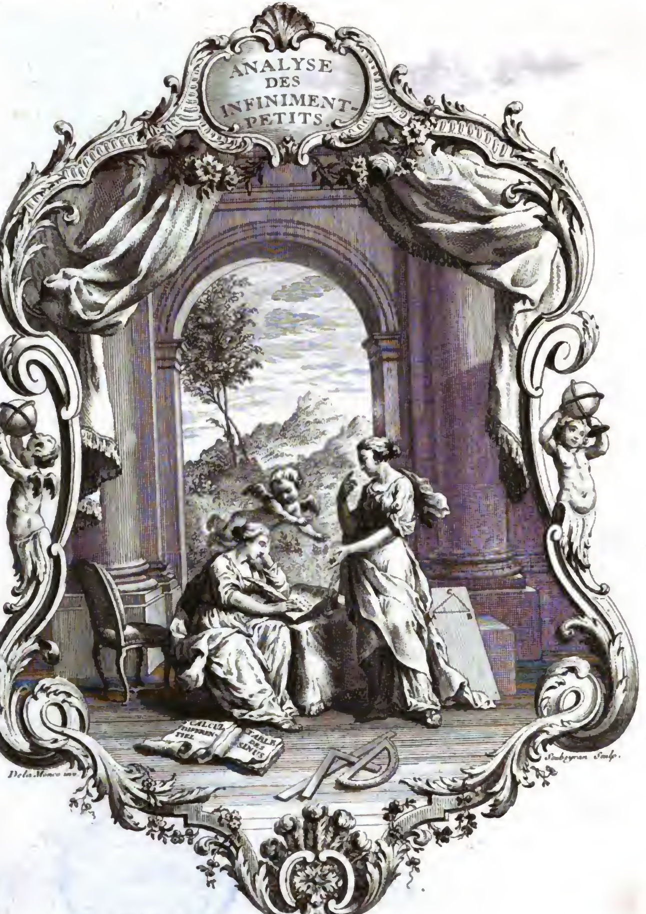

ANALYSE
DES
INFINIMENT-
PETITS.

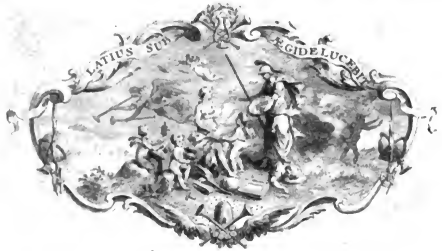
INTRODUCTIO
IN ANALYSIN
INFINITORUM.

AUCTORE

LEONHARDO EULERO,

Professore Regio BEROLINENSI, *& Academiæ Imperialis Scientiarum* PETROPOLITANÆ *Socio.*

TOMUS PRIMUS.

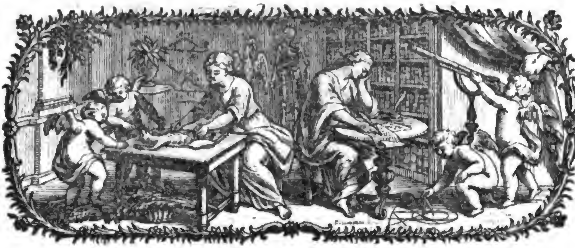


LAUSANNÆ,

Apud MARCUM-MICHAELEM BOUSQUET & Socios.

MDCCXLVIII.

A Lausanne et Geneve, chez MARC-MICHEL BOUSQUET *et Comp.e 1748.*



ILLUSTRISSIMO VIRO

JOHANNI JACOBO

DORTOUS DE MAIRAN,

UNI EX XLVIRIS

ACADEMIÆ GALLICÆ,

REGIÆ ETIAM SCIENTIARUM

PARISIENSIS,

IN QUA SECRETARII PERPETUI MUNUS NUPER

ABDICAVIT,

* NEC

NEC NON

ALIARUM BENE MULTARUM,

LONDINENSIS, PETROPOLITANÆ,

&c.

SOCIETATUM, ACADEMIARUMVE

SOCIO DIGNISSIMO.

MARCUS-MICHAEL BOUSQUET.

VIR ILLUSTRISSIME,

P*Atronos* Euleriano *ſcripto quærere neceſſe neutiquam eſſe, Mathematicarum Diſciplinarum cultoribus ſatis conſtat. Sciunt utique illi, varias earum partes novis eum luminibus ſic illuſtraſſe, ut indè meritò clariſ-*

ſimi

ſimi rerum in his abſtruſiſſimarum interpretis locum ſit conſequutus. Quem quin egregiè tueatur, immò tollat ſe altiùs quoque opere iſthoc, nemo dubitabit, certior hiſce factus, indulſiſſe Te *mihi, ut illuſtriſſimo nomini* Tuo *dicatum publicè prodiret. Pertinere autem hoc in me collatum beneficium ad Auctoris decus probe intelligens, Ipſe, ut eo uterer, lubens conceſſit; & cum in rem meam faciat omnimodò, quî neglexiſſem?*

Ab his equidem, quibus Libros inſcribunt, ſibi neſcio quid ideò deberi, plerique tacitè conſtituunt; acceptaque beneficia quodammodo remunerari, ut ſeſe ferè nexu liberent omni. Ego verò ſecus ſentio. Mihi certè merum eſt beneficium patroni, quòd ſcriptoris aut excuſoris *opera id genus honore condecorari patiantur. Hac mente utique* Tibi, VIR ILLUSTRISSIME, *animi gratissimi summæque observantiæ professionem hisce publicam excipias, rogo.*

Paratum promtumque semper juvandis litterarum studiis qui Te *novit, &* *notus vel hoc nomine es cuicumque in Republica doctorum Europæ totius non hospiti, plurimis officiis meæ etiam conditionis homines à* Te *affectos fuisse statuat necesse est. Nempe, tanquam* Tibi *uni esset injunctum curare, ut floreant humanum ingenium illustrantes scientiæ omnes, hominumque in usus adinventæ artes, ad singulis inservientium artifices etiam* Te *demittere dignaris, vel ab illa sublimium rerum perscrutatione, Cælive ipsius*

ſius Tibi *tam nota regione, ut quæ huc-uſque mentes hominum metu complebant Phænomena minùs intellecta, per* Te *jam grato tantùm admirationis ſenſu contemplentur, earumque cauſas habeant perſpectas.*

Hinc ille veluti ex condicto Academiarum Orbis eruditi concurſus, ut adlectum Te *cætui ſuo conſequerentur, ornamento aliàs carituro inſigni, quo cæteras nollent præ ſe frui. Hinc inprimis Illuſtriſſimæ Pariſienſis de* Te *judicium, cum ageretur de ſucceſſore ſufficiendo in locum emeriti* Fontenellii, *Viri, cujus ex ore calamoque fluere Scientiarum Artiumque omnium exquiſitiores divitiæ, elegantiæque univerſæ perpetuo viſæ ſunt, & videbuntur dum ſani ſenſus quicquam hu-*

mano ingenio erit. Tibi, *scilicet*, *Commentariorum Academiæ conscribendorum provincia, cui præfectus ille erat, demandabatur continuò; quam, ut ornare diutiùs voluisses, docti omnes optabant: hoc uno minùs dolentes* Te *aliter censuisse, quòd aliis* Tibi *magis placituris, profuturisque nihilominus litteris in universum eruditionis ingeniive thesauros impenderes. Quod ut ad ultimas usque metas hominum vitæ positas incolumis, florens, atque beatus præstes, omni votorum contentione precor. Vale!*

Dabam *Lausannæ* die 1. Aprilis
Anni Æræ Dionys. 1748.

PRÆFA-

PRÆFATIO.

SÆpenumero animadverti, maximam difficultatum partem, quas Matheseos cultores in addiscenda Analysi infinitorum offendere solent, inde oriri, quod, Algebra communi vix apprehensa, animum ad illam sublimiorem artem appellant; quo fit, ut non solum quasi in limine subsistant, sed etiam perversas ideas illius infiniti, cujus notio in subsidium vocatur, sibi forment. Quanquam autem Analysis infinitorum non perfectam Algebræ communis, omniumque artificiorum adhuc inventorum cognitionem requirit; tamen plurimæ extant quæstiones, quarum evolutio discentium animos ad sublimiorem scientiam præparare valet, quæ tamen in communibus Algebræ elementis, vel omittuntur, vel non satis accurate pertractantur. Hanc ob rem non dubito, quin ea, quæ in his libris congessi, hunc defectum abunde supplere queant. Non solum enim operam dedi, ut eas res, quas Analysis

infini-

infinitorum absolute requirit, uberius atque distinctius exponerem, quam vulgo fieri solet; sed etiam satis multas quæstiones enodavi, quibus Lectores sensim & quasi præter expectationem ideam infiniti sibi familiarem reddent. Plures quoque quæstiones per præcepta communis Algebræ hic resolvi, quæ vulgo in Analysi infinitorum tractantur: quo facilius deinceps utriusque Methodi summus consensus eluceat.

Divisi hoc Opus in duos Libros, in quorum priori, quæ ad meram Analysin pertinent, sum complexus: in posteriori vero, quæ ex Geometria sunt scitu necessaria, explicavi, quoniam Analysis infinitorum ita quoque tradi solet, ut simul ejus applicatio ad Geometriam ostendatur. In utroque autem prima Elementa prætermisi, eaque tantum exponenda duxi, quæ alibi, vel omnino non, vel minus commode tractata, vel ex diversis principiis petita reperiuntur.

In primo igitur Libro, cum universa Analysis infinitorum circa quantitates variabiles earumque Functiones versetur, hoc argumentum de Functionibus inprimis fusius exposui; atque Functionum tam transformationem, quam resolutionem & evolutionem per series infinitas demonstravi. Complures enumeravi Functionum species, quarum in Analysi sublimiori præcipue ratio est habenda. Primum eas distinxi in algebraicas & transcendentes; quarum illæ per operationes in Algebra communi usitatas ex quantitatibus variabilibus formantur, hæ vero vel per alias rationes componuntur, vel ex iisdem operationibus infinities repetitis efficiuntur. Algebraicarum functionum primaria subdivisio fit in rationales & irrationales, priores docui cum in partes simpliciores, tum in factores resolvere; quæ operatio in Calculo integrali maximum adjumentum affert; posteriores vero, quemadmodum idoneis substitutionibus ad formam rationalem perduci queant ostendi. Evolutio autem per series infinitas ad utrumque genus æque pertinet, atque etiam ad Fun-

ctiones

ctiones transcendentes summa cum utilitate applicari solet; at quantopere doctrina de seriebus infinitis Analysin sublimiorem amplificaverit, nemo est qui ignoret. Nonnulla igitur adjunxi Capita, quibus plurium serierum infinitarum proprietates, atque summas sum scrutatus; quarum quædam ita sunt comparatæ, ut sine subsidio Analysis infinitorum vix investigari posse videantur. Hujusmodi series sunt, quarum summæ exprimuntur, vel per Logarithmos vel Arcus circulares: quæ quantitates cum sint transcendentes, dum per quadraturam Hyperbolæ & Circuli exhibentur, maximam partem demum in Analysi infinitorum tractari sunt solitæ. Postquam autem a potestatibus ad quantitates exponentiales essem progressus, quæ nil aliud sunt nisi potestates, quarum exponentes sunt variabiles; ex earum conversione maxime naturalem ac fœcundam Logarithmorum ideam sum adeptus: unde non solum amplissimus eorum usus sponte est consecutus, sed etiam ex ea cunctas series infinitas, quibus vulgo istæ quantitates repræsentari solent, elicere licuit: hincque adeo facillimus se prodidit modus Tabulas Logarithmorum construendi. Simili modo in contemplatione Arcuum circularium sum versatus; quod quantitatum genus, etsi a Logarithmis maxime est diversum, tamen tam arcto vinculo est connexum, ut dum alterum imaginarium fieri videtur, in alterum transeat. Repetitis autem ex Geometria quæ de inventione Sinuum & Cosinuum Arcuum multiplorum ac submultiplorum traduntur, ex Sinu vel Cosinu cujusque Arcus expressi Sinum Cosinumque Arcus minimi & quasi evanescentis, quo ipso ad series infinitas sum deductus: unde, cum Arcus evanescens Sinui suo sit æqualis, Cosinus vero radio, quemvis Arcum cum suo Sinu & Cosinu ope serierum infinitarum comparavi. Tum vero tam varias expressiones cum finitas tum infinitas pro hujus generis quantitatibus obtinui, ut ad earum naturam perspiciendam Calculo infinitesimali prorsus non amplius eſſet opus. Atque quemadmodum Logarithmi peculiarem Algorithmum requirunt, cujus in univerſa Analyſi ſummus extat uſus, ita quantitates circulares ad certam quoque Algorithmi normam perduxi; ut in calculo æque commode ac Logarithmi & ipſæ quantitates algebraicæ tractari poſſent. Quantum autem hinc utilitatis ad reſolutionem difficillimarum quæſtionum redundet, cum nonnulla Capita hujus Libri luculenter declarant, tum ex Analyſi infinitorum plurima ſpecimina proferri poſſent, niſi jam ſatis eſſent cognita, & indies magis multiplicarentur. Maximum autem hæc inveſtigatio attulit adjumentum ad Functiones fractas in factores reales reſolvendas; quod argumentum, cum in Calculo integrali ſit prorſus neceſſarium, diligentius enucleavi. Series poſtmodum infinitas, quæ ex hujuſmodi Functionum evolutione naſcuntur, & quæ recurrentium nomine innotuerunt, examini ſubjeci; ubi earum tam ſummas quam terminos generales, aliasque inſignes proprietates exhibui: & quoniam ad hæc reſolutio in factores manuduxit, ita viciſſim, quemadmodum producta ex pluribus, imo etiam infinitis, factoribus conflata per multiplicationem in ſeries explicentur, perpendi. Quod negotium non ſolum ad cognitionem innumerabilium ſerierum viam aperuit, ſed quia hoc modo ſeries in producta ex infinitis factoribus conſtantia reſolvere licebat, ſatis commodas inveni expreſſiones numericas, quarum ope Logarithmi Sinuum, Coſinuum, & Tangentium faciliime ſupputari poſſunt. Præterea quoque ex eodem fonte ſolutiones plurium quæſtionum, quæ circa partitionem numerorum proponi poſſunt, derivavi; cujuſmodi quæſtiones ſine hoc ſubſidio vires Analyſeos ſuperare videantur. Hæc tanta materiarum diverſitas in plura volumina facile excreſcere potuiſſet; ſed omnia, quantum fieri potuit, tam ſuccincte propoſui, ut ubique fundamentum clariſſime quidem explicaretur, uberior vero amplificatio induſtriæ Lectorum relinqueretur; quo habeant, quibus

quibus vires ſuas exerceant, fineſque Analyſeos ulterius promoveant. Neque enim vereor profiteri, in hoc Libro non ſolum multa plane nova contineri; ſed etiam fontes eſſe detectos, unde plurima inſignia inventa adhuc hauriri queant.

Eodem inſtituto ſum uſus in altero Libro, ubi, quæ vulgo ad Geometriam ſublimiorem referri ſolent, pertractavi. Antequam autem de Sectionibus Conicis, quæ alias fere ſolæ hunc locum occupant, agerem; Theoriam Linearum Curvarum in genere ita propoſui, ut ad ſcrutationem naturæ quarumvis Linearum Curvarum cum utilitate adhiberi poſſet. Ad hoc nullum aliud ſubſidium affero, præter æquationem, qua cujuſque Lineæ Curvæ natura exprimitur; ex eaque cum figuram, tum primarias proprietates deducere doceo: id quod potiſſimum in Sectionibus Conicis præſtitiſſe mihi ſum viſus; quæ antehac vel ſecundum ſolam Geometriam vel per Analyſin quidem, ſed nimis imperfecte ac minus naturaliter, tractari ſunt ſolitæ. Ex æquatione ſcilicet generali pro Lineis ſecundi ordinis primum earum proprietates generales explicavi, tum eas in genera ſeu ſpecies ſubdiviſi; reſpiciendo utrum habeant ramos in infinitum excurrentes, an vero tota Curva finito ſpatio includatur. Priori autem caſu inſuper diſpiciendum erat, quot ſint rami in infinitum excurrentes, & cujus naturæ ſint ſinguli; an habeant Lineas rectas aſymtotas, an minus. Sicque obtinui tres conſuetas Sectionum Conicarum ſpecies; quarum prima eſt Ellipſis, tota in ſpatio finito contenta; ſecunda autem Hyperbola, quæ quatuor habet ramos infinitos ad duas rectas aſymptotas convergentes; tertia vero ſpecies prodiit Parabola duos habens ramos infinitos aſymtotis deſtitutos. Simili porro ratione Lineas tertii ordinis ſum perſecutus, quas, poſt expoſitas earum proprietates generales, diviſi in ſedecim genera; ad eaque omnes ſeptuaginta duas ſpecies Newtoni revocavi. Ipſam vero methodum ita clare deſcripſi, ut pro

 quovis

quovis Linearum ordine sequente divisio in genera facillime institui queat; cujus negotii periculum quoque feci in Lineis quarti ordinis. His deinde, quæ ad ordines Linearum pertinent, expeditis, reversus sum ad generales omnium Linearum affectiones eruendas. Explicavi itaque methodum definiendi tangentes curvarum, earum normales, atque etiam ipsam curvaturam, quæ per radium osculi æstimari solet: quæ etsi nunc quidem plerumque Calculo differentiali absolvuntur, tamen idem per solam communem Algebram hic præstiti, ut deinceps transitus ab Analysi finitorum ad Analysin infinitorum eo facilior reddatur. Perpendi etiam curvarum puncta flexus contrarii, cuspides, puncta duplicia, ac multiplicia; modumque exposui hæc omnia ex æquationibus sine ulla difficultate definiendi. Interim tamen non nego, has quæstiones multo facilius Calculi differentialis ope enodari posse. Attigi quoque controversiam de cuspide secundi ordinis, ubi ambo arcus in cuspidem coeuntes curvaturam in eandem partem vertunt; eamque ita composuisse mihi videor, ut nullum dubium amplius superesse possit. Denique adjunxi aliquot Capita, in quibus Lineas Curvas, quæ datis proprietatibus gaudeant, invenire docui; pluraque tandem Problemata circa singulares Circuli sectiones soluta dedi. Quæ cum sint ea ex Geometria, quæ ad Analysin infinitorum addiscendam maximum adminiculum afferre videntur, Appendicis loco ex Stereometria Theoriam solidorum eorumque superficierum per Calculum proposui, & quemadmodum cujusque superficiei natura per æquationem inter tres variabiles exponi queat, ostendi. Hinc, superficiebus instar linearum in ordines digestis, secundum dimensionum quas variabiles in æquatione constituunt numerum, in primo ordine solam superficiem planam contineri ostendi. Superficies vero secundi ordinis, ratione habita partium in infinitum expansarum, in sex genera divisi; similique modo pro ceteris ordinibus divisio institui poterit.

poterit. Contemplatus ſum quoque interſectiones duarum ſuperficierum; quæ cum plerumque ſint curvæ non in eodem plano ſitæ, quemadmodum æquationibus comprehendi queant, monſtravi. Tandem etiam poſitionem planorum tangentium, atque rectarum, quæ ad ſuperficies ſint normales, determinavi.

De cetero, cum non paucæ res hic occurrant ab aliis jam tractatæ, veniam rogare me oportet, quod non ubique honorificam mentionem eorum, qui ante me in eodem genere elaborarunt, fecerim. Cum enim mihi propoſitum eſſet omnia quam breviſſime pertractare, Hiſtoria cujuſque Problematis magnitudinem operis non mediocriter auxiſſet. Interim tamen pleræque quæſtiones, quæ alibi quoque ſolutæ reperiuntur, hic ſolutiones ex aliis principiis ſunt nactæ; ita ut non exiguam partem mihi vindicare poſſem. Spero autem cum iſta, tum ea potiſſimum, quæ prorſus nova hic proferuntur, pleriſque, qui hoc ſtudio delectantur, non ingrata eſſe futura.

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TOMI SECUNDI.

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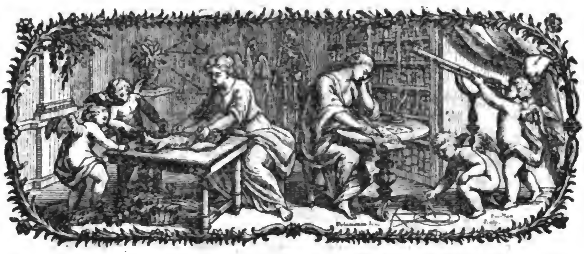
INTRO-

INTRODUCTIO
IN
ANALYSIN INFINITORUM.

LIBER PRIMUS,

Continens

Explicationem de Functionibus quantitatum variabilium; earum resolutione in Factores, atque evolutione per Series infinitas: una cum doctrina de Logarithmis, Arcubus circularibus, eorumque Sinubus & Tangentibus; pluribusque aliis rebus, quibus Analysis infinitorum non mediocriter adjuvatur.



LIBER PRIMUS.

CAPUT PRIMUM.

De Functionibus in genere.

1. *Quantitas constans est quantitas determinata, perpetuo eumdem valorem servans.*

Ejusmodi quantitates sunt numeri cujusvis generis, quippe qui eumdem, quem semel obtinuerunt, valorem constanter conservant: atque si hujusmodi quantitates constantes per characteres indicare convenit, adhibentur litteræ Alphabethi initiales *a*, *b*, *c*, &c. In Analysi quidem communi, ubi tantum quantitates determinatæ considerantur, hæ litteræ Alphabethi priores quantitates cognitas denotare solent, posteriores vero quantitates incognitas; at in Analysi sublimiori hoc discrimen non tantopere spectatur, cum hic ad illud quantitatum discrimen præcipue respiciatur, quo aliæ constantes, aliæ vero variabiles statuuntur.

 2. *Quantitas variabilis est quantitas indeterminata seu universalis, quæ omnes omnino valores determinatos in se complectitur.*

Cum ergo omnes valores determinati numeris exprimi queant, quantitas variabilis omnes numeros cujusvis generis involvit. Quemadmodum scilicet ex ideis individuorum formantur ideæ specierum & generum; ita quantitas variabilis est genus, sub quo omnes quantitates determinatæ continentur. Hujusmodi autem quantitates variabiles per litteras Alphabethi postremas z, y, x, &c. repræsentari solent.

3. *Quantitas variabilis determinatur, dum ei valor quicunque determinatus tribuitur.*

Quantitas ergo variabilis innumerabilibus modis determinari potest, cum omnes omnino numeros ejus loco substituere liceat. Neque significatus quantitatis variabilis exhauritur, nisi omnes valores determinati ejus loco fuerint substituti. Quantitas ergo variabilis in se complectitur omnes prorsus numeros, tam affirmativos quam negativos, tam integros quam fractos, tam rationales quam irrationales & transcendentes. Quinetiam cyphra & numeri imaginarii a significatu quantitatis variabilis non excluduntur.

4. *Functio quantitatis variabilis, est expressio analytica quomodocunque composita ex illa quantitate variabili, & numeris seu quantitatibus constantibus.*

Omnis ergo expressio analytica, in qua præter quantitatem variabilem z omnes quantitates illam expressionem componentes sunt constantes, erit Functio ipsius z: Sic $a+3z$; $az-4zz$; $az+b\sqrt{(aa-zz)}$; c^z; &c. sunt Functiones ipsius z.

5. *Functio ergo quantitatis variabilis ipsa erit quantitas variabilis.*

Cum enim loco quantitatis variabilis omnes valores determinatos substituere liceat, hinc Functio innumerabiles valores determinatos induet; neque ullus valor determinatus excipietur, quem Functio induere nequeat, cum quantitas variabilis quoque valores imaginarios involvat. Sic etsi hæc Functio $\sqrt{(9-zz)}$, numeris realibus loco z substituendis, nunquam valorem ternario majorem recipere potest; tamen ipsi z valores imaginarios tribuendo

tribuendo ut $5\sqrt{-1}$, nullus aſſignari poterit valor determinatus quin ex formula $\sqrt{(9-zz)}$ elici queat. Occurrunt autem nonnunquam Functiones tantum apparentes, quæ, utcunque quantitas variabilis varietur, tamen uſque eumdem valorem retinent, ut z^0; 1^z; $\frac{aa-az}{a-z}$, quæ, etſi ſpeciem Functionis mentiuntur, tamen revera ſunt quantitates conſtantes. CAP. I.

6. *Præcipuum Functionum diſcrimen in modo compoſitionis, quo ex quantitate variabili & quantitatibus conſtantibus formantur, poſitum eſt.*

Pendet ergo ab Operationibus quibus quantitates inter ſe componi & permiſceri poſſunt: quæ Operationes ſunt Additio & Subtractio; Multiplicatio & Diviſio: Evectio ad Poteſtates & Radicum Extractio; quo etiam Reſolutio Æquationum eſt referenda. Præter has Operationes, quæ algebraïcæ vocari ſolent, dantur complures aliæ tranſcendentes, ut Exponentiales, Logarithmicæ, atque innumerabiles aliæ, quas Calculus integralis ſuppeditat.

Interim ſpecies quædam Functionum notari poſſunt; ut multipla $2z$; $3z$; $\frac{3}{5}z$; az; &c. & Poteſtates ipſius z, ut z^2; z^3; $z^{\frac{1}{2}}$; z^{-1}; &c. quæ, uti ex unica operatione ſunt deſumtæ, ita expreſſiones quæ ex operationibus quibuſcunque naſcuntur, Functionum nomine inſigniuntur.

7. *Functiones dividuntur in Algebraïcas & Tranſcendentes; illæ ſunt, quæ componuntur per operationes algebraïcas ſolas, hæ vero in quibus operationes tranſcendentes inſunt.*

Sunt ergo multiplæ ac Poteſtates ipſius z Functiones algebraïcæ; atque omnes omnino expreſſiones, quæ per operationes algebraïcas ante memoratas formantur, cujuſmodi eſt

$\frac{a+bz^n-c\sqrt{(2z-zz)}}{aaz-3bz^3}$. Quin-etiam Functiones algebraïcæ ſæpenumero nequidem explicite exhiberi poſſunt, cujuſmodi Functio ipſius z eſt Z, ſi definiatur per hujuſmodi æquationem; $Z^5 = azzZ^3 - bz^4Z^2 + cz^3Z - 1$. Quanquam enim hæc

 æquatio

LIB. I. æquatio resolvi nequit; tamen constat Z æquari expressioni cuipiam ex variabili z & constantibus compositæ; ac propterea fore Z Functionem quamdam ipsius z. Cæterum de Functionibus transcendentibus notandum est, eas demum fore transcendentes, si operatio transcendens non solum ingrediatur, sed etiam quantitatem variabilem afficiat. Si enim operationes transcendentes tantum ad quantitates constantes pertineant, Functio nihilominus algebraïca est censenda: uti si c denotet circumferentiam Circuli, cujus radius sit $= 1$, erit utique c quantitas transcendens, verumtamen hæ expressiones $c + z$; cz^2; $4z^c$ &c. erunt Functiones algebraïcæ ipsius z. Parvi quidem est momenti dubium quod a quibusdam movetur, utrum ejusmodi expressiones z^c Functionibus algebraïcis annumerari jure possint, necne; quinetiam Potestates ipsius z, quarum exponentes sint numeri irrationales, uti $z^{\sqrt{2}}$ nonnulli maluerunt Functiones interscendentes quam algebraïcas appellare.

8. *Functiones algebraïcæ subdividuntur in Rationales & Irrationales: illæ sunt, si quantitas variabilis in nulla irrationalitate involvitur; hæ vero, in quibus signa radicalia quantitatem variabilem afficiunt.*

In Functionibus ergo rationalibus aliæ operationes præter Additionem, Subtractionem, Multiplicationem, Divisionem, & Evectionem ad Potestates, quarum exponentes sint numeri integri, non insunt: erunt adeo $a + z$; $a - z$; az; $\frac{aa + zz}{a + z}$; $az^3 - bz^5$; &c. Functiones rationales ipsius z. At hujusmodi expressiones $\sqrt{z}$; $a + \sqrt{(aa - zz)}$; $\sqrt[3]{(a - 2z + zz)}$: $\frac{aa - z\sqrt{(aa + zz)}}{a + z}$ erunt Functiones irrationales ipsius z.

Hæ commode distinguuntur in Explicitas & Implicitas.

Explicitæ sunt, quæ per signa radicalia sunt evolutæ, cujusmodi exempla modo sunt data. Implicitæ vero Functiones irrationales sunt quæ ex resolutione æquationum ortum habent. Sic Z erit Functio irrationalis implicita ipsius z, si per hujusmodi æqua-

æquationem $Z^7 = az Z^2 - bz^5$ definiatur; quoniam valorem explicitum pro Z, admissis etiam signis radicalibus, exhibere non licet; propterea quod Algebra communis nondum ad hunc perfectionis gradum est evecta.

9. *Functiones rationales denuo subdividuntur in Integras & Fractas.*

In illis neque z usquam habet exponentes negativos, neque expressiones continent fractiones, in quarum denominatores quantitas variabilis z ingrediatur: unde intelligitur Functiones fractas esse, in quibus denominatores z continentes, vel exponentes negativi ipsius z occurrant. Functionum integrarum hæc ergo erit Formula generalis: $a + bz + cz^2 + dz^3 + ez^4 + fz^5 +$ &c. nulla enim Functio ipsius z integra excogitari potest, quæ non in hac expressione contineatur. Functiones autem fractæ omnes, quia plures fractiones in unam cogi possunt, continebuntur in hac Formula:

$$\frac{a + bz + cz^2 + dz^3 + ez^4 + fz^5 + \&c.}{\alpha + \beta z + \gamma z^2 + \delta z^3 + \epsilon z^4 + \zeta z^5 + \&c.}$$

ubi notandum est quantitates constantes a, b, c, d, &c. α, β, γ, δ, &c. sive sint affirmativæ, sive negativæ, sive integræ sive fractæ, sive rationales sive irrationales, sive etiam transcendentes, naturam Functionum non mutare.

10. *Deinde potissimum tenenda est Functionum divisio in Uniformes ac Multiformes.*

Functio autem uniformis est, quæ si quantitati variabili z valor determinatus quicunque tribuatur, ipsa quoque unicum valorem determinatum obtineat. Functio autem Multiformis est, quæ, pro unoquoque valore determinato in locum variabilis z substituto, plures valores determinatos exhibet. Sunt igitur omnes Functiones rationales, sive integræ sive fractæ, Functiones uniformes; quoniam ejusmodi expressiones, quicunque valor quantitati variabili tribuatur, non nisi unicum valorem præbent. Functiones autem irrationales omnes sunt multiformes; propterea quod signa radicalia sunt ambigua, & geminum valorem involvunt. Dantur autem quoque inter Functiones transcenden-

tes,

 tes, & uniformes, & multiformes: quin-etiam habentur Functiones infinitiformes; cujusmodi est Arcus Circuli Sinui z respondens; dantur enim Arcus circulares innumerabiles qui omnes eumdem habeant Sinum. Denotent autem hæ litteræ P, Q, R, S, T &c. singulæ Functiones uniformes ipsius z.

11. *Functio biformis ipsius* z *est ejusmodi Functio, quæ pro quovis ipsius* z *valore determinato, geminum valorem præbeat.*

Hujusmodi Functiones radices quadratæ exhibent, ut $\sqrt{(2z+zz)}$: quicunque enim valor pro z statuatur expressio $\sqrt{(2z+zz)}$ duplicem habet significatum, vel affirmativum vel negativum. Generatim vero Z erit Functio biformis ipsius z, si determinetur per æquationem quadraticam $Z^2 - PZ + Q = 0$: si quidem P & Q fuerint Functiones uniformes ipsius z. Erit namque $Z = \frac{1}{2}P \pm \sqrt{(\frac{1}{4}P^2 - Q)}$; ex quo patet cuique valori determinato ipsius z duplicem valorem determinatum ipsius Z respondere. Hic autem notandum est, vel utrumque valorem Functionis Z esse realem, vel utrumque imaginarium. Tum vero erit semper, uti constat ex natura æquationum, binorum valorum ipsius Z summa $= P$, ac productum $= Q$.

12. *Functio triformis ipsius* z *est, qua pro quovis ipsius* z *valore, tres valores determinatos exhibet.*

Hujusmodi Functiones ex resolutione æquationum cubicarum originem trahunt. Si enim fuerint P, Q, & R Functiones uniformes, sitque $Z^3 - PZ^2 + QZ - R = 0$, erit Z Functio triformis ipsius z; quia pro quolibet valore determinato ipsius z triplicem valorem obtinet. Tres isti ipsius Z valores unicuique valori ipsius z respondentes, vel erunt omnes reales, vel unicus erit realis, dum bini reliqui sunt imaginarii. Cæterum constat horum trium valorum summam perpetuo esse $= P$; summam factorum ex binis esse $= Q$, & productum ex omnibus tribus esse $= R$.

13. *Functio quadriformis ipsius* z *est, qua pro quovis ipsius* z *valore quatuor valores determinatos exhibet.*

Hujusmodi Functiones ex resolutione æquationum biquadraticarum

ticarum nascuntur. Quod si enim P, Q, R, & S denotent Functiones uniformes ipsius z, fueritque $Z^4 - PZ^3 + QZ^2 - RZ + S = 0$, erit Z Functio quadriformis ipsius z; eo quod cuique valori ipsius z quadruplex valor ipsius Z respondet. Quatuor horum valorum ergo, vel omnes erunt reales, vel duo reales duoque imaginarii, vel omnes quatuor erunt imaginarii. Ceterum perpetuo summa horum quatuor valorum ipsius Z est $= P$, summa factorum ex binis $= Q$, summa factorum ex ternis $= R$, ac productum omnium $= S$. Simili autem modo comparata est ratio Functionum quinqueformium & sequentium. CAP. I.

14. *Erit ergo* Z *Functio multiformis ipsius* z, *quæ, pro quovis valore ipsius* z, *tot exhibet valores quot numerus* n *continet unitates; si* Z *definiatur per hanc æquationem* $Z^n - PZ^{n-1} + QZ^{n-2} - RZ^{n-3} + SZ^{n-4} - \&c. = 0$.

Ubi quidem notandum est n esse oportere numerum integrum; atque perpetuo, ut dijudicari possit quam multiformis sit Functio Z ipsius z, æquatio, per quam Z definitur, reduci debet ad rationalitatem; quo facto exponens maximæ potestatis ipsius Z indicabit quæsitum valorum numerum cuique ipsius z valori respondentium. Deinde quoque tenendum est litteras P, Q, R, S, &c. denotare debere Functiones uniformes ipsius z: si enim aliqua earum jam esset Functio multiformis, tum Functio Z multo plures præbitura esset valores unicuique valori ipsius z respondentes, quam quidem numerus dimensionum ipsius Z indicaret. Semper autem, si qui valores ipsius fuerint imaginarii, eorum numerus erit par; unde intelligitur, si fuerit n numerus impar, perpetuo unum ad minimum valorem ipsius Z fore realem: contra autem fieri posse, si numerus n fuerit par, ut nullus prorsus valor ipsius Z sit realis.

15. *Si* Z *ejusmodi fuerit Functio multiformis ipsius* z *ut perpetuo nonnisi unicum valorem exhibeat realem; tum* Z *Functionem uniformem ipsius* z *mentietur, ac plerumque loco Functionis uniformis usurpari poterit.*

Ejusmodi Functiones erunt $\sqrt[3]{P}$, $\sqrt[5]{P}$, $\sqrt[7]{P}$, &c. quippe quæ perpetuo nonnisi unicum valorem realem præbent, reliquis omnibus existentibus imaginariis, dummodo P fuerit Functio uniformis ipsius z. Hanc ob rem hujusmodi expressio $P^{\frac{m}{n}}$, quoties n fuerit numerus impar, Functionibus uniformibus annumerari poterit; sive m fuerit numerus par sive impar. Quod si autem n fuerit numerus par, tum $P^{\frac{m}{n}}$ vel nullum habebit valorem realem, vel duos; ex quo ejusmodi expressiones $P^{\frac{m}{n}}$, existente n numero pari, eodem jure Functionibus biformibus accenseri poterunt: siquidem fractio $\frac{m}{n}$ ad minores terminos non fuerit reducibilis.

16. *Si fuerit* y *Functio quæcunque ipsius* z; *tum vicissim* z *erit Functio ipsius* y.

Cum enim y sit Functio ipsius z, sive uniformis sive multiformis; dabitur æquatio, qua y per z & constantes quantitates definitur. Ex eadem vero æquatione vicissim z per y & constantes definiri poterit; unde quoniam y est quantitas variabilis, z æquabitur expressioni ex y & constantibus compositæ, eritque adeo Functio ipsius y. Hinc quoque patebit quam multiformis Functio futura sit z ipsius y: fierique potest ut, etiamsi y fuerit Functio uniformis ipsius z, tamen z futura sit Functio multiformis ipsius y. Sic si y ex hac æquatione per z definiatur; $y^3 = ayz - bzz$; erit utique y Functio triformis ipsius z, contra vero z Functio tantum biformis ipsius y.

17. *Si fuerint* y *&* x *Functiones ipsius* z, *erit quoque* y *Functio ipsius* x, *& vicissim* x *Functio ipsius* y.

Cum enim sit y Functio ipsius z, erit quoque Functio ipsius y: similique modo erit etiam z Functio ipsius x. Hanc ob rem Functio ipsius y æqualis erit Functioni ipsius x; ex qua æquatione & y per x & viceversa x per y definiri poterit: quocirca manifestum est esse y Functionem ipsius x, atque x Functionem ipsius

ipsius y. Sæpissime quidem has Functiones explicite exhibere non licet ob defectum Algebræ; interim tamen nihilo minus, quasi omnes æquationes resolvi possent, hæc Functionum reciprocatio perspicitur. Ceterum per methodum in Algebra traditam, ex datis binis æquationibus, quarum altera continet y & z, altera vero x & z, per eliminationem quantitatis z formabitur una æquatio relationem inter x & y exprimens.

18 *Species denique quædam Functionum peculiares sunt notandæ; sic Functio par ipsius* z *est, quæ eundem dat valorem, sive pro* z *ponatur valor determinatus* + k *sive* — k.

Hujusmodi ergo Functio par ipsius z erit zz; sive enim ponatur $z = +k$, sive $z = -k$, eundem valorem præbebit expressio zz, nempe $zz = +kk$. Simili modo Functiones pares ipsius z erunt hæ ipsius z potestates z^4, z^6, z^8, & generatim omnis potestas z^m, si fuerit m numerus par, sive affirmativus sive negativus. Quin etiam cum $z^{\frac{m}{n}}$ mentiatur Functionem ipsius z uniformem, si n sit numerus impar, perspicuum est $z^{\frac{m}{n}}$ fore Functionem parem ipsius z, si m fuerit numerus par, n vero numerus impar. Hanc ob rem, expressiones ex hujusmodi potestatibus utcunque compositæ præbebunt Functiones pares ipsius z; sic Z erit Functio par ipsius z, si fuerit $Z = a + bz^2 + cz^4 + dz^6 + \&c.$ item si fuerit $Z = \frac{a + bz^2 + cz^4 + dz^6 + \&c.}{\alpha + \beta z^2 + \gamma z^4 + \delta z^6 + \&c.}$; Similique modo exponentes fractos ipsius z introducendo, erit Z Functio par ipsius z si fuerit $Z = a + bz^{\frac{2}{3}} + cz^{\frac{2}{5}} + dz^{\frac{4}{7}}$ &c. vel $Z = a + bz^{-\frac{2}{3}} + cz^{-\frac{4}{3}} + dz^{-\frac{2}{5}} + \&c.$ vel $Z = \frac{a + bz^{\frac{2}{7}} + cz^{-\frac{4}{5}} + dz^{\frac{8}{3}}}{\alpha + \beta z^{\frac{2}{3}} + \gamma z^{-\frac{2}{5}} + \delta z^{\frac{4}{7}}}$. Cujusmodi expressiones, cum omnes sint Functiones uniformes ipsius z, appellari poterunt Functiones pares uniformes ipsius z.

 19. *Func-*

LIB. I.

19. *Functio multiformis par ipsius* z *est, quæ etiam si pro quovis valore ipsius* z *plures exhibeat valores determinatos, tamen eosdem valores præbet, sive ponatur* $z = +k$, *sive* $z = -k$.

Sit Z ejusmodi Functio multiformis par ipsius z; quoniam natura Functionis multiformis exprimitur per æquationem inter Z & z, in qua Z tot habeat dimensiones, quot varios valores complectatur; manifestum est Z fore Functionem multiformem parem, si in æquatione naturam ipsius Z exprimente quantitas variabilis z ubique pares habeat dimensiones. Sic, si fuerit $Z^2 = azZ^4 + bz^2$, erit Z Functio biformis par ipsius z; sin autem sit $Z^3 - az^2Z^2 + bz^4Z - cz^8 = 0$, erit Z Functio triformis par ipsius z; atque generatim, si P, Q, R, S &c. denotent Functiones uniformes pares ipsius z, erit Z Functio biformis par ipsius z si sit $Z^2 - PZ + Q = 0$. At Z erit Functio triformis par ipsius z si sit $Z^3 - PZ^2 + QZ - R = 0$, & ita porro.

20 *Functio ergo, sive uniformis sive multiformis, par ipsius* z *erit ejusmodi expressio ex quantitate variabili* z *& constantibus conflata, in qua ubique numerus dimensionum ipsius* z *sit par.*

Hujusmodi ergo Functiones, præter uniformes quarum exempla ante sunt allata, erunt hæ expressiones $a + \sqrt{(bb - zz)}$; $azz + \sqrt[3]{(a^6z^4 - bz^2)}$ item $az^{\frac{2}{3}} + \sqrt[7]{(z^2 + \sqrt{(a^4 - z^4)})}$ &c.

Unde patet Functiones pares ita definiri posse, ut dicantur esse Functiones ipsius zz.

Si enim ponatur $y = zz$, fueritque Z Functio quæcunque ipsius y; restituto ubique zz loco y, erit Z ejusmodi Functio ipsius z, in qua z ubique parem habeat dimensionum numerum. Excipiendi tamen sunt ii casus, quibus in expressione ipsius Z occurrunt $\sqrt{y}$: ac hujusmodi aliæ formæ, quæ, facto $y = zz$ signa radicalia amittunt. Quamvis enim sit $y + \sqrt{ay}$ Functio ipsius y, tamen posito $y = zz$, eadem expressio non erit Functio par ipsius z; cum fiat $y + \sqrt{ay} = zz + z\sqrt{a}$. Exclusis ergo his casibus, definitio ultima Functio-

num

num parium erit bona, atque ad ejusmodi Functiones formandas idonea. CAP. I.

21. *Functio impar ipsius z est ejusmodi Functio, cujus valor, si loco z ponatur — z, fit quoque negativus.*

Hujusmodi Functiones ergo impares erunt omnes potestates ipsius z, quarum exponentes sunt numeri impares, ut z^1, z^3, z^5, z^7; &c. item z^{-1}, z^{-3}, z^{-5}; &c. tum vero etiam $z^{\frac{m}{n}}$ erit Functio impar, si ambo numeri, m & n fuerint numeri impares. Generatim vero omnis expressio ex hujusmodi potestatibus composita erit Functio impar ipsius z; cujusmodi sunt, $az + bz^3 : az + az^{-1}$; item $z^{\frac{1}{3}} + az^{\frac{3}{5}} + bz^{-\frac{5}{3}}$; &c. Harum autem Functionum natura & inventio ex Functionibus paribus facilius perspicietur.

22. *Si Functio par ipsius z multiplicetur per z vel per ejusdem Functionem imparem quamcunque, productum erit Functio impar ipsius z.*

Sit P Functio par ipsius z, quæ idcirco manet eadem si loco z ponatur $-z$; quod si ergo in producto Pz, ponatur $-z$ loco z, prodibit $-Pz$; unde Pz erit Functio impar ipsius z. Sit jam P Functio par ipsius z, & Q functio impar ipsius z; atque ex Definitione patet si loco z ponatur $-z$, valorem ipsius P manere eundem, at valorem ipsius Q abire in sui negativum $-Q$; quare productum PQ, posito $-z$ loco z, abibit in $-PQ$, hoc est in sui negativum; eritque ideo PQ Functio impar ipsius z. Sic cum sit $a + \surd(aa + zz)$ functio par, & z^3 Functio impar ipsius z, erit productum $az^3 + z^3 \surd(aa + zz)$ Functio impar ipsius z; similique modo $z \times \frac{a + bzz}{\alpha + \mathfrak{C}zz} = \frac{az + bz^3}{\alpha + \mathfrak{C}zz}$ Functio impar ipsius z. Ex his vero etiam intelligitur, si duarum Functionum P & Q, quarum altera P est par, altera Q, impar, altera per alteram dividatur, quotum fore Functionem imparem; erit ergo $\frac{P}{Q}$ itemque $\frac{Q}{P}$ Functio impar ipsius z.

 23. *Si Functio impar per Functionem imparem vel multiplicetur, vel dividatur; quod resultat erit Functio par.*

Sint Q & S Functiones impares ipsius z; ita ut, posito $-z$ loco z, Q abeat in $-Q$, & S in $-S$; atque perspicuum est tam productum QS, quam quotum $\frac{Q}{S}$ eundem valorem retinere, etiamsi pro z ponatur $-z$; ideoque esse utrumque Functionem parem ipsius z. Manifestum itaque porro est cujusque Functionis imparis quadratum esse Functionem parem; cubum vero Functionem imparem; biquadratum iterum Functionem parem, atque ita porro.

24. *Si fuerit* y *Functio impar ipsius* z; *erit vicissim* z *Functio impar ipsius* y.

Cum enim sit y Functio impar ipsius z; si ponatur $-z$ loco z, abibit y in $-y$. Quod si ergo z per y definiatur, necesse est ut posito $-y$ loco y, quoque z abeat in $-z$; eritque ideo z Functio impar ipsius y. Sic quia, posito $y = z^3$, est y Functio impar ipsius z; erit quoque, ex æquatione $z^3 = y$ seu $z = y^{\frac{1}{3}}$, z Functio impar ipsius y. Et quia si fuerit $y = az + bz^3$, est y Functio impar ipsius z, erit vicissim, ex æquatione $bz^3 + az = y$, valor ipsius z per y expressus Functio impar ipsius y.

25. *Si natura Functionis* y *per ejusmodi æquationem definiatur, in cujus singulis terminis numerus dimensionum, quas* y *&* z *occupant conjunctim, sit vel par ubique, vel impar; tum erit* y *Functio impar ipsius* z.

Quod si enim in ejusmodi æquatione ubique loco z scribatur $-z$; simulque $-y$ loco y; omnes æquationis termini vel manebunt iidem, vel fient negativi, utroque vero casu æquatio manebit eadem. Unde patet $-y$ eodem modo per $-z$ determinatum iri, quo $+y$ per $+z$ determinatur; & hanc ob rem, si loco z ponatur $-z$, valor ipsius y abibit in $-y$, seu y erit Functio impar ipsius z. Sic si fuerit vel $yy = ayz + bzz + c$; vel $y^3 + ayyz = byzz + cy + dz$, ex utraque æquatione y erit Functio impar ipsius z.

26. *Si*

26. *Si* Z *fuerit Functio ipsius* z, *&* Y *Functio ipsius* y, *atque* Y *eodem modo definiatur per variabilem* y *& constantes, quo* Z *definitur per variabilem* z *& constantes; tum hæ Functiones* Y *et* Z *vocantur Functiones similes ipsarum* y *&* z.

Si scilicet fuerit $Z = a + bz + cz^2$, & $Y = a + by + cy^2$, erunt Z & Y Functiones similes ipsarum z & y, similique modo in multiformibus, si fuerit $Z^3 = azzZ + b$ & $Y^3 = ayyY + b$; erunt Z & Y Functiones similes ipsarum z & y. Hinc sequitur, si Y & Z fuerint hujusmodi Functiones similes ipsarum y & z; tum si loco z scribatur y, Functionem Z abituram esse in Functionem Y. Solet hæc similitudo etiam hoc modo verbis exprimi, ut Y talis Functio dicatur ipsius y, qualis Functio sit Z ipsius z. Hæ locutiones perinde occurrent, sive quantitates variabiles z & y a se invicem pendeant, sive secus: sic qualis Functio est $ay + by^2$ ipsius y, talis Functio erit $a(y+n) + b(y+n)^2$ ipsius $y+n$, existente scilicet $z = y + n$: tum qualis Functio est $\frac{a + bz + czz}{\alpha + \beta z + \gamma zz}$ ipsius z, talis Functio erit $\frac{azz + bz + c}{\alpha zz + \beta z + \gamma}$ ipsius $\frac{1}{z}$; posito $y = \frac{1}{z}$. Atque ex his luculenter perspicitur ratio similitudinis Functionum, cujus per universam Analysin sublimiorem uberrimus est usus. Ceterum hæc in genere de natura Functionum unius variabilis sufficere possunt; cum plenior expositio in applicatione sequente tradatur.

CAPUT II.

De transformatione Functionum.

