas, Phys. Lett. B **264**, 440 (1991).