27. F*Unctiones in alias formas transmutantur, vel loco quantitatis variabilis aliam introducendo, vel eandem quantitatem variabilem retinendo.*

Quod si eadem quantitas variabilis servatur, Functio proprie mutari non potest. Sed omnis transformatio consistit in alio

alio modo eandem Functionem exprimendi, quemadmodum ex Algebra constat eandem quantitatem per plures diversas formas exprimi posse. Hujusmodi transformationes sunt, si loco hujus Functionis $2 - 3z + zz$ ponatur $(1 - z)(2 - z)$, vel $(a + z)^3$ loco $a^3 + 3aaz + 3azz + z^3$, vel $\frac{a}{a - z} + \frac{a}{a + z}$ loco $\frac{2aa}{aa - zz}$; vel $\sqrt{(1 + zz)} + z$ loco $\frac{1}{\sqrt{(1+zz)} - z}$; quæ expressiones, etsi forma differunt, tamen revera congruunt. Sæpe numero autem harum plurium formarum idem significantium una aptior est ad propositum efficiendum quam reliquæ, & hanc ob rem formam commodissimam eligi oportet.

Alter transformationis modus, quo loco quantitatis variabilis z alia quantitas variabilis y introducitur, quæ quidem ad z datam teneat relationem, per substitutionem fieri dicitur; hocque modo ita uti convenit, ut Functio proposita succinctius & commodius exprimatur, uti si ista proposita fuerit ipsius z Functio, $a^4 - 4a^3z + 6aazz - 4az^3 + z^4$; si loco $a - z$ ponatur y, prodibit ista multo simplicior ipsius y Functio y^4: &, si habeatur hæc Functio irrationalis $\sqrt{(aa + zz)}$ ipsius z, si ponatur $z = \frac{aa - yy}{2y}$, ista Functio per y expressa fiet rationalis $= \frac{aa + yy}{2y}$. Hunc autem transformationis modum in sequens Caput differam, hoc Capite illum, qui sine substitutione procedit, expositurus.

28. *Functio integra ipsius z sæpenumero commode in suos factores resolvitur, sicque in productum transformatur.*

Quando Functio integra hoc pacto in factores resolvitur, ejus natura multo facilius perspicitur; casus enim statim innotescunt, quibus Functionis valor fit $= 0$. Sic hæc ipsius z Functio $6 - 7z + z^3$ transformatur in hoc productum $(1 - z)(2 - z)(3 + z)$ ex quo statim liquet Functionem propositam tribus casibus fieri $= 0$; scilicet si $z = 1$, & $z = 2$, & $z = -3$, quæ proprietates ex forma $6 - 7z + z^3$ non tam facile intelliguntur. Istiusmodi Factores, in quibus variabiles z nulla

nulla occurrit poteſtas, vocantur Factores ſimplices, ut diſtinguantur a Factoribus compoſitis, in quibus ipſius z ineſt quadratum vel cubus, vel alia poteſtas altior. Erit ergo in genere $f+gz$ forma Factorum ſimplicium, $f+gz+hzz$ forma Factorum duplicium; $f+gz+hzz+iz^3$ forma Factorum triplicium, & ita porro. Perſpicuum autem eſt Factorem duplicem duos complecti valores ſimplices, Factorem triplicem tres ſimplices, & ita porro. Hinc Functio ipſius z integra, in qua exponens ſummæ poteſtatis ipſius z eſt $=n$, continebit n Factores ſimplices; ex quo ſimul, ſi qui Factores fuerint vel duplices vel triplices, &c. numerus Factorum cognoſcetur.

29. *Factores ſimplices Functionis cujuſcunque integræ* Z *ipſius* z *reperiuntur, ſi Functio* Z *nihilo æqualis ponatur, atque ex hac æquatione omnes ipſius* z *radices inveſtigentur: ſingulæ enim ipſius* z *radices dabunt totidem Factores ſimplices Functionis* Z.

Quod ſi enim ex æquatione $Z=0$, fuerit quæpiam radix $z=f$, erit $z-f$ diviſor, ac proinde Factor Functionis Z, ſic igitur inveſtigandis omnibus radicibus æquationis $Z=0$, quæ ſint $z=f$, $z=g$, $z=h$; &c., Functio Z reſolvetur in ſuos Factores ſimplices, atque transformabitur in productum $Z=(z-f)(z-g)(z-h)$ &c.: ubi quidem notandum eſt ſi ſummæ poteſtatis ipſius z in Z non fuerit coefficiens $=+1$, tum productum $(z-f)(z-g)$ &c. inſuper per illum coefficientem multiplicari debere. Sic ſi fuerit $Z=Az^n+Bz^{n-1}+Cz^{n-2}+$ &c. erit $Z=A.(z-f).(z-g).(z-h).$ &c. At ſi fuerit $Z=A+Bz+Cz^2+Dz^3+Ez^4+$ &c. atque æquationis $Z=0$ radices z repertæ ſint; f; g; h; i; &c. erit $Z=A(1-\frac{z}{f})(1-\frac{z}{g})(1-\frac{z}{h}).$ &c. Ex his autem viciſſim intelligitur, ſi Functionis Z Factor fuerit $z-f$, ſeu $1-\frac{z}{f}$; tum valorem Functionis in nihilum abire, ſi loco z ponatur f. Facto enim $z=f$, unus Factor $z-f$, ſeu $1-\frac{z}{f}$, Functionis Z, ideoque ipſa Functio Z evaneſcere debet.

LIB. I. 30. *Factores simplices ergo erunt vel reales, vel imaginarii; &, si Functio Z habeat Factores imaginarios eorum numerus semper erit par.*

Cum enim Factores simplices nascantur ex radicibus æquationis $Z = 0$, radices reales præbebunt Factores reales, & imaginariæ imaginarios; in omni autem æquatione numerus radicum imaginariarum semper est par: quamobrem Functio Z, vel nullos habebit Factores imaginarios, vel duos, vel quatuor, vel sex, &c. Quod si Functio Z duos tantum habeat Factores imaginarios, eorum productum erit reale, ideoque præbebit Factorem duplicem realem. Sit enim $P =$ producto ex omnibus Factoribus realibus, erit productum duorum Factorum imaginariorum $= \frac{Z}{P}$; hincque reale. Simili modo si Functio Z habeat quatuor, vel sex, vel octo &c. Factores imaginarios; erit eorum productum semper reale: nempe æquale quoto, qui oritur, si Functio Z dividatur per productum omnium Factorum realium.

31. *Si fuerit Q productum reale ex quatuor Factoribus simplicibus imaginariis, tum idem hoc productum Q resolvi poterit in duos Factores duplices reales.*

Habebit enim Q ejusmodi formam $z^4 + Az^3 + Bz^2 + Cz + D$; quæ si negetur in duos Factores duplices reales resolvi posse, resolubilis erit statuenda in duos Factores duplices imaginarios; qui hujusmodi formam habebunt $zz - 2(p + q\sqrt{-1})z + r + s\sqrt{-1}$. & $zz - 2(p - q\sqrt{-1})z + r - s\sqrt{-1}$; aliæ enim formæ imaginariæ concipi non possunt, quarum productum fiat reale, nempe $= z^4 + Az^3 + Bz^2 + Cz + D$. Ex his autem Factoribus imaginariis duplicibus sequentes emergent quatuor Factores simplices imaginarii ipsius Q,

I. $z - (p + q\sqrt{-1}) + \sqrt{(pp + 2pq\sqrt{-1} - qq - r - s\sqrt{-1})}$
II. $z - (p + q\sqrt{-1}) - \sqrt{(pp + 2pq\sqrt{-1} - qq - r - s\sqrt{-1})}$
III. $z - (p - q\sqrt{-1}) + \sqrt{(pp - 2pq\sqrt{-1} - qq - r + s\sqrt{-1})}$
IV. $z - (p - q\sqrt{-1}) - \sqrt{(pp - 2pq\sqrt{-1} - qq - r + s\sqrt{-1})}$

Horum

Horum Factorum multiplicentur primus ac tertius in ſe invicem, posito brevitatis gratia, $t = pp - qq - r$, & $u = 2pq - s$; eritque horum Factorum productum $= zz - (2p - \sqrt{2t + 2\sqrt{(tt+uu)}})z$

$$+ pp + qq - p\sqrt{2t + 2\sqrt{(tt+uu)}} + \sqrt{(tt+uu)};$$
$$+ q\sqrt{-2t + 2\sqrt{(tt+uu)}}$$

quod utique eſt reale. Simili autem modo productum ex Factoribus ſecundo & quarto erit reale nempe $= zz - (2p + \sqrt{2t + 2\sqrt{(tt+uu)}})z$

$$+ pp + qq + p\sqrt{2t + 2\sqrt{(tt+uu)}} + \sqrt{(tt+uu)}.$$
$$+ q\sqrt{-2t + 2\sqrt{(tt+uu)}}$$

Quocirca productum propoſitum Q, quod in duos Factores duplices reales reſolvi poſſe negabatur, nihilo minus actu in duos Factores duplices reales eſt reſolutum.

32. *Si Functio integra Z ipſius z quotcunque habeat Factores ſimplices imaginarios, bini ſemper ita conjungi poſſunt, ut eorum productum fiat reale.*

Quoniam numerus radicum imaginariarum ſemper eſt par, ſit is $= 2n$; ac primo quidem patet productum harum radicum imaginariarum omnium eſſe reale. Quod ſi ergo duæ tantum radices imaginariæ habeantur, erit earum productum utique reale; ſin autem quatuor habeantur Factores imaginarii, tum, uti vidimus, eorum productum reſolvi poteſt in duos Factores duplices reales formæ $fzz + gz + h$. Quanquam autem eundem demonſtrandi modum ad altiores poteſtates extendere non licet, tamen extra dubium videtur eſſe poſitum eandem proprietatem in quotcunque Factores imaginarios competere; ita ut ſemper loco $2n$ Factorum ſimplicium imaginariorum induci queant n Factores duplices reales. Hinc omnis Functio integra ipſius z reſolvi poterit in Factores reales vel ſimplices vel duplices. Quod quamvis non ſummo rigore ſit demonſtratum, tamen ejus veritas in ſequentibus magis corroborabitur, ubi hujus generis Functiones $a + bz^n$; $a + bz^n + cz^{2n}$; $a + bz^n + cz^{2n} + dz^{3n}$ &c. actu in iſtiuſmodi Factores duplices reales reſolventur.

 33. *Si*

LIB. I. 33. *Si Functio integra Z, posito z = a induat valorem A, & posito z = b, induat valorem B; tum, loco z valores medios inter a & b ponendo, Functio Z quosvis valores medios inter A & B accipere potest.*

Cum enim Z sit Functio uniformis ipsius z, quicunque valor realis ipsi z tribuatur, Functio quoque Z hinc valorem realem obtinebit. Cum igitur Z, priore casu $z = a$, nanciscatur valorem A; posteriore casu $z = b$, autem, valorem B; ab A ad B transire non poterit, nisi per omnes valores medios transeundo. Quod si ergo æquatio $Z - A = 0$ habeat radicem realem, simulque $Z - B = 0$ radicem realem suppeditet; tum æquatio quoque $Z - C = 0$ radicem habebit realem; si quidem C intra valores A & B contineatur. Hinc si expressiones $Z - A$ & $Z - B$ habeant Factorem simplicem realem, tum expressio quæcunque $Z - C$ Factorem simplicem habebit realem, dummodo C intra valores A & B contineatur.

34. *Si in Functione integra Z exponens maxima ipsius z potestatis fuerit numerus impar $2n+1$, tum ea Functio Z unicum ad minimum habebit Factorem simplicem realem.*

Habebit scilicet Z hujusmodi formam $z^{2n+1} + \alpha z^{2n} + \beta z^{2n-1} + \gamma z^{2n-2} +$ &c. in qua si ponatur $z = \infty$, quia valores singulorum terminorum præ primo evanescunt, fiet $Z = (\infty)^{2n+1} = \infty$; ideoque $Z - \infty$ Factorem simplicem habebit realem nempe $z - \infty$. Sin autem ponatur $z = -\infty$, fiet $Z = (-\infty)^{2n+1} = -\infty$, ideoque habebit $Z + \infty$ Factorem simplicem realem $z + \infty$. Cum igitur tam $Z - \infty$, quam $Z + \infty$ habeat Factorem simplicem realem; sequitur etiam $Z + C$ habiturum esse Factorem simplicem realem, siquidem C contineatur intra limites $+\infty$ & $-\infty$; hoc est si C fuerit numerus realis quicunque, sive affirmativus, sive negativus. Hanc ob rem, facto $C = 0$, habebit quoque ipsa Functio Z Factorem simplicem realem $z - c$; atque quantitas c

contine-

continebitur intra limites $+\infty$ & $-\infty$, eritque idcirco vel quantitas affirmativa, vel negativa, vel nihil. CAP. II.

35. *Functio igitur integra Z, in qua exponens maximae potestatis ipsius z est numerus impar, vel unum habebit Factorem simplicem realem, vel tres, vel quinque, vel septem &c.*

Cum enim demonstratum sit Functionem Z certo unum habere Factorem simplicem realem $z-c$; ponamus eam praeterea unum Factorem habere $z-d$, atque dividatur Functio Z, in qua maxima ipsius z potestas sit z^{2n+1}, per $(z-c).(z-d)$; erit quoti maxima potestas $= z^{2n-1}$, cujus exponens, cum sit numerus impar, indicat denuo ipsius Z dari Factorem simplicem realem. Si ergo Z plures uno habeat Factores simplices reales, habebit vel tres, vel (quoniam eodem modo progredi licet) quinque, vel septem, &c. Erit scilicet numerus Factorum simplicium realium impar, & quia numerus omnium Factorum simplicium est $= 2n+1$, erit numerus Factorum imaginariorum par.

36. *Functio integra Z, in qua exponens maximae potestatis ipsius z est numerus par 2n, vel duos habebit Factores simplices reales vel quatuor, vel sex, vel &c.*

Ponamus ipsius Z constare Factorum simplicium realium numerum imparem $2m+1$; si ergo per horum omnium productum dividatur Functio Z, quoti maxima potestas erit $= z^{2n-2m-1}$, ejusque ideo exponens numerus impar; habebit ergo Functio Z praeterea unum certo Factorem simplicem realem, ex quo numerus omnium Factorum simplicium realium ad minimum erit $= 2m+2$, ideoque par; ac numerus Factorum imaginariorum pariter par. Omnis ergo Functionis integrae Factores simplices imaginarii sunt numero pares; quemadmodum quidem jam ante statuimus.

37. *Si in Functione integra Z exponens maximae potestatis ipsius z fuerit numerus par, atque terminus absolutus, seu constans, signo $-$ affectus, tum Functio Z ad minimum duos habet Factores simplices reales.*

LIB. I. Functio ergo Z, de qua hic sermo est, hujusmodi formam habebit $z^{2n} \pm \alpha z^{2n-1} \pm \beta z^{2n-2} \pm \ldots\ldots \pm \nu z - A$. Si jam ponatur $z = \infty$, fiet, uti supra vidimus, $Z = \infty$; atque, si ponatur $z = 0$, fiet $Z = -A$. Habebit ergo $Z - \infty$ Factorem realem $z - \infty$, & $Z + A$ Factorem $z - 0$: unde cum o contineatur intra limites $-\infty$ & $+A$, sequitur $Z + 0$ habere Factorem simplicem realem $z - c$, ita ut c contineatur intra limites o & ∞. Deinde, cum posito $z = -\infty$, fiat $Z = \infty$, ideoque $Z - \infty$ Factorem habeat $z + \infty$, & $Z + A$ Factorem $z + 0$, sequitur quoque $Z + 0$ Factorem simplicem realem habere $z + d$; ita ut d intra limites o & ∞ contineatur; unde constat propositum. Ex his igitur perspicitur si Z talis fuerit Functio, qualis hic est descripta, æquationem $Z = 0$, duas ad minimum habere debere radices reales, alteram affirmativam, alteram negativam. Sic æquatio hæc $z^4 + \alpha z^3 + \beta z^2 + \gamma z - aa = 0$, duas habet radices reales, alteram affirmativam, alteram negativam.

38. *Si in Functione fracta, quantitas variabilis* z *tot vel plures habeat dimensiones in numeratore, quam in denominatore; tum ista Functio resolvi poterit in duas partes, quarum altera est Functio integra, altera fracta; in cujus numeratore quantitas variabilis* z *pauciores habeat dimensiones quam in denominatore.*

Si enim exponens maximæ potestatis ipsius z minor fuerit in denominatore quam in numeratore; tum numerator per denominatorem dividatur more solito, donec in quoto ad exponentes negativos ipsius z perveniatur; hoc ergo loco abrupta divisionis operatione quotus constabit ex parte integra atque fractione, in cujus numeratore minor erit dimensionum numerus ipsius z quam in denominatore; hic autem quotus Functioni propositæ est æqualis. Sic, si hæc proposita fuerit Functio fracta $\frac{1+z^4}{1+zz}$, ea per divisionem ita resolvetur.

$$\begin{array}{l} zz+1)\ z^4+1\ (zz-1+\frac{2}{1+zz} \\ \underline{\quad z^4+zz} \\ \quad -zz+1 \\ \underline{\quad -zz-1} \\ \qquad +2 \end{array}$$

eritque $\frac{1+z^4}{1+zz} = zz - 1 + \frac{2}{1+zz}$. Hujusmodi Functiones fractæ, in quibus quantitas variabilis z tot vel plures habet dimensiones in numeratore quam in denominatore, ad similitudinem Arithmeticæ vocari possunt fractiones spuriæ, vel Functiones fractæ spuriæ, quo distinguantur a Functionibus fractis genuinis, in quarum numeratore quantitas variabilis z pauciores habet dimensiones quam in denominatore. Functio itaque fracta spuria resolvi poterit in Functionem integram, & Functionem fractam genuinam; hæcque resolutio per vulgarem divisionis operationem absolvetur.

39. *Si denominator Functionis fractæ duos habeat Factores inter se primos; tum ipsa Functio fracta resolvetur in duas fractiones, quarum denominatores sint illis binis Factoribus respective æquales.*

Quanquam hæc resolutio ad Functiones fractas spurias æque pertinet atque ad genuinas, tamen eam ad genuinas potissimum accomodabimus. Resoluto autem denominatore hujusmodi Functionis fractæ in duos Factores inter se primos, ipsa Functio resolvetur in duas alias Functiones fractas genuinas, quarum denominatores sint illis binis Factoribus respective æquales; hæcque resolutio, si quidem fractiones sint genuinæ, unico modo fieri potest; cujus rei veritas ex exemplo clarius quam per ratiocinium perspicietur. Sit ergo proposita hæc Functio fracta $\frac{1-2z+3zz-4z^3}{1+4z^4}$, cujus denominator $1+4z^4$ cum sit æqualis huic producto $(1+2z+2zz)(1-2z+2zz)$, fractio proposita in duas fractiones resolvetur, quarum alterius denominator erit $1+2z+2zz$, alterius $1-2z+2zz$: ad quas inveniendas, quia sunt genuinæ, statuantur numeratores illius $= \alpha + \beta z$, hujus $= \gamma + \delta z$, eritque per hypothesin

$\frac{1 - 2z + 3zz - 4z^3}{1 + 4z^4} = \frac{\alpha + \beta z}{1 + 2z + 2zz} + \frac{\gamma + \delta z}{1 - 2z + 2zz}$: addantur actu hæ duæ fractiones, eritque summæ

$$\begin{array}{l|l} \text{Numerator} & \text{Denominator} \\ +\alpha - 2\alpha z + 2\alpha zz & \\ \quad + \beta z - 2\beta zz + 2\beta z^3 & 1 + 4z^4 \\ +\gamma + 2\gamma z + 2\gamma zz & \\ \quad + \delta z + 2\delta zz + 2\delta z^3 & \end{array}$$

Cum ergo denominator æqualis sit denominatori fractionis propositæ, numeratores quoque æquales reddi debent: quod, ob tot litteras incognitas $\alpha, \beta, \gamma, \delta$, quot sunt termini æquales efficiendi, utique fieri, idque unico modo poterit: nancifcimur scilicet has quatuor æquationes

$$\begin{array}{ll} \text{I.} \quad \alpha + \gamma = 1 & \text{III.} \; 2\alpha - 2\beta + 2\gamma + 2\delta = 3 \\ \text{II.} \; -2\alpha + \beta + 2\gamma + \delta = -2 & \text{IV.} \; 2\beta + 2\delta = -4. \end{array}$$

Hinc ob $\alpha + \gamma = 1$, & $\beta + \delta = -2$; æquationes II. & III. dabunt $\alpha - \gamma = 0$ & $\delta - \beta = \frac{1}{2}$; ex quibus fit $\alpha = \frac{1}{2}$; $\gamma = \frac{1}{2}$; $\beta = \frac{-5}{4}$; $\delta = \frac{-3}{4}$; ideoque fractio proposita $\frac{1 - 2z + 3zz - 4z^3}{1 + 4z^4}$, transformatur in has duas

$\frac{\frac{1}{2} - \frac{5}{4}z}{1 + 2z + 2zz} + \frac{\frac{1}{2} - \frac{3}{4}z}{1 - 2z + 2zz}$. Simili autem modo facile perspicietur resolutionem semper succedere debere: quoniam semper tot litteræ incognitæ introducuntur, quot opus est ad numeratorem propositum eliciendum. Ex doctrina vero fractionum communi intelligitur hanc resolutionem succedere non posse, nisi isti denominatoris Factores fuerint inter se primi.

40. *Functio igitur fracta* $\frac{M}{N}$ *in tot fractiones simplices formæ* $\frac{A}{p - qz}$ *resolvi poterit, quot Factores simplices habet denominator* N *inter se inæquales.*

Repræ-

Repræſentat hic fractio $\frac{M}{N}$ Functionem quamcunque fractam genuinam, ita ut M & N ſint Functiones integræ ipſius z, atque ſumma poteſtas ipſius z in M minor ſit quam in N. Quod ſi ergo denominator N in ſuos Factores ſimplices reſolvatur, hique inter ſe fuerint inæquales, expreſſio $\frac{M}{N}$ in tot fractiones reſolvetur, quot Factores ſimplices in denominatore N continentur; propterea quod quiſque Factor abit in denominatorem fractionis partialis. Si ergo $p - qz$ fuerit Factor ipſius N, is erit denominator fractionis cujuſdam partialis, &, cum in numeratore hujus fractionis numerus dimenſionum ipſius z minor eſſe debeat quam in denominatore $p - qz$, numerator neceſſario erit quantitas conſtans. Hinc ex unoquoque Factore ſimplici $p - qz$ denominatoris N naſcetur fractio ſimplex $\frac{A}{p - qz}$; ita ut ſumma omnium harum fractionum ſit æqualis fractioni propoſitæ $\frac{M}{N}$.

EXEMPLUM.

Sit, exempli cauſa, propoſita hæc Functio fracta $\frac{1 + zz}{z - z^3}$; quia Factores ſimplices denominatoris ſunt z, $1 - z$, & $1 + z$, iſta Functio reſolvetur in has tres fractiones ſimplices $\frac{A}{z} + \frac{B}{1 - z} + \frac{C}{1 + z} = \frac{1 + zz}{z - z^3}$; ubi numeratores conſtantes A, B, & C definire oportet. Reducantur hæ fractiones ad communem denominatorem, qui erit $z - z^3$; atque numeratorum ſumma æquari debebit ipſi $1 + zz$, unde iſta æquatio oritur:

$$\begin{aligned} A + Bz &- Azz = 1 + zz = 1 + 0z + zz. \\ + Cz &+ Bzz \\ &- Czz \end{aligned}$$

LIB. I. quæ totidem comparationes præbet, quot sunt litteræ incognitæ A, B, C; erit scilicet,

I°. $A = 1$.

II°. $B + C = 0$.

III°. $-A + B - C = 1$:

Hinc fit $B - C = 2$; & porro $A = 1$; $B = 1$ & $C = -1$. Functio ergo proposita $\frac{1 + zz}{z - z^3}$ resolvitur in hanc formam $\frac{1}{z} + \frac{1}{1 - z} - \frac{1}{1 + z}$. Simili autem modo intelligitur, quotcunque habuerit denominator N Factores simplices inter se inæquales, semper fractionem $\frac{M}{N}$ in totidem fractiones simplices resolvi. Sin autem aliquot Factores fuerint æquales inter se, tum alio modo post-explicando resolutio institui debet.

41. *Cum igitur quilibet Factor simplex denominatoris* N *suppeditet fractionem simplicem pro resolutione Functionis propositæ* $\frac{M}{N}$; *ostendendum est quomodo ex Factore simplice denominatoris* N *cognito, fractio simplex respondens reperiatur.*

Sit $p - qz$ Factor simplex ipsius N, ita ut sit $N = (p - qz)S$; atque S Functio integra ipsius z; ponatur fractio ex Factore $p - qz$ orta $= \frac{A}{p - qz}$, & sit fractio ex altero Factore denominatoris S oriunda $= \frac{P}{S}$, ita ut, secundum §. 39., futurum sit $\frac{M}{N} = \frac{A}{p - qz} + \frac{P}{S} = \frac{M}{(p - qz)S}$; hinc erit $\frac{P}{S} = \frac{M - AS}{(p - qz)S}$; quæ fractiones cum congruere debeant, necesse est ut $M - AS$ sit divisibile per $p - qz$; quoniam Functio integra P ipsi quoto æquatur. Quando vero $p - qz$ Divisor existit ipsius $M - AS$, hæc expressio posito $z = \frac{p}{q}$ evanescit. Ponatur ergo ubique loco z hic valor constans $\frac{p}{q}$ in M

&

& S, erit $M - AS = 0$, ex quo fiet $A = \frac{M}{S}$; hocque ergo CAP. II. modo reperitur numerator A fractionis quaesitae $\frac{A}{p - qz}$; atque si ex singulis denominatoris N Factoribus simplicibus, dummodo sint inter se inaequales, hujusmodi fractiones simplices formentur, harum fractionum simplicium omnium summa erit aequalis Functioni propositae $\frac{M}{N}$.

EXEMPLUM.

Sic, si in Exemplo praecedente $\frac{1 + zz}{z - z^3}$, ubi est $M = 1 + zz$, & $N = z - z^3$, sumatur z pro Factore simplice, erit $S = 1 - zz$, atque fractionis simplicis $\frac{A}{z}$ hinc ortae erit numerator $A = \frac{1 + zz}{1 - zz} = 1$ posito $z = 0$, quem valorem z obtinet si ipse hic Factor simplex z nihilo aequalis ponatur. Simili modo si pro denominatoris Factore sumatur $1 - z$, ut sit $S = z + zz$ erit $A = \frac{1 + zz}{z + zz}$, facto $1 - z = 0$, unde erit $A = 1$, & ex Factore $1 - z$ nascitur fractio $\frac{1}{1 - z}$. Tertius denique Factor $1 + z$, ob $S = z - zz$, & $A = \frac{1 + zz}{z - zz}$, posito $1 + z = 0$, seu $z = -1$, dabit $A = -1$, & fractionem simplicem $= \frac{-1}{1 + z}$. Quare per hanc regulam reperitur $\frac{1 + zz}{z - z^3} = \frac{1}{z} + \frac{1}{1 - z} - \frac{1}{1 + z}$, ut ante.

42. *Functio fracta hujus formae* $\frac{P}{(p - qz)^n}$, *cujus numerator* P *non tantam ipsius* z *potestatem involvit quantam denominator* $(p - qz)^n$, *transmutari potest in hujusmodi fractiones partiales*

 $\frac{A}{(p-qz)^n} + \frac{B}{(p-qz)^{n-1}} + \frac{C}{(p-qz)^{n-2}} + \ldots + \frac{K}{p-qz}$; *quarum omnium numeratores sint quantitates constantes.*

Quoniam maxima potestas ipsius z in P minor est quam z^n, erit z^{n-1}, ideoque P hujusmodi habebit formam:

$$\alpha + \beta z + \gamma z^2 + \delta z^3 + \ldots\ldots\ldots + \kappa z^{n-1}$$

existente terminorum numero $= n$, cui æquari debet numerator summæ omnium fractionum partialium, postquam singulæ ad eundem denominatorem $(p-qz)^n$ fuerint perductæ: qui numerator propterea erit $= A + B(p-qz) + C(p-qz)^2 + D(p-qz)^3 + \ldots\ldots + K(p-qz)^{n-1}$. Hujus maxima ipsius z potestas est, ut ibi, z^{n-1}, atque tot habentur litteræ incognitæ $A, B, C, \ldots\ldots K$, (quarum numerus est $= n$,) quot sunt termini congruentes reddendi. Quamobrem litteræ constantes A, B, C, &c. ita definiri poterunt, ut fiat Functio fracta genuina $\frac{P}{(p-qz)^n} = \frac{A}{(p-qz)^n} + \frac{B}{(p-qz)^{n-1}} + \frac{C}{(p-qz)^{n-2}} + \frac{D}{(p-qz)^{n-3}} + \ldots\ldots + \frac{K}{p-qz}$. Ipsa autem horum numeratorum inventio mox facilis aperietur.

43. *Si Functionis fractæ* $\frac{M}{N}$ *denominator* N *Factorem habeat* $(p-qz)^2$, *sequenti modo fractiones partiales ex hoc Factore oriundæ reperientur.*

Cujusmodi fractiones partiales ex singulis Factoribus denominatoris simplicibus, qui alios sibi æquales non habeant, oriantur, ante est ostensum: nunc igitur ponamus duos Factores inter se esse æquales, seu, iis conjunctis, denominatoris N Factorem esse $(p-qz)^2$. Ex hoc ergo Factore per §. præced. duæ nascentur fractiones partiales hæ $\frac{A}{(p-qz)^2} + \frac{B}{p-qz}$. Sit autem

tem $N = (p - qz)^2 S$, eritque $\frac{M}{N} = \frac{M}{(p-qz)^2 S} = \frac{A}{(p-qz)^2} + \frac{B}{p-qz} + \frac{P}{S}$, denotante $\frac{P}{S}$ omnes fractiones simplices junctim sumptas ex denominatoris Factore S ortas. Hinc erit $\frac{P}{S} = \frac{M - AS - B(p-qz)S}{(p-qz)^2 S}$, & $P = \frac{M - AS - B(p-qz)S}{(p-qz)^2}$ = Functioni integræ. Debet ergo $M - AS - B(p-qz)S$ divisibile esse per $(p-qz)^2$: sit primum divisibile per $p-qz$, atque tota expressio $M - AS - B(p-qz)S$ evanescet, posito $p - qz = 0$, seu $z = \frac{p}{q}$; ponatur ergo ubique $\frac{p}{q}$ loco z, eritque $M - AS = 0$, ideoque $A = \frac{M}{S}$; scilicet fractio $\frac{M}{S}$, si loco z ubique ponatur $\frac{p}{q}$, dabit valorem ipsius A constantem. Hoc invento quantitas $M - AS - B(p-qz)S$ etiam per $(p-qz)^2$ divisibilis esse debet, seu $\frac{M-AS}{p-qz} - BS$ denuo per $p - qz$ divisibile esse debet. Posito ergo ubique $z = \frac{p}{q}$ erit $\frac{M-AS}{p-qz} = BS$, ideoque $B = \frac{M-AS}{(p-qz)S} = \frac{1}{p-qz}\left(\frac{M}{S} - A\right)$, ubi notandum est, cum $M - AS$ divisibile sit per $p - qz$, hanc divisionem prius institui debere, quam loco z substituatur $\frac{p}{q}$. Vel ponatur $\frac{M-AS}{p-qz} = T$, eritque $B = \frac{T}{S}$ posito $z = \frac{p}{q}$; inventis ergo numeratoribus A & B, erunt fractiones partiales ex denominatoris N Factore $(p-qz)^2$ ortæ hæ $\frac{A}{(p-qz)^2} + \frac{B}{p-qz}$.

EXEMPLUM I.

Sit hæc proposita Functio fracta $\frac{1 - zz}{zz(1+zz)}$ erit, ob denomina-

LIB. I. nominatoris Factorem quadratum zz; $S = 1 + zz$ & $M = 1 - zz$. Sint fractiones partiales ex zz ortæ $\frac{A}{zz} + \frac{B}{z}$, erit $A = \frac{M}{S} = \frac{1 - zz}{1 + zz}$, posito Factore $z = 0$; hincque $A = 1$. Tum erit $M - AS = - 2zz$ quod divisum per Factorem simplicem z, dabit $T = - 2z$, hincque $B = \frac{T}{S} = \frac{-2z}{1 + zz}$, posito $z = 0$; unde erit $B = 0$; atque ex Factore denominatoris zz orietur unica hæc fractio partialis $\frac{1}{zz}$.

EXEMPLUM II.

Sit hæc proposita Functio fracta $\frac{z^3}{(1 - z)^2 (1 + z^4)}$, cujus, ob denominatoris Factorem quadratum $(1 - z)^2$, fractiones partiales sint $\frac{A}{(1 - z)^2} + \frac{B}{1 - z}$. Erit ergo $M = z^3$ & $S = 1 + z^4$; ideoque $A = \frac{M}{S} = \frac{z^3}{1 + z^4}$, posito $1 - z = 0$, seu $z = 1$: unde fit $A = \frac{1}{2}$. Prodibit ergo $M - AS = z^3 - \frac{1}{2} - \frac{1}{2} z^4 = - \frac{1}{2} + z^3 - \frac{1}{2} z^4$, quod per $1 - z$ divisum dat $T = - \frac{1}{2} - \frac{1}{2} z - \frac{1}{2} zz + \frac{1}{2} z^3$; ideoque $B = \frac{T}{S} = \frac{-1 - z - zz + z^3}{2 + 2z^4}$, posito $z = 1$; ita ut sit $B = \frac{-1}{2}$; fractiones ergo partiales quæsitæ sunt $\frac{1}{2(1 - z)^2} - \frac{1}{2(1 - z)}$.

44. *Si Functionis fractæ* $\frac{M}{N}$ *denominator* N *Factorem habeat* $(p - qz)^3$ *sequenti modo fractiones partiales ex hoc Factore oriundæ* $\frac{A}{(p - qz)^3} + \frac{B}{(p - qz)^2} + \frac{C}{p - qz}$ *reperientur.*

Ponatur

Ponatur $N = (p - qz)^3 S$, sitque fractio ex Factore S orta $= \frac{P}{S}$, erit $P = \frac{M - AS - B(p - qz)S - C(p - qz)^2 S}{(p - qz)^3}$ = Functioni integræ. Numerator ergo $M - AS - B(p - qz)S - C(p - qz)^2 S$ ante omnia divisibilis esse debet per $(p - qz)$; unde is, posito $p - qz = 0$, seu $z = \frac{p}{q}$, evanescere debet, eritque adeo $M - AS = 0$, ideoque $A = \frac{M}{S}$, posito $z = \frac{p}{q}$. Invento hoc pacto A erit $M - AS$ divisibile per $p - qz$ ponatur ergo $\frac{M - AS}{p - qz} = T$, atque $T - BS - C(p - qz)S$ adhuc per $(p - qz)^2$ erit divisibile; fiet ergo $= 0$, posito $p - qz = 0$; ex quo prodit $B = \frac{T}{S}$ posito $z = \frac{p}{q}$. Sic autem invento B erit $T - BS$ divisibile per $p - qz$. Hanc ob rem, posito $\frac{T - BS}{p - qz} = V$, superest ut $V - CS$ divisibile sit per $p - qz$; eritque ergo $V - CS = 0$, posito $p - qz = 0$, atque $C = \frac{V}{S}$, posito $z = \frac{p}{q}$. Inventis ergo hoc modo numeratoribus A, B, C, fractiones partiales ex denominatoris N Factore $(p - qz)^3$ ortæ erunt $\frac{A}{(p - qz)^3} + \frac{B}{(p - qz)^2} + \frac{C}{p - qz}$.

CAP. II.

EXEMPLUM.

Sit proposita hæc fracta Functio $\frac{zz}{(1 - z)^3(1 + zz)}$, ex cujus denominatoris Factore cubico $(1 - z)^3$ oriantur hæ fractiones partiales: $\frac{A}{(1 - z)^3} + \frac{B}{(1 - z)^2} + \frac{C}{1 - z}$. Erit ergo $M = zz$ & $S = 1 + zz$; unde fit primum $A = \frac{zz}{1 + zz}$ posito

LIB. I. posito $1 - z = 0$ seu $z = 1$; ex quo prodit $A = \frac{1}{2}$. Jam ponatur $T = \frac{M - AS}{1 - z}$, erit $T = \frac{\frac{1}{2}zz - \frac{1}{2}}{1 - z} = -\frac{1}{2} - \frac{1}{2}z$; unde oritur $B = \frac{-\frac{1}{2} - \frac{1}{2}z}{1 + zz}$, posito $z = 1$, ita ut sit $B = -\frac{1}{2}$. Ponatur porro $V = \frac{T - BS}{1 - z} = \frac{T + \frac{1}{2}S}{1 - z}$; erit $V = \frac{-\frac{1}{2}z + \frac{1}{2}zz}{1 - z} = -\frac{1}{2}z$; unde fit $C = \frac{V}{S} = \frac{-\frac{1}{2}z}{1 + zz}$ posito $z = 1$, ita ut sit $C = -\frac{1}{4}$. Quo circa fractiones partiales ex denominatoris Factore $(1 - z)^3$ ortæ erunt $\frac{1}{2(1 - z)^3} - \frac{1}{2(1 - z)^2} - \frac{1}{4(1 - z)}$.

45. *Si Functionis fractæ* $\frac{M}{N}$ *denominator* N *Factorem habeat* $(p - qz)^n$; *fractiones partiales hinc ortæ* $\frac{A}{(p - qz)^n} + \frac{B}{(p - qz)^{n-1}} + \frac{C}{(p - qz)^{n-2}} + \ldots\ldots + \frac{K}{p - qz}$ *sequenti modo invenientur.*

Ponatur denominator $N = (p - qz)^n Z$, atque, ratiocinium ut ante instituendo, reperietur ut sequitur:

Primo $A = \frac{M}{Z}$, posito $z = \frac{p}{q}$. Ponatur $P = \frac{M - AZ}{p - qz}$

Secundo $B = \frac{P}{Z}$, posito $z = \frac{p}{q}$. Ponatur $Q = \frac{P - BZ}{p - qz}$

Tertio $C = \frac{Q}{Z}$, posito $z = \frac{p}{q}$. Ponatur $R = \frac{Q - CZ}{p - qz}$

Quarto $D = \frac{R}{Z}$, posito $z = \frac{p}{q}$. Ponatur $S = \frac{R - DZ}{p - qz}$

Quinto $E = \frac{S}{Z}$, posito $z = \frac{p}{q}$. &c.

Hoc ergo modo si definiantur singuli numeratores constan-

tes

tes A, B, C, D, &c. invenientur omnes fractiones partiales, quæ ex denominatoris N Factore $(p - qz)^n$ nascuntur. CAP. II.

EXEMPLUM.

Sit proposita ista Functio fracta $\frac{1 + zz}{z^5(1 + z^3)}$ ex cujus denominatoris Factore z^5 nascantur hæ fractiones partiales $\frac{A}{z^5} + \frac{B}{z^4} + \frac{C}{z^3} + \frac{D}{z^2} + \frac{E}{z}$. Ad quarum numeratores constantes inveniendos, erit $M = 1 + zz$ atque $Z = 1 + z^3$; & $\frac{p}{q} = 0$. Sequens ergo calculus ineatur.

Primum est $A = \frac{M}{Z} = \frac{1 + zz}{1 + z^3}$, posito $z = 0$; ergo $A = 1$.

Ponatur $P = \frac{M - AZ}{z} = \frac{zz - z^3}{z} = z - zz$. Eritque secundo $B = \frac{P}{Z} = \frac{z - zz}{1 + z^3}$, posito $z = 0$; ergo $B = 0$.

Ponatur $Q = \frac{P - BZ}{z} = \frac{z - zz}{z} = 1 - z$; eritque tertio $C = \frac{Q}{Z} = \frac{1 - z}{1 + z^3}$, posito $z = 0$: ergo $C = 1$.

Ponatur $R = \frac{Q - CZ}{z} = \frac{-z - z^3}{z} = -1 - zz$; erit quarto $D = \frac{R}{Z} = \frac{-1 - zz}{1 + z^3}$, posito $z = 0$; ex quo fit $D = -1$.

Ponatur $S = \frac{R - DZ}{z} = \frac{-zz + z^3}{z} = -z + zz$; erit quinto $E = \frac{S}{Z} = \frac{-z + zz}{1 + z^3}$, posito $z = 0$; unde fit $E = 0$.

Quo circa fractiones partiales quæsitæ erunt hæ:
$\frac{1}{z^5} + \frac{0}{z^4} + \frac{1}{z^3} - \frac{1}{z^2} + \frac{0}{z}$.

LIB. I.

46. *Quaecunque ergo proposita fuerit Functio rationalis fracta* $\frac{M}{N}$, *ea sequenti modo in partes resolvetur, atque in formam simplicissimam transmutabitur.*

Quaerantur denominatoris N omnes Factores simplices sive reales sive imaginarii; quorum qui sibi pares non habeant, seorsim tractentur & ex unoquoque per §. 41, fractio partialis eruatur. Quod si idem Factor simplex bis vel pluries occurrat, ii conjunctim sumantur atque ex eorum producto, quod erit potestas formae $(p-qz)^n$, quaerantur fractiones partiales convenientes per §. 45. Hocque modo cum ex singulis Factoribus simplicibus denominatoris erutae fuerint fractiones partiales, tum harum omnium aggregatum aequabitur Functioni propositae $\frac{M}{N}$, nisi fuerit spuria; si enim fuerit spuria, pars integra insuper extrahi atque ad istas fractiones partiales inventas adjici debebit, quo prodeat valor Functionis $\frac{M}{N}$ in forma simplicissima expressus. Perinde autem est sive fractiones partiales ante extractionem partis integrae, sive post quaerantur. Eaedem enim ex singulis denominatoris N Factoribus prodeunt fractiones partiales, sive adhibeatur ipse numerator M, sive idem quocunque denominatoris N multiplo vel auctus vel minutus; id quod regulas datas contemplanti facile patebit.

EXEMPLUM.

Quaeratur valor Functionis $\frac{1}{z^3(1-z)^2(1+z)}$ in forma simplicissima expressus. Sumatur primum Factor denominatoris solitarius $1+z$, qui dat $\frac{p}{q}=-1$. erit $M=1$ & $Z=z^3-2z^4+z^5$. Hinc ad fractionem $\frac{A}{1+z}$ inveniendam, erit $A=\frac{1}{z^3-2z^4+z^5}$, posito $z=-1$; ideoque fit $A=$

$-\frac{1}{4}$, atque ex Factore $1+z$ oritur hæc fractio partialis $\frac{-1}{4(1+z)}$. Jam sumatur Factor quadratus $(1-z)^2$ qui dat $\frac{p}{q}=1$. $M=1$, & $Z=z^3+z^4$; positis ergo fractionibus partialibus hinc ortis $\frac{A}{(1-z)^2}+\frac{B}{1-z}$, erit $A=\frac{1}{z^3+z^4}$, posito $z=1$; ergo $A=\frac{1}{2}$; fiat $P=\frac{M-\frac{1}{2}Z}{1-z}$ $=\frac{1-\frac{1}{2}z^3-\frac{1}{2}z^4}{1-z}=1+z+zz+\frac{1}{2}z^3$; eritque $B=\frac{P}{Z}=\frac{1+z+zz+\frac{1}{2}z^3}{z^3+z^4}$, posito $z=1$; ergo $B=\frac{7}{4}$ & fractiones partiales quæsitæ $\frac{1}{2(1-z)^2}+\frac{7}{4(1-z)}$. Denique tertius Factor cubicus z^3 dat $\frac{p}{q}=0$; $M=1$; & $Z=1-z-zz+z^3$. Positis ergo fractionibus partialibus his $\frac{A}{z^3}+\frac{B}{z^2}+\frac{C}{z}$; erit primum. $A=\frac{M}{N}=\frac{1}{1-z-zz+z^3}$ posito $z=0$; ergo $A=1$. Ponatur $P=\frac{M-Z}{z}=1+z-zz$, erit $B=\frac{P}{Z}$, posito $z=0$; ergo $B=1$. Ponatur $Q=\frac{P-Z}{z}=2-zz$; erit $C=\frac{Q}{Z}$, posito $z=0$; ergo $C=2$. Hanc ob rem Functio proposita $\frac{1}{z^3(1-z)^2(1+z)}$ in hanc formam resolvitur $\frac{1}{z^3}+\frac{1}{z^2}+\frac{2}{z}+\frac{1}{2(1-z)^2}+\frac{7}{4(1-z)}-\frac{1}{4(1+z)}$: nulla enim pars integra insuper accedit, quia fractio proposita non est spuria.

CAPUT III.

De transformatione Functionum per substitutionem.

46. *Si fuerit* y *Functio quaecunque ipsius* z, *atque* z *definiatur per novam variabilem* x, *tum quoque* y *per* x *definiri poterit.*

Cum ergo antea y fuisset Functio ipsius z, nunc nova quantitas variabilis x inducitur, per quam utraque priorum y & z definiatur. Sic, si fuerit $y = \frac{1 - zz}{1 + zz}$, atque ponatur $z = \frac{1 - x}{1 + x}$; hoc valore loco z substituto, erit $y = \frac{2x}{1 + xx}$. Sumpto ergo pro x valore quocunque determinato, ex eo reperientur valores determinati pro z & y, sicque invenitur valor ipsius y respondens illi valori ipsius z qui simul prodiit. Uti si sit $x = \frac{1}{2}$, fiet $z = \frac{1}{3}$, & $y = \frac{4}{5}$; reperitur autem quoque $y = \frac{4}{5}$, si in $\frac{1 - zz}{1 + zz}$, cui expressioni y aequatur, ponatur $z = \frac{1}{3}$.

Adhibetur autem haec novae variabilis introductio ad duplicem finem: vel enim hoc modo irrationalitas, qua expressio ipsius y per z data laborat, tollitur; vel quando ob aequationem altioris gradus, qua relatio inter y & z exprimitur, non licet Functionem explicitam ipsius z ipsi y aequalem exhibere, nova variabilis x introducitur, ex qua utraque y & z commode definiri queat: unde insignis substitutionum usus jam satis elucet, ex sequentibus vero multo clarius perspicietur.

47. *Si fuerit* $y = \surd(a + bz)$; *nova variabilis* x *per quam utraque* z *&* y *rationaliter exprimatur, sequenti modo invenietur.*

Quoniam tam z quam y debet esse Functio rationalis ipsius x; perspicuum est hoc obtineri si ponatur $\surd(a + bz) = bx$: Fiet enim primo $y = bx$; & $a + bz = bbxx$; hincque $z = bxx - \frac{a}{b}$. Quocirca utraque quantitas y & z per Functionem rationalem ipsius x exprimitur; scilicet cum sit $y = \surd(a + bz)$ fiat $z = bxx - \frac{a}{b}$; erit $y = bx$.

48. *Si*

48. *Si fuerit* $y = (a + bz)^{m:n}$; *nova variabilis* x, *per quam tam* y *quam* z *rationaliter exprimatur, sic reperietur.*

Ponatur $y = x^m$, fietque $(a + bz)^{m:n} = x^m$ ideoque $(a + bz)^{1:n} = x$: ergo $a + bz = x^n$ & $z = \frac{x^n - a}{b}$. Sic ergo utraque quantitas y & z rationaliter per x definietur, ope scilicet substitutionis $z = \frac{x^n - a}{b}$, quæ præbet $y = x^m$. Quamvis igitur neque y per z, neque vicissim z per y rationaliter exprimi possit; tamen utraque reddita est Functio rationalis novæ quantitatis variabilis x per substitutionem introductæ, scopo substitutionis omnino convenienter.

49. *Si fuerit* $y = \left(\frac{a + bz}{f + gz}\right)^{m:n}$; *requiritur nova quantitas variabilis* x *per quam utraque* y *&* z *rationaliter exprimatur.*

Manifestum primo est si ponatur $y = x^m$, quæsito satisfieri; erit enim $\left(\frac{a + bz}{f + gz}\right)^{m:n} = x^m$, ideoque $\frac{a + bz}{f + gz} = x^n$; ex qua æquatione elicitur $z = \frac{a - fx^n}{gx^n - b}$; quæ substitutio præbet $y = x^m$.

Hinc quoque intelligitur si fuerit $\left(\frac{\alpha + \beta y}{\gamma + \delta y}\right)^n = \left(\frac{a + bz}{f + gz}\right)^m$; tam y quam z rationaliter per x expressum iri, si utraque formula ponatur $= x^{mn}$; reperietur enim $y = \frac{\alpha - \gamma x^m}{\delta x^m - \beta}$ & $z = \frac{a - fx^n}{gx^n - b}$: qui casus nil habent difficultatis.

50. *Si fuerit* $y = \sqrt{((a + bz)(c + dz))}$; *substitutio idonea invenietur, qua* y *&* z *rationaliter exprimuntur, hoc modo.*

Ponatur $\sqrt{((a + bz)(c + dz))} = (a + bz)x$, facile enim perspicitur hinc valorem rationalem pro z esse proditurum; quia valor ipsius z per æquationem simplicem determinatur. Erit

 ergo

LIB. I. ergo $c+dz=(a+bz)xx$, hincque $z=\frac{c-axx}{bxx-d}$. Quare porro fiet $a+bz=\frac{bc-ad}{bxx-d}$; & ob $y=\sqrt{((a+bz)(c+dz))}=(a+bz)x$ habebitur $y=\frac{(bc-ad)x}{bxx-d}$. Functio ergo irrationalis $y=\sqrt{((a+bz)(c+dz))}$ ad rationalitatem perducitur ope substitutionis $z=\frac{c-axx}{bxx-d}$, quippe quæ dabit $y=\frac{(bc-ad)x}{bxx-d}$. Sic, si fuerit $y=\sqrt{(aa-zz)}=\sqrt{((a+z)(a-z))}$; ob $b=+1$; $c=a$, $d=-1$, ponatur $z=\frac{a-axx}{1+xx}$, eritque $y=\frac{2ax}{1+xx}$. Quoties ergo quantitas post signum $\sqrt{}$ habuerit duos Factores simplices reales, hoc modo reductio ad rationalitatem absolvetur; sin autem Factores bini simplices fuerint imaginarii, sequenti modo uti præstabit.

51. *Sit* $y=\sqrt{(p+qz+rzz)}$; *atque requiritur substitutio idonea pro* z *facienda, ut valor ipsius* y *fiat rationalis.*

Pluribus modis hoc fieri potest, prout p & q fuerint quantitates vel affirmativæ vel negativæ. Sit primo p quantitas affirmativa, ac ponatur aa pro p; etiamsi enim p non sit quadratum, tamen irrationalitas quantitatum constantium præsens negotium non turbat. Sit igitur

I. $y=\sqrt{(aa+bz+czz)}$; ac ponatur $\sqrt{(aa+bz+czz)}=a+xz$; erit $b+cz=2ax+xxz$; unde fit $z=\frac{b-2ax}{xx-c}$: tum vero erit $y=a+xz=\frac{bx-axx-ac}{xx-c}$; ubi z & y sunt Functiones rationales ipsius x. Sit jam

II. $y=\sqrt{(aazz+bz+c)}$; ac ponatur $\sqrt{(aazz+bz+c)}=az+x$; erit $bz+c=2axz+xx$, & $z=\frac{xx-c}{b-2ax}$. Tum autem fit $y=az+x=\frac{-ac+bx-axx}{b-2ax}$.

III. Si fuerint p & r quantitates negativæ; tum, nisi sit $qq>4pr$, valor ipsius y semper erit imaginarius. Quod si autem fuerit $qq>4pr$; expressio $p+qz+rzz$ in duos Factores resolvi

resolvi poterit, qui casus ad §. præced. reducitur. Sæpenumero autem commodius ad hanc formam reducitur, $y = \sqrt{(aa + (b + cz)(d + ez))}$; pro qua ad rationalitatem perducenda ponatur $y = a + (b + cz)x$, eritque $d + ez = 2ax + bxx + cxxz$; unde fit $z = \frac{d - 2ax - bxx}{cxx - e}$, & $y = \frac{-ae + (cd - be)x - acxx}{cxx - e}$. Interdum commodius fieri potest reductio ad hanc formam, $y = \sqrt{(aazz + (b + cz)(d + ez))}$. Tum ponatur $y = az + (b + cz)x$; erit $d + ez = 2axz + bxx + cxxz$ & $z = \frac{bxx - d}{e - 2ax - cxx}$, atque $y = \frac{-ad + (be - cd)x - abxx}{e - 2ax - cxx}$. CAP. III.

EXEMPLUM.

Si habeatur ista ipsius z Functio irrationalis $y = \sqrt{(-1 + 3z - zz)}$; quæ cum reduci queat ad hanc formam $y = \sqrt{(1 - 2 + 3z - zz)} = \sqrt{(1 - (1 - z)(2 - z))}$; ponatur $y = 1 - (1 - z)x$, erit $-2 + z = -2x + xx - xxz$ & $z = \frac{2 - 2x + xx}{1 + xx}$. Deinde est $1 - z = \frac{-1 + 2x}{1 + xx}$ & $y = 1 - (1 - z)x = \frac{1 + x - xx}{1 + xx}$. Atque hi sunt fere casus, quos Algebra indeterminata, seu methodus *Diophantæa*, suppeditat; neque alios casus in his tractatis non comprehensos per substitutionem rationalem ad rationalitatem reducere licet. Quocirca ad alterum substitutionis usum monstrandum progredior.

52. *Si* y *ejusmodi fuerit Functio ipsius* z *ut sit* $ay^{\alpha} + bz^{\beta} + cy^{\gamma}z^{\delta} = 0$, *invenire novam variabilem* x, *per quam valores ipsarum* y *&* z *explicite assignari queant.*

Quoniam resolutio æquationum generalis non habetur, ex æquatione proposita $ay^{\alpha} + bz^{\beta} + cy^{\gamma}z^{\delta} = 0$ neque y per z neque

LIB. I. z neque viciſſim z per y exhiberi poteſt. Quo igitur huic incommodo remedium afferatur; ponatur $y = x^m z^n$, eritque $a x^{\alpha m} z^{\alpha n} + b z^{\beta} + c x^{\gamma m} z^{\gamma n + \delta} = 0$. Determinetur nunc exponens n ita ut ex hac æquatione valor ipſius z definiri queat; quod tribus modis præſtari poteſt.

I. Sit $\alpha n = \beta$; ideoque $n = \frac{\beta}{\alpha}$; erit, æquatione per $z^{\alpha n} = z^{\beta}$ diviſa, $a x^{\alpha m} + b + c x^{\gamma m} z^{\gamma n - \beta + \delta} = 0$; unde oritur $z = \left(\frac{-a x^{\alpha m} - b}{c x^{\gamma m}}\right)^{\frac{1}{\gamma n - \beta + \delta}}$, ſive

$$z = \left(\frac{-a x^{\alpha m} - b}{c x^{\gamma m}}\right)^{\frac{\alpha}{\beta\gamma - \alpha\beta + \alpha\delta}}, \&$$

$$y = x^m \left(\frac{-a x^{\alpha m} - b}{c x^{\gamma m}}\right)^{\frac{\beta}{\beta\gamma - \alpha\beta + \alpha\delta}}.$$

II. Sit $\beta = \gamma n + \delta$ ſeu $n = \frac{\beta - \delta}{\gamma}$; erit, æquatione per z^{β} diviſa, $a x^{\alpha m} z^{\alpha n - \beta} + b + c x^{\gamma m} = 0$; unde oritur

$$z = \left(\frac{-b - c x^{\gamma m}}{a x^{\alpha m}}\right)^{\frac{1}{\alpha n - \beta}} = \left(\frac{-b - c x^{\gamma m}}{a x^{\alpha m}}\right)^{\frac{\gamma}{\alpha\beta - \alpha\delta - \beta\gamma}},$$

atque $y = x^m \left(\frac{-b - c x^{\gamma m}}{a x^{\alpha m}}\right)^{\frac{\beta - \delta}{\alpha\beta - \alpha\delta - \beta\gamma}}$.

III. Sit $\alpha n = \gamma n + \delta$, ſeu $n = \frac{\delta}{\alpha - \gamma}$; erit, æquatione per $z^{\alpha n}$ diviſa, $a x^{\alpha m} + b z^{\beta - \alpha n} + c x^{\gamma m} = 0$; unde oritur

$$z = \left(\frac{-a x^{\alpha m} - c x^{\gamma m}}{b}\right)^{\frac{1}{\beta - \alpha n}} =$$

$$\left(\frac{-ax^{\alpha m}-cx^{\gamma m}}{b}\right)^{\frac{\alpha-\gamma}{\alpha\beta-\beta\gamma-\alpha\delta}};\text{ atque}$$

$$y=x^m\left(\frac{-ax^{\alpha m}-cx^{\gamma m}}{b}\right)^{\frac{\delta}{\alpha\beta-\beta\gamma-\alpha\delta}}.$$

Tribus igitur diversis modis erutæ sunt Functiones ipsius x; quæ ipsis z & y sunt æquales. Præterea vero pro m numerum pro lubitu substituere licet cyphra excepta; sicque formulæ ad commodissimam expressionem reduci poterunt.

EXEMPLUM.

Exprimatur natura Functionis y per hanc æquationem $y^3+z^3-cyz=0$; atque quærantur Functiones ipsius x ipsis y & z æquales. Erit ergo $a=-1$; $b=-1$; $\alpha=3$; $\beta=3$; $\gamma=1$; & $\delta=1$. Hinc primus modus dabit, posito $m=1$, $z=\left(\frac{x^3+1}{cx}\right)^{-1}$ & $y=x\left(\frac{x^3+1}{cx}\right)^{-1}$, sive $z=\frac{cx}{1+x^3}$ & $y=\frac{cxx}{1+x^3}$; quarum expressionum utraque adeo est rationalis.

Secundus modus vero dabit hos valores:

$$z=\left(\frac{cx-1}{x^3}\right)^{1:3},\ \&\ y=x\left(\frac{cx-1}{x^3}\right)^{2:3},\ \text{sive}$$

$$z=\frac{1}{x}\sqrt[3]{(cx-1)},\ \&\ y=\frac{1}{x}\sqrt[3]{(cx-1)^2}.$$

Tertius modus ita rem expediet ut sit

$$z=(cx-x^3)^{2:3},\ \&\ y=x(cx-x^3)^{1:3}.$$

53. *Hinc a posteriori intelligitur cujusmodi æquationes, quibus valor Functionis* y *per* z *determinatur, hoc modo novam variabilem* x *introducendo resolvi queant.*

Ponamus enim resolutione jam instituta prodiisse has deter-

LIB. I. minationes $z = \left(\frac{ax^{\alpha} + bx^{\beta} + cx^{\gamma} + \&c.}{A + Bx^{\mu} + Cx^{\nu} + \&c.}\right)^{p:r}$, atque $y = x$
$\left(\frac{ax^{\alpha} + bx^{\beta} + cx^{\gamma} + \&c.}{A + Bx^{\mu} + Cx^{\nu} + \&c.}\right)^{q:r}$; eritque $y^p = x^p z^q$; & hinc
$x = yz^{-q:p}$. Cum igitur fit $z^{r:p} = \frac{ax^{\alpha} + bx^{\beta} + cx^{\gamma} + \&c.}{A + Bx^{\mu} + Cx^{\nu} + \&c.}$, fi
loco x ejus valorem $yz^{-q:p}$ fubftituamus; prodibit ifta æquatio $z^{r:p} = \frac{ay^{\alpha}z^{-\alpha q:p} + by^{\beta}z^{-\beta q:p} + cy^{\gamma}z^{-\gamma q:p} + \&c.}{A + By^{\mu}z^{-\mu q:p} + Cy^{\nu}z^{-\nu q:p} + \&c.}$;
quæ reducitur ad hanc $Az^{r:p} + By^{\mu}z^{(r-\mu q):p} + Cy^{\nu}$ $z^{(r-\nu q):q} + \&c. = ay^{\alpha}z^{-\alpha q:p} + by^{\beta}z^{-\beta q:p}$ $+ cy^{\gamma}z^{-\gamma q:p} + \&c.$ quæ multiplicata per $z^{\alpha q:p}$ tranfibit in hanc: $Az^{(\alpha q + r):p} + By^{\mu}z^{(\alpha q - \mu q + r):p} + Cy^{\nu}z^{(\alpha q - \nu q + r):p}$ $+ \&c. = ay^{\alpha} + by^{\beta}z^{(\alpha q - \beta q):p} + cy^{\gamma}z^{(\alpha q - \gamma q):p} + \&c.$ Ponatur $\frac{\alpha q + r}{p} = m$ & $\frac{\alpha q - \beta q}{p} = n$: fiet $p = \alpha - \beta$; $q = n$, & $r = \alpha m - \beta m - \alpha n$: atque nafcetur ifta æquatio: $Az^{m} + By^{\mu}z^{m - \mu n:(\alpha - \beta)} + Cy^{\nu}z^{m - \nu n:(\alpha - \beta)} + \&c. =$ $ay^{\alpha} + by^{\beta}z^{n} + cy^{\gamma}z^{(\alpha - \gamma)n:(\alpha - \beta)} + \&c.$ quæ propterea ita refolvetur ut fit:

$$z = \left(\frac{ax^{\alpha} + bx^{\beta} + cx^{\gamma} + \&c.}{A + Bx^{\mu} + Cx^{\nu} + \&c.}\right)^{\frac{\alpha - \beta}{\alpha m - \beta m - \alpha n}} \quad \&$$

$$y = x\left(\frac{ax^{\alpha} + bx^{\beta} + cx^{\gamma} + \&c.}{A + Bx^{\mu} + Cx^{\nu} + \&c.}\right)^{\frac{n}{\alpha m - \beta m - \alpha n}}.$$

Vel ponatur $\frac{\alpha q + r}{p} = m$, & $\frac{\alpha q - \mu q + r}{p} = n$, erit $m - n$

=

$= \frac{\mu q}{p}$; & $\frac{q}{p} = \frac{m-n}{\mu}$, atque $\frac{r}{p} = m - \frac{\alpha m + \alpha n}{\mu}$. Hinc fit $p = \mu$; $q = m - n$; & $r = \mu m - \alpha m + \alpha n$; atque hæc æquatio reſultabit: Cap. III.

$Az^m + By^\mu z^n + Cy^\nu z^{\mu m - \nu(m-n):\mu} + \&c. = ay^\alpha + by^\beta z^{(\alpha-\beta)(m-n):\mu} + cy^\gamma z^{(\alpha-\gamma)(m-n):\mu}$ + &c. quæ ita reſolvetur ut ſit:

$$z = \left(\frac{ax^\alpha + bx^\beta + cx^\gamma + \&c.}{A + Bx^\mu + Cx^\nu + \&c.}\right)^{\frac{\mu}{\mu m - \alpha m + \alpha n}} \&$$

$$y = x\left(\frac{ax^\alpha + bx^\beta + cx^\gamma + \&c.}{A + Bx^\mu + Cx^\nu + \&c.}\right)^{\frac{m-n}{\mu m - \alpha m + \alpha n}}.$$

54. *Si* y *ita pendeat a* z *ut ſit* $ayy + byz + czz + dy + ez = 0$, *ſequenti modo tam* y *quam* z *rationaliter per novam variabilem* x *exprimetur.*

Ponatur $y = xz$, erit diviſione per z facta: $axxz + bxz + cz + dx + e = 0$, ex qua reperitur $z = \frac{-dx - e}{axx + bx + c}$, & $y = \frac{-dxx - ex}{axx + bx + c}$.

At vero ad formam propoſitam reduci poteſt hæc æquatio inter y & z: $ayy + byz + czz + dy + ez + f = 0$ diminuendo vel augendo utramque variabilem certa quadam quantitate conſtante, unde & hæc æquatio per novam variabilem x rationaliter explicari poteſt.

55. *Si* y *ita pendeat a* z, *ut ſit* $ay^3 + by^2z + cyz^2 + dz^3 + eyy + fyz + gzz = 0$; *ſequenti modo tam* y *quam* z *rationaliter per novam variabilem* x *exprimi poterit.*

Ponatur $y = xz$, & facta ſubſtitutione tota æquatio per zz dividi poterit: prodibit autem $ax^3z + bxxz + cxz + dz + exx + fx + g = 0$. Unde oritur $z = \frac{-exx - fx - g}{ax^3 + bxx + cx + d}$ ex quo erit $y = \frac{-ex^3 - fxx - gx}{ax^3 + bxx + cx + d}$.

LIB. I. Ex his casibus facile intelligitur quemadmodum æquationes altiorum graduum, quibus y per z definitur, comparatæ esse debeant, ut hujusmodi resolutio locum habere queat. Ceterum hi casus in superioribus formulis §. 53. continentur: at, quia formulæ generales non tam facile ad hujusmodi casus sæpius occurrentes accommodantur, visum est horum aliquos seorsim evolvere.

56. *Si* y *ita pendeat a* z *ut sit* $ayy + byz + czz = d$ *hoc modo utraque quantitas* y *&* z *per novam variabilem* x *exprimetur.*

Ponatur $y = xz$, eritque $(axx + bx + c)zz = d$, ideoque $z = \sqrt{\frac{d}{axx + bx + c}}$ & $y = x\sqrt{\frac{d}{axx + bx + c}}$.

Simili modo si fuerit, $ay^3 + by^2z + cyz^2 + dz^3 = ey + fz$; posito $y = xz$, tota æquatio per z divisa dabit $(ax^3 + bxx + cx + d)zz = ex + f$; unde oritur $z = \sqrt{\frac{ex + f}{ax^3 + bxx + cx + d}}$; & $y = x\sqrt{\frac{ex + f}{ax^3 + bxx + cx + d}}$. Hi autem casus aliique similes resolutiones admittentes comprehenduntur in sequente paragrapho.

57. *Si* y *ita pendeat a* z *ut sit* $ay^m + by^{m-1}z + cy^{m-2}z^2 + dy^{m-3}z^3 + \&c. = \alpha y^n + \beta y^{n-1}z + \gamma y^{n-2}z^2 + \delta y^{n-3}z^3 + \&c.$ *Sequenti modo tam* z *quam* y *commode per novam variabilem* x *exprimetur.*

Sit $y = xz$, atque facta substitutione tota æquatio dividi poterit per z^n, siquidem exponens m sit major quam n; eritque $(ax^m + bx^{m-1} + cx^{m-2} + \&c.)z^{m-n} = \alpha x^n + \beta x^{n-1} + \gamma x^{n-2} + \&c.$ unde obtinebitur

$z =$

$$z = \left(\frac{\alpha x^n + \beta x^{n-1} + \gamma x^{n-2} + \delta x^{n-3} + \&c.}{ax^m + bx^{m-1} + cx^{m-2} + dx^{m-3} + \&c.}\right)^{1:(m-n)} \quad \&$$

$$y = x\left(\frac{\alpha x^n + \beta x^{n-1} + \gamma x^{n-2} + \delta x^{n-3} + \&c.}{ax^m + bx^{m-1} + cx^{m-2} + dx^{m-3} + \&c.}\right)^{1:(m-n)}.$$

Hæc ſcilicet reſolutio locum habet, ſi in æquatione naturam Functionis y per z exprimente, duplex tantum ubique occurrit dimenſionum ab y & z ſumptarum numerus; uti in caſu tractato in ſingulis terminis numerus dimenſionum vel eſt m vel n.

58. *Si in æquatione inter* y *&* z *triplicis generis dimenſiones occurrant, quarum ſumma tantum ſuperet mediam, quantum hæc media infimam, ope reſolutionis æquationis quadratæ variabiles* y *&* z *per novam* x *exprimi poterunt.*

Si enim ponatur $y = xz$, diviſione per minimam ipſius z poteſtatem facta, valor ipſius z per x, ope extractionis radicis quadratæ exhibebitur, id quod ex ſequentibus exemplis erit manifeſtum.

EXEMPLUM I.

Sit $ay^3 + by^2z + cyzz + dz^3 = 2eyy + 2fyz + 2gzz + hy + iz$; ac ponatur $y = xz$: erit, diviſione per z facta, $(ax^3 + bxx + cx + d)zz = 2(exx + fx + g)z + hx + i$; ex qua ſequens ipſius z obtinebitur valor:

$$z = \frac{exx + fx + g \pm \sqrt{((exx + fx + g)^2 + (ax^3 + bxx + cx + d)(hx + i))}}{ax^3 + bxx + cx + d};$$

quo invento erit $y = xz$.

EXEMPLUM II.

Sit $y^5 = 2az^3 + by + cz$; ac, poſito $y = xz$, erit $x^5z^4 = 2azz + bx + c$; ex qua reperitur $zz = \frac{a \pm \sqrt{(aa + bx^6 + cx^5)}}{x^5}$; &

$$z = \frac{\sqrt{(a \pm \sqrt{(aa + bx^6 + cx^5)})}}{xx\sqrt{x}} \ \& \ y = \frac{\sqrt{(a \pm \sqrt{(aa + bx^6 + cx^5)})}}{x\sqrt{x}}.$$

EXEMPLUM III.

Sit $y^{10} = 2ayz^6 + byz^3 + cz^4$, in qua cum dimenſiones ſint 10, 7, & 4, ponatur $y = xz$; atque æquatio per z^4 diviſa abibit in hanc: $x^{10} z^6 = 2axz^3 + bx + c$ ſeu $z^6 = \frac{2axz^3 + bx + c}{x^{10}}$; unde invenitur $z^3 = \frac{ax \pm x\sqrt{(aa + bx^9 + cx^8)}}{x^{10}}$; ideoque erit $z = \frac{\sqrt[3]{(a \pm \sqrt{(aa + bx^9 + cx^8)})}}{x^3}$; atque $y = \frac{\sqrt[3]{(a \pm \sqrt{(aa + bx^9 + cx^8)})}}{x^2}$. Ex quibus exemplis uſus hujuſmodi ſubſtitutionum abunde perſpicitur.

CAPUT IV.

De explicatione Functionum per ſeries infinitas.

59. Cum Functiones fractæ atque irrationales ipſius z non in forma integra $A + Bz + Cz^2 + Dz^3 +$ &c. continentur, ita ut terminorum numerus ſit finitus, quæri ſolent hujuſmodi expreſſiones in infinitum excurrentes, quæ valorem cujuſvis Functionis ſive fractæ ſive irrationalis exhibeant. Quin etiam natura Functionum tranſcendentium melius intelligi cenſetur, ſi per ejuſmodi formam, etſi infinitam, exprimantur. Cum enim natura Functionis integræ optime perſpiciatur, ſi ſecundum diverſas poteſtates ipſius z explicetur, atque adeo ad formam $A + Bz + Cz^2 + Dz^3 +$ &c. reducatur, ita eadem forma aptiſſima videtur ad reliquarum Functionum omnium indolem menti repræſentandam, etiamſi terminorum numerus ſit revera infinitus. Perſpicuum autem eſt nullam Functionem non integram ipſius z per numerum hujuſmodi terminorum $A + Bz + Cz^2 +$ &c. finitum exponi poſſe; eo ipſo enim Functio

Functio foret integra; num vero per hujusmodi terminorum seriem infinitam exhiberi possit, si quis dubitet, hoc dubium per ipsam evolutionem cujusque Functionis tolletur. Quo autem hæc explicatio latius pateat, præter potestates ipsius z exponentes integros affirmativos habentes, admitti debent potestates quæcunque. Sic dubium erit nullum quin omnis Functio ipsius z in hujusmodi expressionem infinitam transmutari possit: $Az^{\alpha} + Bz^{\beta} + Cz^{\gamma} + Dz^{\delta} +$ &c. denotantibus exponentibus α, β, γ, δ, &c. numeros quoscunque.

60. *Per divisionem autem continuam intelligitur fractionem* $\frac{a}{\alpha + \beta z}$ *resolvi in hanc seriem infinitam* $\frac{a}{\alpha} - \frac{a\beta z}{\alpha^2} + \frac{a\beta^2 z^2}{\alpha^3} - \frac{a\beta^3 z^3}{\alpha^4} + \frac{a\beta^4 z^4}{\alpha^5} -$ *&c., quæ, cum quilibet terminus ad sequentem habeat rationem constantem* $1 : \frac{\beta z}{\alpha}$, *vocatur series geometrica.*

Potest vero quoque hæc series ita inveniri, ut ipsa initio pro incognita habeatur: ponatur enim $\frac{a}{\alpha + \beta z} = A + Bz + Cz^2 + Dz^3 + Ez^4 +$ &c. atque ad æqualitatem producendam quærantur coëfficientes A, B, C, D, &c. Erit ergo $a = (\alpha + \beta z)(A + Bz + Cz^2 + Dz^3 +$ &c.$)$, & multiplicatione actu peracta fiet

$$\begin{aligned} a = \alpha A &+ \alpha Bz + \alpha Cz^2 + \alpha Dz^3 + \alpha Ez^4 + \&c. \\ &+ \beta Az + \beta Bz^2 + \beta Cz^3 + \beta Dz^4 + \&c. \end{aligned}$$

Quamobrem esse debet $a = \alpha A$, ideoque $A = \frac{a}{\alpha}$, & coëfficientium uniuscujusque potestatis ipsius z summa nihilo æqualis est ponenda: unde prodibunt hæ æquationes,

$\alpha B + \beta A = 0$	cognito ergo quovis coëfficiente
$\alpha C + \beta B = 0$	facile reperitur sequens; si enim
$\alpha D + \beta C = 0$	fuerit coëfficiens termini cujusque $= P$
$\alpha E + \beta D = 0$	& sequens $= Q$ erit $\alpha Q + \beta P = 0$
&c.	sive $Q = \frac{\beta P}{\alpha}$.

Cum

LIB. I. Cum igitur terminus primus A ſit determinatus $= \frac{a}{\alpha}$ ex eo ſequentes litteræ B, C, D, &c. definiuntur eodem modo, quo ex diviſione ſunt orti. Ceterum ex inſpectione perſpicitur in ſerie infinita pro $\frac{a}{\alpha + \beta z}$ inventa poteſtatis z^n coëfficientem fore $= \pm \frac{a\beta^n}{\alpha^{n+1}}$, ubi ſignum $+$ locum habet ſi n ſit numerus par, ſignum $-$ autem ſi n ſit numerus impar: ſeu coëfficiens erit $= \frac{a}{\alpha}\left(\frac{-\beta}{\alpha}\right)^n$.

61. *Simili modo ope diviſionis continuatæ hæc Functio fracta* $\frac{a + bz}{\alpha + \beta z + \gamma zz}$ *in ſeriem infinitam converti poteſt.*

Cum autem diviſio ſit tædioſa, neque tam facile naturam ſeriei infinitæ oſtendat, commodius erit ſeriem quæſitam fingere, atque modo ante tradito determinare. Sit igitur

$$\frac{a + bz}{\alpha + \beta z + \gamma zz} = A + Bz + Cz^2 + Dz^3 + Ez^4 + \&c.$$

multiplicetur utrinque per $\alpha + \beta z + \gamma zz$, atque fiet

$$\begin{aligned} a + bz = \alpha A &+ \alpha Bz + \alpha Cz^2 + \alpha Dz^3 + \alpha Ez^4 + \&c. \\ &+ \beta Az + \beta Bz^2 + \beta Cz^3 + \beta Dz^4 + \&c. \\ &+ \gamma Az^2 + \gamma Bz^3 + \gamma Cz^4 + \&c. \end{aligned}$$

Hinc erit $\alpha A = a$; $\alpha B + \beta A = b$; unde reperitur $A = \frac{a}{\alpha}$ & $B = \frac{b}{\alpha} - \frac{a\beta}{\alpha\alpha}$; reliquæ vero litteræ ex ſequentibus æquationibus determinabuntur:

$\alpha C + \beta B + \gamma A = 0$
$\alpha D + \beta C + \gamma B = 0$
$\alpha E + \beta D + \gamma C = 0$
$\alpha F + \beta E + \gamma D = 0$
&c.

hinc ergo ex binis quibuſque coëfficientibus contiguis ſequens reperitur. Sic ſi duo coëfficientes contigui fuerint P, Q & ſequens R, erit $\alpha R + \beta Q + \gamma P = 0$ ſeu $R = \frac{-\beta Q - \gamma P}{\alpha}$.

Cum igitur duæ litteræ primæ A & B jam ſint inventæ ſequentes C, D, E, F &c. omnes ſucceſſive ex iis invenientur,

tur, sicque reperietur Series infinita $A+Bz+Cz^2+Dz^3+$ &c. Functioni fractæ propositæ $\frac{a+bz}{\alpha+\beta z+\gamma z^2}$ æqualis.

EXEMPLUM.

Si fuerit proposita hæc fractio $\frac{1+2z}{1-z-zz}$, huicque æqualis statuatur Series $A+Bz+Cz^2+Dz^3+$ &c. ob $a=1$; $b=2$; $\alpha=1$; $\beta=-1$; $\gamma=-1$; erit $A=1$; $B=3$; tum vero erit

$C=B+A$ quilibet ergo coëfficiens æqualis est summæ duorum præcedentium; quare si cogniti fuerint duo coëfficientes contigui P & Q, erit sequens $R=P+Q$.

$D=C+B$

$E=D+C$

$F=E+D$

&c.

Cum igitur duo coëfficientes primi A & B sint cogniti, fractio proposita $\frac{1+2z}{1-z-zz}$ in hanc Seriem infinitam transmutatur $1+3z+4z^2+7z^3+11z^4+18z^5+$ &c., quæ nullo negotio quousque libuerit continuari potest.

62. Ex his jam satis intelligitur indoles Serierum infinitarum, in quas Functiones fractæ transmutantur; tenent enim ejusmodi legem, ut quilibet terminus ex aliquot præcedentibus determinari possit. Scilicet, si denominator fractionis propositæ fuerit $\alpha+\beta z$, atque Series infinita statuatur

$$A+Bz+Cz^2+\ldots\ldots+Pz^n+Qz^{n+1}+Rz^{n+2}+Sz^{n+3}+\&c.;$$

quilibet coëfficiens Q ex præcedente P solo ita definietur ut sit $\alpha Q+\beta P=0$. Sin denominator fuerit trinomium $\alpha+\beta z+\gamma zz$, quilibet coëfficiens Seriei R ex duobus præcedentibus Q & P ita definietur ut sit $\alpha R+\beta Q+\gamma P=0$: simili modo si denominator fuerit quadrinomium, ut $\alpha+\beta z+\gamma zz+\delta z^3$, quilibet coëfficiens seriei S ex tribus antecedentibus R, Q & P ita determinabitur, ut sit $\alpha S+\beta R+\gamma Q+\delta P=0$,

 sicque

LIB. I. ſicque de ceteris. In his ergo Seriebus quilibet terminus determinatur ex aliquot antecedentibus ſecundum legem quandam conſtantem, quæ lex ex denominatore fractionis hanc Seriem producentis ſponte apparet. Vocari autem hæ Series a Celeb. MOIVRÆO, qui earum naturam maxime eſt ſcrutatus, ſolent *recurrentes*, propterea quod ad terminos antecedentes eſt recurrendum, ſi ſequentes inveſtigare velimus.

63. Ad harum porro Serierum formationem requiritur ut denominatoris terminus conſtans α non ſit $= 0$: cum enim inventus ſit terminus Seriei primus $A = \frac{a}{\alpha}$, tum is, tum omnes ſequentes fierent infiniti, ſi eſſet $\alpha = 0$. Hoc ergo caſu excluſo, quem deinceps evolvam, Functio fracta in Seriem infinitam recurrentem tranſmutanda, hujuſmodi habebit formam $\frac{a + bz + cz^2 + dz^3 + \&c.}{1 - \alpha z - \beta z^2 - \gamma z^3 - \delta z^4 - \&c.}$; ubi primum denominatoris terminum pono $= 1$, huc enim ſemper fractio reduci poteſt, niſi is ſit $= 0$; reliquos autem denominatoris terminos omnes tanquam negativos contemplor, ut Seriei hinc formatæ omnes termini fiant affirmativi. Quod ſi enim Series recurrens hinc orta ponatur $A + Bz + Cz^2 + Dz^3 + Ez^4 + \&c.$ coëfficientes ita determinabuntur ut ſit

$$\begin{aligned} A &= a \\ B &= \alpha A + b \\ C &= \alpha B + \beta A + c \\ D &= \alpha C + \beta B + \gamma A + d \\ E &= \alpha D + \beta C + \gamma B + \delta A + e \\ &\&c. \end{aligned}$$

Quilibet ergo coëfficiens æqualis eſt aggregato ex multiplis aliquot præcedentium una cum numero quodam, quem numerator præbet. Niſi autem numerator in infinitum progrediatur, hæc additio mox ceſſabit, atque quivis terminus ſecundum legem conſtantem ex aliquot præcedentibus determinabitur. Ne ergo lex progreſſionis uſquam turbetur conveniet

Functio-

Functionem fractam genuinam adhibere: si enim fractio spuria accipiatur, tum pars integra in ea contenta ad Seriem accedet, atque in illis terminis, quos vel auget vel minuit, legem progressionis interrumpet. Exempli gratia hæc fractio spuria $\frac{1+2z-z^3}{1-z-zz}$, præbebit hanc Seriem $1+3z+4zz+6z^3+10z^4+16z^5+26z^6+42z^7+$ &c. ubi a lege, qua quivis coëfficiens est summa duorum præcedentium, terminus quartus $6z^3$ excipitur.

64. Peculiarem contemplationem Series recurrentes merentur, si denominator fractionis, unde oriuntur, fuerit potestas. Sic, si ista fractio $\frac{a+bz}{(1-\alpha z)^2}$ in Seriem resolvatur, prodit

$$\begin{array}{llllll} a & +2\alpha a\,z & +3\alpha^2 a\,z^2 & +4\alpha^3 a\,z^3 & +5\alpha^4 a\,z^4 & +\ \&c. \\ & +\ b & +2\alpha b & +3\alpha^2 b & +4\alpha^3 b & \end{array}$$

in qua coëfficiens potestatis z^n erit $(n+1)\alpha^n a+n\alpha^{n-1}b$. Erit tamen hæc Series recurrens, quia quilibet terminus ex duobus præcedentibus determinatur, cujus determinationis lex perspicitur ex denominatore evoluto $1-2\alpha z+\alpha\alpha zz$. Si ponatur $\alpha=1$ & $z=1$, abit Series in progressionem arithmeticam generalem $a+(2a+b)+(3a+2b)+(4a+3b)+$ &c. cujus differentiæ sunt constantes. Omnis ergo progressio arithmetica est Series recurrens: si enim sit $A+B+C+D+E+F+$ &c. progressio arithmetica, erit $C=2B-A$; $D=2C-B$; $E=2D-C$, &c.

65. Deinde hæc fractio $\frac{a+bz+czz}{(1-\alpha z)^3}$ ob $\frac{1}{(1-\alpha z)^3}=(1-\alpha z)^{-3}$ $=1+3\alpha z+6\alpha^2z^2+10\alpha^3z^3+15\alpha^4z^4+$ &c. transmutabitur in hanc Seriem infinitam:

$$\begin{array}{llllll} a & +3\alpha a\,z & +6\alpha^2 a & +10\alpha^3 a & +15\alpha^4 a & +\ \&c. \\ & +\ b & +3\alpha b\,z^2 & +6\alpha^2 b\,z^3 & +10\alpha^3 b\,z^4 & \\ & & +\ c & +3\alpha c & +6\alpha^2 c & \end{array}$$

LIB. I. in qua poteſtas z^n coëfficientem habebit $\frac{(n+1)(n+2)}{1.\ 2}\alpha^n a + \frac{n(n+1)}{1.\ 2}\alpha^{n-1} b + \frac{(n-1)n}{1.\ 2}\alpha^{n-2} c$. Quod ſi autem ponatur $\alpha = 1$ & $z = 1$, Series hæc abibit in progreſſionem generalem ſecundi ordinis, cujus differentiæ ſecundæ ſunt conſtantes. Deſignet $A + B + C + D + E +$ &c. hujuſmodi progreſſionem, erit ea ſimul Series recurrens, cujus quilibet terminus ex tribus antecedentibus ita determinatur ut ſit $D = 3C - 3B + A$; $E = 3D - 3C + B$; $F = 3E - 3D + C$ &c. Cum igitur terminorum in progreſſione arithmetica procedentium ſecundæ differentiæ quoque ſint æquales, nempe $= 0$, hæc proprietas quoque ad progreſſiones arithmeticas extenditur.

66. Simili modo hæc fractio $\frac{a + bz + czz + dz^3}{(1 - \alpha z)^4}$ dabit Seriem infinitam, in qua poteſtas ipſius z quæcunque z^n hunc habebit coëfficientem $\frac{(n+1)(n+2)(n+3)}{1.\ 2.\ 3}\alpha^n a + \frac{n(n+1)(n+2)}{1.\ 2.\ 3}\alpha^{n-1} b + \frac{(n-1)n(n+1)}{1.\ 2.\ 3}\alpha^{n-2} c + \frac{(n-2)(n-1)n}{1.\ 2.\ 3}\alpha^{n-3} d$: poſito ergo $\alpha = 1$ & $z = 1$; hæc Series in ſe complectetur omnes progreſſiones algebraicas tertii ordinis, quarum differentiæ tertiæ ſunt conſtantes: omnes ergo hujus ordinis progreſſiones, cujuſmodi ſit $A + B + C + D + E + F +$ &c. erunt ſimul recurrentes ex denominatore $1 - 4z + 6zz - 4z^3 + z^4$ ortæ; unde erit $E = 4D - 6C + 4B - A$; $F = 4E - 6D + 4C - B$ &c., quæ proprietas ſimul in omnes progreſſiones inferiorum ordinum competit.

67. Hoc modo oſtendentur omnes progreſſiones algebraicas cujuſcunque ordinis, quæ tandem ad differentias conſtantes deducunt, eſſe Series recurrentes, quarum lex definiatur ex denominatore $(1 - z)^n$, exiſtente n numero majore quam is, qui ordinem progreſſionis indicat. Cum igitur $a^m + (a+b)^m +$

$(a + 2b)$

$(a+2b)^m + (a+3b)^m +$ &c. exhibeat progressio- CAP. IV.
nem ordinis m; erit ob naturam Serierum recurrentium

$$o = a^m - \frac{n}{1}(a+b)^m + \frac{n(n-1)}{1.\ 2}(a+2b)^m$$
$$- \frac{n(n-1)(n-2)}{1.\ 2.\ 3}(a+3b)^m + \ldots\ldots\ldots\ldots \pm$$
$$\frac{n}{1}(a+(n-1)b)^m \mp (a+nb)^m;$$ ubi signa superiora valent si n sit numerus par; inferiora autem si n sit numerus impar. Hæc ergo æquatio semper est vera si fuerit n numerus integer major quam m. Hinc ergo intelligitur quam late pateat doctrina de Seriebus recurrentibus.

68. Si denominator fuerit potestas non binomii sed multinomii, natura Seriei quoque alio modo explicari potest. Sit nempe hæc fractio $\frac{1}{(1 - \alpha z - \beta z^2 - \gamma z^3 - \delta z^4 - \&c.)^{m+1}}$ proposita, erit Series infinita hinc nata

$$1 + \frac{(m+1)}{1}\alpha z + \frac{(m+1)(m+2)}{1.\ 2}\alpha^2 z^2 + \frac{(m+1)(m+2)(m+3)}{1.\ 2.\ 3}\alpha^3 z^3$$
$$+ \frac{(m+1)}{1}\beta \qquad + \frac{(m+1)(m+2)}{1.\ 2}2\alpha\beta z^3 + \&c.$$
$$+ \frac{(m+1)}{1}\gamma$$

Ad naturam hujus Seriei penitius inspiciendam, exponatur hæc Series per litteras generales hoc modo:

$1 + Az + Bz^2 + Cz^3 + \ldots + Kz^{n-3} + Lz^{n-2} + Mz^{n-1} + Nz^n +$ &c., ac quilibet coëfficiens N ex tot procedentibus, quot sunt litteræ $\alpha . \beta . \gamma . \delta$ &c. ita determinabitur ut sit: $N = \frac{m+n}{n}\alpha M + \frac{2m+n}{n}\beta L + \frac{3m+n}{n}\gamma K + \frac{4m+n}{n}\delta I +$ &c. quæ lex continuationis etsi non est constans, sed ab exponente potestatis z pendet, tamen eidem Seriei alia convenit lex progressionis constans, quam denominator evolutus præbet, naturæ Serierum recurrentium consentaneam. Illa vero lex non constans tantum locum habet si numerator fractionis fuerit unitas seu quantitas constans; si enim quoque aliquot potestates ipsius z contineret, tum illa lex multo magis fieret complicata, id quod post tradita calculi differentialis principia facilius patebit.

69. Quoniam hactenus posuimus primum denominatoris terminum constantem non esse $= 0$, ejusque loco unitatem collocavimus; nunc videamus cujusmodi Series oriantur, si in denominatore terminus constans evanescat. His casibus ergo Functio fracta hujusmodi formam habebit $\frac{a + bz + czz + \&c.}{z(1 - \alpha z - \beta zz - \&c.)}$, convertatur ergo, neglecto denominatoris Factore z, reliqua fractio $\frac{a + bz + czz + \&c.}{1 - \alpha z - \beta z^2 - \&c.}$ in Seriem recurrentem $A + Bz + Cz^2 + Dz^3 + \&c.$ atque manifestum est fore $\frac{a + bz + czz + \&c.}{z(1 - \alpha z - \beta z^2 - \gamma z^3 - \&c.)} = \frac{A}{z} + B + Cz + Dz^2 + Ez^3 + \&c.$ Simili modo erit $\frac{a + bz + cz^2 + \&c.}{z^2(1 - \alpha z - \beta z^2 - \&c.)} = \frac{A}{zz} + \frac{B}{z} + C + Dz + Ez^2 + \&c.$, atque generatim erit $\frac{a + bz + cz^2 + \&c.}{z^m(1 - \alpha z - \beta z^2 - \gamma z^3 - \&c.)} = \frac{A}{z^m} + \frac{B}{z^{m-1}} + \frac{C}{z^{m-2}} + \frac{D}{z^{m-3}} + \&c.$ quicunque numerus fuerit exponens m.

70. Quoniam per substitutionem loco z alia variabilis x in Functionem fractam introduci, hocque pacto Functio fracta quævis in innumerabiles formas diversas transmutari potest; hoc modo eadem Functio fracta infinitis modis per Series recurrentes explicari poterit. Sit scilicet proposita hæc fractio $y = \frac{1 + z}{1 - z - zz}$ & per Seriem recurrentem $y = 1 + 2z + 3z^2 + 5z^3 + 8z^4 + \&c.$: ponatur $z = \frac{1}{x}$ erit y $=$

$= \frac{xx+x}{xx-x-1} = \frac{-x(1+x)}{1+x-xx}$. Jam $\frac{1+x}{1+x-xx} = 1 +$
$0x + xx - x^3 + 2z^4 - 3x^5 + 5x^6 -$ &c.; unde erit $y = -x + 0x^2 - x^3 + x^4 - 2x^5 + 3x^6 - 5x^7 +$ &c. Vel ponatur $z = \frac{1-x}{1+x}$, erit $y = \frac{-2-2x}{1-4x-xx}$; unde fit $y = -2 - 10x - 42xx - 178x^3 - 754x^4 -$ &c. cujusmodi Series recurrentes pro y innumerabiles inveniri possunt.

71. Functiones irrationales ex hoc theoremate universali in Series infinitas transformari solent, quod sit $(P+Q)^{\frac{m}{n}}$
$= P^{\frac{m}{n}} + \frac{m}{n} P^{\frac{m-n}{n}} Q + \frac{m(m-n)}{n.\ 2n} P^{\frac{m-2n}{n}} Q^2 +$
$\frac{m(m-n)(m-2n)}{n.\ 2n.\ 3n} P^{\frac{m-3n}{n}} Q^3 +$ &c.: hi enim termini, nisi fuerit $\frac{m}{n}$ numerus integer affirmativus, in infinitum excurrunt. Sic erit pro m & n numeros definitos scribendo.

$$(P+Q)^{\frac{1}{2}} = P^{\frac{1}{2}} + \frac{1}{2} P^{-\frac{1}{2}} Q - \frac{1.1}{2.4} P^{-\frac{3}{2}} Q^2 +$$
$$\frac{1.1.3}{2.4.6} P^{-\frac{5}{2}} Q^3 - \text{\&c.}$$

$$(P+Q)^{-\frac{1}{2}} = P^{-\frac{1}{2}} - \frac{1}{2} P^{-\frac{3}{2}} Q + \frac{1.3}{2.4} P^{-\frac{5}{2}} Q^2 -$$
$$\frac{1.3.5}{2.4.6} P^{-\frac{7}{2}} Q^3 + \text{\&c.}$$

$$(P+Q)^{\frac{1}{3}} = P^{\frac{1}{3}} + \frac{1}{3} P^{-\frac{2}{3}} Q - \frac{1.2}{3.6} P^{-\frac{5}{3}} Q^2 +$$
$$\frac{1.2.5}{3.6.19} P^{-\frac{8}{3}} Q^3 - \text{\&c.}$$

(P

LIB. I. $(P+Q)^{-\frac{1}{3}} = P^{-\frac{1}{3}} - \frac{1}{3} P^{-\frac{4}{3}} Q + \frac{1.4}{3.6} P^{-\frac{7}{3}} Q^2 -$
$\frac{1.4.7}{3.6.9} P^{-\frac{10}{3}} Q^3 +$ &c.

$(P+Q)^{\frac{2}{3}} = P^{\frac{2}{3}} + \frac{2}{3} P^{-\frac{1}{3}} Q - \frac{2.1}{3.6} P^{-\frac{4}{3}} Q^2 +$
$\frac{2.1.4}{3.6.9} P^{-\frac{7}{3}} Q^3 -$&c.

&c.

72. Hujusmodi ergo Serierum termini ita progrediuntur ut quilibet ex antecedente formari possit: sit enim Seriei, quæ ex $(P+Q)^{\frac{m}{n}}$ nascitur, terminus quilibet $= MP^{\frac{m-kn}{n}} Q^k$ erit sequens $= \frac{m-kn}{(k+1)n} MP^{\frac{m-(k+1)n}{n}} Q^{k+1}$. Notandum autem est in quovis termino sequente exponentem ipsius P unitate decrescere, contra vero exponentem ipsius Q unitate crescere. Quo autem hæc facilius ad quemvis casum accommodentur, forma generalis $(P+Q)^{\frac{m}{n}}$ ita exponi potest $P^{\frac{m}{n}} (1+\frac{Q}{P})^{\frac{m}{n}}$: evoluta enim formula $(1+\frac{Q}{P})^{\frac{m}{n}}$ Serieque resultante per $P^{\frac{m}{n}}$ multiplicata, prodibit ipsa Series ante data. Tum vero si m non solum numeros integros denotet, sed etiam fractos, pro n tuto unitas collocari poterit. Quibus factis, si pro $\frac{Q}{P}$, quæ est Functio ipsius z, ponatur Z, habebitur

$(1+Z)^m = 1 + \frac{m}{1} Z + \frac{m(m-1)}{1.2} Z^2 + \frac{m(m-1)(m-2)}{1.2.3} Z^3$
+ &c. Ad sequentes progressionum leges autem observandas conveniet hanc formulæ generalis in Seriem conversionem notasse

tasse $(1+Z)^{m-1} = 1 + \frac{(m-1)}{1} Z + \frac{(m-1)(m-2)}{1.\ 2} Z^2 + \frac{(m-1)(m-2)(m-3)}{1.\ 2.\ 3} Z^3 + \&c.$

73. Sit igitur primum $Z = \alpha z$, eritque $(1+\alpha z)^{m-1} = 1 + \frac{m-1}{1} \alpha z + \frac{(m-1)(m-2)}{1\ \ 2} \alpha^2 z^2 + \frac{(m-1)(m-2)(m-3)}{1.\ 2.\ 3} \alpha^3 z^3 + \&c.$ Scribatur pro hac Serie ista forma generalis

$1 + Az + Bz^2 + Cz^3 + \ldots\ldots + Mz^{n-1} + Nz^n + \&c.$

atque quilibet coëfficiens N ex præcedente M ita determinabitur ut sit $N = \frac{m-n}{n} \alpha M$. Sic, posito $n = 1$, cum sit $M = 1$, erit $N = A = \frac{m-1}{1} \alpha$; tum facto $n = 2$, ob $M = A = \frac{m-1}{1} \alpha$, erit $N = B = \frac{m-2}{2} \alpha M = \frac{(m-1)(m-2)}{1.\ \ 2} \alpha^2$ similique modo porro $C = \frac{m-3}{3} \alpha B = \frac{(m-1)(m-2)(m-3)}{1.\ 2.\ 3} \alpha^3$, uti Series ante inventa declarat.

74. Sit $Z = \alpha z + \beta zz$, eritque $(1 + \alpha z + \beta zz)^{m-1} = 1 + \frac{(m-1)}{1} (\alpha z + \beta zz) + \frac{(m-1)(m-2)}{1.\ 2} (\alpha z + \beta zz)^2 + \&c.$ Quod si ergo termini secundum potestates ipsius z disponantur erit $(1 + \alpha z + \beta zz)^{m-1} =$

$$1 + \frac{(m-1)}{1} \alpha z + \frac{(m-1)(m-2)}{1.\ 2} \alpha^2 z^2 + \frac{(m-1)(m-2)(m-3)}{1.\ 2.\ 3} \alpha^3 z^3 + \&c.$$
$$+ \frac{(m-1)}{1} \beta z^2 + \frac{(m-1)(m-2)}{1.\ 2} 2\alpha\beta z^3 + \&c.$$

LIB. I. Scribatur pro hac Serie ista forma generalis:

$$1+Az+Bz^2+Cz^3+\dots+Lz^{n-2}+Mz^{n-1}+Nz^n+\&c.$$

atque quilibet coëfficiens ex duobus antecedentibus ita definietur ut sit $N=\frac{m-n}{n}\alpha M+\frac{2m-n}{n}\beta L$, unde omnes termini ex primo, qui est 1, definiri poterunt. Erit nempe

$$A=\frac{m-1}{1}\alpha;$$

$$B=\frac{(m-2)}{2}\alpha A+\frac{(2m-2)}{2}\beta$$

$$C=\frac{(m-3)}{3}\alpha B+\frac{(2m-3)}{3}\beta A$$

$$D=\frac{(m-4)}{4}\alpha C+\frac{(2m-4)}{4}\beta B$$

&c.

75. Si fuerit $Z=\alpha z+\beta z^2+\gamma z^3$, erit $(1+\alpha z+\beta z^2+\gamma z^3)^{m-1}=1+\frac{(m-1)}{1}(\alpha z+\beta z^2+\gamma z^3)+\frac{(m-1)(m-2)}{1.\ 2}(\alpha z+\beta z^2+\gamma z^3)^2+\&c.$, quæ expressio, si omnes termini secundum potestates ipsius z ordinentur, abibit in hanc Seriem:

$$1+\frac{(m-1)}{1}\alpha z+\frac{(m-1)(m-2)}{1.\ 2}\alpha^2z^2+\frac{(m-1)(m-2)(m-3)}{1.\ 2.\ 3}\alpha^3z^3$$
$$+\frac{(m-1)}{1}\beta z^2+\frac{(m-1)(m-2)}{1.\ 2}2\alpha\beta z^3+\&c.$$
$$+\frac{(m-1)}{1}\gamma z^3$$

cujus lex progressionis ut melius patescat, ponatur ejus loco

$$1+Az+Bz^2+Cz^3+\dots+Kz^{n-3}+Lz^{n-2}+Mz^{n-1}+Nz^n,$$

cujus Seriei quilibet coëfficiens ex tribus antecedentibus ita determinatur ut sit $N=\frac{(m-n)}{n}\alpha M+\frac{(2m-n)}{n}\beta L+\frac{(3m-n)}{n}\gamma K.$

Cum

Cum igitur primus terminus sit $=1$, & antecedentes nulli, erit

$$A = \frac{m-1}{1}\alpha$$
$$B = \frac{(m-2)}{2}\alpha A + \frac{(2m-2)}{2}\beta$$
$$C = \frac{(m-3)}{3}\alpha B + \frac{(2m-3)}{3}\beta A + \frac{(3m-3)}{3}\gamma$$
$$D = \frac{(m-4)}{4}\alpha C + \frac{(2m-4)}{4}\beta B + \frac{(3m-4)}{4}\gamma A$$
$$E = \frac{(m-5)}{5}\alpha D + \frac{(2m-5)}{5}\beta C + \frac{(3m-5)}{5}\gamma B$$

&c.

76. Generaliter ergo si ponatur $(1+\alpha z+\beta z^2+\gamma z^3+\delta z^4+\&c.)^{m-1} = 1+Az+Bz^2+Cz^3+Dz^4+Ez^5+$ &c., hujus Seriei singuli termini ita ex præcedentibus definientur, ut sit

$$A = \frac{m-1}{1}\alpha$$
$$B = \frac{(m-2)}{2}\alpha A + \frac{(2m-2)}{2}\beta$$
$$C = \frac{(m-3)}{3}\alpha B + \frac{(2m-3)}{3}\beta A + \frac{(3m-3)}{3}\gamma$$
$$D = \frac{(m-4)}{4}\alpha C + \frac{(2m-4)}{4}\beta B + \frac{(3m-4)}{4}\gamma A + \frac{(4m-4)}{4}\delta$$
$$E = \frac{(m-5)}{5}\alpha D + \frac{(2m-5)}{5}\beta C + \frac{(3m-5)}{5}\gamma B + \frac{(4m-5)}{5}\delta A + \frac{(5m-5)}{5}\varepsilon$$

&c.

quilibet scilicet terminus per tot præcedentes determinatur, quot habentur litteræ α, β, γ, δ, &c. in Functione ipsius z cujus potestas in Seriem convertitur. Ceterum ratio hujus legis convenit cum ea, quam supra §. 68. ubi similem formam $(1-\alpha z-\beta z^2-\gamma z^3-\&c.)^{-m-1}$ in Seriem infinitam

H 2 tam

LIB. I. tam resolvimus; si enim loco m scribatur $-m$ atque litteræ α, β, γ, δ, &c. negative accipiantur, Series inventæ prorsus congruent. Interim hoc loco non licet rationem hujus progressionis legis a priori demonstrare, id quod per principia calculi differentialis demum commode fieri poterit; interea ergo sufficiet veritatem per applicationem ad omnis generis exempla comprobasse.

CAPUT V.

De Functionibus duarum pluriumve variabilium.

77. Quanquam plures hactenus quantitates variabiles sumus contemplati, tamen eæ ita erant comparatæ, ut omnes unius essent Functiones, unaque determinata reliquæ simul determinarentur. Nunc autem ejusmodi considerabimus quantitates variabiles, quæ a se invicem non pendeant, ita ut quamvis uni determinatus valor tribuatur, reliquæ tamen nihilominus maneant indeterminatæ ac variabiles. Ejusmodi ergo quantitates variabiles, cujusmodi sint x, y, z, ratione significationis convenient, cum quælibet omnes valores determinatos in se complectatur; at, si inter se comparentur maxime erunt diversæ, cum, licet pro una z valor quicunque determinatus substituatur, reliquæ tamen x & y æque late pateant, atque ante. Discrimen ergo inter quantitates variabiles a se pendentes, & non pendentes in hoc versatur, ut priori casu, si una determinetur, simul reliquæ determinentur; posteriori vero determinatio unius significationes reliquarum minime restringat.

78. *Functio ergo duarum pluriumve quantitatum variabilium,* x, y, z, *est expressio quomodocunque ex his quantitatibus composita.*

Ita erit $x^3 + xyz + az^2$ Functio quantitatum variabilium trium x, y, z. Hæc ergo Functio, si una determinetur variabilis,

riabilis, puta z, hoc est ejus loco constans numerus substituatur, manebit adhuc quantitas variabilis, scilicet Functio ipsarum x & y. Atque si, præter z, quoque y determinetur, tum erit adhuc Functio ipsius x. Hujusmodi ergo plurium variabilium Functio non ante valorem determinatum obtinebit, quam singulæ quantitates variabiles fuerint determinatæ. Cum igitur una quantitas variabilis infinitis modis determinari possit, Functio duarum variabilium, quia pro quavis determinatione unius infinitas determinationes suscipere potest, omnino infinities infinitas determinationes admittet. Atque in Functione trium variabilium numerus determinationum erit adhuc infinities major; sicque porro crescet pro pluribus variabilibus. CAP. V.

79. *Hujusmodi Functiones plurium variabilium perinde atque Functiones unius variabilis, commodissime dividuntur in algebraicas ac transcendentes.*

Quarum illæ sunt, in quibus ratio compositionis in solis Algebræ operationibus est posita; hæ vero, in quarum formationem quoque operationes transcendentes ingrediuntur. In his denuo species notari possent, prout operationes transcendentes vel omnes quantitates variabiles implicant, vel aliquot, vel tantum unicam. Sic ista expressio $zz + y \log. z$, quia Logarithmus ipsius z inest, erit quidem Functio transcendens ipsarum y & z, verum ideo minus transcendens est putanda, quod si variabilis z determinetur, supersit Functio algebraica ipsius y. Interim tamen non expedit hujusmodi subdivisionibus tractationem amplificari.

80. *Functiones deinde algebraicæ subdividuntur in rationales & irrationales; rationales autem porro in integras ac fractas.*

Ratio harum denominationum ex Capite primo jam abunde intelligitur. Functio scilicet rationalis omnino est libera ab omni irrationalitate quantitates variabiles, quarum Functio dicitur, afficiente; hæcque erit integra si nullis fractionibus inquinetur, contra vero fracta. Sic Functionis integræ duarum variabilium y & z hæc erit forma generalis: $\alpha + \beta y + \gamma z + \delta y^2 + \epsilon yz + \xi z^2 + \eta z^3 + \theta y^2 z + \iota y z^2 + \kappa z^3 + \&c.$ Quod

LIB. I. si ergo P & Q denotent hujusmodi Functiones integras, sive duarum sive plurium variabilium, erit $\frac{P}{Q}$ forma generalis Functionum fractarum. Functio denique irrationalis est vel explicita, vel implicita; illa per signa radicalia jam penitus est evoluta, hæc autem per æquationem irresolubilem exhibetur: sic V erit Functio implicita irrationalis ipsarum y & z, si fuerit $V^3 = (ayz + z^3)V^2 + (y^4 + z^4)V + y^5 + 2ayz^3 + z^5$.

81. *Multiformitas deinde in his Functionibus æque notari debet, atque in iis, quæ ex unica variabili constant.*

Sic Functiones rationales erunt uniformes, quia singulis quantitatibus variabilibus determinatis, unicum valorem determinatum exibent. Denotent P, Q, R, S, &c. Functiones rationales seu uniformes variabilium x, y, z, eritque V Functio biformis earundem variabilium, si fuerit $V^2 - PV + Q = 0$; quicunque enim valores determinati quantitatibus x, y, & z tribuuntur, Functio V non unum sed duplicem perpetuo habebit valorem determinatum. Simili modo erit V Functio triformis si fuerit $V^3 - PV^2 + QV - R = 0$: atque Functio quadriformis si fuerit $V^4 - PV^3 + QV^2 - RV + S = 0$: hocque modo ratio Functionum multiformium ulteriorum erit comparata.

82. Quemadmodum si Functio unius variabilis z nihilo æqualis ponitur, quantitas variabilis z valorem consequitur determinatum vel simplicem vel multiplicem; ita si Functio duarum variabilium y & z nihilo æqualis ponitur, tum altera variabilis per alteram definitur, ejusque ideo Functio evadit, cum ante a se mutuo non penderent. Simili modo si Functio trium variabilium x, y, z, nihilo æqualis statuatur, tum una variabilis per duas reliquas definitur, earumque Functio existit. Idem evenit si Functio non nihilo sed quantitati constanti vel etiam alii Functioni æqualis ponatur; ex omni enim æquatione, quotcunque variabiles involvat, semper una variabilis per reliquas definitur, earumque fit Functio; duæ autem æqua-

tiones

tiones diversæ inter easdem variabiles ortæ binas per reliquas definient, atque ita porro. CAP. V.

83. *Functionum autem duarum pluriumve variabilium divisio maxime notatu digna est in homogeneas & heterogeneas.*

Functio homogenea est per quam ubique idem regnat variabilium numerus dimensionum: Functio autem heterogenea est, in qua diversi occurrunt dimensionum numeri. Censetur vero unaquæque variabilis unam dimensionem constituere; quadratum uniuscujusque atque productum ex duabus, duas; productum ex tribus variabilibus, sive iisdem sive diversis, tres & ita porro; quantitates autem constantes ad dimensionum numerationem non admittuntur. Ita in his formulis αy; βz, unica dimensio inesse dicitur; in his vero αy^2; βyz; γz^2 duæ insunt dimensiones: in his αy^3; $\beta y^2 z$; $\gamma y z^2$; δz^3, tres; in his vero αy^4; $\beta y^3 z$; $\gamma y^2 z^2$; $\delta y z^3$; εz^4; quatuor, sicque porro.

84. Applicemus primum hanc distinctionem ad Functiones integras, atque duas tantum variabiles inesse ponamus, quoniam plurium par est ratio.

Functio igitur integra erit homogenea in cujus singulis terminis idem existit dimensionum numerus.

Subdividentur ergo hujusmodi Functiones commodissime secundum numerum dimensionum, quem variabiles in ipsis ubique constituunt. Sic erit $\alpha y + \beta z$ forma generalis Functionum integranum unius dimensionis: hæc vero expressio $\alpha y^2 + \beta y z + \gamma z^2$ erit forma generalis Functionum duarum dimensionum, tum forma generalis Functionum trium dimensionum erit: $\alpha y^3 + \beta y^2 z + \gamma y z^2 + \delta z^3$; quatuor dimensionum vero hæc: $\alpha y^4 + \beta y^3 z + \gamma y^2 z^2 + \delta y z^3 + \varepsilon z^4$; & ita porro. Ad analogiam igitur erit quantitas constans sola α Functio nullius dimensionis.

85. *Functio porro fracta erit homogenea, si ejus Numerator ac Denominator fuerint Functiones homogeneæ.*

Sic hæc Fractio $\frac{ayy + bzz}{\alpha y + \beta z}$ erit Functio homogenea ipsa-

rum

 rum y & z; numerus dimensionum autem habebitur, si a numero dimensionum Numeratoris subtrahatur numerus dimensionum Denominatoris: atque ob hanc rationem Fractio allata erit Functio unius dimensionis. Hæc vero Fractio $\frac{y^5+z^5}{yy+zz}$ erit Functio trium dimensionum. Quando ergo in Numeratore ac Denominatore idem dimensionum numerus inest, tum Fractio erit Functio nullius dimensionis, uti evenit in hac Fractione $\frac{y^3+z^3}{yyz}$, vel etiam in his $\frac{y}{z}$; $\frac{\alpha zz}{yy}$; $\frac{\beta y^3}{z^3}$. Quod si igitur in Denominatore plures sint dimensiones quam in Numeratore, numerus dimensionum Fractionis erit negativus: sic $\frac{y}{zz}$ erit Functio -1 dimensionis: $\frac{y+z}{y^4+z^4}$ erit Functio -3 dimensionum: $\frac{1}{y^5+ayz^4}$ erit Functio -5 dimensionum, quia in Numeratore nulla inest dimensio. Ceterum sponte intelligitur plures Functiones homogeneas, in quibus singulis idem regnat dimensionum numerus, sive additas sive subtractas præbere Functionem quoque homogeneam ejusdem dimensionum numeri. Sic hæc expressio $\alpha y + \frac{\beta zz}{y} + \frac{\gamma y^4 - \delta z^4}{yyz + yzz}$ erit Functio unius dimensionis: hæc autem $\alpha + \frac{\beta y}{z} + \frac{\gamma zz}{yy} + \frac{yy+zz}{yy-zz}$ erit Functio nullius dimensionis.

86. Natura Functionum homogenearum quoque ad expressiones irrationales extenditur. Si enim fuerit P Functio quæcunque homogenea, puta n dimensionum, tum $\sqrt{P}$ erit Functio $\frac{1}{2}n$ dimensionum; $\sqrt[3]{P}$ erit Functio $\frac{1}{3}n$ dimensionum, & generatim $P^{\frac{\mu}{\nu}}$ erit Functio $\frac{\mu}{\nu}n$ dimensionum. Sic $\sqrt{(yy+zz)}$ erit Functio unius dimensionis; $\sqrt[3]{(y^9+z^9)}$ erit Functio trium dimensionum: $(yz+zz)^{\frac{3}{4}}$ erit Functio $\frac{3}{2}$ dimensionum: at-

que

que $\frac{yy + zz}{\sqrt{(y^4 + z^4)}}$ erit Functio nullius dimensionis. His ergo cum præcedentibus conjunctis intelligetur hæc expressio $\frac{1}{y}$ $+ \frac{y\sqrt{(yy+zz)}}{z^3} - \frac{y}{\sqrt[3]{(y^6 - z^6)}} + \frac{y\sqrt{z}}{zz\sqrt{y} + \sqrt{(y^3+z^3)}}$ esse Functio homogenea -1 dimensionis.

87. Utrum Functio irrationalis implicita sit homogenea necne, ex his facile colligi potest. Sit V hujusmodi Functio implicita ac $V^3 + PV^2 + QV + R = 0$, existentibus P, Q & R Functionibus ipsarum y & z. Primum igitur patet V Functionem homogeneam esse non posse, nisi P, Q, & R sint Functiones homogeneæ. Præterea vero si ponamus V esse Functionem n dimensionum, erit V^2 Functio $2n$, & V^3 Functio $3n$ dimensionum; cum igitur ubique idem debeat esse numerus dimensionum, oportet, ut P sit Functio n dimensionum, Q Functio $2n$ dimensionum, & R Functio $3n$ dimensionum. Si ergo vicissim litteræ P, Q, R Functiones homogeneæ respective n, $2n$, $3n$ dimensionum, hinc concludetur fore V Functionem n dimensionum. Ita si fuerit $V^3 + (y^4 + z^4)V^2 + ay^8V - z^{10} = 0$ erit V Functio homogenea duarum dimensionum, ipsarum y & z.

88. *Si fuerit* V *Functio homogenea* n *dimensionum ipsarum* y *&* z, *in eaque ponatur ubique* $y = uz$, *Functio* V *abibit in productum ex potestate* z^n *in Functionem quandam variabilis* u.

Per hanc enim substitutionem $y = uz$, in singulos terminos tantæ inducentur potestates ipsius z, quantæ ante inerant ipsius y. Cum igitur in singulis terminis dimensiones ipsarum y & z conjunctim æquassent numerum n, nunc sola variabilis z ubique habebit n dimensiones, ideoque ubique inerit ejus potestas z^n. Per hanc ergo potestatem Functio V fiet divisibilis & quotus erit Functio variabilem tantum u involvens. Hoc primum patebit in Functionibus integris; si enim sit $V = ay^3 + \beta y^2 z + \gamma yz^2 + \delta z^3$, posito $y = uz$, fiet $V = z^3$ $(\alpha u^3 +$

LIB. I. $(\alpha u^3 + \beta u^2 + \gamma u + \delta)$. Deinde vero idem manifeſtum eſt in fractis: ſit enim $V = \frac{\alpha y + \beta z}{yy + zz}$, nempe Functio -1 dimenſionis, facto $y = uz$ fiet $V = z^{-1}\left(\frac{\alpha u + \beta}{uu + 1}\right)$. Neque etiam Functiones irrationales hinc excipiuntur, ſi enim ſit $V = \frac{y + \sqrt{(yy + zz)}}{z\sqrt{(y^3 + z^3)}}$, quæ eſt Functio $-\frac{3}{2}$ dimenſionum; poſito $y = uz$, prodibit $V = z^{-\frac{3}{2}}\left(\frac{u + \sqrt{(uu + 1)}}{\sqrt{(u^3 + 1)}}\right)$. Hoc itaque modo Functiones homogeneæ duarum tantum variabilium reducentur ad Functiones unius variabilis; neque enim poteſtas ipſius z, quia eſt Factor, Functionem illam ipſius u inquinat.

89. *Functio ergo homogenea* V *duarum variabilium* y *&* z *nullius dimenſionis, poſito* y = uz, *tranſmutabitur in Functionem unicæ variabilis* u *puram.*

Cum enim numerus dimenſionum ſit nullus, Poteſtas ipſius z, quæ Functionem ipſius u multiplicabit, erit $z^0 = 1$; hocque caſu variabilis z prorſus ex computo egredietur. Ita ſi fuerit $V = \frac{y + z}{y - z}$, facto $y = uz$, orietur $V = \frac{u + 1}{u - 1}$: atque in irrationalibus ſi ſit $V = \frac{y - \sqrt{(yy - zz)}}{z}$ poſito $y = uz$ erit $V = u - \sqrt{(uu - 1)}$.

90. *Functio integra homogenea duarum variabilium* y *&* z, *reſolvi poterit in tot Factores ſimplices formæ* αy + βz, *quot habuerit dimenſiones.*

Cum enim Functio ſit homogenea, poſito $y = uz$, tranſibit in productum ex z^n in Functionem quandam ipſius u integram, quæ Functio propterea in Factores ſimplices formæ $\alpha u + \beta$ reſolvi poterit. Multiplicentur ſinguli Factores hi per z, eritque uniuſcujuſque forma $\alpha uz + \beta z = \alpha y + \beta z$ ob $uz = y$. Propter multiplicatorem autem z^n, tot hujuſmodi Factores naſcentur quot exponens n contineat unitates; Factores autem hi hi simplices erunt vel reales vel imaginarii, hoc est coëfficientes α, & β erunt vel reales vel imaginarii.

Ex hoc itaque sequitur Functionem duarum dimensionum $ayy + byz + czz$ duos habere Factores simplices formæ $\alpha y + \beta z$; Functio autem $ay^3 + by^2z + cyz^2 + dz^3$ habebit tres Factores simplices formæ $\alpha y + \beta z$; sicque porro Functionum homogenearum integrarum, quæ plures habent dimensiones, natura erit comparata.

91. Quemadmodum ergo hæc expressio $\alpha y + \beta z$ continet formam generalem Functionum integrarum unius dimensionis - ita $(\alpha y + \beta z)(\gamma y + \delta z)$ erit forma generalis Functionum integrarum duarum dimensionum: atque in hac forma $(\alpha y + \beta z)(\gamma y + \delta z)(\epsilon y + \zeta z)$ continebuntur omnes Functiones integræ trium dimensionum, sicque omnes Functiones integræ homogeneæ per producta ex tot hujusmodi Factoribus $\alpha y + \beta z$ exhiberi poterunt, quot Functiones illæ contineant dimensiones. Isti autem Factores eodem modo per resolutionem æquationum reperiuntur, quo supra Factores simplices Functionum integrarum unius variabilis invenire docuimus. Ceterum hæc proprietas Functionum homogenearum duarum variabilium non extenditur ad Functiones homogeneas trium pluriumve variabilium: forma enim generalis hujusmodi Functionum duarum tantum dimensionum, quæ est $ayy + byz + cyx + dxy + exx + fzz$ generaliter non reduci potest ad hujusmodi productum $(\alpha y + \beta z + \gamma x)(\delta y + \epsilon z + \zeta x)$; multoque minus Functiones plurium dimensionum ad hujusmodi producta revocari possunt.

92. Ex his, quæ de Functionibus homogeneis sunt dicta, simul intelligitur, quid sit Functio heterogenea: in cujus scilicet terminis non ubique idem dimensionum numerus deprehenditur. Possunt autem Functiones heterogeneæ subdividi pro multiplicitate dimensionum, quæ in ipsis occurrunt. Sic Functio bifida erit, in qua duplex dimensionum numerus occurrit, eritque adeo aggregatum duarum Functionum homo-

LIB. I. genearum, quarum numeri dimensionum differunt; ita $y^5 + 2y^3z^2 + yy + zz$ erit Functio bifida, quia partim quinque, partim duas continet dimensiones. Functio autem trifida est, in qua tres diversi dimensionum numeri insunt, seu quæ in tres Functiones homogeneas distribui possunt, uti $y^6 + y^2z^2 + z^4 + y - z$.

Præterea autem dantur Functiones heterogeneæ fractæ vel irrationales tantopere permixtæ, quæ in Functiones homogeneas resolvi non possunt, cujusmodi sunt $\frac{y^3 + ayz}{by + zz}$, $\frac{a + \sqrt{(yy + zz)}}{yy - bz}$.

93. Interdum Functio heterogenea ope substitutionis idoneæ, vel loco unius vel utriusque variabilis factæ, ad homogeneam reduci potest; quod quibus casibus fieri queat, non tam facile indicare licet. Sufficiet ergo exempla quædam attulisse, quibus ejusmodi reductio locum habet. Si scilicet hæc proposita sit Functio $y^5 + zzy + y^3z + \frac{z^3}{y}$; post levem attentionem apparebit, eam ad homogeneitatem perduci, posito $z = xx$: prodibit enim $y^5 + x^4y + y^3xx + \frac{x^6}{y}$, Functio homogenea 5 dimensionum ipsarum x & y. Deinde hæc Functio $y + y^2x + y^3xx + y^5x^4 + \frac{a}{x}$ ad homogeneitatem reducitur ponendo $x = \frac{1}{z}$, prodit enim Functio unius dimensionis $y + \frac{yy}{z} + \frac{y^3}{zz} + \frac{y^5}{z^4} + az$. Multo difficiliores autem sunt casus, quibus non per tam simplicem substitutionem ad homogeneitatem pervenire licet.

94. Tandem inprimis notari meretur Functionum integrarum secundum ordines divisio satis usitata, secundum quam ordo definitur ex maximo dimensionum numero qui in Functione inest. Sic $xx + yy + zz + ay - aa$ est Functio secundi ordinis, quia duæ dimensiones occurrunt. Et $y^4 + yz^3 - ay^2z +$

abyz —

$abyz - aayy + b^4$ pertinet ad Functiones quarti ordinis. Ad hanc divisionem potissimum in doctrina de lineis curvis respici solet; unde adhuc una Functionum integrarum divisio commemoranda venit. CAP. V.

95. Supereſt ſcilicet diviſio Functionum integrarum in complexas atque incomplexas. Functio autem complexa eſt, quæ in Factores rationales reſolvi poteſt, ſeu quæ eſt productum ex duabus Functionibus pluribusve rationalibus; cujuſmodi eſt $y^4 - z^4 + 2az^3 - 2byzz - aazz + 2abzy - bbyy$, quæ eſt productum ex his duabus Functionibus $(yy + zz - az + by)$ $(yy - zz + az - by)$. Ita vidimus omnem Functionem integram homogeneam, quæ tantum duas variabiles complectatur, eſſe Functionem complexam, quoniam tot Factores ſimplices formæ $\alpha y + \beta z$ habet, quot continet dimenſiones. Functio igitur integra erit incomplexa, ſi in Factores rationales reſolvi omnino nequeat; uti $yy + zz - aa$, cujus nullos dari Factores rationales facile intelligitur. Ex inquiſitione Diviſorum patebit, utrum Functio propoſita ſit complexa an incomplexa.

CAPUT VI.

De Quantitatibus exponentialibus ac Logarithmis.

96. Quanquam notio Functionum tranſcendentium in calculo integrali demum perpendetur, tamen antequam eo perveniamus, quaſdam ſpecies magis obvias, atque ad plures inveſtigationes aditum aperientes, evolvere conveniet. Primum ergo conſiderandæ ſunt quantitates exponentiales, ſeu Poteſtates, quarum Exponens ipſe eſt quantitas variabilis. Perſpicuum enim eſt hujuſmodi quantitates ad Functiones algebraicas referri non poſſe, cum in his Exponentes non niſi conſtantes locum habeant. Multiplices autem ſunt quan-

LIB. I. titates exponentiales, prout vel solus Exponens est quantitas variabilis, vel præterea etiam ipsa quantitas elevata; prioris generis est a^z, hujus vero y^z; quin etiam ipse Exponens potest esse quantitas exponentialis uti in his formis a^{a^z}; a^{y^z}; y^{a^z}; x^{y^z}. Hujusmodi autem quantitatum non plura constituemus genera, cum earum natura satis clarè intelligi queat, si primam tantum speciem a^z evolverimus.

97. Sit igitur proposita hujusmodi quantitas exponentialis a^z, quæ est Potestas quantitatis constantis a, Exponentem habens variabilem z. Cum igitur iste Exponens z, omnes numeros determinatos in se complectatur, primum patet si loco z omnes numeri integri affirmativi successive substituantur, loco a^z hos prodituros esse valores determinatos a^1; a^2; a^3; a^4; a^5; a^6; &c. Sin autem pro z ponantur successive numeri negativi -1, -2, -3, &c. prodibunt $\frac{1}{a}$; $\frac{1}{a^2}$; $\frac{1}{a^3}$; $\frac{1}{a^4}$; &c. ac, si fuerit $z = 0$, habebitur semper $a^0 = 1$. Quod si loco z numeri fracti ponantur, ut $\frac{1}{2}$; $\frac{1}{3}$; $\frac{2}{3}$; $\frac{1}{4}$; $\frac{3}{4}$; &c. orientur isti valores $\sqrt{a}$; $\sqrt[3]{a}$; $\sqrt[3]{aa}$; $\sqrt[4]{a}$; $\sqrt[4]{a^3}$; &c., qui in se spectati geminos pluresve induunt valores, cum radicum extractio semper valores multiformes producat. Interim tamen hoc loco valores tantum primarii, reales scilicet atque affirmativi admitti solent; quia quantitas a^z tanquam Functio uniformis ipsius z spectatur. Sic $a^{\frac{5}{2}}$ medium quendam tenebit locum inter a^2 & a^3, eritque ideo quantitas ejusdem generis; & quamvis valor $a^{\frac{5}{2}}$ sit æque $= -aa\sqrt{a}$, ac $= +aa\sqrt{a}$; tamen posterior tantum in censum venit. Eodem modo res se habet, si Exponens z valores irrationales accipiat, quibus casibus cum difficile sit numerum valorum involutorum concipere,

pere, unicus tantum realis consideratur. Sic $a^{\sqrt{7}}$ erit valor determinatus intra limites a^2 & a^3 comprehensus. CAP. VI.

98. Maxime autem valores quantitatis exponentialis a^z a magnitudine numeri constantis a pendebunt. Si enim fuerit $a = 1$, semper erit $a^z = 1$, quicunque valores Exponenti z tribuatur; sin autem fuerit $a > 1$, tum valor ipsius a^z eo erunt majores, quo major numerus loco z substituatur, atque adeo, posito $z = \infty$, in infinitum excrescunt; si fuerit $z = 0$, fiet $a^z = 1$, &, si fit $z < 0$ valores a^z fient unitate minores, quoad posito $z = -\infty$ fiat $a^z = 0$. Contrarium evenit si fit $a < 1$, verum tamen quantitas affirmativa; tum enim valores ipsius a^z decrescent, crescente z supra 0; crescent vero, si pro z numeri negativi substituantur. Cum enim sit $a < 1$, erit $\frac{1}{a} > 1$; posito ergo $\frac{1}{a} = b$; erit $a^z = b^{-z}$, unde posterior casus ex priori dijudicari poterit.

99. Si fit $a = 0$, ingens saltus in valoribus ipsius a^z deprehenditur, quamdiu enim fuerit z numerus affirmativus seu major nihilo, erit perpetuo $a^z = 0$: si fit $z = 0$ erit $a^0 = 1$; sin autem fuerit z numerus negativus, tum a^z obtinebit valorem infinite magnum. Sit enim $z = -3$; erit $a^z = 0^{-3} = \frac{1}{0^3} = \frac{1}{0}$, ideoque infinitum. Multo majores autem saltus occurrent, si quantitas constans a habeat valorem negativum, puta -2; tum enim ponendis loco z numeris integris valores ipsius a^z alternatim erunt affirmativi & negativi, ut ex hac Serie intelligitur

$$a^{-4};\ a^{-3};\ a^{-2};\ a^{-1};\ a^0;\ a^1;\ a^2;\ a^3;\ a^4;\ \&c.$$

+

LIB. I. $+\frac{1}{16}$; $-\frac{1}{8}$; $+\frac{1}{4}$; $-\frac{1}{2}$; 1; -2; $+4$; -8; $+16$. Præterea vero si Exponenti z valores tribuantur fracti, Potestas $a^z = (-2)^z$ mox reales mox imaginarios induet valores, erit enim $a^{\frac{1}{2}} = \sqrt{-2}$, imaginarium; at erit $a^{\frac{1}{3}} = \sqrt[3]{-2} = -\sqrt[3]{2}$ reale: utrum autem, si Exponenti z tribuantur valores irrationales, Potestas a^z exhibeat quantitates reales an imaginarias, ne quidem definiri licet.

100. His igitur incommodis numerorum negativorum loco a substituendorum commemoratis, statuamus a esse numerum affirmativum, & unitate quidem majorem, quia huc quoque illi casus, quibus a est numerus affirmativus unitate minor, facile reducuntur. Si ergo ponatur $a^z = y$, loco z substituendo omnes numeros reales, qui intra limites $+\infty$ & $-\infty$ continentur, y adipiscetur omnes valores affirmativos intra limites $+\infty$ & 0 contentos. Si enim sit $z = \infty$ erit $y = \infty$; si $z = 0$ erit $y = 1$, & si $z = -\infty$ fiet $y = 0$. Vicissim ergo quicunque valor affirmativus pro y accipiatur, dabitur quoque valor realis respondens pro z ita ut sit $a^z = y$; sin autem ipsi y tribueretur valor negativus, Exponens z valorem realem habere non poterit.

101. Si igitur fuerit $y = a^z$, erit y Functio quædam ipsius z, &, quemadmodum y a z pendeat, ex natura Potestatum facile intelligitur; hinc enim quicunque valor ipsi z tribuatur, valor ipsius y determinatur. Erit autem $yy = a^{2z}$; $y^3 = a^{3z}$: & generaliter erit $y^n = a^{nz}$; unde sequitur fore $\sqrt{y} = a^{\frac{1}{2}z}$; $\sqrt[3]{y} = a^{\frac{1}{3}z}$ & $\frac{1}{y} = a^{-z}$; $\frac{1}{yy} = a^{-2z}$; & $\frac{1}{\sqrt{y}} = a^{-\frac{1}{2}z}$, & ita porro. Præterea, si fuerit $v = a^x$ erit $vy = a^{x+z}$ & $\frac{v}{y} = a^{x-z}$, quorum subsidiorum beneficio eo facilius valor ipsius y ex dato valore ipsius z inveniri potest.

EXEM-

EXEMPLUM.

Si fuerit $a = 10$, ex Arithmetica, qua utimur, denaria in promtu erit valores ipsius y exhibere, si quidem pro z numeri integri ponantur. Erit enim $10^1 = 10$; $10^2 = 100$; $10^3 = 1000$; $10^4 = 10000$; & $10^0 = 1$; item $10^{-1} = \frac{1}{10} = 0,1$; $10^{-2} = \frac{1}{100} = 0,01$; $10^{-3} = \frac{1}{1000} = 0,001$: sin autem pro z Fractiones ponantur, ope radicum extractionis valores ipsius y indicari possunt: sic erit $10^{\frac{1}{2}} = \sqrt{10} = 3,162277$, &c.

102. Quemadmodum autem, dato numero a, ex quovis valore ipsius z reperiri potest valor ipsius y, ita vicissim, dato valore quocunque affirmativo ipsius y, conveniens dabitur valor ipsius z; ut sit $a^z = y$; iste autem valor ipsius z, quatenus tanquam Functio ipsius y spectatur, vocari solet LOGARITHMUS ipsius y. Supponit ergo doctrina Logarithmorum numerum certum constantem loco a substituendum, qui propterea vocatur *basis* Logarithmorum; qua assumta erit Logarithmus cujusque numeri y Exponens Potestatis a^z, ita ut ipsa Potestas a^z aequalis sit numero illi y; indicari autem Logarithmus numeri y solet hoc modo ly. Quod si ergo fuerit $a^z = y$, erit $z = ly$: ex quo intelligitur, basin Logarithmorum, etiamsi ab arbitrio nostro pendeat, tamen esse debere numerum unitate majorem: hincque nonnisi numerorum affirmativorum Logarithmos realiter exhiberi posse.

103. Quicunque ergo numerus pro basi Logarithmica a accipiatur, erit semper $l1 = 0$; si enim in aequatione $a^z = y$, quae convenit cum hac $z = ly$, ponatur $y = 1$, erit $z = 0$. Deinde numerorum unitate majorum Logarithmi erunt affirmativi, pendentes a valore basis a, sic erit $la = 1$; $laa = 2$; $la^3 = 3$; $la^4 = 4$;

LIB. I. $la^4 = 4$; &c., unde a posteriori intelligi potest, quantus numerus pro basi sit assumtus, scilicet ille numerus, cujus Logarithmus est $= 1$, erit basis Logarithmica. Numerorum autem unitate minorum, affirmativorum tamen, Logarithmi erunt negativi; erit enim $l\frac{1}{a} = -1$; $l\frac{1}{aa} = -2$; $l\frac{1}{a^3} = -3$, &c.; numerorum autem negativorum Logarithmi non erunt reales, sed imaginarii, uti jam notavimus.

104. Simili modo si fuerit $ly = z$; erit $lyy = 2z$; $ly^3 = 3z$; & generaliter $ly^n = nz$, seu $ly^n = nly$, ob $z = ly$. Logarithmus igitur cujusque Potestatis ipsius y æquatur Logarithmo ipsius y per Exponentem Potestatis multiplicato; sic erit $l\sqrt{y} = \frac{1}{2}z = \frac{1}{2}ly$; $l\frac{1}{\sqrt{y}} = ly^{-\frac{1}{2}} = -\frac{1}{2}ly$; & ita porro; unde ex dato Logarithmo cujusque numeri inveniri possunt Logarithmi quarumcunque ipsius Potestatum. Sin autem jam inventi sint duo Logarithmi, nempe $ly = z$ & $lv = x$: cum sit $y = a^z$ & $v = a^x$ erit $lvy = x + z = lv + ly$; hinc Logarithmus Producti duorum numerorum æquatur summæ Logarithmorum Factorum; simili vero modo erit $l\frac{y}{v} = z - x = ly - lv$; hincque Logarithmus Fractionis æquatur Logarithmo Numeratoris dempto Logarithmo Denominatoris, quæ regulæ inserviunt Logarithmis plurium numerorum inveniendis, ex cognitis jam aliquot Logarithmis.

105. Ex his autem patet aliorum numerorum non dari Logarithmos rationales, nisi Potestatum baseos a; nisi enim numerus alius b fuerit Potestas basis a, ejus Logarithmus numero rationali exprimi non poterit. Neque vero etiam Logarithmus ipsius b erit numerus irrationalis; si enim foret $lb = \sqrt{n}$, tum esset $a^{\sqrt{n}} = b$; id quod fieri nequit, si quidem numeri a & b rationales statuantur; solent autem imprimis numerorum rationa-

tionalium & integrorum Logarithmi desiderari, quia ex his Logarithmi Fractionum ac numerorum surdorum inveniri possunt. Cum igitur Logarithmi numerorum, qui non sunt Potestates basis a, neque rationaliter neque irrationaliter exhiberi queant, merito ad quantitates transcendentes referuntur, hincque Logarithmi quantitatibus transcendentibus annumerari solent.

106. Hanc ob rem Logarithmi numerorum vero tantum proxime per Fractiones decimales exprimi solent, qui eo minus à veritate discrepabunt, ad quo plures figuras fuerint exacti. Atque hoc modo per solam radicis quadratæ extractionem cujusque numeri Logarithmus vero proxime determinari poterit. Cum enim, posito $ly = z$ & $lv = x$, fit $l\sqrt{vy} = \frac{x+z}{2}$; si numerus propositus b contineatur intra limites a^2 & a^3, quorum Logarithmi sunt 2 & 3, quæratur valor ipsius $a^{2\frac{1}{2}}$ seu $a^2\sqrt{a}$, atque b vel intra limites a^2 & $a^{2\frac{1}{2}}$ vel $a^{2\frac{1}{2}}$ & a^3 continebitur, utrumvis accidat, sumendo medio proportionali, denuo limites propiores prodibunt, hocque modo ad limites pervenire licebit, quorum intervallum data quantitate minus evadat, & quibuscum numerus propositus b sine errore confundi possit. Quoniam vero horum singulorum limitum Logarithmi dantur, tandem Logarithmus numeri b reperietur.

EXEMPLUM.

Ponatur basis Logarithmica $a = 10$, quod in tabulis usu receptis fieri solet; & quæratur vero tantum proxime Logarithmus numeri 5; quia hic continetur intra limites 1 & 10 quorum Logarithmi sunt 0 & 1; sequenti modo radicum extractio continua instituatur, quoad ad limites à numero proposito 5 non amplius discrepantes perveniatur.

Lib. I.

$A = 1,000000$;	$lA = 0,0000000$	fit
$B = 10,000000$;	$lB = 1,0000000$;	$C = \sqrt{AB}$
$C = 3,162277$;	$lC = 0,5000000$;	$D = \sqrt{BC}$
$D = 5,623413$;	$lD = 0,7500000$;	$E = \sqrt{CD}$
$E = 4,216964$;	$lE = 0,6250000$;	$F = \sqrt{DE}$
$F = 4,869674$;	$lF = 0,6875000$;	$G = \sqrt{DF}$
$G = 5,232991$;	$lG = 0,7187500$;	$H = \sqrt{FG}$
$H = 5,048065$;	$lH = 0,7031250$;	$I = \sqrt{FH}$
$I = 4,958069$;	$lI = 0,6953125$;	$K = \sqrt{HI}$
$K = 5,002865$;	$lK = 0,6992187$;	$L = \sqrt{IK}$
$L = 4,980416$;	$lL = 0,6972656$;	$M = \sqrt{KL}$
$M = 4,991627$;	$lM = 0,6982421$;	$N = \sqrt{KM}$
$N = 4,997242$;	$lN = 0,6987304$;	$O = \sqrt{KN}$
$O = 5,000052$;	$lO = 0,6989745$;	$P = \sqrt{NO}$
$P = 4,998647$;	$lP = 0,6988525$;	$Q = \sqrt{OP}$
$Q = 4,999350$;	$lQ = 0,6989135$;	$R = \sqrt{OQ}$
$R = 4,999701$;	$lR = 0,6989440$;	$S = \sqrt{OR}$
$S = 4,999876$;	$lS = 0,6989592$;	$T = \sqrt{OS}$
$T = 4,999963$;	$lT = 0,6989668$;	$V = \sqrt{OT}$
$V = 5,000008$;	$lV = 0,6989707$;	$W = \sqrt{TV}$
$W = 4,999984$;	$lW = 0,6989687$;	$X = \sqrt{WV}$
$X = 4,999997$;	$lX = 0,6989697$;	$Y = \sqrt{VX}$
$Y = 5,000003$;	$lY = 0,6989702$;	$Z = \sqrt{XY}$
$Z = 5,000000$;	$lZ = 0,6989700$;	

Sic ergo mediis proportionalibus ſumendis tandem perventum eſt ad $Z = 5,000000$, ex quo Logarithmus numeri 5 quæſitus eſt 0,*698970*, poſita baſi Logarithmica $= 10$. Quare erit proxime $10^{\frac{69897}{100000}} = 5$. Hoc autem modo computatus eſt canon Logarithmorum vulgaris à Briggio & Vlacquio, quamquam poſtea eximia inventa ſunt compendia, quorum ope multo expeditius Logarithmi ſupputari poſſunt.

107. Dantur ergo tot diverſa Logarithmorum ſyſtemata quot varii numeri pro baſi *a* accipi poſſunt, atque ideo numerus ſyſtema-

tematum Logarithmicorum erit infinitus. Perpetuo autem in duobus ſyſtematis Logarithmi ejuſdem numeri eandem inter ſe ſervant rationem. Sit baſis unius ſyſtematis $= a$, alterius $= b$, atque numeri n Logarithmus in priori ſyſtemate $= p$, in poſteriori $= q$; erit $a^p = n$ & $b^q = n$; unde $a^p = b^q$; ideoque $a = b^{\frac{q}{p}}$. Oportet ergo ut Fractio $\frac{q}{p}$ conſtantem obtineat valorem, quicunque numerus pro n fuerit aſſumtus. Quod ſi ergo pro uno ſyſtemate Logarithmi omnium numerorum fuerint computati, hinc facili negotio per regulam auream Logarithmi pro quovis alio ſyſtemate reperiri poſſunt. Sic, cum dentur Logarithmi pro baſi 10, hinc Logarithmi pro quavis alia baſi, puta 2, inveniri poſſunt; quæratur enim Logarithmus numeri n pro baſi 2, qui ſit $= q$, cum ejuſdem numeri n Logarithmus ſit $= p$ pro baſi 10. Quoniam pro baſi 10 eſt $l2 = 0, 3010300$, & pro baſi 2, eſt $l2 = 1$, erit $0, 3010300 : 1 = p : q$ ideoque $q = \frac{p}{0, 3010300} = 3, 3219277 . p$, ſi ergo omnes Logarithmi communes multiplicentur per numerum 3, 3219277, prodibit tabula Logarithmorum pro baſi 2.

108. *Hinc ſequitur duorum numerorum Logarithmos in quocunque ſyſtemate eandem tenere rationem.*

Sint enim duo numeri M & N, quorum pro baſi a Logarithmi ſint m & n, erit $M = a^m$ & $N = a^n$: hinc fiet $a^{nm} = M^n = N^m$, ideoque $M = N^{\frac{m}{n}}$; in qua æquatione cum baſis a non amplius inſit, perſpicuum eſt Fractionem $\frac{m}{n}$ habere valorem à baſi a non pendentem. Sint enim pro alia baſi b numerorum eorundem M & N Logarithmi μ & ν, pari modo colligetur fore $M = N^{\frac{\mu}{\nu}}$. Erit ergo $N^{\frac{m}{n}} = N^{\frac{\mu}{\nu}}$, hincque $\frac{m}{n} = \frac{\mu}{\nu}$, ſeu $m : n = \mu : \nu$. Ita jam vidimus in omni Logarithmorum systemate Logarithmos diversarum ejusdem numeri Potestatum ut y^m & y^n tenere rationem Exponentium $m : n$.

109. Ad canonem ergo Logarithmorum pro basi quacunque a condendum sufficit numerorum tantum primorum Logarithmos methodo ante tradita, vel alia commodiori, supputasse. Cum enim Logarithmi numerorum compositorum sint æquales summis Logarithmorum singulorum Factorum, Logarithmi numerorum compositorum per solam additionem reperientur. Sic, si habeantur Logarithmi numerorum 3 & 5, erit $l\,15 = l\,3 + l\,5$; $l\,45 = 2\,l\,3 + l\,5$. Atque, cum supra pro basi $a = 10$, inventus sit $l\,5 = 0,6989700$, præterea autem sit $l\,10 = 1$ erit $l\,\frac{10}{5} = l\,2 = l\,10 - l\,5$, ideoque orietur $l\,2 = 1 - 0,6989700 = 0,3010300$; ex his autem numerorum primorum 2 & 5 Logarithmis inventis reperientur Logarithmi omnium numerorum ex his 2 & 5 compositorum; cujusmodi sunt isti 4, 8, 16, 32, 64, &c; 20, 40, 80, 25, 50; &c.

110. Tabularum autem Logarithmicarum amplissimus est usus in contrahendis calculis numericis, propterea quod ex ejusmodi tabulis non solum dati cujusque numeri Logarithmus, sed etiam cujusque Logarithmi propositi numerus conveniens reperiri potest. Sic, si c, d, e, f, g, h, denotent numeros quoscunque, citra multiplicationem reperiri poterit valor istius expressionis $\frac{ccd\sqrt{e}}{f\sqrt[3]{gh}}$, erit enim hujus expressionis Logarithmus $= 2\,lc + ld + \frac{1}{2}\,le - lf - \frac{1}{3}\,lg - \frac{1}{3}\,lh$, cui Logarithmo si quæratur numerus respondens, habebitur valor quæsitus. Inprimis autem inserviunt tabulæ Logarithmicæ dignitatibus atque radicibus intricatissimis inveniendis, quarum operationum loco in Logarithmis tantum multiplicatio & divisio adhibetur.

Exem-

EXEMPLUM I.

Quæratur valor hujus Poteſtatis $2^{\frac{7}{12}}$: quoniam ejus Logarithmus eſt $= \frac{7}{12} l 2$, multiplicetur Logarithmus binarii ex tabulis qui eſt 0, 3010300 per $\frac{7}{12}$ hoc eſt per $\frac{1}{2} + \frac{1}{12}$ erit, $l 2^{\frac{7}{12}} = 0, 1756008$, cui Logarithmo reſpondet numerus 1, 498307, qui ergo proxime exhibet valorem $2^{\frac{7}{12}}$.

EXEMPLUM II.

Si numerus incolarum cujuſpiam provinciæ quotannis ſui parte trigeſima augeatur, initio autem in provincia habitaverint 100000 hominum, quæritur poſt 100 annos incolarum numerus. Sit brevitatis gratia initio incolarum numerus $= n$, ita ut ſit $n =$ 100000, anno elapſo uno erit incolarum numerus $= (1 + \frac{1}{30})n = \frac{31}{30} n$: poſt duos annos $= (\frac{31}{30})^2 n$: poſt tres annos $= (\frac{31}{30})^3 n$, hincque poſt centum annos $= (\frac{31}{30})^{100} n = (\frac{31}{30})^{100} 100000$; cujus Logarithmus eſt $= 100 \, l \frac{31}{30} + l\, 100000$. At eſt $l \frac{31}{30} = l\, 31 - l\, 30 = 0, 014240439$, unde $100 \, l \frac{31}{30} = 1, 4240439$, ad quem ſi addatur $l 100000 = 5$, erit Logarithmus numeri incolarum quæſiti $= 6\ 4240439$, cui reſpondet numerus $= 2654874$. Poſt centum ergo annos numerus incolarum fit plus quam vicies ſexies cum ſemiſſe major.

EXEM-

EXEMPLUM III.

Cum post diluvium à sex hominibus genus humanum sit propagatum, si ponamus ducentis annis post, numerum hominum jam ad 1000000 excrevisse, quaeritur quanta sui parte numerus hominum quotannis augeri debuerit. Ponamus hoc tempore numerum hominum parte sua $\frac{1}{x}$ quotannis increvisse, atque post ducentos annos prodierit necesse est numerus hominum $= (\frac{1+x}{x})^{200} 6 = 1000000$, unde fit $\frac{1+x}{x} = (\frac{1000000}{6})^{\frac{1}{200}}$. Erit ergo $l\frac{1+x}{x} = \frac{1}{200} l\frac{1000000}{6} = \frac{1}{200}$. $5,2218487 = 0,0261092$, ideoque $\frac{1+x}{x} = \frac{1061963}{1000000}$, & $1000000 = 61963x$, unde fit $x = 16$ circiter. Ad tantam ergo hominum multiplicationem suffecisset, si quotannis decima sexta sui parte increverint; quae multiplicatio ob longaevam vitam non nimis magna censeri potest. Quod si autem eadem ratione per intervallum 400 annorum numerus hominum crescere perrexisset, tum numerus hominum ad $1000000.\frac{1000000}{6} = 166666666666$ ascendere debuisset, quibus sustentandis universus orbis terrarum nequaquam par fuisset.

EXEMPLUM IV.

Si singulis seculis numerus hominum duplicetur, quaeritur incrementum annuum. Si quotannis hominum numerum parte sua $\frac{1}{x}$ crescere ponamus, & initio numerus hominum fuerit $= n$, erit is post centum annos $= (\frac{1+x}{x})^{100} n$, qui cum esse debeat

beat $= 2n$, erit $\frac{1+x}{x} = 2^{\frac{1}{100}}$ & $l\frac{1+x}{x} = \frac{1}{100} \cdot l2 =$ $0,0030103$; hinc $\frac{1+x}{x} = \frac{10069555}{10000000}$; ergo $x =$ $\frac{10000000}{69555} = 144$, circiter; ſufficit ergo ſi numerus hominum quotannis parte ſua $\frac{1}{144}$ augeatur. Quam ob cauſam maxime ridiculæ ſunt eorum incredulorum hominum objectiones, qui negant tam brevi temporis ſpatio ab uno homine univerſam terram incolis impleri potuiſſe.

111. Potiſſimum autem Logarithmorum uſus requiritur ad ejuſmodi æquationes reſolvendas, in quibus quantitas incognita in Exponentem ingreditur. Sic, ſi ad hujuſmodi perveniatur æquationem $a^x = b$, ex qua incognitæ x valorem erui oporteat, hoc non niſi per Logarithmos effici poterit. Cum enim ſit $a^x = b$ erit $la^x = xla = lb =$ ideoque $x =$ $\frac{lb}{la}$, ubi quidem perinde eſt, quonam ſyſtemate Logarithmico utatur, cum in omni ſyſtemate Logarithmi numerorum a & b eandem inter ſe teneant rationem.

EXEMPLUM I.

Si numerus hominum quotannis centeſima ſui parte augeatur; quæritur poſt quot annos numerus hominum fiat decuplo major. Ponamus hoc evenire poſt x annos, & initio hominum numerum fuiſſe $= n$, erit is ergo elapſis x annis $=$ $(\frac{101}{100})^x n$, qui cum æqualis ſit $10n$, fiet $(\frac{101}{100})^x = 10$; ideoque $xl\frac{101}{100} = l10$ & $x = \frac{l10}{l101 - l100}$. Prodibit itaque $x = \frac{10000000}{43214} = 231$. Poſt annos ergo 231 fiet homi-

 num

 num numerus, quorum incrementum annuum tantum centesimam partem efficit, decuplo major; hinc post *462* annos fiet centies, & post *693* annos millies major.

Exemplum. II.

Quidam debet 400000 florenos hac conditione ut quotannis usuram 5 de centenis solvere teneatur; exsolvit autem singulis annis 25000 florenos: quæritur post quot annos debitum penitus extinguatur. Scribamus a pro debita summa 400000 fl. & b pro summa 25000 fl. quotannis soluta; debebit ergo elapso uno anno $\frac{105}{100}a - b$; elapsis duobus annis $(\frac{105}{100})^2 a - \frac{105}{100}b - b$; elapsis tribus annis $(\frac{105}{100})^3 a - (\frac{105}{100})^2 b - \frac{105}{100}b - b$; hinc, posito brevitatis causa, n pro $\frac{105}{100}$, elapsis x annis adhuc debebit $n^x a - n^{x-1}b - n^{x-2}b - n^{x-3}b - \ldots\ldots\ldots - b = n^x a - b(1 + n + n^2 + \ldots\ldots + n^{x-1})$. Cum igitur sit ex natura progressionum geometricarum, $1 + n + n^2 + \ldots\ldots\ldots + n^{x-1} = \frac{n^x - 1}{n - 1}$, post x annos debitor adhuc debebit $n^x a - \frac{n^x b + b}{n - 1}$ flor., quod debitum nihilo æquale positum dabit hanc æquationem $n^x a = \frac{n^x b - b}{n - 1}$, seu $(n - 1)n^x a = n^x b - b$, ideoque $(b - na + a)n^x = b$ & $n^x = \frac{b}{b - (n - 1)a}$, unde fit $x = \frac{lb - l(b - (n - 1)a)}{ln}$. Cum jam sit $a = 400000$, $b = 25000$, $n = \frac{105}{100}$, erit $(n - 1)a = 20000$ & $b - (n - 1)a = 5000$, atque annorum, quibus debitum penitus extinguitur, numerus $x =$

$\frac{l\,25000 - l\,5000}{l\frac{105}{100}} = \frac{l\,5}{l\frac{21}{20}} = \frac{6989700}{211893}$; erit ergo x aliquanto minor quam 33; ſcilicet elapſis annis 33 non ſolum debitum extinguetur, ſed creditor debitori reddere tenebitur $\frac{(n^{33} - 1)b}{n - 1}$

$$- n^{33}a = \frac{(\frac{21}{20})^{33} \cdot 5000 - 25000}{\frac{1}{20}} = 100000 (\frac{21}{20})^{33} - 500000$$

flor. Quia vero eſt $l\,\frac{21}{20} = 0{,}0211892991$, erit $l(\frac{21}{20})^{33} = 0{,}69924687$, & $l\,100000\,(\frac{21}{20})^{33} = 5{,}6992469$, cui reſpondet hic numerus 500318, 8; unde creditor debitori poſt 33 annos reſtituere debet $318\frac{4}{5}$ florenos.

112. Logarithmi autem vulgares ſuper baſi = 10 extracti, præter hunc uſum, quem Logarithmi in genere præſtant, in Arithmetica decimali uſu recepta ſingulari gaudent commodo, atque ob hanc cauſam præ aliis ſyſtematibus inſignem afferunt utilitatem. Cum enim Logarithmi omnium numerorum, præter denarii Poteſtates, in Fractionibus decimalibus exhibeantur, numerorum inter 1 & 10 contentorum Logarithmi intra limites 0 & 1, numerorum autem inter 10 & 100 contentorum Logarithmi inter limites 1 & 2, & ita porro, continebuntur. Conſtat ergo Logarithmus quiſque ex numero integro & Fractione decimali; & ille numerus integer vocari ſolet CHARACTERISTICA; Fractio decimalis autem MANTISSA. Characteriſtica itaque unitate deficiet a numero notarum, quibus numerus conſtat; ita Logarithmi numeri 78509 Characteriſtica erit 4, quia is ex quinque notis ſeu figuris conſtat. Hinc ex Logarithmo cujuſvis numeri ſtatim intelligitur, ex quot figuris numerus ſit compoſitus. Sic numerus Logarithmo 7, 5804631 reſpondens ex 8 figuris conſtabit.

113. Si ergo duorum Logarithmorum Mantiſſæ conveniant, Characteriſticæ vero tantum diſcrepent, tum numeri his Logarithmis

LIB. I. rithmis respondentes rationem habebunt, ut Potestas denarii ad unitatem, ideoque ratione figurarum, quibus constant, convenient. Ita horum Logarithmorum 4, 9130187 & 6, 9130187 numeri erunt 81850 & 8185000; Logarithmo autem 3, 9130187 conveniet 8185, & Logarithmo huic 0, 9130187 convenit 8, 185. Sola ergo Mantissa indicabit figuras numerum componentes, quibus inventis, ex Characteristica patebit, quot figuræ a sinistra ad integra referri debeant, reliquæ ad dextram vero dabunt Fractiones decimales. Sic, si hic Logarithmus fuerit inventus 2, 7603429, Mantissa indicabit has figuras 5758945, Characteristica 2 autem numerum illi Logarithmo determinat, ut sit 575, 8945; si Characteristica esset 0, foret numerus 5, 758945; sin denuo unitate minuatur ut sit — 1, erit numerus respondens decies minor, nempe 0, 5758945; & Characteristicæ — 2 respondebit 0, 05758945 &c.: loco Characteristicarum autem hujusmodi negativarum — 1, — 2, — 3, &c. scribi solent 9, 8, 7, &c., atque subintelligitur hos Logarithmos denario minui debere. Hæc vero in manductionibus ad tabulas Logarithmorum fusius exponi solent.

EXEMPLUM.

Si hæc progressio 2, 4, 16, 256, &c., cujus quisque terminus est quadratum præcedentis, continuetur usque ad terminum vigesimum quintum; quæritur magnitudo hujus termini ultimi. Termini hujus progressionis per Exponentes ita commodius exprimuntur: 2^1, 2^2, 2^4, 2^8, &c. ubi patet Exponentes progressionem geometricam constituere, atque termini vigesimi quinti exponentem fore $2^{24} = 16777216$, ita ut ipse terminus quæsitus sit $= 2^{16777216}$, hujus ergo Logarithmus erit $= 16777216. l2$. Cum ergo sit $l2 = 0, 30102999566398119$5, erit numeri quæsiti Logarithmus $= 5050445,25973367$, ex cujus Characteristica patet numerum quæsitum more solito expressum constare ex 5050446 figuris. Mantissa autem 259733675932 in tabula la Logarithmorum quæsita dabit figuras initiales numeri quæsiti, quæ erunt 181858. Quanquam ergo iste numerus nullo modo exhiberi potest, tamen affirmari potest eum omnino ex 5050446 figuris constare, atque figuras initiales sex esse 181858, quas dextrorsum adhuc 5050440 figuræ sequantur, quarum insuper nonnullæ ex majori Logarithmorum canone definiri possent, undecim scilicet figuræ initiales erunt 18185852986.

CAPUT VII.

De quantitatum exponentialium ac Logarithmorum per Series explicatione.

114. Quia est $a^0 = 1$, atque crescente Exponente ipsius a simul valor Potestatis augetur, si quidem a est numerus unitate major; sequitur si Exponens infinite parum cyphram excedat, Potestatem ipsam quoque infinite parum unitatem esse superaturam. Sit ω numerus infinite parvus, seu Fractio tam exigua, ut tantum non nihilo sit æqualis, erit $a^\omega = 1 + \psi$, existente ψ quoque numero infinite parvo. Ex præcedente enim capite constat nisi ψ esset numerus infinite parvus, neque ω talem esse posse. Erit ergo vel $\psi = \omega$, vel $\psi > \omega$, vel $\psi < \omega$, quæ ratio utique a quantitate litteræ a pendebit, quæ cum adhuc sit incognita, ponatur $\psi = k\omega$; ita ut sit $a^\omega = 1 + k\omega$; &, sumta a pro basi Logarithmica, erit $\omega = l(1 + k\omega)$.

EXEMPLUM.

Quo clarius appareat, quemadmodum numerus k pendeat a basi a, ponamus esse $a = 10$; atque ex tabulis vulgaribus quæramus Logarithmum numeri quam minime unitatem superantis, puta $1 + \frac{1}{1000000}$, ita ut sit $k\omega = \frac{1}{1000000}$; erit $l(1 + \frac{1}{1000000}) = l\frac{1000001}{1000000} = 0{,}00000043429 = \omega$. Hinc, ob $k\omega = 0,\ 00000100000$, erit $\frac{1}{k} = \frac{43429}{100000}$ & $k = \frac{100000}{43429} = 2,\ 30258$: unde patet k esse numerum finitum pendentem a valore basis a. Si enim alius numerus pro basi a statuatur, tum Logarithmus ejusdem numeri $1 + k\omega$ ad priorem datam tenebit rationem, unde simul alius valor litteræ k prodiret.

115. Cum sit $a^\omega = 1 + k\omega$, erit $a^{i\omega} = (1 + k\omega)^i$, quicunque numerus loco i substituatur. Erit ergo $a^{i\omega} = 1 + \frac{i}{1}k\omega + \frac{i(i-1)}{1.\ 2}k^2\omega^2 + \frac{i(i-1)(i-2)}{1.\ 2.\ 3}k^3\omega^3 + \&c.$ Quod si ergo statuatur $i = \frac{z}{\omega}$, & z denotet numerum quemcunque finitum, ob ω numerum infinite parvum, fiet i numerus infinite magnus, hincque $\omega = \frac{z}{i}$, ita ut sit ω Fractio denominatorem habens infinitum, adeoque infinite parva, qualis est assumta. Substituatur ergo $\frac{z}{i}$ loco ω, eritque $a^z = (1 + \frac{kz}{i})^i = 1 + \frac{1}{1}kz + \frac{1(i-1)}{1.\ 2i}k^2z^2 + \frac{1(i-1)(i-2)}{1.\ 2i.\ 3i}k^3z^3 + \frac{1(i-1)(i-2)(i-3)}{1.\ 2i.\ 3i.\ 4i.}k^4z^4 + \&c.$, quæ æquatio erit vera si pro i numerus infinite magnus substituatur. Tum vero est k numerus definitus ab a pendens, uti modo vidimus.

116. Cum autem i sit numerus infinite magnus, erit $\frac{i-1}{i} = 1$; patet enim quo major numerus loco i substituatur, eo propius valorem Fractionis $\frac{i-1}{i}$ ad unitatem esse accessurum, hinc si

i sit

i sit numerus omni assignabili major, Fractio quoque $\frac{i-1}{i}$ ipsam unitatem adæquabit. Ob similem autem rationem erit $\frac{i-2}{i} = 1$; $\frac{i-3}{i} = 1$; & ita porro; hinc sequitur fore $\frac{i-1}{2i} = \frac{1}{2}$; $\frac{i-2}{3i} = \frac{1}{3}$; $\frac{i-3}{4i} = \frac{1}{4}$; & ita porro. His igitur valoribus substitutis, erit $a^z = 1 + \frac{kz}{1} + \frac{k^2z^2}{1.2} + \frac{k^3z^3}{1.2.3} + \frac{k^4z^4}{1.2.3.4} +$ &c. in infinitum. Hæc autem æquatio simul relationem inter numeros a & k ostendit, posito enim $z = 1$, erit $a = 1 + \frac{k}{1} + \frac{k^2}{1.2} + \frac{k^3}{1.2.3} + \frac{k^4}{1.2.3.4} +$ &c., atque ut a sit $= 10$, necesse est ut sit circiter $k = 2,30258$, uti ante invenimus. CAP. VII.

117. Ponamus esse $b = a^n$, erit, sumto numero a pro basi Logarithmica, $lb = n$. Hinc, cum sit $b^z = a^{nz}$, erit per Seriem infinitam $b^z = 1 + \frac{knz}{1} + \frac{k^2n^2z^2}{1.2} + \frac{k^3n^3z^3}{1.2.3} + \frac{k^4n^4z^4}{1.2.3.4} +$ &c., posito vero lb pro n, erit $b^z = 1 + \frac{kz}{1}lb + \frac{k^2z^2}{1.2}(lb)^2 + \frac{k^3z^3}{1.2.3}(lb)^3 + \frac{k^4z^4}{1.2.3.4}(lb)^4 +$ &c.. Cognito ergo valore litteræ k ex dato valore basis a, quantitas exponentialis quæcunque b^z per Seriem infinitam exprimi poterit, cujus termini secundum Potestates ipsius z procedant. His expositis ostendamus quoque quomodo Logarithmi per Series infinitas explicari possint.

118. Cum sit $a^\omega = 1 + k\omega$, existente ω Fractione infinite parva, atque ratio inter a & k definiatur per hanc æquationem $a = 1 + \frac{k}{1} + \frac{k^2}{1.2} + \frac{k^3}{1.2.3} +$ &c., si a sumatur pro basi Logarithmica, erit $\omega = l(1+k\omega)$ & $i\omega = l(1+k\omega)^i$.

Mani-

Manifestum autem est, quo major numerus pro i sumatur, eo magis Potestatem $(1+k\omega)^i$ unitatem esse superaturam; atque statuendo $i =$ numero infinito, valorem Potestatis $(1+k\omega)^i$ ad quemvis numerum unitate majorem ascendere. Quod si ergo ponatur $(1+k\omega)^i = 1+x$, erit $l(1+x) = i\omega$, unde, cum sit $i\omega$ numerus finitus, Logarithmus scilicet numeri $1+x$, perspicuum est, i esse debere numerum infinite magnum, alioquin enim $i\omega$ valorem finitum habere non posset.

119. Cum autem positum sit $(1+k\omega)^i = 1+x$, erit $1+k\omega = (1+x)^{\frac{1}{i}}$ & $k\omega = (1+x)^{\frac{1}{i}} - 1$, unde fit $i\omega = \frac{i}{k}((1+x)^{\frac{1}{i}} - 1)$. Quia vero est $i\omega = l(1+x)$, erit $l(1+x) = \frac{i}{k}(1+x)^{\frac{1}{i}} - \frac{i}{k}$, posito i numero infinite magno. Est autem $(1+x)^{\frac{1}{i}} = 1 + \frac{1}{i}x - \frac{1(i-1)}{i.\ 2i}x^2 + \frac{1(i-1)(2i-1)}{i.\ 2i.\ 3i}x^3 - \frac{1(i-1)(2i-1)(3i-1)}{i.\ 2i.\ 3i.\ 4i}x^4 + \&c.$ Ob i autem numerum infinitum, erit $\frac{i-1}{2i} = \frac{1}{2}$; $\frac{2i-1}{3i} = \frac{2}{3}$; $\frac{3i-1}{4i} = \frac{3}{4}$, &c.; hinc erit $i(1+x)^{\frac{1}{i}} = i + \frac{x}{1} - \frac{xx}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \&c.$, & consequenter $l(1+x) = \frac{1}{k}(\frac{x}{1} - \frac{xx}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \&c.)$, posita basi Logarithmica $= a$ ac denotante k numerum huic basi convenientem, ut scilicet sit $a = 1 + \frac{k}{1} + \frac{k^2}{1.2} + \frac{k^3}{1.2.3} + \&c.$

120. Cum igitur habeamus Seriem Logarithmo numeri $1+x$ æqualem, ejus ope ex data basi a definire poterimus valorem numeri

numeri k. Si enim ponamus $1+x=a$, ob $la=1$, erit CAP. VII.

$1=\frac{1}{k}\left(\frac{a-1}{1}-\frac{(a-1)^2}{2}+\frac{(a-1)^3}{3}-\frac{(a-1)^4}{4}+\&c.\right)$,
hincque habebitur $k=\frac{a-1}{1}-\frac{(a-1)^2}{2}+\frac{(a-1)^3}{3}-\frac{(a-1)^4}{4}+\&c.$, cujus ideo Seriei infinitæ valor, si ponatur $a=10$, circiter esse debebit $=2,30258$; quanquam difficulter intelligi potest esse $2,30258=\frac{9}{1}-\frac{9^2}{2}+\frac{9^3}{3}-\frac{9^4}{4}+\&c.$, quoniam hujus Seriei termini continuo fiunt majores, neque adeo aliquot terminis sumendis summa vero propinqua haberi potest: cui incommodo mox remedium afferetur.

121. Quoniam igitur est $l(1+x)=\frac{1}{k}\left(\frac{x}{1}-\frac{x^2}{2}+\frac{x^3}{3}-\&c.\right)$, erit, posito x negativo, $l(1-x)=-\frac{1}{k}\left(\frac{x}{1}+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\&c.\right)$. Subtrahatur Series posterior a priori, erit $l(1+x)-l(1-x)=l\frac{1+x}{1-x}=\frac{2}{k}\times\left(\frac{x}{1}+\frac{x^3}{3}+\frac{x^5}{5}+\frac{x^7}{7}+\&c.\right)$. Nunc ponatur $\frac{1+x}{1-x}=a$, ut sit $x=\frac{a-1}{a+1}$, ob $la=1$ erit $k=2\left(\frac{a-1}{a+1}+\frac{(a-1)^3}{3(a+1)^3}+\frac{(a-1)^5}{5(a+1)^5}+\&c.\right)$, ex qua æquatione valor numeri k ex basi a inveniri poterit. Si ergo basis a ponatur $=10$ erit $k=2\left(\frac{9}{11}+\frac{9^3}{3.11^3}+\frac{9^5}{5.11^5}+\frac{9^7}{7.11^7}+\&c.\right)$, cujus Seriei termini sensibiliter decrescunt, ideoque mox valorem pro k satis propinquum exhibent.

122. Quoniam ad systema Logarithmorum condendum basin a pro lubitu accipere licet, ea ita assumi poterit ut fiat $k=1$. Ponamus ergo esse $k=1$, eritque per Seriem supra

LIB. I. (116) inventam, $a = 1 + \frac{1}{1} + \frac{1}{1.2} + \frac{1}{1.2.3} + \frac{1}{1.2.3.4} + \&c.,$ qui termini, ſi in fractiones decimales convertantur atque actu addantur, præbebunt hunc valorem pro $a =$ $2,71828182845904523536028$, cujus ultima adhuc nota veritati eſt conſentanea. Quod ſi jam ex hac baſi Logarithmi conſtruantur, ii vocari ſolent Logarithmi *naturales* ſeu *hyperbolici*, quoniam quadratura hyperbolæ per iſtiuſmodi Logarithmos exprimi poteſt. Ponamus autem brevitatis gratia pro numero hoc $2,718281828459$ &c. conſtanter litteram e, quæ ergo denotabit baſin Logarithmorum naturalium ſeu hyperbolicorum, cui reſpondet valor litteræ $k = 1$; ſive hæc littera e quoque exprimet ſummam hujus Seriei $1 + \frac{1}{1} + \frac{1}{1.2} + \frac{1}{1.2.3} + \frac{1}{1.2.3.4} +$ &c. in infinitum.

123. Logarithmi ergo hyperbolici hanc habebunt proprietatem, ut numeri $1 + \omega$ Logarithmus ſit $= \omega$, denotante ω quantitatem infinite parvam, atque cum ex hac proprietate valor $k = 1$ innoteſcat, omnium numerorum Logarithmi hyperbolici exhiberi poterunt. Erit ergo, poſita e pro numero ſupra invento, perpetuo $e^z = 1 + \frac{z}{1} + \frac{z^2}{1.2} + \frac{z^3}{1.2.3} + \frac{z^4}{1.2.3.4} + \&c.$ ipſi vero Logarithmi hyperbolici ex his Seriebus invenientur, quibus eſt $l(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \&c.$, & $l\,\frac{1+x}{1-x} = \frac{2x}{1} + \frac{2x^3}{3} + \frac{2x^5}{5} + \frac{2x^7}{7} + \frac{2x^9}{9} + \&c.,$ quæ Series vehementer convergunt, ſi pro x ſtatuatur fractio valde parva: ita ex Serie poſteriori facili negotio inveniuntur Logarithmi numerorum unitate non multo majorum. Poſito namque $x = \frac{1}{5}$, erit $l\,\frac{6}{4} = l\,\frac{3}{2} = \frac{2}{1.5} + \frac{2}{3.5^3} + \frac{2}{5.5^5} + \frac{2}{7.5^7} + \&c.$, & facto $x = \frac{1}{7}$, erit $l\,\frac{4}{3} = \frac{2}{1.7} + \frac{2}{3.7^3} + \frac{2}{5.7^5} + \frac{2}{7.7^7}$

$\frac{2}{7 \cdot 7^7} + \&c.$, facto $x = \frac{1}{9}$, erit $l\frac{5}{4} = \frac{2}{1 \cdot 9} + \frac{2}{3 \cdot 9^3} + \frac{2}{5 \cdot 9^5} +$ CAP. VII.
$\frac{2}{7 \cdot 9^7} + \&c.$. Ex Logarithmis vero harum fractionum reperientur Logarithmi numerorum integrorum, erit enim ex natura Logarithmorum $l\frac{3}{2} + l\frac{4}{3} = l2$; tum $l\frac{3}{2} + l2 = l3$; & $2l2 = l4$; porro $l\frac{5}{4} + l4 = l5$; $l2 + l3 = l6$; $3l2 = l8$; $2l3 = l9$; & $l2 + l5 = l10$.

EXEMPLUM.

Hinc Logarithmi hyperbolici numerorum ab 1 usque ad 10 ita se habebunt, ut sit

$l1$	=	0, 00000 00000 00000 00000 00000
$l2$	=	0, 69314 71805 59945 30941 72321
$l3$	=	1, 09861 22886 68109 69139 52452
$l4$	=	1, 38629 43611 19890 61883 44642
$l5$	=	1, 60943 79124 34100 37460 07593
$l6$	=	1, 79175 94692 28055 00081 24773
$l7$	=	1, 94591 01490 55313 30510 54639
$l8$	=	2, 07944 15416 79835 92825 16964
$l9$	=	2, 19722 45773 36219 38279 04905
$l10$	=	2, 30258 50929 94045 68401 79914

Hi scilicet Logarithmi omnes ex superioribus tribus Seriebus sunt deducti, præter $l7$, quem hoc compendio sum assecutus. Posui nimirum in Serie posteriori $x = \frac{1}{99}$ sicque obtinui $l\frac{100}{98} = l\frac{50}{49} = 0,0202027073175194484078230$, qui subtractus a $l50 = 2l5 + l2 = 3,9120230054281460586187508$, relinquit $l49$, cujus semissis dat $l7$.

LIB. I. 124. Ponatur Logarithmus hyperbolicus ipsius $1+x$ seu $l(1+x)=y$; erit $y=\frac{x}{1}-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+$&c.. Sumto autem numero a pro basi Logarithmica, sit numeri ejusdem $1+x$ Logarithmus $=v$; erit, ut vidimus, $v=\frac{1}{k}(x-\frac{xx}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\&c.)=\frac{y}{k}$; hincque $k=\frac{y}{v}$; ex quo commodissime valor ipsius k basi a respondens ita definitur ut sit æqualis cujusvis numeri Logarithmo hyperbolico diviso per Logarithmum ejusdem numeri ex basi a formati. Posito ergo numero hoc $=a$, erit $v=1$, hincque fit $k=$ Logarithmo hyperbolico basis a. In systemate ergo Logarithmorum communium, ubi est $a=10$, erit $k=$ Logarithmo hyperbolico ipsius 10, unde fit $k=2,30258509299404568401799914$, quem valorem jam supra satis prope collegimus. Si ergo singuli Logarithmi hyperbolici per hunc numerum k dividantur, vel, quod eodem redit, multiplicentur per hanc fractionem decimalem 0,43429448190325182765112891, prodibunt Logarithmi vulgares basi $a=10$ convenientes.

125. Cum sit $e^z=1+\frac{z}{1}+\frac{z^2}{1.2}+\frac{z^3}{1.2.3}+$&c., si ponatur $a^y=e^z$, erit, sumtis Logarithmis hyperbolicis, $yla=z$, quia est $le=1$, quo valore loco z substituto, erit $a^y=1+\frac{yla}{1}+\frac{y^2(la)^2}{1.2}+\frac{y^3(la)^3}{1.2.3}+$&c., unde quælibet quantitas exponentialis ope Logarithmorum hyperbolicorum per Seriem infinitam explicari potest. Tum vero, denotante i numerum infinite magnum, tam quantitates exponentiales quam Logarithmi per potestates exponi possunt. Erit enim $e^z=(1+\frac{z}{i})^i$, hincque $a^y=(1+\frac{yla}{i})^i$, deinde pro Logarithmis hyperbolicis habetur $l(1+x)=i((1+x)^{\frac{1}{i}}-1)$. De cetero

tero Logarithmorum hyperbolicorum uſus in calculo integrali fuſius demonſtrabitur. CAP. VII.

CAPUT VIII.

De quantitatibus transſcendentibus ex Circulo ortis.

126. POſt Logarithmos & quantitates exponentiales conſiderari debent Arcus circulares eorumque Sinus & Coſinus, quia non ſolum aliud quantitatum tranſcendentium genus conſtituunt, ſed etiam ex ipſis Logarithmis & exponentialibus, quando imaginariis quantitatibus involvuntur, proveniunt, id quod infra clarius patebit.

Ponamus ergo Radium Circuli ſeu Sinum totum eſſe $= 1$, atque ſatis liquet Peripheriam hujus Circuli in numeris rationalibus exacte exprimi non poſſe, per approximationes autem inventa eſt Semicircumferentia hujus Circuli eſſe $=$ 3, 14159265358979323846264338327950288419716939937510582097494459230781640628620899862803482534211706798214808651327230664709384 46 $+$, pro quo numero, brevitatis ergo, ſcribam π, ita ut ſit $\pi =$ Semicircumferentiæ Circuli, cujus Radius $= 1$, ſeu π erit longitudo Arcus 180 graduum.

127. Denotante z Arcum hujus Circuli quemcunque, cujus Radium perpetuo aſſumo $= 1$; hujus Arcus z conſiderari potiſſimum ſolent Sinus & Coſinus. Sinum autem Arcus z in poſterum hoc modo indicabo, *ſin. A.* z, ſeu tantum *ſin.* z. Coſinum vero hoc modo *coſ. A.* z, ſeu tantum *coſ.* z. Ita, cum π ſit Arcus 180°, erit *ſin.* $0\pi = 0$; *coſ.* $0\pi = 1$; & *ſin.* $\frac{1}{2}\pi = 1$, *coſ.* $\frac{1}{2}\pi = 0$; *ſin.* $\pi = 0$; *coſ.* $\pi = -1$; *ſin.* $\frac{3}{2}\pi = -1$; *coſ.* $\frac{3}{2}\pi = 0$; *ſin.* $2\pi = 0$; & *coſ.* $2\pi = 1$. Omnes ergo Sinus & Coſinus intra limites $+1$ & -1 continen-

LIB. I. tinentur. Erit autem porro $cos. z = sin. (\frac{1}{2}\pi - z)$, & $sin. z = cos. (\frac{1}{2}\pi - z)$, atque $(sin. z)^2 + (cos. z)^2 = 1$. Præter has denominationes notandæ sunt quoque hæ: $tang. z$, quæ denotat Tangentem Arcus z; $cot. z$ Cotangentem Arcus z; constatque esse $tang. z = \frac{sin. z}{cos. z}$ & $cot. z = \frac{cos. z}{sin. z} = \frac{1}{tang. z}$; quæ omnia ex Trigonometria sunt nota.

128. Hinc vero etiam constat si habeantur duo Arcus y & z, fore $sin. (y + z) = sin. y. cos. z + cos. y. sin. z$, & $cos. (y + z) = cos. y. cos. z - sin. y. sin. z$, itemque $sin. (y - z) = sin. y. cos. z - cos. y. sin. z$ & $cos. (y - z) = cos. y. cos. z + sin. y. sin. z$.

Hinc loco y substituendo Arcus $\frac{1}{2}\pi$; π; $\frac{3}{2}\pi$, &c., erit

$sin. (\frac{1}{2}\pi + z) = + cos. z$	$sin. (\frac{1}{2}\pi - z) = + cos. z$
$cos. (\frac{1}{2}\pi + z) = - sin. z$	$cos. (\frac{1}{2}\pi - z) = + sin. z$
$sin. (\pi + z) = - sin. z$	$sin. (\pi - z) = + sin. z$
$cos. (\pi + z) = - cos. z$	$cos. (\pi - z) = - cos. z$
$sin. (\frac{3}{2}\pi + z) = - cos. z$	$sin. (\frac{3}{2}\pi - z) = - cos. z$
$cos. (\frac{3}{2}\pi + z) = + sin. z$	$cos. (\frac{3}{2}\pi - z) = - sin. z$
$sin. (2\pi + z) = + sin. z$	$sin. (2\pi - z) = - sin. z$
$cos. (2\pi + z) = + cos. z$	$cos. (2\pi - z) = + cos. z$

Si

Si ergo n denotet numerum integrum quemcunque, erit

$\sin.\left(\frac{4n+1}{2}\pi+z\right)=+\cos.z$	$\sin.\left(\frac{4n+1}{2}\pi-z\right)=+\cos.z$
$\cos.\left(\frac{4n+1}{2}\pi+z\right)=-\sin.z$	$\cos.\left(\frac{4n+1}{2}\pi-z\right)=-\sin.z$
$\sin.\left(\frac{4n+2}{2}\pi+z\right)=-\sin.z$	$\sin.\left(\frac{4n+2}{2}\pi-z\right)=+\sin.z$
$\cos.\left(\frac{4n+2}{2}\pi+z\right)=-\cos.z$	$\cos.\left(\frac{4n+2}{2}\pi-z\right)=-\cos.z$
$\sin.\left(\frac{4n+3}{2}\pi+z\right)=-\cos.z$	$\sin.\left(\frac{4n+3}{2}\pi-z\right)=-\cos.z$
$\cos.\left(\frac{4n+3}{2}\pi+z\right)=+\sin.z$	$\cos.\left(\frac{4n+3}{2}\pi-z\right)=-\sin.z$
$\sin.\left(\frac{4n+4}{2}\pi+z\right)=+\sin.z$	$\sin.\left(\frac{4n+4}{2}\pi-z\right)=-\sin.z$
$\cos.\left(\frac{4n+4}{2}\pi+z\right)=+\cos.z$	$\cos.\left(\frac{4n+4}{2}\pi-z\right)=+\cos.z$

Quæ formulæ veræ ſunt ſive n ſit numerus affirmativus ſive negativus integer.

129. Sit $\sin.z=p$ & $\cos.z=q$ erit $pp+qq=1$; & $\sin.y=m$; $\cos.y=n$; ut ſit quoque $mm+nn=1$; Arcuum ex his compoſitorum Sinus & Coſinus ita ſe habebunt.

$\sin.\ z=p$	$\cos.\ z=q$
$\sin.(y+z)=mq+np$	$\cos.(y+z)=nq-mp$
$\sin.(2y+z)=2mnq+(nn-mm)p$	$\cos.(2y+z)=(nn-mm)q-2mnp$
$\sin.(3y+z)=(3mn^2-m^3)q+(n^3-3m^2n)p$	$\cos.(3y+z)=(n^3-3m^2n)q-(3mn^2-m^3)p$
&c.	&c.

Arcus iſti z, $y+z$, $2y+z$, $3y+z$, &c., in arithmetica progreſſione progrediuntur; eorum vero tam Sinus quam Coſinus progreſſionem recurrentem conſtituunt, qualis ex denominatore $1-2nx+(mm+nn)xx$ oritur; eſt enim

ſin.

LIB. I. $\sin.(2y+z) = 2n\sin.(y+z) - (mm+nn)\sin.z$ five
$\sin.(2y+z) = 2\cos.y.\sin.(y+z) - (\sin.z)$; atque fimili modo
$\cos.(2y+z) = 2\cos.y.\cos.(y+z) - \cos.z$. Eodem modo erit porro
$\sin.(3y+z) = 2\cos.y.\sin.(2y+z) - \sin.(y+z)$, &
$\cos.(3y+z) = 2\cos.y.\cos.(2y+z) - \cos.(y+z)$, itemque
$\sin.(4y+z) = 2\cos.y.\sin.(3y+z) - \sin.(2y+z)$, &
$\cos.(4y+z) = 2\cos.y.\cos.(3y+z) - \cos.(2y+z)$ &c.
Cujus legis beneficio Arcuum in progreffione arithmetica progredientium tam Sinus quam Cofinus quoufque libuerit expedite formari poffunt.

130. Cum fit $\sin.(y+z) = \sin.y.\cos.z + \cos.y\sin.z$ atque $\sin.(y-z) = \sin.y.\cos.z - \cos.y.\sin.z$, erit his expreffionibus vel addendis vel fubtrahendis:

$$\sin.y.\cos.z = \frac{\sin.(y+z) + \sin.(y-z)}{2}$$

$$\cos.y.\sin.z = \frac{\sin.(y+z) - \sin.(y-z)}{2}$$

Quia porro eft $\cos.(y+z) = \cos.y.\cos.z - \sin.y.\sin.z$, atque $\cos.(y-z) = \cos.y.\cos.z + \sin.y.\sin.z$, erit pari modo

$$\cos.y.\cos.z = \frac{\cos.(y-z) + \cos.(y+z)}{2}$$

$$\sin.y.\sin.z = \frac{\cos.(y-z) - \cos.(y+z)}{2}.$$

Sit $y = z = \frac{1}{2}v$, erit ex his poftremis formulis:

$$(\cos.\tfrac{1}{2}v)^2 = \frac{1+\cos.v}{2}, \& \cos.\tfrac{1}{2}v = \sqrt{\frac{1+\cos.v}{2}}$$

$$(\sin.\tfrac{1}{2}v)^2 = \frac{1-\cos.v}{2}, \& \sin.\tfrac{1}{2}v = \sqrt{\frac{1-\cos.v}{2}}$$

unde, ex dato Cofinu cujufque anguli reperiuntur ejus femiffis Sinus & Cofinus.

131. Ponatur Arcus $y+z=a$, & $y-z=b$; erit $y = \frac{a+b}{2}$ & $z = \frac{a-b}{2}$, quibus in fuperioribus formulis fubftitutis

tutis, habebuntur hæ æquationes, tanquam totidem Theoremata. CAP. VIII.

$$fin.\,a + fin.\,b = 2\,fin.\,\frac{a+b}{2}\,cof.\,\frac{a-b}{2}$$

$$fin.\,a - fin.\,b = 2\,cof.\,\frac{a+b}{2}\,fin.\,\frac{a-b}{2}$$

$$cof.\,a + cof.\,b = 2\,cof.\,\frac{a+b}{2}\,cof.\,\frac{a-b}{2}$$

$$cof.\,b - cof.\,a = 2\,fin.\,\frac{a+b}{2}\,fin.\,\frac{a-b}{2}.$$

ex his porro nafcuntur, ope divifionis, hæc Theoremata

$$\frac{fin.\,a + fin.\,b}{fin.\,a - fin.\,b} = tang.\,\frac{a+b}{2}.\,cot.\,\frac{a-b}{2} = \frac{tang.\,\frac{a+b}{2}}{tang.\,\frac{a-b}{2}}$$

$$\frac{fin.\,a + fin.\,b}{cof.\,a + cof.\,b} = tang.\,\frac{a+b}{2}$$

$$\frac{fin.\,a + fin.\,a}{cof.\,b - cof.\,a} = cot.\,\frac{a-b}{2}$$

$$\frac{fin.\,a - fin.\,b}{cof.\,a + cof.\,b} = tang.\,\frac{a-b}{2}$$

$$\frac{fin.\,a - fin.\,a}{cof.\,b - cof.\,a} = cot.\,\frac{a+b}{2}$$

$$\frac{cof.\,a + cof.\,b}{cof.\,b - cof.\,a} = cot.\,\frac{a+b}{2}.\,cot.\,\frac{a-b}{2}$$

Ex his denique deducuntur ifta Theoremata

$$\frac{fin.\,a + fin.\,b}{cof.\,a + cof.\,b} = \frac{cof.\,b - cof.\,a}{fin.\,a - fin.\,b};$$

$$\frac{fin.\,a + fin.\,b}{fin.\,a - fin.\,b} \times \frac{cof.\,a + cof.\,b}{cof.\,b - cof.\,a} = \left(cot.\,\frac{a-b}{2}\right)^2.$$

$$\frac{fin.\,a + fin.\,b}{fin.\,a - fin.\,b} \times \frac{cof.\,b - cof.\,a}{cof.\,a + cof.\,b} = \left(tang.\,\frac{a+b}{2}\right)^2$$

132. Cum fit $(fin.\,z)^2 + (cof.\,z)^2 = 1$ erit, Factoribus fumendis, $(cof.\,z + \sqrt{-1}.\,fin.\,z)(cof.\,z - \sqrt{-1}.\,fin.\,z) = 1$; qui Factores, etfi imaginarii, tamen ingentem præftant ufum in Arcubus combinandis & multiplicandis. Quæratur enim productum horum Factorum $(cof.\,z + \sqrt{-1}.\,fin.\,z)(cof.\,y + \sqrt{-1}.\,fin.\,y)$ ac reperietur $cof.\,y.\,cof.\,z - fin.\,y.\,fin.\,z + (cof.\,y.\,fin.\,z + fin.\,y.\,cof.\,z)$

LIB. I. $\sqrt{-1}$. Cum autem sit $cos.y.cos.z - sin.y.sin.z = cos.(y+z)$ & $cos.y.sin.z + sin.y.cos.z = sin.(y+z)$ erit hoc productum $(cos.y + \sqrt{-1}.sin.y)(cos.z + \sqrt{-1}.sin.z) = cos.(y+z) + \sqrt{-1}.sin.(y+z)$

& simili modo

$(cos.y - \sqrt{-1}.sin.y)(cos.z - \sqrt{-1}.sin.z) = cos.(y+z) - \sqrt{-1}.sin.(y+z)$

item

$(cos.x \pm \sqrt{-1}.sin.x)(cos.y \pm \sqrt{-1}.sin.y)(cos.z \pm \sqrt{-1}.sin.z) = cos.(x+y+z) \pm \sqrt{-1}.sin.(x+y+z)$.

133. Hinc itaque sequitur fore $(cos.z \pm \sqrt{-1}.sin z)^2 = cos.2z \pm \sqrt{-1}.sin.2z$, & $(cos.z \pm \sqrt{-1}.sin.z)^3 = cos.3z \pm \sqrt{-1}.sin.3z$.

ideoque generaliter erit $(cos.z \pm \sqrt{-1}.sin.z)^n = cos.nz \pm \sqrt{-1}.sin.nz$:

Unde, ob signorum ambiguitatem, erit

$$cos.nz = \left(\frac{(cos.z + \sqrt{-1}.sin.z)^n + (cos.z - \sqrt{-1}.sin.z)^n}{2}\right) \&$$

$$sin.nz = \frac{(cos.z + \sqrt{-1}.sin.z)^n - (cos.z - \sqrt{-1}.sin.z)^n}{2\sqrt{-1}}$$

Evolutis ergo binomiis hisce erit per Series:

$$cos.nz = (cos.z)^n - \frac{n(n-1)}{1.2}(cos.z)^{n-2}(sin.z)^2 + \frac{n(n-1)(n-2)(n-3)}{1.2.3.4}(cos.z)^{n-4}(sin.z)^4 - \frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{1.2.3.4.5.6}(cos.z)^{n-6}(sin.z)^6 + \&c., \&$$

$$sin.nz = \frac{n}{1}(cos.z)^{n-1}sin.z - \frac{n(n-1)(n-2)}{1.2.3}(cos.z)^{n-3}(sin.z)^3 + \frac{n(n-1)(n-2)(n-3)(n-4)}{1.2.3.4.5}(cos.z)^{n-5}(sin.z)^5 - \&c.$$

134.

134. Sit Arcus z infinite parvus, erit $\mathit{fin.}\, z = z$ & $\mathit{cof.}\, z = 1$: fit autem n numerus infinite magnus, ut fit Arcus nz finitæ magnitudinis, puta, $nz = v$; ob $\mathit{fin.}\, z = z = \frac{v}{n}$ erit CAP. VIII.

$$\mathit{cof.}\, v = 1 - \frac{v^2}{1.2} + \frac{v^4}{1.2.3.4} - \frac{v^6}{1.2.3.4.5.6} + \&c., \&$$

$$\mathit{fin.}\, v = v - \frac{v^3}{1.2.3} + \frac{v^5}{1.2.3.4.5} - \frac{v^7}{1.2.3.4.5.6.7} + \&c.$$

Dato ergo Arcu v, ope harum Serierum ejus Sinus & Cofinus inveniri poterunt; quarum formularum ufus quo magis pateat, ponamus Arcum v effe ad quadrantem, feu 90°, ut m ad n, feu effe $v = \frac{m}{n}.\frac{\pi}{2}$; Quia nunc valor ipfius π conftat, fi is ubique fubftituatur, prodibit

$$\mathit{fin.}\, A. \frac{m}{n} 90° =$$

$$+ \frac{m}{n}.\ 1,570796326794896619231321691 6$$
$$- \frac{m^3}{n^3}.\ 0,6459640975062462536557565636$$
$$+ \frac{m^5}{n^5}.\ 0,0796926262461670451205055488$$
$$- \frac{m^7}{n^7}.\ 0,0046817541353186881006854632$$
$$+ \frac{m^9}{n^9}.\ 0,0001604411847873598218726605$$
$$- \frac{m^{11}}{n^{11}}.\ 0,0000035988432352120853404580$$
$$+ \frac{m^{13}}{n^{13}}.\ 0,0000000569217292196792681171$$
$$- \frac{m^{15}}{n^{15}}.\ 0,0000000006688035109811467224$$
$$+ \frac{m^{17}}{n^{17}}.\ 0,0000000000060669357311061950$$

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$- \frac{m^{19}}{n^{19}}$. 0, 0000000000000437706546731370

$+ \frac{m^{21}}{n^{21}}$. 0, 0000000000000002571422892856

$- \frac{m^{23}}{n^{23}}$. 0, 0000000000000000012538995403

$+ \frac{m^{25}}{n^{25}}$. 0, 0000000000000000000051564550

$- \frac{m^{27}}{n^{27}}$. 0, 0000000000000000000000181239

$+ \frac{m^{29}}{n^{29}}$. 0, 0000000000000000000000000549

atque *cos.* A. $\frac{m}{n} 90^{\circ} =$

$+$ 1, 0000000000000000000000000000

$- \frac{m^{2}}{n^{2}}$. 1, 2337005501361698273543113745

$+ \frac{m^{4}}{n^{4}}$. 0, 2536695079010480136365633659

$- \frac{m^{6}}{n^{6}}$. 0, 0208634807633529608730516364

$+ \frac{m^{8}}{n^{8}}$. 0, 0009192602748394265802417158

$- \frac{m^{10}}{n^{10}}$. 0, 0000251020423730606054810526

$+ \frac{m^{12}}{n^{12}}$. 0, 0000047108747778818171503665

$- \frac{m^{14}}{n^{14}}$. 0, 0000000063866030837918522408

$+ \frac{m^{16}}{n^{16}}$. 0, 0000000000656596311497947230

$- \frac{m^{18}}{n^{18}}$. 0, 0000000000005294400200734620

$+ \frac{m^{20}}{n^{20}}$. 0, 0000000000000034377391790981

$- \frac{m^{22}}{n^{22}}$. 0, 0000000000000000183599165212

+

$$+\frac{m^{24}}{n^{24}}.\ 0,00000000000000000000820675327$$
$$-\frac{m^{26}}{n^{26}}.\ 0,00000000000000000000003115285$$
$$+\frac{m^{28}}{n^{28}}.\ 0,00000000000000000000000010165$$
$$-\frac{m^{30}}{n^{30}}.\ 0,00000000000000000000000000026$$

Cum igitur ſufficiat Sinus & Coſinus angulorum ad 45° noſſe, fractio $\frac{m}{n}$ ſemper minor erit quam $\frac{1}{2}$, hincque etiam ob Poteſtates fractionis $\frac{m}{n}$, Series exhibitæ maxime convergent, ita ut plerumque aliquot tantum termini ſufficiant, præcipue, ſi Sinus & Coſinus non ad tot figuras deſiderentur.

135. Inventis Sinibus & Coſinibus inveniri quidem poſſunt Tangentes & Cotangentes, per analogias conſuetas, at quia in hujuſmodi ingentibus numeris multiplicatio & diviſio vehementer eſt moleſta, peculiari modo eas exprimere convenit. Erit ergo

$$\text{tang.}\ v = \frac{\text{ſin.}\ v}{\text{coſ.}\ v} = \frac{v - \frac{v^3}{1.2.3} + \frac{v^5}{1.2.3.4.5} - \frac{v^7}{1.2.3\ldots.7} + \&c.}{1 - \frac{v^2}{1.2} + \frac{v^4}{1.2.3.4} - \frac{v^6}{1.2.0\ldots.6} + \&c.}$$

$$\&\ \text{cot.}\ v = \frac{\text{coſ.}\ v}{\text{ſin.}\ v} = \frac{1 - \frac{v^2}{1.2} + \frac{v^4}{1.2.3.4} - \frac{v^6}{1.2.3\ldots.6} + \&c.}{v - \frac{v^3}{1.2.3} + \frac{v^5}{1.2.3.4.5} - \frac{v^7}{1.2.3\ldots.7} + \&c.}$$

LIB. I. si jam sit Arcus $v = \frac{m}{n}\, 90^\circ$ erit eodem modo quo ante

$tang.\ A. \frac{m}{n}\, 90^\circ =$	$cot.\ A. \frac{m}{n}\, 90^\circ =$
$+ \frac{2mn}{nn-mm} \cdot 0{,}6366197723675$	$+ \frac{n}{m} \cdot 0{,}6366197723675$
$+ \frac{m}{n} \cdot 0{,}2975567820597$	$- \frac{4mn}{4nn-mm} \cdot 0{,}3183098861837$
$+ \frac{m^3}{n^3} \cdot 0{,}0186886502773$	$- \frac{m}{n} \cdot 0{,}2052888894145$
$+ \frac{m^5}{n^5} \cdot 0{,}0018424752034$	$- \frac{m^3}{n^3} \cdot 0{,}0065510747882$
$+ \frac{m^7}{n^7} \cdot 0{,}0001975800714$	$- \frac{m^5}{n^5} \cdot 0{,}0003450292554$
$+ \frac{m^9}{n^9} \cdot 0{,}0000216977245$	$- \frac{m^7}{n^7} \cdot 0{,}0000202791060$
$+ \frac{m^{11}}{n^{11}} \cdot 0{,}0000024011370$	$- \frac{m^9}{n^9} \cdot 0{,}0000012366527$
$+ \frac{m^{13}}{n^{13}} \cdot 0{,}0000002664132$	$- \frac{m^{11}}{n^{11}} \cdot 0{,}0000000764959$
$+ \frac{m^{15}}{n^{15}} \cdot 0{,}0000000295864$	$- \frac{m^{13}}{n^{13}} \cdot 0{,}0000000047597$
$+ \frac{m^{17}}{n^{17}} \cdot 0{,}0000000032867$	$- \frac{m^{15}}{n^{15}} \cdot 0{,}0000000002969$
$+ \frac{m^{19}}{n^{19}} \cdot 0{,}0000000003651$	$- \frac{m^{17}}{n^{17}} \cdot 0{,}0000000000185$
$+ \frac{m^{21}}{n^{21}} \cdot 0{,}0000000000405$	$- \frac{m^{19}}{n^{19}} \cdot 0{,}0000000000011$
$+ \frac{m^{23}}{n^{23}} \cdot 0{,}0000000000045$	
$+ \frac{m^{25}}{n^{25}} \cdot 0{,}0000000000005$	

quarum Serierum ratio infra fusius exponetur.

136. Ex superioribus quidem constat, si cogniti fuerint omnium angulorum semirecto minorum Sinus & Cosinus, inde simul omnium angulorum majorum Sinus & Cosinus haberi. Verum si tantum angulorum 30° minorum habeantur Sinus

Sinus & Cosinus, ex iis, per solam additionem & subtractionem, omnium angulorum majorum Sinus & Cosinus inveniri possunt. Cum enim sit $\mathit{sin}.\ 30° = \frac{1}{2}$, erit, posito $y = 30°$ ex (130) $\mathit{cos}.\ z = \mathit{sin}.\ (30 + z) + \mathit{sin}.\ (30 - z)$; & $\mathit{sin}.\ z = \mathit{cos}.\ (30 - z) - \mathit{cos}.\ (30 + z)$, ideoque ex Sinibus & Cosinibus angulorum z & $30 - z$, reperiuntur $\mathit{sin}.\ (30 + z) = \mathit{cos}.\ z - \mathit{sin}.\ (30 - z)$ & $\mathit{cos}.\ (30 + z) = \mathit{cos}.\ (30 - z) - \mathit{sin}.\ z$, unde Sinus & Cosinus angulorum a 30° ad 60°, hincque omnes majores definiuntur. CAP. VIII.

137. In Tangentibus & Cotangentibus simile subsidium usu venit. Cum enim sit $\mathit{tang}.\ (a + b) = \frac{\mathit{tang}.\ a + \mathit{tang}.\ b}{1 - \mathit{tang}.\ a.\ \mathit{tang}.\ b}$, erit $\mathit{tang}.\ 2a = \frac{2\ \mathit{tang}.\ a}{1 - \mathit{tang}.\ a.\ \mathit{tang}.\ a}$, & $\mathit{cot}.\ 2a = \frac{\mathit{cot}.\ a - \mathit{tang}.\ a}{2}$ unde ex Tangentibus & Cotangentibus Arcuum 30° minorum inveniuntur Cotangentes usque ad 60°.

Sit jam $a = 30 - b$ erit $2a = 60 - 2b$ & $\mathit{cot}.\ 2a = \mathit{tang}.\ (30 + 2b)$; erit ergo $\mathit{tang}.\ (30 + 2b) = \frac{\mathit{cot}.\ (30 - b) - \mathit{tang}.\ (30 - b)}{2}$, unde etiam Tangentes Arcuum 30° majorum obtinentur.

Secantes autem & Cosecantes ex Tangentibus per solam subtractionem inveniuntur; est enim $\mathit{cosec}.\ z = \mathit{cot}.\ \frac{1}{2} z - \mathit{cot}.\ z$, & hinc $\mathit{sec}.\ z = \mathit{cot}.\ (45° - \frac{1}{2} z) - \mathit{tang}.\ z$. Ex his ergo luculenter perspicitur, quomodo canones Sinuum construi potuerint.

138. Ponatur denuo in formulis §. 133, Arcus z infinite parvus, & sit n numerus infinite magnus i, ut iz obtineat valorem finitum v. Erit ergo $nz = v$; & $z = \frac{v}{i}$, unde $\mathit{sin}.\ z = \frac{v}{i}$ & $\mathit{cos}.\ z = 1$; his substitutis fit $\mathit{cos}.\ v =$

(1 +

LIB. I.

$$\frac{\left(1+\frac{v\sqrt{-1}}{i}\right)^i+\left(1-\frac{v\sqrt{-1}}{i}\right)^i}{2}$$; atque $fin.\, v =$

$$\frac{\left(1+\frac{v\sqrt{-1}}{i}\right)^i-\left(1-\frac{v\sqrt{-1}}{i}\right)^i}{2\sqrt{-1}}.$$ In Capite autem præcedente vidimus esse $\left(1+\frac{z}{i}\right)^i = e^z$, denotante e basin Logarithmorum hyperbolicorum: scripto ergo pro z partim $+v\sqrt{-1}$ partim $-v\sqrt{-1}$ erit $cof.\, v =$
$$\frac{e^{+v\sqrt{-1}}+e^{-v\sqrt{-1}}}{2} \;\&\; fin.\, v = \frac{e^{+v\sqrt{-1}}-e^{-v\sqrt{-1}}}{2\sqrt{-1}}.$$
Ex quibus intelligitur quomodo quantitates exponentiales imaginariæ ad Sinus & Cosinus Arcuum realium reducantur. Erit vero $e^{+v\sqrt{-1}} = cof.\, v + \sqrt{-1}.\, fin.\, v$ & $e^{-v\sqrt{-1}} = cof.\, v - \sqrt{-1}.\, fin.\, v$.

139. Sit jam in iisdem formulis §. 130. n numerus infinite parvus, seu $n = \frac{1}{i}$, existente i numero infinite magno, erit $cof.\, nz = cof.\, \frac{z}{i} = 1$ & $fin.\, nz = fin.\, \frac{z}{i} = \frac{z}{i}$; Arcus enim evanescentis $\frac{z}{i}$ Sinus est ipsi æqualis, Cosinus vero $= 1$. His positis habebitur

$$1 = \frac{(cof.\, z + \sqrt{-1}.\, fin.\, z)^{\frac{1}{i}} + (cof.\, z - \sqrt{-1}.\, fin.\, z)^{\frac{1}{i}}}{2} \;\&$$

$$\frac{z}{i} = \frac{(cof.\, z + \sqrt{-1}.\, fin.\, z)^{\frac{1}{i}} - (cof.\, z - \sqrt{-1}.\, fin.\, z)^{\frac{1}{i}}}{2\sqrt{-1}}.$$ Sumendis autem Logarithmis hyperbolicis supra (125) ostendimus esse $l(1+x) = i(1+x)^{\frac{1}{i}} - i$, seu $y^{\frac{1}{i}} = 1 + \frac{1}{i} ly$, posito

posito y loco $1+x$. Nunc igitur, posito loco y, partim $cos.z + \sqrt{-1}.\,sin.\,z$ partim $cos.\,z - \sqrt{-1}.\,sin.\,z$, prodibit $1 =$ CAP. VIII.

$$\frac{1+\frac{1}{i}l(cos.z+\sqrt{-1}.\,sin.\,z)+1+\frac{1}{i}l(cos.z-\sqrt{-1}.\,sin.z)}{2}$$

$= 1$, ob Logarithmos evanescentes; ita ut hinc nil sequatur. Altera vero æquatio pro Sinu suppeditat:

$$\frac{z}{i} = \frac{\frac{1}{i}l(cos.z+\sqrt{-1}.\,sin.\,z) - \frac{1}{i}l(cos.z-\sqrt{-1}.\,sin.\,z)}{2\sqrt{-1}}$$

ideoque $z = \frac{1}{2\sqrt{-1}} \times l\frac{cos.\,z + \sqrt{-1}.\,sin.\,z}{cos.\,z - \sqrt{-1}.\,sin.\,z}$, unde patet quemadmodum Logarithmi imaginarii ad Arcus circulares revocentur.

140. Cum sit $\frac{sin.z}{cos.z} = tang.\,z$, Arcus z per suam Tangentem ita exprimetur ut sit $z = \frac{1}{2\sqrt{-1}}\,l\frac{1+\sqrt{-1}.\,tang.z}{1-\sqrt{-1}.\,tang.z}$. Supra vero (§. 123) vidimus esse $l\frac{1+x}{1-x} = \frac{2x}{1} + \frac{2x^3}{3} + \frac{2x^5}{5} + \frac{2x^7}{7} +$ &c.. Posito ergo $x = \sqrt{-1}.\,tang.z$, fiet $z = \frac{tang.\,z}{1} - \frac{(tang.\,z)^3}{3} + \frac{(tang.\,z)^5}{5} - \frac{(tang.\,z)^7}{7} +$ &c.. Si ergo ponamus $tang.\,z = t$, ut sit z Arcus, cujus Tangens est t, quem ita indicabimus $A.\,tang.\,t$, ideoque erit $z = A\,tang.\,t$. Cognita ergo Tangente t erit Arcus respondens $z = \frac{t}{1} - \frac{t^3}{3} + \frac{t^5}{5} - \frac{t^7}{7} + \frac{t^9}{9} -$ &c.. Cum igitur, si Tangens t æquetur Radio 1, fiat Arcus $z =$ Arcui 45° seu $z = \frac{\pi}{4}$, erit $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} +$ &c., quæ est Series a LEIBNITZIO primum producta, ad valorem Peripheriæ Circuli exprimendum.

141. Quo autem ex hujusmodi Serie longitudo Arcus Cir-

culi expedite definiri possit, perspicuum est pro Tangente t fractionem satis parvam substitui debere. Sic ope hujus Seriei facile reperietur longitudo Arcus z, cujus Tangens t æquetur $\frac{1}{10}$, foret enim iste Arcus $z = \frac{1}{10} - \frac{1}{3000} + \frac{1}{500000} -$ &c., cujus Seriei valor per approximationem non difficulter in fractione decimali exhiberetur. At vero ex tali Arcu cognito nihil pro longitudine totius Circumferentiæ concludere licebit, cum ratio, quam Arcus, cujus Tangens est $= \frac{1}{10}$, ad totam Peripheriam tenet, non sit assignabilis. Hanc ob rem ad Peripheriam indagandam, ejusmodi Arcus quæri debet, qui sit simul pars aliquota Peripheriæ, & cujus Tangens satis exigua commode exprimi queat. Ad hoc ergo institutum sumi solet Arcus 30°. cujus Tangens est $= \frac{1}{\sqrt{3}}$, quia minorum Arcuum cum Peripheria commensurabilium Tangentes nimis fiunt irrationales. Quare, ob Arcum $30^\circ = \frac{\pi}{6}$, erit $\frac{\pi}{6} = \frac{1}{\sqrt{3}} - \frac{1}{3 \cdot 3\sqrt{3}} + \frac{1}{5 \cdot 3^2 \sqrt{3}} -$ &c., & $\pi = \frac{2\sqrt{3}}{1} - \frac{2\sqrt{3}}{3 \cdot 3} + \frac{2\sqrt{3}}{5 \cdot 3^2} - \frac{2\sqrt{3}}{7 \cdot 3^3} +$ &c., cujus Seriei ope valor ipsius π ante exhibitus incredibili labore fuit determinatus.

142. Hic autem labor eo major est, quod primum singuli termini sint irrationales, tum vero quisque tantum, circiter, triplo sit minor quam præcedens. Huic itaque incommodo ita occurri poterit: sumatur Arcus 45° seu $\frac{\pi}{4}$ cujus valor, etsi per Seriem vix convergentem $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} +$ &c., exprimitur, tamen is retineatur, atque in duos Arcus a & b dispertiatur ut sit $a + b = \frac{\pi}{4} = 45^\circ$. Cum igitur sit $tang. (a+b) = 1 = \frac{tang.\, a + tang.\, b}{1 - tang.\, a.\, tang.\, b}$ erit $1 - tang.a.\, tang.b =$

tang. a

$tang.\, a + tang.\, b$ & $tang.\, b = \frac{1 - tang.\, a}{1 + tang.\, a}$. Sit nunc $tang.\, a = \frac{1}{2}$, erit $tang.\, b = \frac{1}{3}$, hinc uterque Arcus a & b per Seriem rationalem multo magis, quam ſuperior, convergentem exprimetur, eorumque ſumma dabit valorem Arcus $\frac{\pi}{4}$; hinc itaque erit

$$\pi = 4 \left\{ \begin{array}{l} \frac{1}{1.2} - \frac{1}{3.2^3} + \frac{1}{5.2^5} - \frac{1}{7.2^7} + \frac{1}{9.2^9} - \&c. \\ \frac{1}{1.3} - \frac{1}{3.3^3} + \frac{1}{5.3^5} - \frac{1}{7.3^7} + \frac{1}{9.3^9} - \&c. \end{array} \right\}$$

hoc ergo modo multo expeditius longitudo ſemicircumferentiæ π inveniri potuiſſet, quam quidem factum eſt ope Seriei ante commemoratæ.

CAPUT IX.

De inveſtigatione Factorum trinomialium.

143. Quemadmodum Factores ſimplices cujuſque Functionis integræ inveniri oporteat, ſupra quidem oſtendimus hoc fieri per reſolutionem æquationum. Si enim propoſita ſit Functio quæcunque integra $\alpha + \beta z + \gamma z^2 + \delta z^3 + \epsilon z^4 +$ &c., hujusque quærantur Factores ſimplices formæ $p - qz$, manifeſtum eſt, ſi $p - qz$ fuerit Factor Functionis $\alpha + \beta z + \gamma z^2 +$ &c., tum, poſito $z = \frac{p}{q}$, quo caſu Factor $p - qz$ fit $= 0$, etiam ipſam Functionem propoſitam evaneſcere debere. Hinc $p - qz$ erit Factor vel diviſor Functionis $\alpha + \beta z + \gamma z^2 + \delta z^3 + \epsilon z^4 +$ &c., ſequitur fore hanc

LIB. I. expressionem $\alpha + \frac{\beta p}{q} + \frac{\gamma p^2}{q^2} + \frac{\delta p^3}{q^3} + \frac{\epsilon p^4}{q^4} + \&c. = 0$. Unde vicissim, si omnes radices $\frac{p}{q}$ hujus æquationis eruantur, singulæ dabunt totidem Factores simplices Functionis integræ propositæ $\alpha + \beta z + \gamma z^2 + \delta z^3 + \&c.$, nempe $p - qz$. Patet autem simul numerum Factorum hujusmodi simplicium ex maxima Potestate ipsius z definiri.

144. Hoc autem modo plerumque difficulter Factores imaginarii eruuntur, quamobrem hoc Capite methodum peculiarem tradam, cujus ope sæpenumero Factores simplices imaginarii inveniri queant. Quoniam vero Factores simplices imaginarii ita sunt comparati, ut binorum productum fiat reale, hos ipsos Factores imaginarios reperiemus, si Factores investigemus duplices, seu hujus formæ $p - qz + rzz$, reales quidem, sed quorum Factores simplices sint imaginarii. Quod si enim Functionis $\alpha + \beta z + \gamma z^2 + \delta z^3 + \&c.$, constent omnes Factores reales duplices hujus formæ trinomialis $p - qz + rzz$, simul omnes Factores imaginarii habebuntur.

145. Trinomium autem $p - qz + rzz$ Factores simplices habebit imaginarios, si fuerit $4pr > qq$ seu $\frac{q}{2\sqrt{pr}} < 1$. Cum igitur Sinus & Cosinus Angulorum sint unitate minores, formula $p - qz + rzz$ Factores simplices habebit imaginarios si fuerit $\frac{q}{2\sqrt{pr}}$ = Sinui vel Cosinui cujuspiam Anguli. Sit ergo $\frac{q}{2\sqrt{pr}} = cos. A\phi$, seu $q = 2\sqrt{pr}. cos. \phi$, atque trinomium $p - qz + rzz$ continebit Factores simplices imaginarios. Ne autem irrationalitas molestiam facessat, assumo hanc formam $pp - 2pqz. cos. \phi + qqzz$, cujus Factores simplices imaginarii erunt hi, $qz - p(cos. \phi + \sqrt{-1}. sin. \phi)$ & $qz - p(cos. \phi - \sqrt{-1}. sin. \phi)$. Ubi quidem patet si fuerit $cos. \phi = \pm 1$, tum ambos Factores, ob $sin. \phi = 0$, fieri æquales & reales.

146. Proposita ergo Functione integra $\alpha + \beta z + \gamma z^2 + \delta z^3 + \&c.$,

* $\cos\varphi^2 + \sin\varphi^2 = (\cos.\varphi + \sin\varphi.\sqrt{-1}) \times (\cos\varphi - \sin\varphi.\sqrt{-1}) = 1$

En resolvant l'equation $qqzz - 2pqz\cos\varphi = -pp$, on trouve

$qz - p.\cos.\varphi = \pm\sqrt{(pp \times (\cos\varphi^2 - 1))} = \pm\sqrt{(pp(\cos.\varphi^2 - \cos.\varphi^2 - \sin.\varphi^2))} = \pm\sqrt{-pp.\sin}$

Donc $qz - p\cos\varphi \mp p.\sin.\varphi.\sqrt{-1} = qz - p(\cos.\varphi \mp \sin\varphi.\sqrt{-1})$ dont les

deux facteurs de $qqzz - 2pqz.\cos\varphi$ ...

&c. ; ejus Factores simplices imaginarii eruentur, si determinentur litteræ p & q cum Angulo ϕ, ut hoc trinomium $pp - 2pqz.\cos.\phi + qqzz$ fiat Factor Functionis. Tum enim simul inerunt isti Factores simplices imaginarii $qz - p(\cos.\phi + \sqrt{-1}.\sin.\phi)$ & $qz - p(\cos.\phi - \sqrt{-1}.\sin.\phi)$. Quam ob rem Functio proposita evanescet, si ponatur tam $z = \frac{p}{q} \times (\cos.\phi + \sqrt{-1}.\sin.\phi)$ quam $z = \frac{p}{q}(\cos.\phi - \sqrt{-1}.\sin.\phi)$. Hinc, facta substitutione utraque, duplex nascetur æquatio, ex quibus tam fractio $\frac{p}{q}$ quam Arcus ϕ definiri poterunt.

147. Hæ autem substitutiones loco z faciendæ, etiamsi primo intuitu difficiles videantur, tamen per ea, quæ in Capite præcedente sunt tradita, satis expedite absolventur. Cum enim fuerit ostensum esse $(\cos.\phi \pm \sqrt{-1}.\sin.\phi)^n = \cos.n\phi \pm \sqrt{-1} \times \sin.n\phi$, sequentes formulæ loco singularum ipsius z Potestatum habebuntur substituendæ.

pro priori Factore	pro altero Factore
$z = \frac{p}{q}(\cos.\phi + \sqrt{-1}.\sin.\phi)$	$z = \frac{p}{q}(\cos.\phi - \sqrt{-1}.\sin.\phi)$
$z^2 = \frac{p^2}{q^2}(\cos.2\phi + \sqrt{-1}.\sin.2\phi)$	$z^2 = \frac{p^2}{q^2}(\cos.2\phi - \sqrt{-1}.\sin.2\phi)$
$z^3 = \frac{p^3}{q^3}(\cos.3\phi + \sqrt{-1}.\sin.3\phi)$	$z^3 = \frac{p^3}{q^3}(\cos.3\phi - \sqrt{-1}.\sin.3\phi)$
$z^4 = \frac{p^4}{q^4}(\cos.4\phi + \sqrt{-1}.\sin.4\phi)$	$z^4 = \frac{p^4}{q^4}(\cos.4\phi - \sqrt{-1}.\sin.4\phi)$
&c.	&c.

Ponatur brevitatis gratia $\frac{p}{q} = r$, factaque substitutione sequentes duæ nascentur æquationes.

$$0 = \left\{ \begin{array}{l} \alpha + \beta r.\cos.\phi + \gamma r^2.\cos.2\phi + \delta r^3.\cos.3\phi + \&c. \\ \quad + \beta r\sqrt{-1}.\sin.\phi + \gamma r^2\sqrt{-1}.\sin.2\phi + \delta r^3\sqrt{-1}.\sin.3\phi + \&c. \end{array} \right\}$$

$$0 = \left\{ \begin{array}{l} \alpha + \beta r.\cos.\phi + \gamma r^2.\cos.2\phi + \delta r^3.\cos.3\phi + \&c. \\ \quad - \beta r\sqrt{-1}.\sin.\phi - \gamma r^2\sqrt{-1}.\sin.2\phi - \delta r^3\sqrt{-1}.\sin.3\phi - \&c. \end{array} \right\}$$

148. Quod si hæ duæ æquationes invicem addantur & subtrahantur, & posteriori casu per $2\sqrt{-1}$ dividantur, prodibunt hæ duæ æquationes reales:

$$0 = \alpha + \beta r. cos. \phi + \gamma r^2. cos. 2\phi + \delta r^3. cos. 3\phi + \&c.$$
$$0 = \beta r. sin. \phi + \gamma r^2. sin. 2\phi + \delta r^3. sin. 3\phi + \&c.$$

quæ statim ex forma Functionis propositæ

$$\alpha + \beta z + \gamma z^2 + \delta z^3 + \epsilon z^4 + \&c.$$

formari possunt, ponendo primum pro unaquaque ipsius z potestate $z^n = r^n\ cos. n\phi$, deinceps $z^n = r^n\ sin. n\phi$. Sic enim ob $sin. 0\phi = 0$ & $cos. 0\phi = 1$., pro z^0 seu 1 in termino constanti priori casu ponitur 1, posteriori autem 0. Si ergo ex his duabus æquationibus definiantur incognitæ r & ϕ, ob $r = \frac{p}{q}$, habebitur Factor Functionis trinomialis $pp - 2pqz. cos. \phi + qqzz$, duos Factores simplices imaginarios involvens.

149. Si æquatio prior multiplicetur per $sin. m\phi$; posterior per $cos. m\phi$, atque producta vel addantur vel subtrahantur, prodibunt istæ duæ æquationes:

$$0 = \alpha. sin. m\phi + \beta r. sin. (m+1)\phi + \gamma r^2. sin. (m+2)\phi + \delta r^3. sin. (m+3)\phi + \&c.$$
$$0 = \alpha. sin. m\phi + \beta r. sin. (m+1)\phi + \gamma r^2. sin. (m-2)\phi + \delta r^3. sin. (m-3)\phi + \&c.$$

Sin autem æquatio prior multiplicetur per $cos. m\phi$ & posterior per $sin. m\phi$, per additionem ac subtractionem sequentes emergent æquationes.

$$0 = \alpha. cos. m\phi + \beta r. cos. (m-1)\phi + \gamma r^2. cos. (m-2)\phi + \delta r^3. cos. (m-3)\phi + \&c.$$
$$0 = \alpha. cos. m\phi + \beta r. cos. (m+1)\phi + \gamma r^2. cos. (m+2)\phi + \delta r^3. cos. (m+3)\phi + \&c.$$

Hujus-

Hujusmodi ergo duæ æquationes quæcunque conjunctæ determinabunt incognitas r & Φ; quod cum plerumque pluribus modis fieri possit, simul plures Factores trinomiales obtinentur, iique adeo omnes, quos Functio proposita in se complectitur.

150. Quo usus harum regularum clarius appareat, quarumdam Functionum sæpius occurrentium Factores trinomiales hic indagabimus, ut eos, quoties occasio postulaverit, hinc depromere liceat. Sit itaque proposita hæc Functio $a^n + z^n$, cujus Factores trinomiales formæ $pp - 2pqz.cof.\Phi + qqzz$ determinari oporteat; posito ergo $r = \frac{p}{q}$, habebuntur hæ duæ æquationes: $o = a^n + r^n.cof.n\Phi$ & $o = r^n.fin.n\Phi$, quarum posterior dat $fin.n\Phi = o$; unde erit $n\Phi$ Arcus vel hujus formæ $(2k+1)\pi$ vel $2k\pi$, denotante k numerum integrum. Casus hos ideo distinguo, quod eorum Cosinus sint differentes; priori enim casu erit $cof.(2k+1)\pi = -1$ posteriori casu autem $cof.2k\pi = +1$. Patet autem priorem formam $n\Phi = (2k+1)\pi$ sumi debere, quippe quæ dat $cof.n\Phi = -1$, unde fit $o = a^n - r^n$, hincque porro $r = a = \frac{p}{q}$. Erit ergo $p = a$, $q = 1$, & $\Phi = \frac{(2k+1)\pi}{n}$, unde Functionis $a^n + z^n$ Factor erit $aa - 2az.cof.\frac{(2k+1)\pi}{n} + zz$. Cum igitur pro k numerum quemque integrum ponere liceat, prodeunt hoc modo plures Factores, neque tamen infiniti, quoniam si $2k+1$, ultra n augetur, Factores priores recurrunt, quod ex exemplis clarius patebit, cum sit $cof.(2\pi \pm \Phi) = cof.\Phi$. Deinde si n est numerus impar, posito $2k+1 = n$, erit Factor quadratus $aa + 2az + zz$ neque vero hinc sequitur quadratum $(a+z)^2$ esse Factorem Functionis $a^n + z^n$, quoniam (in §. 148) unica æquatio resultat, qua tantum patet $a + z$ esse Divisorem formulæ

LIB. I. formulæ $a^n + z^n$; quæ regula ſemper eſt tenenda quoties $cof. \Phi$ ſit vel $+1$ vel -1.

EXEMPLUM.

Evolvamus aliquot caſus, quo iſti Factores clarius ob oculos ponantur, atque hos caſus in duas claſſes diſtribuamus, prout n fuerit numerus vel par vel impar.

Si $n = 1$ Formulæ $a + z$ Factor eſt $a + z$	Si $n = 2$ Formulæ $a^2 + z^2$ Factor eſt $a^2 + z^2$
Si $n = 3$ Formulæ $a^3 + z^3$ Factores ſunt $aa - 2az.cof.\frac{1}{3}\pi + zz$ $a + z$	Si $n = 4$ Formulæ $a^4 + z^4$ Factores ſunt $aa - 2az.cof.\frac{1}{4}\pi + zz$ $aa - 2az.cof.\frac{3}{4}\pi + zz$
Si $n = 5$ Formulæ $a^5 + z^5$ Factores ſunt $aa - 2az.cof.\frac{1}{5}\pi + zz$ $aa - 2az.cof.\frac{3}{5}\pi + zz$ $a + z$	Si $n = 6$ Formulæ $a^6 + z^6$ Factores ſunt $aa - 2az.cof.\frac{1}{6}\pi + zz$ $aa - 2az.cof.\frac{3}{6}\pi + zz$ $aa - 2az.cof.\frac{5}{6}\pi + zz$

Ex quibus exemplis patet omnes Factores obtineri, ſi loco $2k+1$ omnes numeri impares non majores, quam Exponens

n

n, substituantur, iis vero casibus quibus Factor quadratus prodit, tantum ejus radicem Factoribus annumerari debere. Cap. IX.

151. Si proposita sit hæc Functio $a^n - z^n$, ejus Factor trinomialis erit $pp - 2pqz.\,cos.\,\phi + qqzz$, si posito $r = \frac{p}{q}$, fuerit $0 = a^n - r^n.\,cos.\,n\phi$ & $0 = r^n.\,sin.\,n\phi$. Erit ergo iterum $sin.\,n\phi = 0$, ideoque $n\phi = (2k+1)\pi$ vel $n\phi = 2k\pi$. Hoc autem casu valor posterior sumi debet, ut sit $cos.\,n\phi = +1$, qui dat $0 = a^n - r^n$ & $r = \frac{p}{q} = a$. Habebitur itaque $p = a$; $q = 1$; & $\phi = \frac{2k\pi}{n}$; unde Factor trinomialis formulæ propositæ erit $= aa - 2az.\,cos.\,\frac{2k}{n}\pi + zz$; quæ forma, si loco $2k$ omnes numeri pares non majores quam n ponantur, simul dabit omnes Factores; ubi de Factoribus quadratis idem est tenendum quod ante monuimus. Ac primo quidem, posito $k = 0$, prodit Factor $aa - 2az + zz$, pro quo vero radix $a - z$ capi debet. Similiter, si n fuerit numerus par & ponatur $2k = n$, prodit $aa + 2az + zz$, unde patet $a + z$ esse divisorem formæ $a^n - z^n$.

EXEMPLUM.

Casus Exponentis n ut ante tractati ita se habebunt, prout n fuerit numerus vel impar vel par.

 Si

Si $n = 1$ Formulæ $a - z$ ipsa erit Factor $a - z$	Si $n = 2$ Formulæ $a^2 - z^2$ Factores erunt $a - z$ $a + z$
Si $n = 3$ Formulæ $a^3 - z^3$ Factores erunt $a - z$ $aa - 2az.\cos.\frac{2}{3}\pi + zz$	Si $n = 4$ Formulæ $a^4 - z^4$ Factores erunt $a - z$ $aa - 2az.\cos.\frac{2}{4}\pi + zz$ $a + z$
Si $n = 5$ Formulæ $a^5 - z^5$ Factores erunt $a - z$ $aa - 2az.\cos.\frac{2}{5}\pi + zz$ $aa - 2az.\cos.\frac{4}{5}\pi + zz$	Si $n = 6$ Formulæ $a^6 - z^6$ Factores erunt $a - z$ $aa - 2az.\cos.\frac{2}{6}\pi + zz$ $aa - 2az.\cos.\frac{4}{6}\pi + zz$ $a + z$

152. His igitur confirmatur id, quod supra jam innuimus, omnem Functionem integram, si non in Factores simplices reales, tamen in Factores duplices reales resolvi posse. Vidimus enim hanc Functionem indefinitæ dimensionis $a^n \pm z^n$ semper in Factores duplices reales, præter simplices reales, resolvi posse. Progrediamur ergo ad Functiones magis compositas, uti: $\alpha + \beta z^n + \gamma z^{2n}$, cujus quidem, si duos habeat Factores formæ $\eta + \theta z^n$, resolutio ex præcedentibus abunde patet. Hoc ergo tantum erit efficiendum, ut formæ $\alpha + \beta z^n + \gamma z^{2n}$, eo casu, quo non habet duos Factores reales formæ $\eta + \theta z^n$, reso-

resolutionem in Factores reales, vel simplices vel duplices, doceamus.

153. Consideremus ergo hanc Functionem: $a^{2n} - 2a^n z^n \times cos. g + z^{2n}$, quæ in duos Factores formæ $\eta + \theta z^n$ reales resolvi nequit. Quod si ergo ponamus hujus Functionis Factorem duplicem realem esse $pp - 2pqz. cos. \phi + qqzz$, posito $r = \frac{p}{q}$, duæ sequentes æquationes erunt resolvendæ: $0 = a^{2n} - 2a^n r^n \cdot cos. g. cos. n\phi + r^{2n}. cos. 2n\phi$ & $0 = - 2a^n r^n \cdot cos. g. sin. n\phi + r^{2n} sin. 2n\phi$. Vel, loco prioris æquationis sumatur ex §. 149, (ponendo $m = 2n$), hæc $0 = a^{2n}. sin. 2n\phi - 2a^n r^n \cdot cos. g. sin. n\phi$, quæ cum posteriori collata dat $r = a$; tum vero erit $sin. 2n\phi = 2 cos. g. sin. n\phi$: At est $sin. 2n\phi = 2 sin. n\phi. cos. n\phi$. unde fit $cos. n\phi = cos. g$. At est semper $cos. (2k\pi \pm g) = cos. g$, ex quo habetur $n\phi = 2k\pi \pm g$ & $\phi = \frac{2k\pi \pm g}{n}$. Hinc ergo Factor generalis duplex formæ propositæ erit $= aa - 2az. cos. \frac{2k\pi \pm g}{n} + zz$; atque omnes Factores prodibunt, si pro $2k$ omnes numeri pares non majores quam n successive substituantur, uti ex applicatione ad casus videre licebit.

EXEMPLUM.

Consideremus ergo casus quibus n est 1, 2, 3, 4, &c., ut ratio Factorum appareat. Erit ergo

Formulæ

$a^2 - 2az. cos. g + zz$

Unicus Factor

$aa - 2az. cos. g + zz$

Formulæ

$a^4 - 2a^2z^2. cos. g + z^4$

P 2 Facto-

Factores duo

$$aa - 2az.\cos.\frac{g}{2} + z^2$$

$$aa - 2az\cos.\left(\frac{2\pi \pm g}{2}\right) + zz \text{ seu } aa + 2az\cos.\frac{g}{2} + zz)$$

Formulæ

$$a^6 - 2a^3z^3.\cos.g + z^6$$

Factores tres

$$aa - 2az.\cos.\frac{g}{3} + z^2$$

$$aa - 2az.\cos.\frac{2\pi - g}{3} + z^2$$

$$aa - 2az.\cos.\frac{2\pi + g}{3} + z^2$$

Formulæ

$$a^8 - 2a^4.z^4\cos.g + z^8$$

Factores quatuor

$$aa - 2az.\cos.\frac{g}{4} + zz$$

$$aa - 2az.\cos.\frac{2\pi - g}{4} + zz$$

$$aa - 2az.\cos.\frac{2\pi + g}{4} + zz$$

$$aa - 2az.\cos.\frac{4\pi \pm g}{4} + zz \text{ seu } aa + 2az.\cos.\frac{g}{4} + zz$$

Formulæ

$$a^{10} - 2a^5z^5.\cos.g + z^{10}$$

Factores quinque

$$aa - 2az.\cos.\frac{g}{5} + zz$$

$$aa - 2az.\cos.\frac{2\pi - g}{5} + zz$$

$$aa - 2az.\cos.\frac{2\pi + g}{5} + zz$$

$$aa - 2az.\cos.\frac{4\pi - g}{5} + zz$$

$$aa - 2az.\cos.\frac{4\pi + g}{5} + zz$$

Con-

Confirmatur ergo etiam his exemplis omnem Functionem integram in Factores reales, sive simplices sive duplices, resolvi posse. CAP. IX.

154. Hinc ulterius progredi licebit ad Functionem hanc $\alpha + \beta z^n + \gamma z^{2n} + \delta z^{3n}$, quæ certo habebit unum Factorem realem formæ $\eta + \theta z^n$, cujus igitur Factores reales, vel simplices vel duplices, exhiberi possunt; alter vero multiplicator formæ $\iota + \varkappa z^n + \lambda z^{2n}$, utcumque fuerit comparatus, per §. præced. pari modo in Factores resolvi poterit. Deinde hæc Functio $\alpha + \beta z^n + \gamma z^{2n} + \delta z^{3n} + \varepsilon z^{4n}$, cum perpetuo habeat duos Factores reales formæ hujus $\eta + \theta z^n + \iota z^{2n}$, similiter in Factores, vel simplices vel duplices, reales resolvitur. Quin etiam progredi licet ad formam $\alpha + \beta z^n + \gamma z^{2n} + \delta z^{3n} + \varepsilon z^{4n} + \zeta z^{5n}$ quæ, cum certo habeat unum Factorem formæ $\eta + \theta z^n$, alter Factor erit formæ præcedentis; unde etiam hæc Functio resolutionem in Factores reales, vel simplices vel duplices, admittet. Quare si ullum dubium mansisset circa hujusmodi resolutionem omnium Functionum integrarum, hoc nunc fere penitus tolletur.

155. Traduci vero etiam potest hæc in Factores resolutio ad Series infinitas; scilicet, quia vidimus supra esse $1 + \frac{x}{1} + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} + \frac{x^4}{1.2.3.4} + \&c. = e^x$; at vero esse $e^x = (1 + \frac{x}{i})^i$, denotante i numerum infinitum, perspicuum est Seriem $1 + \frac{1}{x} + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} + \&c.$ habere Factores infinitos simplices inter se æquales nempe $1 + \frac{x}{i}$. At si ab eadem Serie primus terminus dematur, erit $\frac{x}{1} + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} +$ &c.

LIB. I. &c. $= e^x - 1 = (1 + \frac{x}{i})^i - 1$, cujus formæ cum §. 151 comparatæ, quo fit $a = 1 + \frac{x}{i}$; $n = i$ & $z = 1$; Factor quicunque erit $= (1 + \frac{x}{i})^2 - 2(1 + \frac{x}{i}) cos. \frac{2k}{i}\pi + 1$, unde, substituendo pro $2k$ omnes numeros pares, simul omnes Factores prodibunt. Posito autem $2k = 0$ prodit Factor quadratus $\frac{xx}{ii}$, pro quo autem tantum ob rationes allegatas radix $\frac{x}{i}$ sumi debet, erit ergo x Factor expressionis $e^x - 1$. quod quidem sponte patet. Ad reliquos Factores inveniendos notari oportet esse, ob Arcum $\frac{2k}{i}\pi$ infinite parvum, $cos. \frac{2k}{i}\pi = 1 - \frac{2kk}{ii}\pi\pi$ (134), terminis sequentibus, ob i numerum infinitum, in nihilum abeuntibus. Hinc erit Factor quilibet $= \frac{xx}{ii} + \frac{4kk}{ii}\pi\pi + \frac{4kk\pi\pi}{i^3}x$, atque adeo forma $e^x - 1$ erit divisibilis per $1 + \frac{x}{i} + \frac{xx}{4kk\pi\pi}$. Quare expressio $e^x - 1 = x(1 + \frac{x}{1.2} + \frac{x^2}{1.2.3} + \frac{x^3}{1.2.3.4} + \&c.)$, præter Factorem x, habebit hos infinitos $(1 + \frac{x}{i} + \frac{xx}{4\pi\pi})(1 + \frac{x}{i} + \frac{xx}{16\pi\pi})(1 + \frac{x}{i} + \frac{xx}{36\pi\pi})(1 + \frac{x}{i} + \frac{xx}{64\pi\pi})$ &c.

156. Cum autem hi Factores contineant partem infinite parvam $\frac{x}{i}$, quæ, cum in singulis insit, atque per multiplicationem omnium, quorum numerus est $\frac{1}{2}i$, producat terminum $\frac{x}{2}$, omitti non potest. Ad hoc ergo incommodum vitandum consideremus hanc expressionem $e^x - e^{-x} = (1 + \frac{x}{i})^i - (1 - \frac{x}{i})^i = 2(\frac{x}{1} + \frac{x^3}{1.2.3} + \frac{x^5}{1.2.3.4.5} + \&c.)$ est

est enim $e^{-x} = 1 - \frac{x}{1} + \frac{x^2}{1.2} - \frac{x^3}{1.2.3} + \&c.$; quæ cum §. 151. comparata dat $n = i$, $a = 1 + \frac{x}{i}$ & $z = 1 - \frac{x}{i}$ unde hujus expressionis Factor erit $= aa - 2az.cos.\frac{2k}{n}\pi + zz = 2 + \frac{2xx}{ii} - 2(1 - \frac{xx}{ii})\,cos.\frac{2k}{i}\pi = \frac{4xx}{ii} + \frac{4kk}{ii}\pi\pi - \frac{4kk\pi\pi xx}{i^4}$, ob $cos.\frac{2k}{i}\pi = 1 - \frac{2kk\pi\pi}{ii}$. Functio ergo $e^x - e^{-x}$ divisibilis erit per $1 + \frac{xx}{kk\pi\pi} - \frac{xx}{ii}$, ubi autem terminus $\frac{xx}{ii}$ tuto omittitur, quia etsi per i multiplicetur, tamen manet infinite parvus. Præterea vero ut ante, si $k = 0$, erit primus Factor $= x$. Quocirca, his Factoribus in ordinem redactis, erit $\frac{e^x - e^{-x}}{2} = x(1 + \frac{xx}{\pi\pi})(1 + \frac{xx}{4\pi\pi})(1 + \frac{xx}{9\pi\pi})(1 + \frac{xx}{16\pi\pi})(1 + \frac{xx}{25\pi\pi})\&c. = x(1 + \frac{xx}{1.2.3} + \frac{x^4}{1.2.3.4.5} + \frac{x^6}{1.2\ldots7} + \&c.)$. Singulis scilicet Factoribus per multiplicationem constantis ejusmodi formam dedi, ut per actualem multiplicationem primus terminus x resultet.

157. Eodem modo cum sit $\frac{e^x + e^{-x}}{2} = 1 + \frac{xx}{1.2} + \frac{x^4}{1.2.3.4} + \&c. = \frac{(1 + \frac{x}{i})^i + (1 - \frac{x}{i})^i}{2}$, hujus expressionis cum superiori $a^n + z^n$ comparatio dabit $a = 1 + \frac{x}{i}$; $z = 1 - \frac{x}{i}$ & $n = i$: erit ergo Factor quicunque $= aa - 2az \times cos.\frac{2k+1}{n}\pi + zz = 2 + \frac{2xx}{ii} - 2(1 - \frac{xx}{ii})\,cos.\frac{2k+1}{i}\pi$. Est autem

autem $cos.\ \frac{2k+1}{i}\pi = 1 - \frac{(2k+1)^2\pi\pi}{2ii}$, unde forma Factoris erit $= \frac{4xx}{ii} + \frac{(2k+1)^2\pi^2}{ii}$, evanescente termino cujus denominator est i^4. Quoniam ergo omnis Factor expressionis $1 + \frac{xx}{1.2} + \frac{x^4}{1.2.3.4} +$ &c. hujusmodi formam habere debet $1 + \alpha xx$, quo Factor inventus ad hanc formam reducatur, dividi debet per $\frac{(2k+1)^2\pi^2}{ii}$: hinc Factor formæ propositæ erit $1 + \frac{4xx}{(2k+1)^2\pi\pi}$, ex eoque omnes Factores infiniti invenientur, si loco $2k+1$ successive omnes numeri impares substituantur. Hanc ob rem erit

$$\frac{e^x + e^{-x}}{2} = 1 + \frac{xx}{1.2} + \frac{x^4}{1.2.3.4} + \frac{x^6}{1.2.3.4.5.6} + \&c. =$$

$$\left(1 + \frac{4xx}{\pi\pi}\right)\left(1 + \frac{4xx}{9\pi\pi}\right)\left(1 + \frac{4xx}{25\pi\pi}\right)\left(1 + \frac{4xx}{49\pi\pi}\right) \&c.$$

158. Si x fiat quantitas imaginaria, formulæ hæ exponentiales in Sinum & Cosinum cujuspiam Arcus realis abeunt. Sit enim $x = z\sqrt{-1}$; erit $\frac{e^{z\sqrt{-1}} - e^{-z\sqrt{-1}}}{2\sqrt{-1}} =$ $fin.\ z = z - \frac{z^3}{1.2.3} + \frac{z^5}{1.2.3.4.5} - \frac{z^7}{1.2.3 \ldots 7} + \&c.$, quæ adeo expressio hos habet Factores numero infinitos $z\left(1 - \frac{zz}{\pi\pi}\right)\left(1 - \frac{zz}{4\pi\pi}\right)\left(1 - \frac{zz}{9\pi\pi}\right)\left(1 - \frac{zz}{16\pi\pi}\right)\left(1 - \frac{zz}{25\pi\pi}\right)$ &c., seu erit $fin.\ z = z\left(1 - \frac{z}{\pi}\right)\left(1 + \frac{z}{\pi}\right)\left(1 - \frac{z}{2\pi}\right)$ $\left(1 + \frac{z}{2\pi}\right)\left(1 - \frac{z}{3\pi}\right)\left(1 + \frac{z}{3\pi}\right)$ &c.. Quoties ergo Arcus z ita est comparatus, ut quispiam Factor evanescat, quod fit si $z = 0$, $z = \pm\pi$; $z = \pm 2\pi$, & generaliter si $z = \pm k\pi$, denotante k numerum quemcunque integrum, simul Sinus

nus ejus Arcus debet esse $= 0$, quod quidem ita patet, ut hinc istos Factores a posteriori eruere licuisset. CAP. IX.

Simili modo, cum sit $\frac{e^{z\sqrt{-1}} + e^{-z\sqrt{-1}}}{2} = cos.\ z$; erit quoque $cos.\ z = (1 - \frac{4zz}{\pi\pi})(1 - \frac{4zz}{9\pi\pi})(1 - \frac{4zz}{25\pi\pi})(1 - \frac{4zz}{49\pi\pi})$ &c., seu, his Factoribus in binos resolvendis, erit quoque $cos.\ z = (1 - \frac{2z}{\pi})(1 + \frac{2z}{\pi})(1 - \frac{2z}{3\pi})(1 + \frac{2z}{3\pi})(1 - \frac{2z}{5\pi})(1 + \frac{2z}{5\pi})$ &c., ex qua pari modo patet, si fuerit $z = \pm\frac{(2k+1)}{2}\pi$, fore $cos.\ z = 0$, id quod etiam ex natura Circuli liquet.

159. Ex §. 152. etiam inveniri possunt Factores hujus expressionis $e^x - 2\,cos.\,g + e^{-x} = 2(1 - cos.\,g + \frac{xx}{1.2} + \frac{x^4}{1.2.3.4} + \&c.)$. Transit enim hæc expressio in hanc $(1 + \frac{x}{i})^i - 2\,cos.\,g + (1 - \frac{x}{i})^i$, quæ cum illa forma comparata dat $2n = i$; $a = 1 + \frac{x}{i}$, & $z = 1 - \frac{x}{i}$, unde Factor quicunque hujus formulæ erit $= aa - 2az.\ cos.\ \frac{2k\pi \pm g}{n} + zz = 2 + \frac{2xx}{ii} - 2(1 - \frac{xx}{ii}).\ cos.\ \frac{2(2k\pi \pm g)}{i}$: at est $cos.\ \frac{2(2k\pi \pm g)}{i} = 1 - \frac{2(2k\pi \pm g)^2}{ii}$, unde Factor erit $= \frac{4xx}{ii} + \frac{4(2k\pi \pm g)^2}{ii}$, seu hujus formæ $1 + \frac{xx}{(2k\pi \pm g)^2}$. Si ergo expressio per $2(1 - cos.\,g)$ dividatur, ut in Serie infinita terminus constans sit $= 1$, erit, sumendis omnibus Factoribus, $\frac{e^x - 2\,cos.\,g + e^{-x}}{2(1 - cos.\,g)} = (1 + \frac{xx}{gg})(1 + \frac{xx}{(2\pi - g)^2})$

 $(1 +$

$(1+\frac{xx}{(2\pi+g)^2})(1+\frac{xx}{(4\pi-g)^2})(1+\frac{xx}{(4\pi+g)^2})$ $(1+\frac{xx}{(6\pi-g)^2})(1+\frac{xx}{(6\pi+g)^2})$ &c.. Atque, si loco x ponatur $z\sqrt{-1}$, erit $\frac{cos.z - cos.g}{1 - cos.g} = (1-\frac{z}{g})(1+\frac{z}{g})$ $(1-\frac{z}{2\pi-g})(1+\frac{z}{2\pi-g})(1-\frac{z}{2\pi+g})(1+\frac{z}{2\pi+g})$ $(1-\frac{z}{4\pi-g})(1+\frac{z}{4\pi-g})$ &c., $= 1 - \frac{zz}{1.2(1-cos.g)} + \frac{z^4}{1.2.3.4(1-cos.g)} - \frac{z^6}{1.2\ldots 6(1-cos.g)} +$ &c.. Hujus adeo Seriei in infinitum continuatæ Factores omnes cognoscuntur.

160. Commode etiam hujusmodi Functionis $e^{b+x} \pm e^{c-x}$ Factores inveniri omnesque assignari possunt. Transmutatur enim in hanc formam $(1+\frac{b+x}{i})^i \pm (1+\frac{c-x}{i})^i$, quæ comparata cum forma $a^i \pm z^i$, Factorem habebit $aa - 2az.cos.\frac{m\pi}{i} + zz$; denotante m numerum imparem si valeat signum superius, contra vero numerum parem. Cum autem, ob i numerum infinite magnum, sit $cos.\frac{m\pi}{i} = 1 - \frac{mm\pi\pi}{2ii}$, erit Factor ille generalis $= (a-z)^2 + \frac{mm\pi\pi}{ii}az$. At hoc casu erit $a = 1 + \frac{b+x}{i}$ & $z = 1 + \frac{c-x}{i}$, unde fit $(a-z)^2 = \frac{(b-c+2x)^2}{ii}$ & $az = 1 + \frac{b+c}{i} + \frac{bc+(c-b)x-xx}{ii}$; ideoque Factor erit per ii multiplicatus $= (b-c)^2 + 4(b-c)x + 4xx + mm\pi\pi$, neglectis terminis per i vel ii divisis, quoniam jam omnis generis termini adsunt, præ quibus hi evanescerent. Termino ergo constante ad unitatem per divisionem reducto erit Factor $= 1 + \frac{4(b-c)x + 4xx}{mm\pi\pi + (b-c)^2}$.

161. Nunc,

161. Nunc, quoniam in omnibus Factoribus terminus constans est $=1$, ipsa Functio $e^{b+x} \pm e^{c-x}$ per ejusmodi constantem dividi debet, ut terminus constans fiat $=1$, seu ut ejus valor, posito $x=0$, fiat $=1$; talis Divisor erit $e^b \pm e^c$, & hanc ob rem expressio hæc $\frac{e^{b+x} \pm e^{c-x}}{e^b \pm e^c}$ per Factores numero infinitos exponi poterit. Erit ergo, si valeat signum superius atque m denotet numerum imparem, CAP. IX.

$$\frac{e^{b+x}+e^{c-x}}{e^b+e^c} = \left(1+\frac{4(b-c)x+4xx}{\pi\pi+(b-c)^2}\right)\left(1+\frac{4(b-c)x+4xx}{9\pi\pi+(b-c)^2}\right)$$
$$\left(1+\frac{4(b-c)x+4xx}{25\pi\pi+(b-c)^2}\right) \text{\&c.},$$

sin autem signum inferius valeat, atque ideo m denotet numerum parem, casuque $m=0$ radix Factoris quadrati ponatur, erit $\frac{e^{b+x}-e^{c-x}}{e^b-e^c} =$

$$\left(1+\frac{2x}{b-c}\right)\left(1+\frac{4(b-c)x+4xx}{4\pi\pi+(b-c)^2}\right)\left(1+\frac{4(b-c)x+4xx}{16\pi\pi+(b-c)^2}\right)$$
$$\left(1+\frac{4(b-c)x+4xx}{36\pi\pi+(b-c)^2}\right) \text{\&c.}$$

162. Ponatur $b=0$, quod sine detrimento universalitatis fieri potest, eritque $\frac{e^x+e^c e^{-x}}{1+e^c} = \left(1-\frac{4cx+4xx}{\pi\pi+cc}\right)$

$$\left(1-\frac{4cx+4xx}{9\pi\pi+cc}\right)\left(1-\frac{4cx+4xx}{25\pi\pi+cc}\right) \text{\&c.}; \quad \frac{e^x-e^c e^{-x}}{1-e^c}$$
$$=\left(1-\frac{2x}{c}\right)\left(1-\frac{4cx+4xx}{4\pi\pi+cc}\right)\left(1-\frac{4cx+4xx}{16\pi\pi+cc}\right)$$
$$\left(1-\frac{4cx+4xx}{36\pi\pi+cc}\right) \text{\&c.}.$$

Jam ponatur c negativum, atque habebuntur hæ duæ æquationes: $\frac{e^x+e^{-c}e^{-x}}{1+e^{-c}} =$

Lib. I. $(1 + \frac{4cx + 4xx}{\pi\pi + cc})(1 + \frac{4cx + 4xx}{9\pi\pi + cc})(1 + \frac{4cx + 4xx}{25\pi\pi + cc})$ &c.;

$$\frac{e^{x} - e^{-c} e^{-x}}{1 - e^{-c}} = (1 + \frac{2x}{c})(1 + \frac{4cx + 4xx}{4\pi\pi + cc})$$

$(1 + \frac{4cx + 4xx}{16\pi\pi + cc})(1 + \frac{4cx + 4xx}{36\pi\pi + cc})$ &c.. Multiplicetur forma prima per tertiam, ac prodibit $\frac{e^{2x} + e^{-2x} + e^{c} + e^{-c}}{2 + e^{c} + e^{-c}}$; ponatur vero y loco $2x$, eritque $\frac{e^{y} + e^{-y} + e^{c} + e^{-c}}{2 + e^{c} + e^{-c}} = (1 - \frac{2cy + yy}{\pi\pi + cc})$ $(1 + \frac{2cy + yy}{\pi\pi + cc})(1 - \frac{2cy + yy}{9\pi\pi + cc})(1 + \frac{2cy + yy}{9\pi\pi + cc})(1 - \frac{2cy + yy}{25\pi\pi + cc})$ $(1 + \frac{2cy + yy}{25\pi\pi + cc})$ &c.. Multiplicetur prima forma per quartam, erit productum $= \frac{e^{2x} - e^{-2x} + e^{c} - e^{-c}}{e^{c} - e^{-c}}$; ponatur y pro $2x$, eritque $\frac{e^{y} - e^{-y} + e^{c} - e^{-c}}{e^{c} - e^{-c}} =$ $(1 + \frac{y}{c})(1 - \frac{2cy + yy}{\pi\pi + cc})(1 + \frac{2cy + yy}{4\pi\pi + cc})(1 - \frac{2cy + yy}{9\pi\pi + cc})$ $(1 + \frac{2cy + yy}{16\pi\pi + cc})(1 - \frac{2cy + yy}{25\pi\pi + cc})$ &c.. Si secunda forma per quartam multiplicetur, prodibit eadem æquatio nisi quod c capiendum sit negativum, erit nempe

$$\frac{e^{c} - e^{-c} - e^{y} + e^{-y}}{e^{c} - e^{-c}} = (1 - \frac{y}{c})(1 + \frac{2cy + yy}{\pi\pi + cc})$$

$(1 - \frac{2cy + yy}{4\pi\pi + cc})(1 + \frac{2cy + yy}{9\pi\pi + cc})(1 - \frac{2cy + yy}{16\pi\pi + cc})$ $(1 + \frac{2cy + yy}{25\pi\pi + cc})(1 - \frac{2cy + yy}{36\pi\pi + cc})$ &c.. Multiplicetur denique

nique

nique forma secunda per quartam eritque $\frac{e^{y}+e^{-y}-e^{c}-e^{-c}}{2-e^{c}-e^{-c}}$
$=(1-\frac{yy}{cc})(1-\frac{2cy+yy}{4\pi\pi+cc})(1+\frac{2cy+yy}{4\pi\pi+cc})(1-\frac{2cy+yy}{16\pi\pi+cc})$
$(1+\frac{2cy+yy}{16\pi\pi+cc})(1-\frac{2cy+yy}{36\pi\pi+cc})(1+\frac{2cy+yy}{36\pi\pi+cc})$ &c.

163. Hæ quatuor combinationes nunc commode ad Circulum transferri possunt, ponendo $c=g\sqrt{-1}$ & $y=v\sqrt{-1}$: erit enim $e^{v\sqrt{-1}}+e^{-v\sqrt{-1}}=2\,cos.\,v$; $e^{v\sqrt{-1}}-e^{-v\sqrt{-1}}=2\sqrt{-1}.\,fin.\,v.$ & $e^{g\sqrt{-1}}+e^{-g\sqrt{-1}}=2\,cos.\,g$; $e^{g\sqrt{-1}}-e^{-g\sqrt{-1}}=2\sqrt{-1}.\,fin.\,g.$ Hinc prima combinatio dabit $\frac{cos.\,v+cos.\,g}{1+cos.\,g}=1-\frac{vv}{1.2(1+cos.\,g)}+\frac{v^4}{1.2.3.4(1+cos.\,g)}-\frac{v^6}{1.2\ldots.6(1+cos.\,g)}+$ &c. $=(1+\frac{2gv-vv}{\pi\pi-gg})$
$(1-\frac{2gv-vv}{\pi\pi-gg})(1+\frac{2gv-vv}{9\pi\pi-gg})(1-\frac{2gv-vv}{9\pi\pi-gg})$
$(1+\frac{2gv-vv}{25\pi\pi-gg})(1-\frac{2gv-vv}{25\pi\pi-gg})$ &c. $=(1+\frac{v}{\pi-g})$
$(1-\frac{v}{\pi+g})(1-\frac{v}{\pi-g})(1+\frac{v}{\pi+g})(1+\frac{v}{3\pi-g})$
$(1-\frac{v}{3\pi+g})(1-\frac{v}{3\pi-g})(1+\frac{v}{3\pi+g})$ &c. $=$
$(1-\frac{vv}{(\pi-g)^2})(1-\frac{vv}{(\pi+g)^2})(1-\frac{vv}{(3\pi-g)^2})$
$(1-\frac{vv}{(3\pi+g)^2})(1-\frac{vv}{(5\pi-g)^2})$ &c. Quarta vero combinatio dat $\frac{cos.\,v-cos.\,g}{1-cos.\,g}=1-\frac{vv}{1.2(1-cos.\,g)}+\frac{v^4}{1.2.3.4(1-cos.\,g)}-\frac{v^6}{1.2\ldots.6(1-cos.\,g)}+$ &c. $=(1-\frac{vv}{gg})$
$(1+\frac{2gv-vv}{4\pi\pi-gg})(1-\frac{2gv-vv}{4\pi\pi-gg})(1+\frac{2gv-vv}{16\pi\pi-gg})$

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$(1-\frac{2gv-vv}{16\varpi\varpi-gg})$ &c. $=(1-\frac{v}{g})(1+\frac{v}{g})(1+\frac{v}{2\varpi-g})$
$(1-\frac{v}{2\varpi+g})(1-\frac{v}{2\varpi-g})(1+\frac{v}{2\varpi+g})(1+\frac{v}{4\varpi-g})$
$(1-\frac{v}{4\varpi+g})$ &c. $=(1-\frac{vv}{gg})(1-\frac{vv}{(2\varpi-g)^2})$
$(1-\frac{vv}{(2\varpi+g)^2})(1-\frac{vv}{(4\varpi-g)^2})(1-\frac{vv}{(4\varpi+g)^2})$ &c..
Secunda combinatio dat $\frac{\textit{fin.}\, g+\textit{fin.}\, v}{\textit{fin.}\, g}=1+\frac{v}{\textit{fin.}\, g}-\frac{v^3}{1.2.3\,\textit{fin.}\, g}+$
$\frac{v^5}{1.2\ldots5\,\textit{fin.}\, g}-$ &c. $=(1+\frac{v}{g})(1+\frac{2gv-vv}{\varpi\varpi-gg})$
$(1-\frac{2gv-vv}{4\varpi\varpi-gg})(1+\frac{2gv-vv}{9\varpi\varpi-gg})(1-\frac{2gv-vv}{16\varpi\varpi-gg})$ &c.
$=(1+\frac{v}{g})(1+\frac{v}{\varpi-g})(1-\frac{v}{\varpi+g})(1-\frac{v}{2\varpi-g})$
$(1+\frac{v}{2\varpi+g})(1+\frac{v}{3\varpi-g})(1-\frac{v}{3\varpi+g})(1-\frac{v}{5\varpi-g})$ &c..
Ac ſumto v negativo prodit tertia combinatio.

164. Ipſæ vero etiam expreſſiones in §. 162. primum inventæ ad Arcus circulares traduci poſſunt hoc modo: cum ſit

$$\frac{e^x+e^c e^{-x}}{1+e^c}=\frac{(1+e^{-c})(e^x+e^c e^{-x})}{2+e^c+e^{-c}}=$$
$$\frac{e^x+e^{-x}+e^{c-x}+e^{-c+x}}{2+e^c+e^{-c}},$$ ſi ponamus $c=g\sqrt{-1}$ & $x=z\sqrt{-1}$, hæc expreſſio abit in hanc $\frac{\textit{cof.}\, z+\textit{cof.}\,(g-z)}{1+\textit{cof.}\, g}=$
$\textit{cof.}\, z+\frac{\textit{fin.}\, g.\,\textit{fin.}\, z}{1+\textit{cof.}\, g}$. Erit ergo (ob $\frac{\textit{fin.}\, g}{1+\textit{cof.}\, g}=\textit{tang.}\,\frac{1}{2}g$)
$\textit{cof.}\, z+\textit{tang.}\,\frac{1}{2}g.\,\textit{fin.}\, z=1+\frac{z}{1}\textit{tang.}\,\frac{1}{2}g-\frac{zz}{1.2}-$
$\frac{z^3}{1.2.3}\textit{tang.}\,\frac{1}{2}g+\frac{z^4}{1.2.3.4}+\frac{z^5}{1.2.3.4.5}\textit{tang.}\,\frac{1}{2}g-$ &c.
$=(1+\frac{4gz-4zz}{\varpi\varpi-gg})(1+\frac{4gz-4zz}{9\varpi\varpi-gg})(1+\frac{4gz-4zz}{25\varpi\varpi-gg})$ &c.
$=(1+$

$= \left(1 + \frac{2z}{\pi - g}\right)\left(1 - \frac{2z}{\pi + g}\right)\left(1 + \frac{2z}{3\pi - g}\right)\left(1 - \frac{2z}{3\pi + g}\right)$ $\left(1 + \frac{2z}{5\pi - g}\right)\left(1 - \frac{2z}{5\pi + g}\right)$ &c.. Simili modo altera expressio, si Numerator & Denominator per $1 - e^{-c}$ multiplicetur, abit in $\frac{e^{x} + e^{-x} - e^{c-x} - e^{-c+x}}{2 - e^{c} - e^{-c}}$; quæ, facto $c = g\sqrt{-1}$ & $x = z\sqrt{-1}$, dat $\frac{cof.\, z - cof.\, (g - z)}{1 - cof.\, g} =$ $cof.\, z - \frac{fin.\, g.\, fin.\, z.}{1 - cof.\, g} = cof.\, z - \frac{fin.\, z}{tang.\, \frac{1}{2} g}$. Erit ergo $cof.\, z -$ $cot.\, \frac{1}{2} g.\, fin.\, z = 1 - \frac{z}{1}\, cot.\, \frac{1}{2} g - \frac{zz}{1.2} + \frac{z^3}{1.2.3}\, cot.\, \frac{1}{2} g +$ $\frac{z^4}{1.2.3.4} - \frac{z^5}{1.2...5}\, cot.\, \frac{1}{2} g + \&c. = \left(1 - \frac{2z}{g}\right)\left(1 + \frac{4gz - 4zz}{4\pi\pi - gg}\right)$ $\left(1 + \frac{4gz - 4zz}{16\pi\pi - gg}\right)\left(1 + \frac{4gz - 4zz}{36\pi\pi - gg}\right) \&c. = \left(1 - \frac{2z}{g}\right)$ $\left(1 + \frac{2z}{2\pi - g}\right)\left(1 - \frac{2z}{2\pi + g}\right)\left(1 + \frac{2z}{4\pi - g}\right)\left(1 - \frac{2z}{4\pi + g}\right)$ &c.

Quod si ergo ponatur $v = 2z$ seu $z = \frac{1}{2} v$; habebitur

$$\frac{cof.\, \frac{1}{2}(g - v)}{cof.\, \frac{1}{2} g} = cof.\, \frac{1}{2} v + tang.\, \frac{1}{2} g.\, fin.\, \frac{1}{2} v =$$
$$\left(1 + \frac{v}{\pi - g}\right)\left(1 - \frac{v}{\pi + g}\right)\left(1 + \frac{v}{3\pi - g}\right)\left(1 - \frac{v}{3\pi + g}\right) \&c.;$$
$$\frac{cof.\, \frac{1}{2}(g + v)}{cof.\, \frac{1}{2} g} = cof.\, \frac{1}{2} v - tang.\, \frac{1}{2} g.\, fin.\, \frac{1}{2} v =$$
$$\left(1 - \frac{v}{\pi - g}\right)\left(1 + \frac{v}{\pi + g}\right)\left(1 - \frac{v}{3\pi - g}\right)\left(1 + \frac{v}{3\pi + g}\right) \&c.;$$
$$\frac{fin.\, \frac{1}{2}(g - v)}{fin.\, \frac{1}{2} g} = cof.\, \frac{1}{2} v - cot.\, \frac{1}{2} g.\, fin.\, \frac{1}{2} v =$$
$$\left(1 - \frac{v}{g}\right)\left(1 + \frac{v}{2\pi - g}\right)\left(1 - \frac{v}{2\pi + g}\right)\left(1 + \frac{v}{4\pi - g}\right) \&c.$$

$\frac{fin.\, \frac{1}{2}(g + v)}{fin.\, \frac{1}{2} g}$

LIB. I. $\frac{\sin.\frac{1}{2}(g+v)}{\sin.\frac{1}{2}g} = \cos.\frac{1}{2}v + \cot.\frac{1}{2}g.\sin.\frac{1}{2}v =$

$(1+\frac{v}{g})(1-\frac{v}{2\varpi-g})(1+\frac{v}{2\varpi+g})(1-\frac{v}{4\varpi-g})$ &c.

Quorum Factorum lex progressionis satis est simplex & uniformis; atque ex his expressionibus per multiplicationem oriuntur eæ ipsæ, quæ §. præcedente sunt inventæ.

CAPUT X.

De usu Factorum inventorum in definiendis summis Serierum infinitarum.

165. Si fuerit $1 + Az + Bz^2 + Cz^3 + Dz^4 +$ &c. $= (1+\alpha z)(1+\beta z)(1+\gamma z)(1+\delta z)$ &c., hi Factores, sive sint numero finiti sive infiniti, si in se actu multiplicentur, illam expressionem $1 + A + Bz^2 + Cz^3 + Dz^4 +$ &c., producere debent. Æquabitur ergo coëfficiens A summæ omnium quantitatum $\alpha + \beta + \gamma + \delta + \epsilon +$ &c.. Coëfficiens vero B æqualis erit summæ productorum ex binis, eritque $B = \alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta +$ &c.. Tum vero coëfficiens C æquabitur summæ productorum ex ternis, nempe erit $C = \alpha\beta\gamma + \alpha\beta\delta + \beta\gamma\delta + \alpha\gamma\delta +$ &c.. Atque ita porro erit $D =$ summæ productorum ex quaternis, $E =$ summæ productorum ex quinis, &c., id quod ex Algebra communi constat.

166. Quia summa quantitatum $\alpha + \beta + \gamma + \delta +$ &c., datur una cum summa productorum ex binis, hinc summa Quadratorum $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 +$ &c., inveniri poterit, quippe quæ æqualis est Quadrato summæ demtis duplicibus productis ex binis. Simili modo summa Cuborum, Biquadratorum & altiorum Potestatum definiri potest: si enim ponamus

$P =$

CAP. X.

$$P = \alpha + \beta + \gamma + \delta + \varepsilon + \&c.$$
$$Q = \alpha^2 + \beta^2 + \gamma^2 + \delta^2 + \varepsilon^2 + \&c.$$
$$R = \alpha^3 + \beta^3 + \gamma^3 + \delta^3 + \varepsilon^3 + \&c.$$
$$S = \alpha^4 + \beta^4 + \gamma^4 + \delta^4 + \varepsilon^4 + \&c.$$
$$T = \alpha^5 + \beta^5 + \gamma^5 + \delta^5 + \varepsilon^5 + \&c.$$
$$V = \alpha^6 + \beta^6 + \gamma^6 + \delta^6 + \varepsilon^6 + \&c.$$
&c.

Valores P, Q, R, S, T, V &c. ſequenti modo ex cognitis A, B, C, D, &c., determinabuntur.

$$P = A$$
$$Q = AP - 2B$$
$$R = AQ - BP + 3C$$
$$S = AR - BQ + CP - 4D$$
$$T = AS - BR + CQ - DP + 5E$$
$$V = AT - BS + CR - DQ + EP - 6F$$
&c.

quarum formularum veritas examine inſtituto facile agnoſcitur: interim tamen in calculo differentiali ſummo cum rigore demonſtrabitur.

167. Cum igitur ſupra (§. 156.) invenerimus eſſe: $\frac{e^x - e^{-x}}{2} = x\left(1 + \frac{xx}{1.2.3} + \frac{x^4}{1.2.3.4.5} + \frac{x^6}{1.2.\ldots.7} + \&c.\right) = x\left(1 + \frac{xx}{\pi\pi}\right)\left(1 + \frac{xx}{4\pi\pi}\right)\left(1 + \frac{xx}{9\pi\pi}\right)\left(1 + \frac{xx}{16\pi\pi}\right)\left(1 + \frac{xx}{25\pi\pi}\right)\&c.$, erit $1 + \frac{xx}{1.2.3} + \frac{x^4}{1.2.3.4.5} + \frac{x^6}{1.2.3.\ldots7} + \&c. = \left(1 + \frac{xx}{\pi\pi}\right)\left(1 + \frac{xx}{4\pi\pi}\right)\left(1 + \frac{xx}{9\pi\pi}\right)\left(1 + \frac{xx}{16\pi\pi}\right)\&c.$ Ponatur $xx = \pi\pi z$, eritque $1 + \frac{\pi\pi}{1.2.3} z + \frac{\pi^4}{1.2.3.4.5} z^2 + \frac{\pi^6}{1.2.3\ldots7} z^3 + \&c. = (1+z)\left(1 + \frac{1}{4}z\right)\left(1 + \frac{1}{9}z\right)\left(1 + \frac{1}{16}z\right)$

 $(1 + \frac{1}{25}z)$ &c.. Facta ergo applicatione superioris regulæ ad hunc casum, erit $A = \frac{\pi\pi}{6}$; $B = \frac{\pi^4}{120}$; $C = \frac{\pi^6}{5040}$; $D = \frac{\pi^8}{362880}$ &c.. Quod si ergo ponatur

$$P = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} + \&c.$$
$$Q = 1 + \frac{1}{4^2} + \frac{1}{9^2} + \frac{1}{16^2} + \frac{1}{25^2} + \frac{1}{36^2} + \&c.$$
$$R = 1 + \frac{1}{4^3} + \frac{1}{9^3} + \frac{1}{16^3} + \frac{1}{25^3} + \frac{1}{36^3} + \&c.$$
$$S = 1 + \frac{1}{4^4} + \frac{1}{9^4} + \frac{1}{16^4} + \frac{1}{25^4} + \frac{1}{36^4} + \&c.$$
$$T = 1 + \frac{1}{4^5} + \frac{1}{9^5} + \frac{1}{16^5} + \frac{1}{25^5} + \frac{1}{36^5} + \&c.$$
&c.

atque harum litterarum valores ex A, B, C, D, &c. determinentur, prodibit.

$$P = \frac{\pi\pi}{6}$$
$$Q = \frac{\pi^4}{90}$$
$$R = \frac{\pi^6}{945}$$
$$S = \frac{\pi^8}{9450}$$
$$T = \frac{\pi^{10}}{93555}$$
&c.

168. Patet ergo omnium Serierum infinitarum in hac forma generali $1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \&c.$, contentarum, quoties n fuerit numerus par, ope Peripheriæ Circuli π exhiberi posse; habebit enim semper summa Seriei ad π^n rationem rationalem.

lem. Quo autem valor harum ſummarum clarius perſpiciatur, plures hujuſmodi Serierum ſummas commodiori modo expreſſas hic adjiciam. CAP. X.

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\&c.=\frac{2^0}{1.2.3}.\frac{1}{1}\pi^2$$

$$1+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\frac{1}{5^4}+\&c.=\frac{2^2}{1.2.3.4.5}.\frac{1}{3}\pi^4$$

$$1+\frac{1}{2^6}+\frac{1}{3^6}+\frac{1}{4^6}+\frac{1}{5^6}+\&c.=\frac{2^4}{1.2.3....7}.\frac{1}{3}\pi^6$$

$$1+\frac{1}{2^8}+\frac{1}{3^8}+\frac{1}{4^8}+\frac{1}{5^8}+\&c.=\frac{2^6}{1.2.3....9}.\frac{3}{5}\pi^8$$

$$1+\frac{1}{2^{10}}+\frac{1}{3^{10}}+\frac{1}{4^{10}}+\frac{1}{5^{10}}+\&c.=\frac{2^8}{1.2.3.....11}.\frac{5}{3}\pi^{10}$$

$$1+\frac{1}{2^{12}}+\frac{1}{3^{12}}+\frac{1}{4^{12}}+\frac{1}{5^{12}}+\&c.=\frac{2^{10}}{1.2.3.....13}.\frac{691}{105}\pi^{12}$$

$$1+\frac{1}{2^{14}}+\frac{1}{3^{14}}+\frac{1}{4^{14}}+\frac{1}{5^{14}}+\&c.=\frac{2^{12}}{1.2.3.....15}.\frac{35}{1}\pi^{14}$$

$$1+\frac{1}{2^{16}}+\frac{1}{3^{16}}+\frac{1}{4^{16}}+\frac{1}{5^{16}}+\&c.=\frac{2^{14}}{1.2.3.....17}.\frac{3617}{15}\pi^{16}$$

$$1+\frac{1}{2^{18}}+\frac{1}{3^{18}}+\frac{1}{4^{18}}+\frac{1}{5^{18}}+\&c.=\frac{2^{16}}{1.2.3.....19}.\frac{43867}{21}\pi^{18}$$

$$1+\frac{1}{2^{20}}+\frac{1}{3^{20}}+\frac{1}{4^{20}}+\frac{1}{5^{20}}+\&c.=\frac{2^{18}}{1.2.3....21}.\frac{1222277}{55}\pi^{20}$$

$$1+\frac{1}{2^{22}}+\frac{1}{3^{22}}+\frac{1}{4^{22}}+\frac{1}{5^{22}}+\&c.=\frac{2^{20}}{1.2.3...23}.\frac{854513}{3}\pi^{22}$$

$$1+\frac{1}{2^{24}}+\frac{1}{3^{24}}+\frac{1}{4^{24}}+\frac{1}{5^{24}}+\&c.=\frac{2^{22}}{1.\ 2.\ 3.........25}.\ \frac{1181820455}{273}\pi^{24}$$

$$1+\frac{1}{2^{26}}+\frac{1}{3^{26}}+\frac{1}{4^{26}}+\frac{1}{5^{26}}+\&c.=\frac{2^{24}}{1.\ 2.\ 3.........27}.\ \frac{76977927}{1}\pi^{26}.$$

Hucuſque iſtos Poteſtatum ipſius π Exponentes artificio alibi exponendo continuare licuit, quod ideo hic adjunxi, quod

LIB. I. Seriei fractionum primo intuitu perquam irregularis 1, $\frac{1}{3}$, $\frac{1}{3}$, $\frac{3}{5}$, $\frac{5}{3}$, $\frac{691}{105}$, $\frac{35}{1}$, &c. in plurimis occasionibus eximius est usus.

169. Tractemus eodem modo aequationem §. 157. inventam, ubi erat $\frac{e^x + e^{-x}}{2} = 1 + \frac{xx}{1.2} + \frac{x^4}{1.2.3.4} + \frac{x^6}{1.2.3.4.5.6} +$ &c., $= (1 + \frac{4xx}{\pi\pi})(1 + \frac{4xx}{9\pi\pi})(1 + \frac{4xx}{25\pi\pi})(1 + \frac{4xx}{49\pi\pi})$ &c. Posito ergo $xx = \frac{\pi\pi z}{4}$ erit $1 + \frac{\pi\pi}{1.2.4} z + \frac{\pi^4}{1.2.3.4.4^2} zz + \frac{\pi^6}{1.2....6.4^3} z^3 +$ &c., $= (1 + z)(1 + \frac{1}{9} z)(1 + \frac{1}{25} z)(1 + \frac{1}{49} z)$ &c.. Unde, facta applicatione, erit $A = \frac{\pi\pi}{1.2.4}$; $B = \frac{\pi^4}{1.2.3.4.4^2}$; $C = \frac{\pi^6}{1.2.3....6.4^3}$; &c.. Quod si ergo ponamus

$$P = 1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \frac{1}{81} + \&c.$$
$$Q = 1 + \frac{1}{9^2} + \frac{1}{25^2} + \frac{1}{49^2} + \frac{1}{81^2} + \&c.$$
$$R = 1 + \frac{1}{9^3} + \frac{1}{25^3} + \frac{1}{49^3} + \frac{1}{81^3} + \&c.$$
$$S = 1 + \frac{1}{9^4} + \frac{1}{25^4} + \frac{1}{49^4} + \frac{1}{81^4} + \&c.$$
&c.

reperientur sequentes pro P, Q, R, S, &c., valores:

$$P = \frac{1}{1} \cdot \frac{\pi^2}{2^3}; \qquad Q = \frac{2}{1.2.3} \cdot \frac{\pi^4}{2^5}$$
$$R = \frac{16}{1.2.3.4.5} \cdot \frac{\pi^6}{2^7}; \qquad S = \frac{272}{1.2.3....7} \cdot \frac{\pi^8}{2^9}$$

$T =$

$T = \frac{7936}{1.2.3......9} \cdot \frac{\pi^{10}}{2^{11}}$; $V = \frac{353792}{1.2.3......11} \cdot \frac{\pi^{12}}{2^{13}}$

$W = \frac{22368256}{1.2.3........13} \cdot \frac{\pi^{14}}{2^{15}}$.

170. Eædem ſummæ Poteſtatum numerorum imparium inveniri poſſunt ex ſummis præcedentibus, in quibus omnes numeri occurrunt; ſi enim fuerit $M = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \frac{1}{5^n} +$ &c., erit ubique, per $\frac{1}{2^n}$ multiplicando, $\frac{M}{2^n} = \frac{1}{2^n} + \frac{1}{4^n} + \frac{1}{6^n} + \frac{1}{8^n} +$ &c., quæ Series numeros tantum pares continens, ſi a priori ſubtrahatur, relinquet numeros impares, eritque ideo $M - \frac{M}{2^n} = \frac{2^n - 1}{2^n} M = 1 + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{7^n} + \frac{1}{9^n} +$ &c. Quod ſi autem Series $\frac{M}{2^n}$ bis ſumta ſubtrahatur ab M ſigna prodibunt alternantia, eritque $M - \frac{2M}{2^n} = \frac{2^{n-1} - 1}{2^{n-1}} M = 1 - \frac{1}{2^n} + \frac{1}{3^n} - \frac{1}{4^n} + \frac{1}{5^n} - \frac{1}{6^n} +$ &c.. Per tradita ergo præcepta ſummari poterunt hæ Series

$$1 \pm \frac{1}{2^n} + \frac{1}{3^n} \pm \frac{1}{4^n} + \frac{1}{5^n} \pm \frac{1}{6^n} + \frac{1}{7^n} \pm \text{\&c.}$$

$$1 + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{7^n} + \frac{1}{9^n} + \frac{1}{11^n} + \text{\&c.}$$

Si quidem n ſit numerus par, atque ſumma erit $= A\pi^n$ exiſtente A numero rationali.

171. Præterea vero expreſſiones §. 164 exhibitæ ſimili mo-

do Series notatu dignas suppeditabunt. Cum enim sit $cos. \frac{1}{2} v + tang. \frac{1}{2} g \cdot sin. \frac{1}{2} v = (1 + \frac{v}{\varpi - g})(1 - \frac{v}{\varpi + g})(1 + \frac{v}{3\varpi - g})$ &c., si ponamus $v = \frac{x}{n}\varpi$ & $g = \frac{m}{n}\pi$ erit $(1 + \frac{x}{n-m})(1 - \frac{x}{n+m})(1 + \frac{x}{3n-m})(1 - \frac{x}{3n+m})(1 + \frac{x}{5n-m})(1 - \frac{x}{5n+m})$ &c. $= cos. \frac{x\varpi}{2n} + tang. \frac{m\varpi}{2n} \cdot sin. \frac{x\varpi}{2n} = 1 + \frac{\varpi x}{2n} tang. \frac{m\varpi}{2n} - \frac{\varpi\varpi xx}{2.4nn} - \frac{\varpi^3 x^3}{2.4.6n^3} tang. \frac{m\varpi}{2n} + \frac{\varpi^4 x^4}{2.4.6.8n^4} +$ &c.. Hæc expressio infinita cum §. 165 collata dabit hos valores $A = \frac{\varpi}{2n} tang. \frac{m\varpi}{2n}$; $B = \frac{-\varpi\varpi}{2.4nn}$; $C = \frac{-\varpi^3}{2.4.6n^3} \cdot tang. \frac{m\varpi}{2n}$, $D = \frac{\varpi^4}{2.4.6.8n^4}$; $E = \frac{\varpi^5}{2.4.6.8.10n^5} \cdot tang. \frac{m\varpi}{2n}$ &c. Tum vero erit $\alpha = \frac{1}{n-m}$; $\beta = -\frac{1}{n+m}$; $\gamma = \frac{1}{3n-m}$; $\delta = -\frac{1}{3n+m}$; $\epsilon = \frac{1}{5n-m}$; $\zeta = -\frac{1}{5n+m}$ &c.

172. Hinc ergo ad normam §. 166 sequentes Series exorientur.

$$P = \frac{1}{n-m} - \frac{1}{n+m} + \frac{1}{3n-m} - \frac{1}{3n+m} + \frac{1}{5n-m} - \frac{1}{5n+m} + \&c.$$

$$Q = \frac{1}{(n-m)^2} + \frac{1}{(n+m)^2} + \frac{1}{(3n-m)^2} + \frac{1}{(3n+m)^2} + \frac{1}{(5n-m)^2} + \&c.$$

$$R = \frac{1}{(n-m)^3} - \frac{1}{(n+m)^3} + \frac{1}{(3n-m)^3} - \frac{1}{(3n+m)^3} + \frac{1}{(5n-m)^3} - \&c.$$

$S =$

$S = \frac{1}{(n-m)^4} + \frac{1}{(n+m)^4} + \frac{1}{(3n-m)^4} + \frac{1}{(3n+m)^4} +$
$\frac{1}{(5n-m)^4} +$ &c.

$T = \frac{1}{(n-m)^5} - \frac{1}{(n+m)^5} + \frac{1}{(3n-m)^5} - \frac{1}{(3n+m)^5} +$
$\frac{1}{(5n-m)^5} -$ &c.

$V = \frac{1}{(n-m)^6} + \frac{1}{(n+m)^6} + \frac{1}{(3n-m)^6} + \frac{1}{(3n+m)^6} +$
$\frac{1}{(5n-m)^6} +$ &c.

&c.

Posito autem *tang.* $\frac{m\pi}{2n} = k$ erit, uti ostendimus,

$$P = A = \frac{k\pi}{2n} = \frac{k\pi}{2n}$$
$$Q = \frac{(kk+1)\pi\pi}{4nn} = \frac{(2kk+2)\pi^2}{2.4nn}$$
$$R = \frac{(k^3+k)\pi^3}{8n^3} = \frac{(6k^3+6k)\pi^3}{2.4.6.n^3}$$
$$S = \frac{(3k^4+4kk+1)\pi^4}{48n^4} = \frac{(24k^4+32k^2+8)\pi^4}{2.4.6.8n^4}$$
$$T = \frac{(3k^5+5k^3+2k)\pi^5}{96n^5} = \frac{(120k^5+200k^3+80k)\pi^5}{2.4.6.8.10n^5}$$

173. Pari modo ultima forma §. 164; $cos.\frac{1}{2}v + cot.\frac{1}{2}g \times sin.\frac{1}{2}v = (1+\frac{v}{g})(1-\frac{v}{2\pi-g})(1+\frac{v}{2\pi+g})(1-\frac{v}{4\pi-g})(1+\frac{v}{4\pi+g})$ &c. Si ponamus $v = \frac{x}{n}\pi$, $g = \frac{m}{n}\pi$, & *tang.* $\frac{m\pi}{2n} = k$, ut sit *cot.* $\frac{1}{2}g = \frac{1}{k}$, dabit $cos.\frac{\pi x}{2n} + \frac{1}{k} \times sin.\frac{\pi x}{2n} = 1 + \frac{\pi x}{2nk} - \frac{\pi\pi xx}{2.4nn} - \frac{\pi^3x^3}{2.4.6n^3k} + \frac{\pi^4x^4}{2.4.6.8n^4} + \frac{\pi^5x^5}{2.4.6.8.10n^5k} -$ &c. $= (1+\frac{x}{m})(1-\frac{x}{2n-m})(1+\frac{x}{2n+m})$

(1 —

LIB. I. $(1 - \frac{x}{4n-m})\,(1 + \frac{x}{4n+m})$ &c.. Comparatione ergo cum forma generali (§. 165) instituta erit $A = \frac{\pi}{2nk}$; $B = \frac{-\pi\pi}{2.4n^2}$; $C = \frac{-\pi^3}{2.4.6n^3k}$; $D = \frac{\pi^4}{2.4.6.8n^4}$; $E = \frac{\pi^5}{2.4.6.8.10n^5k}$; &c.; ex Factoribus vero habebitur $\alpha = \frac{1}{m}$; $\beta = \frac{-1}{2n-m}$; $\gamma = \frac{1}{2n+m}$; $\delta = \frac{-1}{4n-m}$; $\epsilon = \frac{1}{4n+m}$ &c.

174. Hinc ergo ad normam §. 166. sequentes Series formabuntur, earumque summæ assignabuntur

$$P = \frac{1}{m} - \frac{1}{2n-m} + \frac{1}{2n+m} - \frac{1}{4n-m} + \frac{1}{4n+m} - \text{\&c.}$$

$$Q = \frac{1}{m^2} + \frac{1}{(2n-m)^2} + \frac{1}{(2n+m)^2} + \frac{1}{(4n-m)^2} + \frac{1}{(4n+m)^2} + \text{\&c.}$$

$$R = \frac{1}{m^3} - \frac{1}{(2n-m)^3} + \frac{1}{(2n+m)^3} - \frac{1}{(4n-m)^3} + \frac{1}{(4n+m)^3} - \text{\&c.}$$

$$S = \frac{1}{m^4} + \frac{1}{(2n-m)^4} + \frac{1}{(2n+m)^4} + \frac{1}{(4n-m)^4} + \frac{1}{(4n+m)^4} + \text{\&c.}$$

$$T = \frac{1}{m^5} - \frac{1}{(2n-m)^5} + \frac{1}{(2n+m)^5} - \frac{1}{(4n-m)^5} + \frac{1}{(4n+m)^5} - \text{\&c.}$$

&c.

Hæ

Hæ autem summæ P, Q, R, S, &c. ita ſe habebunt CAP. X.

$$P = A = \frac{\pi}{2nk} = \frac{1\pi}{2nk}$$

$$Q = \frac{(kk+1)\pi\pi}{4nnkk} = \frac{(2+2kk)\pi^2}{2.\,4\,n^2k^2}$$

$$R = \frac{(kk+1)\pi^3}{8n^3k^3} = \frac{(6+6kk)\pi^3}{2.\,4.\,6\,n^3k^3}$$

$$S = \frac{(k^4+4kk+3)\pi^4}{48n^4k^4} = \frac{(24+32kk+3k^4)\pi^4}{2.\,4.\,6.\,8\,n^4k^4}$$

$$T = \frac{(2k^4+5kk+3)\pi^5}{96n^5k^5} = \frac{(120+200kk+80k^4)\pi^5}{2.\,4.\,6.\,8.\,10\,n^5k^5}$$

$$V = \frac{(2k^6+17k^4+30k^2+15)\pi^6}{960n^6k^6} = \frac{(720+1440kk+816k^4+96k^6)\pi^6}{2.\,4.\,6.\,8.\,10.\,12\,n^6k^6}$$

&c.

175. Series iſtæ generales merentur ut caſus quoſdam particulares inde derivemus, qui prodibunt ſi rationem m ad n in numeris determinemus. Sit igitur primum $m = 1$ & $n = 2$, fiet $k = tang.\ \frac{\pi}{4} = tang.\ 45^\circ = 1$, atque ambæ Serierum claſſes inter ſe congruent. Erit ergo

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \&c.$$

$$\frac{\pi\pi}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \&c.$$

$$\frac{\pi^3}{32} = 1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \frac{1}{9^3} - \&c.$$

$$\frac{\pi^4}{96} = 1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \frac{1}{9^4} + \&c.$$

$$\frac{5\pi^5}{1536} = 1 - \frac{1}{3^5} + \frac{1}{5^5} - \frac{1}{7^5} + \frac{1}{9^5} - \&c.$$

$$\frac{\pi^6}{960} = 1 + \frac{1}{3^6} + \frac{1}{5^6} + \frac{1}{7^6} + \frac{1}{9^6} + \&c.$$

&c.

Harum Serierum primam jam ſupra (§. 140) elicuimus, reliquarum illæ, quæ pares habent Dignitates, modo ante (§. 169)

LIB. I. sunt erutæ; ceteræ, in quibus Exponentes sunt numeri impares, hic primum occurrunt. Constat ergo omnium quoque istarum Serierum:

$$1 - \frac{1}{3^{2n+1}} + \frac{1}{5^{2n+1}} - \frac{1}{7^{2n+1}} + \frac{1}{9^{2n+1}} - \&c.$$

summas per valorem ipsius π assignari posse.

176. Sit nunc $m = 1$, $n = 3$; erit $k = tang. \frac{\pi}{6} = tang. 30° = \frac{1}{\sqrt{3}}$; atque Series §. 172 abibunt in has

$$\frac{\pi}{6\sqrt{3}} = \frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{10} + \frac{1}{14} - \frac{1}{16} + \&c.$$
$$\frac{\pi\pi}{27} = \frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{8^2} + \frac{1}{10^2} + \frac{1}{14^2} + \frac{1}{16^2} + \&c.$$
$$\frac{\pi^3}{162\sqrt{3}} = \frac{1}{2^3} - \frac{1}{4^3} + \frac{1}{8^3} - \frac{1}{10^3} + \frac{1}{14^3} - \frac{1}{16^3} + \&c.$$

&c., sive

$$\frac{\pi}{3\sqrt{3}} = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{5} + \frac{1}{7} - \frac{1}{8} + \&c.$$
$$\frac{4\pi\pi}{27} = 1 + \frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{8^2} + \&c.$$
$$\frac{4\pi^3}{81\sqrt{3}} = 1 - \frac{1}{2^3} + \frac{1}{4^3} - \frac{1}{5^3} + \frac{1}{7^3} - \frac{1}{8^3} + \&c.$$

&c.

in his Seriebus desunt omnes numeri per ternarium divisibiles: hinc pares dimensiones ex jam inventis deducentur hoc modo. Cum sit

$$\frac{\pi\pi}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \&c., \text{ erit}$$
$$\frac{\pi\pi}{6.9} = \frac{1}{3^2} + \frac{1}{6^2} + \frac{1}{9^2} + \frac{1}{12^2} + \&c. = \frac{\pi\pi}{54},$$

quæ posterior Series continens omnes numeros per ternarium divisi-

divisibiles, si subtrahatur a priore, remanebunt omnes numeri non divisibiles per 3: sicque erit $\frac{8\pi\pi}{54} = \frac{4\pi\pi}{27} = 1 + \frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{7^2} + \&c.$, uti invenimus. CAP. X.

177. Eadem hypothesis $m = 1$, $n = 3$, & $k = \frac{1}{\sqrt{3}}$, ad §. 174. accommodata has præbebit summationes

$$\frac{\pi}{2\sqrt{3}} = 1 - \frac{1}{5} + \frac{1}{7} - \frac{1}{11} + \frac{1}{13} - \frac{1}{17} + \frac{1}{19} - \&c.$$
$$\frac{\pi\pi}{9} = 1 + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{11^2} + \frac{1}{13^2} + \frac{1}{17^2} + \frac{1}{19^2} + \&c.$$
$$\frac{\pi^3}{18\sqrt{3}} = 1 - \frac{1}{5^3} + \frac{1}{7^3} - \frac{1}{11^3} + \frac{1}{13^3} - \frac{1}{17^3} + \frac{1}{19^3} - \&c.$$
&c.

in quarum denominatoribus numeri tantum impares occurrunt exceptis iis, qui per ternarium sunt divisibiles. Ceterum pares dimensiones ex jam cognitis deduci possunt, cum enim sit

$$\frac{\pi\pi}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \&c., \text{ erit}$$
$$\frac{\pi\pi}{8.9} = \frac{1}{3^2} + \frac{1}{9^2} + \frac{1}{15^2} + \frac{1}{21^2} + \&c. = \frac{\pi\pi}{72}$$

quæ Series, omnes numeros impares per 3 divisibiles continens, si subtrahatur a superiore, relinquet Seriem quadratorum numerorum imparium per 3 non divisibilium, eritque

$$\frac{\pi\pi}{9} = 1 + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{11^2} + \frac{1}{13^2} + \&c.$$

178. Si Series in §. §. 172. & 174 inventæ vel addantur vel subtrahantur, obtinebuntur aliæ Series notatu dignæ. Erit scilicet $\frac{k\pi}{2n} + \frac{\pi}{2nk} = \frac{1}{m} + \frac{1}{n-m} - \frac{1}{n+m} - \frac{1}{2n-m} + \frac{1}{2n+m}$

LIB. I. $\frac{1}{2n+m} + \&c. = \frac{(kk+1)\pi}{2nk}$: at est $k = tang.\ \frac{m\pi}{2n} = \frac{fin.\ \frac{m\pi}{2n}}{cof.\ \frac{m\pi}{2n}}$, & $1 + kk = \frac{1}{(cof.\ \frac{m\pi}{2n})^2}$, unde $\frac{2k}{1+kk} = 2\,fin.\ \frac{m\pi}{2n} \times cof.\ \frac{m\pi}{2n} = fin.\ \frac{m\pi}{n}$, quo valore substituto habebimus

$$\frac{\pi}{n\,fin.\ \frac{m\pi}{n}} = \frac{1}{m} + \frac{1}{n-m} - \frac{1}{n+m} - \frac{1}{2n-m} + \frac{1}{2n+m} + \frac{1}{3n-m} - \frac{1}{3n+m} - \&c..$$

Simili modo per subtractionem erit $\frac{\pi}{2nk} - \frac{k\pi}{2n} = \frac{(1-kk)\pi}{2nk} = \frac{1}{m} - \frac{1}{n-m} + \frac{1}{n+m} - \frac{1}{2n-m} + \frac{1}{2n+m} - \frac{1}{3n-m} + \frac{1}{3n+m} - \&c.$, at est $\frac{2k}{1-kk} = tang.\ 2.\ \frac{m\pi}{2n} = tang.\ \frac{m\pi}{n} = \frac{fin.\ \frac{m\pi}{n}}{cof.\ \frac{m\pi}{n}}$, hinc erit

$$\frac{\pi\ cof.\ \frac{m\pi}{n}}{n\ fin.\ \frac{m\pi}{n}} = \frac{1}{m} - \frac{1}{n-m} + \frac{1}{n+m} - \frac{1}{2n-m} + \frac{1}{2n+m} - \frac{1}{3n-m} + \&c..$$

Series Quadratorum & altiorum Potestatum hinc ortæ facilius per differentiationem hinc deducentur infra.

179. Quoniam casus, quibus $m = 1$ & $n = 2$ vel 3, jam evolvimus, ponamus $m = 1$ & $n = 4$; erit $fin.\ \frac{m\pi}{n} = fin.\ \frac{\pi}{4} = \frac{1}{\sqrt{2}}$ & $cof.\ \frac{\pi}{4} = \frac{1}{\sqrt{2}}$. Hinc itaque habebitur

$$\frac{\pi}{2\sqrt{2}} = 1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \frac{1}{13} - \frac{1}{15} + \&c..$$

&c.

$\& \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \frac{1}{15} +$ CAP. X.

&c.. Sit $m = 1$ & $n = 8$, erit $\frac{m\pi}{n} = \frac{\pi}{8}$ & $fin.\ \frac{\pi}{8} =$

$\sqrt{\left(\frac{1}{2} - \frac{1}{2\sqrt{2}}\right)}$ & $cof.\ \frac{\pi}{8} = \sqrt{\left(\frac{1}{2} + \frac{1}{2\sqrt{2}}\right)}$ & $\frac{cof.\ \frac{\pi}{8}}{fin.\ \frac{\pi}{8}} =$

$1 + \sqrt{2}$. Hinc itaque erit

$$\frac{\pi}{4\sqrt{(2-\sqrt{2})}} = 1 + \frac{1}{7} - \frac{1}{9} - \frac{1}{15} + \frac{1}{17} + \frac{1}{23} - \&c.$$

$$\frac{\pi}{8(\sqrt{2}-1)} = 1 - \frac{1}{7} + \frac{1}{9} - \frac{1}{15} + \frac{1}{17} - \frac{1}{23} + \&c.$$

Sit nunc $m = 3$ & $n = 8$, erit $\frac{m\pi}{n} = \frac{3\pi}{8}$ & $fin.\ \frac{3\pi}{8} =$

$\sqrt{\left(\frac{1}{2} + \frac{1}{2\sqrt{2}}\right)}$, & $cof.\ \frac{3\pi}{8} = \sqrt{\left(\frac{1}{2} - \frac{1}{2\sqrt{2}}\right)}$, unde $\frac{cof.\ \frac{3}{8}\pi}{fin.\ \frac{3}{8}\pi} =$

$\frac{1}{\sqrt{2}+1}$; ac prodibunt hæ Series

$$\frac{\pi}{4\sqrt{(2+\sqrt{2})}} = \frac{1}{3} + \frac{1}{5} - \frac{1}{11} - \frac{1}{13} + \frac{1}{19} + \frac{1}{21} - \&c.$$

$$\frac{\pi}{8(\sqrt{2}+1)} = \frac{1}{3} - \frac{1}{5} + \frac{1}{11} - \frac{1}{13} + \frac{1}{19} - \frac{1}{21} + \&c.$$

180. Ex his Seriebus per combinationem naſcuntur:

$$\frac{\pi\sqrt{(2+\sqrt{2})}}{4} = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} - \frac{1}{9} - \frac{1}{11} - \frac{1}{13} - \frac{1}{15} + \frac{1}{17} + \frac{1}{19} + \&c.$$

$$\frac{\pi\sqrt{(2-\sqrt{2})}}{4} = 1 - \frac{1}{3} - \frac{1}{5} + \frac{1}{7} - \frac{1}{9} + \frac{1}{11} + \frac{1}{13} - \frac{1}{15} + \frac{1}{17} - \frac{1}{19} + \&c.$$

$$\frac{\pi(\sqrt{(4+2\sqrt{2})}+\sqrt{2}-1)}{8} = 1 + \frac{1}{3} - \frac{1}{5} + \frac{1}{7} - \frac{1}{9} + \frac{1}{11} - \frac{1}{13} - \frac{1}{15} + \frac{1}{17} + \frac{1}{19} + \&c.$$

$$\frac{\pi(\sqrt{(4+2\sqrt{2})}-\sqrt{2}+1)}{8} = 1 - \frac{1}{3} + \frac{1}{5} + \frac{1}{7} - \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \frac{1}{15} + \frac{1}{17} - \frac{1}{19} + \&c.$$

$$\frac{\pi(\sqrt{2}+1+\sqrt{(4-2\sqrt{2})})}{8} = 1 + \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} - \frac{1}{13} - \frac{1}{15} + \frac{1}{17} + \frac{1}{19} + \&c.$$

$$\frac{\pi(\sqrt{2}+1-\sqrt{(4-2\sqrt{2})})}{8} = 1 - \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \frac{1}{13} - \frac{1}{15} + \frac{1}{17} - \frac{1}{19} - \&c.$$

Simili modo, ponendo $n = 16$ & m vel 1 vel 3 vel 5 vel 7, ulterius progredi licet, hocque modo summæ reperientur Serierum 1, $\frac{1}{3}$, $\frac{1}{5}$, $\frac{1}{7}$, $\frac{1}{9}$, &c., in quibus signorum + & — vicissitudines alias leges sequantur.

181. Si in Seriebus §. 178. inventis bini termini in unam summam colligantur, erit

$$\frac{\pi}{n\,\mathit{sin}.\frac{m\pi}{n}} = \frac{1}{m} + \frac{2m}{nn-mm} - \frac{2m}{4nn-mm} + \frac{2m}{9nn-mm} - \frac{2m}{16nn-mm} + \&c.$$

ideoque

$$\frac{1}{nn-mm} - \frac{1}{4nn-mm} + \frac{1}{9nn-mm} - \&c. = \frac{\pi}{2mn.\,\mathit{sin}.\frac{m\pi}{n}} - \frac{1}{2mm}.$$

Altera vero Series dabit

$$\frac{\pi}{n\,\mathit{tang}.\frac{m}{n}\pi} = \frac{1}{m} - \frac{2m}{nn-mm} - \frac{2m}{4nn-mm} - \frac{2m}{9nn-mm} - \&c.$$

hincque

hincque

$$\frac{1}{nn - mm} + \frac{1}{4nn - mm} + \frac{1}{9nn - mm} + \&c. = \frac{1}{2mm} - \frac{\pi}{2mn\ tang.\frac{m}{n}\pi}.$$

Ex his autem conjunctis nascitur hæc

$$\frac{1}{nn - mm} + \frac{1}{9nn - mm} + \frac{1}{25nn - mm} + \&c. = \frac{\pi\ tang.\frac{m}{2n}\pi}{4mn}.$$

Si in hac Serie sit $n = 1$ & m numerus par quicunque $= 2k$, ob $tang.\ k\pi = 0$, erit semper, nisi sit $k = 0$,

$$\frac{1}{1 - 4kk} + \frac{1}{9 - 4kk} + \frac{1}{25 - 4kk} + \frac{1}{49 - 4kk} + \&c. = 0,$$

sin autem in illa Serie fiat $n = 2$ & m fuerit numerus quicunque impar $= 2k + 1$, ob $\frac{1}{tang.\ \frac{m\pi}{n}} = 0$, erit $\frac{1}{4 - (2k+1)^2} +$

$$\frac{1}{16 - (2k+1)^2} + \frac{1}{36 - (2k+1)^2} + \&c. = \frac{1}{2(2k+1)^2}.$$

182. Multiplicentur Series inventæ per nn sitque $\frac{m}{n} = p$, habebuntur istæ formæ

$$\frac{1}{1 - pp} - \frac{1}{4 - pp} + \frac{1}{9 - pp} - \frac{1}{16 - pp} + \&c. = \frac{\pi}{2p\ sin.\ p\pi} - \frac{1}{2pp}$$

$$\frac{1}{1 - pp} + \frac{1}{4 - pp} + \frac{1}{9 - pp} + \frac{1}{16 - pp} + \&c. = \frac{1}{2pp} - \frac{\pi}{2p\ tang.\ p\pi}.$$

Sit $pp = a$, atque nascentur hæ Series

$$\frac{1}{1-a} - \frac{1}{4-a} + \frac{1}{9-a} - \frac{1}{16-a} + \&c. = \frac{\pi\sqrt{a}}{2a\ sin.\ \pi\sqrt{a}} - \frac{1}{2a}$$

$$\frac{1}{1-a} + \frac{1}{4-a} + \frac{1}{9-a} + \frac{1}{16-a} + \&c. = \frac{1}{2a} - \frac{\pi\sqrt{a}}{2a\ tang.\ \pi\sqrt{a}}.$$

Dummodo ergo a non fuerit numerus negativus nec quadratus integer, summa harum Serierum per Circulum exhiberi poterit.

183.

183. Per reductionem autem exponentialium imaginariorum ad Sinus & Cosinus Arcuum circularium supra traditam poterimus quoque summas harum Serierum assignare si a sit numerus negativus. Cum enim sit $e^{x\surd -1} = cos. x + \surd -1 \times sin. x$ & $e^{-x\surd -1} = cos. x - \surd -1 . sin. x$, erit vicissim, posito $y\surd -1$ loco x; $cos. y\surd -1 = \frac{e^{-y} + e^{y}}{2}$ & $sin. y\surd -1 = \frac{e^{-y} - e^{y}}{2\surd -1}$. Quod si ergo $a = -b$ & $y = \pi\surd b$, erit $cos. \pi\surd -b = \frac{e^{-\pi\surd b} + e^{\pi\surd b}}{2}$ & $sin. \pi\surd -b = \frac{e^{-\pi\surd b} - e^{\pi\surd b}}{2\surd -1}$; ideoque $tang. \pi\surd -b = \frac{e^{-\pi\surd b} - e^{\pi\surd b}}{(e^{-\pi\surd b} + e^{\pi\surd b})\surd -1}$. Hinc erit $\frac{\pi\surd -b}{sin. \pi\surd -b} = \frac{-2\pi\surd b}{e^{-\pi\surd b} - e^{\pi\surd b}}$; & $\frac{\pi\surd -b}{tang. \pi\surd -b} = \frac{(e^{-\pi\surd b} + e^{\pi\surd b})\pi\surd b}{e^{-\pi\surd b} - e^{\pi\surd b}}$. His ergo notatis, erit

$$\frac{1}{1+b} - \frac{1}{4+b} + \frac{1}{9+b} - \frac{1}{16+b} + \&c. = \frac{1}{2b} - \frac{\pi\surd b}{(e^{\pi\surd b} - e^{-\pi\surd b})b}$$

$$\frac{1}{1+b} + \frac{1}{4+b} + \frac{1}{9+b} + \frac{1}{16+b} + \&c. = \frac{(e^{\pi\surd b} + e^{-\pi\surd b})\pi\surd b}{2b(e^{\pi\surd b} - e^{-\pi\surd b})} - \frac{1}{2b}.$$

Eædem hæ Series deduci possunt ex §. 162. adhibendo eandem methodum, qua in in hoc capite sum usus. Quoniam vero hoc pacto reductio Sinuum & Cosinuum Arcuum imaginariorum ad quantitates exponentiales reales, non mediocriter illustratur, hanc explicationem alteri præferendam duxi.

CAPUT XI.

De aliis Arcuum atque Sinuum expressionibus infinitis.

184. Quoniam supra (158.), denotante z Arcum Circuli quemcunque, vidimus esse $\sin. z = z(1 - \frac{zz}{\pi\pi})(1 - \frac{zz}{4\pi\pi})(1 - \frac{zz}{9\pi\pi})(1 - \frac{zz}{16\pi\pi})$ &c., & $\cos. z = (1 - \frac{4zz}{\pi\pi})(1 - \frac{4zz}{9\pi\pi})(1 - \frac{4zz}{25\pi\pi})(1 - \frac{4zz}{49\pi\pi})$ &c., ponamus esse Arcum $z = \frac{m\pi}{n}$, erit $\sin. \frac{m\pi}{n} = \frac{m\pi}{n}(1 - \frac{mm}{nn})(1 - \frac{mm}{4nn})(1 - \frac{mm}{9nn})(1 - \frac{mm}{16nn})$ &c., & $\cos. \frac{m}{n}\pi = (1 - \frac{4mm}{nn})(1 - \frac{4mm}{9nn})(1 - \frac{4mm}{25nn})(1 - \frac{4mm}{49nn})$ &c..
Vel ponatur $2n$ loco n, ut prodeant hæ expressiones

$$\sin. \frac{m\pi}{2n} = \frac{m\pi}{2n}.(\frac{4nn - mm}{4nn}.)(\frac{16nn - mm}{16nn}.)(\frac{36nn - mm}{36nn})\ \&c.$$

$$\cos. \frac{m\pi}{2n} = (\frac{nn - mm}{nn}.)(\frac{9nn - mm}{9nn}.)(\frac{25nn - mm}{25nn}.)(\frac{49nn - mm}{49nn})\&c.,$$

quæ, in Factores simplices resolutæ, dant

$$\sin. \frac{m\pi}{2n} = \frac{m\pi}{2n}.(\frac{2n - m}{2n}.)(\frac{2n + m}{2n}.)(\frac{4n - m}{4n}.)(\frac{4n + m}{4n}.)(\frac{6n - m}{6n})\ \&c.$$

$$cos.\ \frac{m\pi}{2n} = \left(\frac{n-m}{n}\right)\left(\frac{n+m}{n}\right)\left(\frac{3n-m}{3n}\right)\left(\frac{3n+m}{3n}\right)\left(\frac{5n-m}{5n}\right)\left(\frac{5n+m}{5n}\right)\ \&c.$$

Ponatur $n-m$ loco m, quia est $fin.\ \frac{(n-m)\pi}{2n} = cof.\ \frac{m\pi}{2n}$ & $cof.\ \frac{(n-m)\pi}{2n} = fin.\ \frac{m\pi}{2n}$, provenient hæ expressiones.

$$cof.\ \frac{m\pi}{2n} = \left(\frac{(n-m)\pi}{2n}\right)\left(\frac{n+m}{2n}\right)\left(\frac{3n-m}{2n}\right)\left(\frac{3n+m}{4n}\right)\left(\frac{5n-m}{4n}\right)\left(\frac{5n+m}{6n}\right)\ \&c.$$

$$fin.\ \frac{m\pi}{2n} = \frac{m}{n}\left(\frac{2n-m}{n}\right)\left(\frac{2n+m}{3n}\right)\left(\frac{4n-m}{3n}\right)\left(\frac{4n+m}{5n}\right)\left(\frac{6n-m}{5n}\right)\ \&c.$$

185. Cum igitur pro Sinu & Cofinu Anguli $\frac{m\pi}{2n}$ binæ habeantur expreffiones, fi eæ inter fe comparentur dividendo, erit $1 = \frac{\pi}{2}\cdot\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{3}{4}\cdot\frac{5}{4}\cdot\frac{5}{6}\cdot\frac{7}{6}\cdot\frac{7}{8}\cdot\frac{9}{8}.$ &c., ideoque $\frac{\pi}{2} = \frac{2.\ 2.\ 4.\ 4.\ 6.\ 6.\ 8.\ 8.\ 10.\ 10.\ 12.\ 12.}{1.\ 3.\ 3.\ 5.\ 5.\ 7.\ 7.\ 9.\ 9.\ 11.\ 11.\ 13.}$ &c., quæ eft expreffio pro Peripheria Circuli, quam WALLISIUS invenit in *Arithmetica infinitorum*. Similes autem huic innumeras expreffiones exhibere licet ope primæ expreffionis pro Sinu; ex ea enim deducitur fore:

$$\frac{\pi}{2} = \frac{n}{m}.fin.\frac{m\pi}{2n}\left(\frac{2n}{2n-m}\right)\left(\frac{2n}{2n+m}\right)\left(\frac{4n}{4n-m}\right)\left(\frac{4n}{4n+m}\right)\left(\frac{6n}{6n-m}\right)\&c.,$$

quæ, pofito $\frac{m}{n} = 1$, præbet illam ipfam WALLISII formulam.

Sit

Sit ergo $\frac{m}{n} = \frac{1}{2}$, ob *fin.* $\frac{1}{4}\pi = \frac{1}{\sqrt{2}}$, erit

$$\frac{\pi}{2} = \frac{\sqrt{2}}{1} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{8}{7} \cdot \frac{8}{9} \cdot \frac{12}{11} \cdot \frac{12}{13} \cdot \frac{16}{15} \cdot \frac{16}{17} \cdot \&c.$$

Sit $\frac{m}{n} = \frac{1}{3}$, ob *fin.* $\frac{1}{6}\pi = \frac{1}{2}$, erit

$$\frac{\pi}{2} = \frac{3}{2} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdot \frac{12}{11} \cdot \frac{12}{13} \cdot \frac{18}{17} \cdot \frac{18}{19} \cdot \frac{24}{23} \ \&c.$$

Quod si Expressio *Wallisiana* dividatur per illam ubi $\frac{m}{n} = \frac{1}{2}$, erit $\sqrt{2} = \frac{2.2.6.6.10.10.14.14.18.18.}{1.3.5.7.\ 9.11.13.15.17.19.}$ &c.

186. Quoniam Tangens cujusque Anguli æquatur Sinui per Cosinum diviso, Tangens quoque per hujusmodi Factores infinitos exprimi poterit. Quod si autem prima Sinus expressio dividatur per alteram Cosinus expressionem, erit

$$tang.\ \frac{m\pi}{2n} = \frac{m}{n-m}\left(\frac{2n-m}{n+m}\right)\left(\frac{2n+m}{3n-m}\right)\left(\frac{4n-m}{3n+m}\right)\left(\frac{4n+m}{5n-m}\right) \&c.,$$

&

$$cot.\ \frac{m\pi}{2n} = \frac{n-m}{m}\left(\frac{n+m}{2n-m}\right)\left(\frac{3n-m}{2n+m}\right)\left(\frac{3n+m}{4n-m}\right)\left(\frac{5n-m}{4n+m}\right) \&c..$$

Simili modo autem Secantes & Cosecantes exprimentur

$$sec.\ \frac{m\pi}{2n} = \left(\frac{n}{n-m}\right)\left(\frac{n}{n+m}\right)\left(\frac{3n}{3n-m}\right)\left(\frac{3n}{3n+m}\right)\left(\frac{5n}{5n-m}\right)\left(\frac{5n}{5n+m}\right) \&c.$$

$$cosec.\ \frac{m\pi}{2n} = \frac{n}{m}\left(\frac{n}{2n-m}\right)\left(\frac{3n}{2n+m}\right)\left(\frac{3n}{4n-m}\right)\left(\frac{5n}{4n+m}\right)\left(\frac{5n}{6n-m}\right) \&c.$$

Sin autem alteræ Sinuum & Cosinuum formulæ combinentur, erit

$$tang.\ \frac{m\pi}{2n} = \frac{\pi}{2}\cdot\frac{m}{n-m}\cdot\frac{1(2n-m)}{2(n+m)}\cdot\frac{3(2n+m)}{2(3n-m)}\cdot\frac{3(4n-m)}{4(3n+m)}.\&c.$$

$$cot.\ \frac{m\pi}{2n} = \frac{\pi}{2}\cdot\frac{n-m}{m}\cdot\frac{1(n+m)}{2(2n-m)}\cdot\frac{3(3n-m)}{2(2n+m)}\cdot\frac{3(3n+m)}{4(4n-m)}.\&c.$$

$$sec.\ \frac{m\pi}{2n} = \frac{2}{\pi}\cdot\frac{n}{n-m}\cdot\frac{2n}{n+m}\cdot\frac{2n}{3n-m}\cdot\frac{4n}{3n+m}\cdot\frac{4n}{5n-m}.\&c.$$

$$cosec.\ \frac{m\pi}{2n} = \frac{2}{\pi}\cdot\frac{n}{m}\cdot\frac{2n}{2n-m}\cdot\frac{2n}{2n+m}\cdot\frac{4n}{4n-m}\cdot\frac{4n}{4n+m}.\&c.$$

187. Si loco m ſcribatur k, ſimilique modo Anguli $\frac{k\pi}{2n}$ Sinus & Coſinus definiantur, ac per has expreſſiones illæ priores dividantur, prodibunt iſtæ formulæ

$$\frac{fin.\ \frac{m\pi}{2n}}{fin.\ \frac{k\pi}{2n}} = \frac{m}{k}\cdot\frac{2n-m}{2n-k}\cdot\frac{2n+m}{2n+k}\cdot\frac{4n-m}{4n-k}\cdot\frac{4n+m}{4n+k}.\ \&c.$$

$$\frac{fin.\ \frac{m\pi}{2n}}{cof.\ \frac{k\pi}{2n}} = \frac{m}{n-k}\cdot\frac{2n-m}{n+k}\cdot\frac{2n+m}{3n-k}\cdot\frac{4n-m}{3n+k}\cdot\frac{4n+m}{5n-k}.\ \&c.$$

$$\frac{cof.\ \frac{m\pi}{2n}}{cof.\ \frac{k\pi}{2n}} = \left(\frac{n-m}{n-k}\right)\left(\frac{n+m}{n+k}\right)\left(\frac{3n-m}{3n-k}\right)\left(\frac{3n+m}{3n+k}\right)\left(\frac{5n-m}{5n-k}\right).\ \&c.$$

$$\frac{cof.\ \frac{m\pi}{2n}}{fin.\ \frac{k\pi}{2n}} = \left(\frac{n-m}{k}\right)\left(\frac{n+m}{2n-k}\right)\left(\frac{3n-m}{2n+k}\right)\left(\frac{3n+m}{4n-k}\right)\left(\frac{5n-m}{4n+k}\right).\ \&c.$$

Sumto ergo pro $\frac{k\pi}{2n}$ ejuſmodi Angulo cujus Sinus & Coſinus dentur, per hos licebit alius cujuſcunque Anguli $\frac{m\pi}{2n}$ Sinum & Coſinum determinare.

188. Viciſſim igitur hujuſmodi expreſſionum, quæ ex Factoribus

ctoribus infinitis constant, valores veri vel per Circuli Peripheriam, vel per Sinus & Cosinus Angulorum datorum assignari possunt, quod ipsum non parvi est momenti, cum etiamnunc aliæ methodi non constent, quarum ope hujusmodi productorum infinitorum valores exhiberi queant. Ceterum vero hujusmodi expressiones parum utilitatis afferunt, ad valores cum ipsius π tum Sinuum Cosinuumve Angulorum $\frac{m\pi}{2n}$ per approximationem eruendos. Quanquam enim isti Factores $\frac{\pi}{2} = 2(1 - \frac{1}{9})(1 - \frac{1}{25})(1 - \frac{1}{49})$ &c., in fractionibus decimalibus non difficulter in se multiplicantur, tamen nimis multi termini in computum duci deberent, si valorem ipsius π ad decem tantum figuras justum invenire vellemus. CAP. XI.

189. Præcipuus autem usus hujusmodi expressionum, etsi infinitarum, in inventione Logarithmorum versatur, in quo negotio Factorum utilitas tanta est, ut sine illis Logarithmorum supputatio esset difficillima. Ac primo quidem, cum sit $\pi = 4(1 - \frac{1}{9})(1 - \frac{1}{25})(1 - \frac{1}{49})$ &c., erit, sumendis Logarithmis, $l\pi = l4 + l(1 - \frac{1}{9}) + l(1 - \frac{1}{25}) + l(1 - \frac{1}{49}) +$ &c., vel $l\pi = l2 - l(1 - \frac{1}{4}) - l(1 - \frac{1}{16}) - l(1 - \frac{1}{36})$ — &c., sive Logarithmi communes sive hyperbolici sumantur. Quoniam vero ex Logarithmis hyperbolicis vulgares facile reperiuntur, insigne compendium adhiberi poterit ad Logarithmum hyperbolicum ipsius π inveniendum.

190. Cum igitur, Logarithmis hyperbolicis sumendis, sit $l(1 - x) = -x - \frac{xx}{2} - \frac{x^3}{3} - \frac{x^4}{4} -$ &c., si hoc modo singuli termini evolvantur, erit

$$l\pi = l4 \begin{cases} -\frac{1}{9} - \frac{1}{2.\ 9^2} - \frac{1}{3.\ 9^3} - \frac{1}{4.\ 9^4} - \&c. \\ -\frac{1}{25} - \frac{1}{2.\ 25^2} - \frac{1}{3.\ 25^3} - \frac{1}{4.\ 25^4} - \&c. \\ -\frac{1}{49} - \frac{1}{2.\ 49^2} - \frac{1}{3.\ 49^3} - \frac{1}{4.\ 49^4} - \&c. \\ \qquad \&c. \end{cases}$$

In his Seriebus numero infinitis verticaliter descendendo ejusmodi prodeunt Series, quarum summas supra jam invenimus, quare si brevitatis gratia ponamus

$$A = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \&c.$$

$$B = 1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \frac{1}{9^4} + \&c.$$

$$C = 1 + \frac{1}{3^6} + \frac{1}{5^6} + \frac{1}{7^6} + \frac{1}{9^6} + \&c.$$

$$D = 1 + \frac{1}{3^8} + \frac{1}{5^8} + \frac{1}{7^8} + \frac{1}{9^8} + \&c.$$

&c.

erit $l\pi = l4 - (A-1) - \frac{1}{2}(B-1) - \frac{1}{3}(C-1) - \frac{1}{4}(D-1) -$ &c.

Est vero, summis supra inventis proxime exprimendis,

$$\begin{aligned} A &= 1,23370055013616982735431 \\ B &= 1,01467803160419205454625 \\ C &= 1,00144707664094212190647 \\ D &= 1,00015517902529611930298 \\ E &= 1,00001704136304482550816 \\ F &= 1,00000188584858311957590 \\ G &= 1,00000020924051921150010 \\ H &= 1,00000002323715737915670 \end{aligned}$$

$I =$

I	$=$	1, 00000000258143755665977
K	$=$	1, 00000000028680769745558
L	$=$	1, 00000000003186677514044
M	$=$	1, 00000000000354072294392
N	$=$	1, 00000000000039341246691
O	$=$	1, 00000000000004371244859
P	$=$	1, 00000000000000485693682
Q	$=$	1, 00000000000000053965957
R	$=$	1, 00000000000000005996217
S	$=$	1, 00000000000000000666246
T	$=$	1, 00000000000000000074027
V	$=$	1, 00000000000000000008225
W	$=$	1, 00000000000000000000913
X	$=$	1, 00000000000000000000101

Hinc fine tædiofo calculo reperitur Logarithmus hyperbolicus ipfius $\pi = 1,\ 14472988584940017414342$, qui fi multiplicetur per $0,\ 43429$ &c., prodit Logarithmus vulgaris ipfius $\pi = 0,\ 49714987269413385435126$.

191. Quia porro tam Sinum quam Cofinum Anguli $\frac{m\pi}{2n}$ expreffum habemus per Factores numero infinitos, utriufque Logarithmum commode exprimere poterimus. Erit autem ex formulis primo inventis

$$l\,\mathit{fin}.\frac{m\pi}{2n} = l\,\pi + l\,\frac{m}{2n} + l\left(1 - \frac{mm}{4nn}\right) + l\left(1 - \frac{mm}{16nn}\right) + l\left(1 - \frac{mm}{36nn}\right) \&c.$$

$$l\,\mathit{cof}.\frac{m\pi}{2n} = l\left(1 - \frac{mm}{nn}\right) + l\left(1 - \frac{mm}{9nn}\right) + l\left(1 - \frac{mm}{25nn}\right) + l\left(1 - \frac{mm}{49nn}\right) + \&c.$$

Hinc primum Logarithmi hyperbolici, ut ante, per Series maxime convergentes facile exprimuntur. Ne autem præter necceffi-

LIB. I. necessitatem Series infinitas multiplicemus, terminos priores actu in Logarithmis involutos relinquamus, eritque

$$l\,sin.\frac{m\pi}{2n} = l\pi + lm + l(2n-m) + l(2n+m) - l8 - 3ln$$
$$-\frac{mm}{16nn} - \frac{m^4}{2.16^2n^4} - \frac{m^6}{3.16^3n^6} - \frac{m^8}{4.16^4n^8} - \&c.$$
$$-\frac{mm}{36nn} - \frac{m^4}{2.36^2n^4} - \frac{m^6}{3.36^3n^6} - \frac{m^8}{4.36^4n^8} - \&c.$$
$$-\frac{mm}{64nn} - \frac{m^4}{2.64^2n^4} - \frac{m^6}{3.64^3n^6} - \frac{m^8}{4.64^4n^8} - \&c.$$
&c.

$$l\,cof.\frac{m\pi}{2n} = l(n-m) + l(n+m) - 2\,ln$$
$$-\frac{mm}{9nn} - \frac{m^4}{2.\ 9^2n^4} - \frac{m^6}{3.\ 9^3n^6} - \frac{m^8}{4.\ 9^4n^8} - \&c.$$
$$-\frac{mm}{25nn} - \frac{m^4}{2.25^2n^4} - \frac{m^6}{3.25^3n^6} - \frac{m^8}{4.25^4n^8} - \&c.$$
$$-\frac{mm}{49nn} - \frac{m^4}{2.49^2n^4} - \frac{m^6}{3.49^3n^6} - \frac{m^8}{4.49^4n^8} - \&c.$$
&c.

192. Occurrunt ergo in his Seriebus singulæ Potestates pares ipsius $\frac{m}{n}$, quæ sunt multiplicatæ per Series, quarum summas jam supra assignavimus. Erit nempe

$$l\,sin.\frac{m\pi}{2n} = lm + l(2n-m) + l(2n+m) - 3\,ln + l\pi - l8$$
$$-\frac{mm}{nn}\left(\frac{1}{4^2} + \frac{1}{6^2} + \frac{1}{8^2} + \frac{1}{10^2} + \frac{1}{12^2} + \&c.\right)$$
$$-\frac{m^4}{2n^4}\left(\frac{1}{4^4} + \frac{1}{6^4} + \frac{1}{8^4} + \frac{1}{10^4} + \frac{1}{12^4} + \&c.\right)$$
$$-\frac{m^6}{3n^6}\left(\frac{1}{4^6} + \frac{1}{6^6} + \frac{1}{8^6} + \frac{1}{10^6} + \frac{1}{12^6} + \&c.\right)$$
$$-\frac{m^8}{4n^8}\left(\frac{1}{4^8} + \frac{1}{6^8} + \frac{1}{8^8} + \frac{1}{10^8} + \frac{1}{12^8} + \&c.\right)$$
&c.

l cof.

$$l\cos.\frac{m\pi}{2n}=l(n-m)+l(n+m)-2ln$$

$$-\frac{mm}{nn}\left(\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{9^2}+\&c.\right)$$
$$-\frac{m^4}{2n^4}\left(\frac{1}{3^4}+\frac{1}{5^4}+\frac{1}{7^4}+\frac{1}{9^4}+\&c.\right)$$
$$-\frac{m^6}{3n^6}\left(\frac{1}{3^6}+\frac{1}{5^6}+\frac{1}{7^6}+\frac{1}{9^6}+\&c.\right)$$
$$-\frac{m^8}{4n^8}\left(\frac{1}{3^8}+\frac{1}{5^8}+\frac{1}{7^8}+\frac{1}{9^8}+\&c.\right)$$
&c.

Serierum posteriorum modo ante (§. 190) summæ sunt exhibitæ; priores Series quidem ex his derivari possent, at, quo facilius ad usum transferri queant, earum summas pariter hic adjiciam.

193. Quod si ergo, brevitatis gratia, ponamus

$$\alpha=\frac{1}{2^2}+\frac{1}{4^2}+\frac{1}{6^2}+\frac{1}{8^2}+\&c.$$
$$\beta=\frac{1}{2^4}+\frac{1}{4^4}+\frac{1}{6^4}+\frac{1}{8^4}+\&c.$$
$$\gamma=\frac{1}{2^6}+\frac{1}{4^6}+\frac{1}{6^6}+\frac{1}{8^6}+\&c.$$
$$\delta=\frac{1}{2^8}+\frac{1}{4^8}+\frac{1}{6^8}+\frac{1}{8^8}+\&c.$$
&c.

erunt summæ in numeris proxime expressæ hæ:

$\alpha = 0,41123351671205660911810$
$\beta = 0,06764520210694613696975$
$\gamma = 0,01589598534350701780804$
$\delta = 0,00392217717264822007570$
$\epsilon = 0,00097753376477325984898$
$\zeta = 0,00024420070472492872274$

$\eta = 0,00006103889453949332915$
$\theta = 0,00001525902225127269977$
$\iota = 0,00000381471182744318008$
$\kappa = 0,00000095367522617534053$
$\lambda = 0,00000023841863595259154$
$\mu = 0,00000005960464832831555$
$\nu = 0,00000001490116141589813$
$\xi = 0,00000000372529031233986$
$o = 0,00000000093132257548284$
$\pi = 0,00000000023283064370807$
$\rho = 0,00000000005820766091685$
$\sigma = 0,00000000001455191522858$
$\tau = 0,00000000000363797880710$
$\upsilon = 0,00000000000090949470177$
$\phi = 0,00000000000022737367544$
$\chi = 0,00000000000005684341886$
$\psi = 0,00000000000001421085471$
$\omega = 0,00000000000000355271367$

reliquæ ſummæ in ratione quadrupla deſcreſcunt.

194. His ergo in ſubſidium vocatis, erit

$$l\,\mathit{ſin}.\frac{m\pi}{2n} = lm + l(2n-m) + l(2n+m) - 3ln + l\pi - l8$$
$$-\frac{mm}{nn}\left(\alpha - \frac{1}{2^2}\right) - \frac{m^4}{2n^4}\left(\beta - \frac{1}{2^4}\right) - \frac{m^6}{3n^6}\left(\gamma - \frac{1}{2^6}\right)$$
$$-\ \&c.$$

$$l\,\mathit{coſ}.\frac{m\pi}{2n} = l(n-m) + l(n+m) - 2ln$$
$$-\frac{mm}{nn}(A-1) - \frac{m^4}{2n^4}(B-1) - \frac{m^6}{3n^6}(C-1) - \&c.,$$

quoniam igitur Logarithmi $l\pi$ & $l8$ dantur, erit

Logarithmi

Logarithmus hyperbolicus Sinus Anguli $\frac{m}{n}\cdot 90^\circ =$

$$lm + l(2n - m) + l(2n + m) - 3ln$$

$$- \quad 0,93471165583043575410$$
$$- \frac{m^2}{n^2}\cdot 0,16123351671205660911$$
$$- \frac{m^4}{n^4}\cdot 0,00257260105347306848$$
$$- \frac{m^6}{n^6}\cdot 0,00009032844783567260$$
$$- \frac{m^8}{n^8}\cdot 0,00000398179316205501$$
$$- \frac{m^{10}}{n^{10}}\cdot 0,00000019425295465196$$
$$- \frac{m^{12}}{n^{12}}\cdot 0,00000001001328748812$$
$$- \frac{m^{14}}{n^{14}}\cdot 0,00000000053404135618$$
$$- \frac{m^{16}}{n^{16}}\cdot 0,00000000002914859658$$
$$- \frac{m^{18}}{n^{18}}\cdot 0,00000000000161797979$$
$$- \frac{m^{20}}{n^{20}}\cdot 0,00000000000009097690$$
$$- \frac{m^{22}}{n^{22}}\cdot 0,00000000000000516827$$
$$- \frac{m^{24}}{n^{24}}\cdot 0,00000000000000029607$$
$$- \frac{m^{26}}{n^{26}}\cdot 0,00000000000000001708$$
$$- \frac{m^{28}}{n^{28}}\cdot 0,00000000000000000099$$
$$- \frac{m^{30}}{n^{30}}\cdot 0,00000000000000000005$$

LIB. I.

At *Logarithmus hyperbolicus Cosinus Ang.* $\frac{m}{n}$ $90^\circ =$

$$l(n-m)+l(n+m)-2\,ln$$

$$-\frac{m^2}{n^2}\cdot 0,23370055013616982735$$
$$-\frac{m^4}{n^4}\cdot 0,00733901580209602727$$
$$-\frac{m^6}{n^6}\cdot 0,00048235888031404063$$
$$-\frac{m^8}{n^8}\cdot 0,00003879475632402982$$
$$-\frac{m^{10}}{n^{10}}\cdot 0,00000340827260896510$$
$$-\frac{m^{12}}{n^{12}}\cdot 0,00000031430809718659$$
$$-\frac{m^{14}}{n^{14}}\cdot 0,00000002989150274450$$
$$-\frac{m^{16}}{n^{16}}\cdot 0,00000000290464467239$$
$$-\frac{m^{18}}{n^{18}}\cdot 0,00000000028682639518$$
$$-\frac{m^{20}}{n^{20}}\cdot 0,00000000002868076974$$
$$-\frac{m^{22}}{n^{22}}\cdot 0,00000000000189697956$$
$$-\frac{m^{24}}{n^{24}}\cdot 0,00000000000029506024$$
$$-\frac{m^{26}}{n^{26}}\cdot 0,00000000000003026249$$
$$-\frac{m^{28}}{n^{28}}\cdot 0,00000000000000312232$$
$$-\frac{m^{30}}{n^{30}}\cdot 0,00000000000000032379$$
$$-\frac{m^{32}}{n^{32}}\cdot 0,00000000000000003373$$
$$-\frac{m^{34}}{n^{34}}\cdot 0,00000000000000000352$$

$$- \frac{m^{36}}{n^{36}} \cdot 0,000000000000000000037$$

$$- \frac{m^{38}}{n^{38}} \cdot 0,0000000000000000000004$$

195. Si isti Sinuum & Cosinuum Logarithmi hyperbolici multiplicentur per 0,4342944819 &c., prodibunt eorundem Logarithmi vulgares ad Radium = 1 relati. Quoniam vero in Tabulis Logarithmus Sinus totius statui solet = 10, quo Logarithmi tabulares Sinuum & Cosinuum obtineantur, post multiplicationem addi debet 10. Hinc erit

Logarithmus tabularis Sinus Anguli $\frac{m}{n} 90^\circ =$

$$l\,m + l(2n - m) + l(2n + m) - 3\,l\,n$$

$+$		$9,594059885702190$
$-$	$\frac{m^2}{n^2}$	$0,070022826605901$
$-$	$\frac{m^4}{n^4}$	$0,001117266441661$
$-$	$\frac{m^6}{n^6}$	$0,000039229146453$
$-$	$\frac{m^8}{n^8}$	$0,000001729270798$
$-$	$\frac{m^{10}}{n^{10}}$	$0,000000084362986$
$-$	$\frac{m^{12}}{n^{12}}$	$0,000000004348715$
$-$	$\frac{m^{14}}{n^{14}}$	$0,000000000231931$
$-$	$\frac{m^{16}}{n^{16}}$	$0,000000000012659$
$-$	$\frac{m^{18}}{n^{18}}$	$0,000000000000702$
$-$	$\frac{m^{20}}{n^{20}}$	$0,000000000000039$

LIB. I.

Logarithmus tabularis Cosinus Anguli $\frac{m}{n}\, 90^{\circ} =$

$$l(n - m) + l(n + m) - 2\, l\, n$$

$+$		10, 000000000000000
$-$	$\frac{m^2}{n^2}$.	0, 101494859341892
$-$	$\frac{m^4}{n^4}$.	0, 003187294065451
$-$	$\frac{m^6}{n^6}$.	0, 000209485800017
$-$	$\frac{m^8}{n^8}$.	0, 000016848348597
$-$	$\frac{m^{10}}{n^{10}}$.	0, 000001480193986
$-$	$\frac{m^{12}}{n^{12}}$.	0, 000000136502272
$-$	$\frac{m^{14}}{n^{14}}$.	0, 000000012981715
$-$	$\frac{m^{16}}{n^{16}}$.	0, 000000001261471
$-$	$\frac{m^{18}}{n^{18}}$.	0, 000000000124567
$-$	$\frac{m^{20}}{n^{20}}$.	0, 000000000012456
$-$	$\frac{m^{22}}{n^{22}}$.	0, 000000000001258
$-$	$\frac{m^{24}}{n^{24}}$.	0, 000000000000128
$-$	$\frac{m^{26}}{n^{26}}$.	0, 000000000000013

196. Harum ergo formularum ope inveniri possunt Logarithmi Sinuum & Cosinuum quorumvis Angulorum tam hyperbolici quam vulgares, etiam ignoratis ipsis Sinibus & Cosinibus. Ex Logarithmis autem Sinuum & Cosinuum per solam subtractionem inveniuntur Logarithmi Tangentium, Co-

tangen-

tangentium, & Secantium, Cosecantiumque, quamobrem pro his peculiaribus formulis non erit opus. Ceterum notandum est numerorum m, n, $n - m$, $n + m$, &c. Logarithmos hyperbolicos accipi oportere, cum Logarithmi hyperbolici Sinuum Cosinuumque quæruntur, vulgares autem, cum tales ope posteriorum formularum sunt indagandi. Præterea $m : n$ denotat rationem, quam Angulus propositus habet ad Angulum rectum; sicque, cum Sinus Angulorum semirecto majorum æquentur Cosinibus Angulorum semirecto minorum ac vicissim, fractio $\frac{m}{n}$ nunquam major accipienda erit quam $\frac{1}{2}$, hancque ob rem termini illi multo magis convergent, ut semissis instituto sufficere possit. CAP. XI.

197. Antequam hoc argumentum relinquamus, aptiorem aperiamus modum Tangentes & Secantes quorumvis Angulorum inveniendi, quam Caput præcedens suppeditat. Quanquam enim Tangentes & Secantes per Sinus & Cosinus determinantur; tamen hoc fit per divisionem, quæ operatio in tantis numeris nimis est operosa. Ac Tangentes quidem & Cotangentes jam supra (§. 136.) exhibuimus, verum illo loco rationem formularum reddere non licuit, quam huic Capiti reservavimus.

198. Ex §. 181. ergo primum expressionem pro Tangente Anguli $\frac{m}{2n}\pi$ elicimus. Cum enim sit $\frac{1}{nn - mm} + \frac{1}{9nn - mm} + \frac{1}{25nn - mm} + \&c. = \frac{\pi}{4mn}.\ tang.\ \frac{m}{2n}\pi$ erit $tang.\ \frac{m}{2n}\pi = \frac{4mn}{\pi}\left(\frac{1}{nn - mm} + \frac{1}{9nn - mm} + \frac{1}{25nn - mm} + \&c.\right)$. Cum deinde sit $\frac{1}{nn - mm} + \frac{1}{4nn - mm} + \frac{1}{9nn - mm} + \&c. = \frac{1}{2mm} - \frac{\pi}{2mn}.cot.\ \frac{m}{n}\pi$, si pro n scribamus $2n$ erit

$$cot.\ \frac{m}{2n}\pi = \frac{2n}{m\pi} - \frac{4nm}{\pi}\left(\frac{1}{4nn - mm} + \frac{1}{16nn - mm} + \frac{1}{36nn - mm}\right.$$

LIB. I. $\frac{1}{36nn - mm}$ + &c.). Convertantur hæ fractiones, præter primas, quippe quæ facile in computum ducuntur, in Series infinitas, erit

$$tang.\ \frac{m}{2n}\pi = \frac{mn}{nn - mm} \cdot \frac{4}{\pi}$$
$$+ \frac{4}{\pi}\left(\frac{m}{3^2 n} + \frac{m^3}{3^4 n^3} + \frac{m^5}{3^6 n^5} + \&c.\right)$$
$$+ \frac{4}{\pi}\left(\frac{m}{5^2 n} + \frac{m^3}{5^4 n^3} + \frac{m^5}{5^6 n^5} + \&c.\right)$$
$$+ \frac{4}{\pi}\left(\frac{m}{7^2 n} + \frac{m^3}{7^4 n^3} + \frac{m^5}{7^6 n^5} + \&c.\right)$$
$$\&c.$$

$$cot.\ \frac{m}{2n}\pi = \frac{n}{m} \cdot \frac{2}{\pi} - \frac{mn}{4nn - mm} \cdot \frac{4}{\pi}$$
$$- \frac{4}{\pi}\left(\frac{m}{4^2 n} + \frac{m^3}{4^4 n^3} + \frac{m^5}{4^6 n^5} + \&c.\right)$$
$$- \frac{4}{\pi}\left(\frac{m}{6^2 n} + \frac{m^3}{6^4 n^3} + \frac{m^5}{6^6 n^5} + \&c.\right)$$
$$- \frac{4}{\pi}\left(\frac{m}{8^2 n} + \frac{m^3}{8^4 n^3} + \frac{m^5}{8^6 n^5} + \&c.\right)$$
$$\&c.$$

198. At ex valore ipsius π cognito reperitur

$\frac{1}{\pi}$ = 0, 318309886183790671537767926745028724, deinde hic eædem Series occurrunt, quas supra litteris A, B, C, D, &c., & α, β, γ, δ, &c., indicavimus. His ergo notatis, erit

$$tang.\ \frac{m}{2n}\pi = \frac{mn}{nn - mm} \cdot \frac{4}{\pi} + \frac{m}{n} \cdot \frac{4}{\pi}(A - 1) + \frac{m^3}{n^3} \times \frac{4}{\pi}(B - 1) + \frac{m^5}{n^5} \cdot \frac{4}{\pi}(C - 1) + \frac{m^7}{n^7} \cdot \frac{4}{\pi}(D - 1)\ \&c.$$

Deinde erit pro Cotangente

cot.

$$\cot. \frac{m}{2n}\varpi = \frac{n}{m}\cdot\frac{2}{\varpi} - \frac{4mn}{4nn - mm}\cdot\frac{1}{\varpi} - \frac{m}{n}\cdot\frac{4}{\pi}(\alpha - \frac{1}{2^2}) - \frac{m^3}{n^3}\cdot\frac{4}{\varpi}(\beta - \frac{1}{2^4}) - \frac{m^5}{n^5}\cdot\frac{4}{\pi}(\gamma - \frac{1}{2^6}) - \&c.,$$ CAP. XI.

atque ex his formulis natæ ſunt expreſſiones, quas ſupra (§. 135.) pro Tangente & Cotangente dedimus; ſimul vero (§. 137.) oſtendimus, quomodo ex Tangentibus & Cotangentibus inventis per ſolam additionem & ſubtractionem Secantes & Coſecantes reperiantur. Harum ergo regularum ope univerſus Canon Sinuum, Tangentium & Secantium, eorumque Logarithmorum multo facilius ſupputari poſſet, quam quidem hoc a primis conditoribus eſt factum.

CAPUT XII.

De reali Functionum fractarum evolutione.

199. JAm ſupra, in Capite ſecundo, methodus eſt tradita Functionem quamcunque fractam in tot partes reſolvendi quot ejus denominator habeat Factores ſimplices; hi enim præbent denominatores fractionum illarum partialium. Ex quo manifeſtum eſt, ſi denominator quos habeat Factores ſimplices imaginarios, fractiones quoque inde ortas fore imaginarias: his ergo caſibus parum juvabit fractionem realem in imaginarias reſolviſſe. Cum igitur oſtendiſſem omnem Functionem integram, qualis eſt denominator cujuſvis fractionis, quantumvis Factoribus ſimplicibus imaginariis ſcateat, tamen in Factores duplices, ſeu ſecundæ dimenſionis, reales ſemper reſolvi poſſe; hoc modo in reſolutione fractionum quantitates imaginariæ evitari poterunt, ſi pro denominatoribus fractionum partialium non Factores denominatoris principalis ſimplices, ſed duplices reales aſſumamus.

200. Sit igitur proposita hæc Functio fracta $\frac{M}{N}$, ex qua tot fractiones simplices secundum methodum supra expositam eliciantur, quot denominator N habuerit Factores simplices reales. Sit autem, loco imaginariorum, hæc expressio $pp - 2pqz.cos.\Phi + qqzz$ Factor ipsius N; &, quoniam in hoc negotio numeratorem & denominatorem in forma evoluta contemplari oportet, sit hæc fractio proposita

$$\frac{A + Bz + Cz^2 + Dz^3 + Ez^4 + \&c.}{(pp - 2pqz.cos.\Phi + qqzz)(\alpha + \beta z + \gamma zz + \delta z^3 + \&c.)},$$

ac ponatur fractio partialis ex denominatoris Factore $pp - 2pqz.cos.\Phi + qqzz$ oriunda hæc: $\frac{\mathrm{A} + \mathfrak{a}z}{pp - 2pqz.cos.\Phi + qqzz}$, quoniam enim variabilis z in denominatore duas habet dimensiones, in numeratore unam habere poterit, non vero plures; alias enim integra Functio contineretur, quam seorsim elici oportet.

201. Sit, brevitatis gratia, numerator $A + Bz + Cz^2 + \&c. = M$ & alter denominatoris Factor $\alpha + \beta z + \gamma z^2 + \&c. = Z$; ponatur altera pars ex denominatoris Factore Z oriunda $= \frac{Y}{Z}$, eritque $Y = \frac{M - \mathrm{A}Z - \mathfrak{a}Zz}{pp - 2pqz.cos.\Phi + qqzz}$, quæ expressio Functio integra ipsius z esse debet, ideoque necesse est ut $M - \mathrm{A}Z - \mathfrak{a}zZ$ divisibile sit per $pp - 2pqz.cos.\Phi + qqzz$. Evanescet ergo $M - \mathrm{A}Z - \mathfrak{a}zZ$, si ponatur $pp - 2pqz \times cos.\Phi + qqzz = 0$, hoc est si ponatur tam $z = \frac{p}{q}(cos.\Phi + \sqrt{-1}.sin.\Phi)$ quam $z = \frac{p}{q}(cos.\Phi - \sqrt{-1}.sin.\Phi)$; sit $\frac{p}{q} = f$, eritque $z^n = f^n(cos.n\Phi \pm \sqrt{-1}.sin.n\Phi)$. Duplex ergo hic valor pro z substitutus duplicem dabit æquationem, unde ambas incognitas constantes A & $\mathfrak{a}$ definire licet.

202. Facta ergo hac substitutione, æquatio $M = \mathrm{A}Z + \mathfrak{a}Zz$ evoluta hanc duplicem dabit æquationem

$A +$

CAP. XII.

$$\left.\begin{array}{l} A + Bf.\,\mathrm{cof.}\,\Phi + Cff.\,\mathrm{cof.}\,2\Phi + Df^3.\,\mathrm{cof.}\,3\Phi + \&c. \\ \pm(Bf.\,\mathrm{fin.}\,\Phi + Cff.\,\mathrm{fin.}\,2\Phi + Df^3.\,\mathrm{fin.}\,3\Phi + \&c.)\sqrt{-1} \end{array}\right\} =$$

$$\left\{\begin{array}{l} \mathrm{A}(\alpha + \beta f.\,\mathrm{cof.}\,\Phi + \gamma ff.\,\mathrm{cof.}\,2\Phi + \delta f^3.\,\mathrm{cof.}\,3\Phi + \&c.) \\ \pm\, \mathrm{A}(\beta f.\,\mathrm{fin.}\,\Phi + \gamma ff.\,\mathrm{fin.}\,2\Phi + \delta f^3.\,\mathrm{fin.}\,3\Phi + \&c.)\sqrt{-1} \\ +\, \mathfrak{A}(\alpha f.\,\mathrm{cof.}\,\Phi + \beta ff.\,\mathrm{cof.}\,2\Phi + \gamma f^3.\,\mathrm{cof.}\,3\Phi + \&c.) \\ \pm\, \mathfrak{A}(\alpha f.\,\mathrm{fin.}\,\Phi + \beta ff.\,\mathrm{fin.}\,2\Phi + \gamma f^3.\,\mathrm{fin.}\,3\Phi + \&c.)\sqrt{-1} \end{array}\right.$$

Sit, ad calculum abbreviandum,

$$\begin{array}{r} A + Bf.\,\mathrm{cof.}\,\Phi + Cff.\,\mathrm{cof.}\,2\Phi + Df^3.\,\mathrm{cof.}\,3\Phi + \&c. = \mathrm{P} \\ Bf.\,\mathrm{fin.}\,\Phi + Cff.\,\mathrm{fin.}\,2\Phi + Df^3.\,\mathrm{fin.}\,3\Phi + \&c. = \mathfrak{P} \\ \alpha + \beta f.\,\mathrm{cof.}\,\Phi + \gamma ff.\,\mathrm{cof.}\,2\Phi + \delta f^3.\,\mathrm{cof.}\,3\Phi + \&c. = \mathrm{Q} \\ \beta f.\,\mathrm{fin.}\,\Phi + \gamma ff.\,\mathrm{fin.}\,2\Phi + \delta f^3.\,\mathrm{fin.}\,3\Phi + \&c. = \mathfrak{Q} \\ \alpha f.\,\mathrm{cof.}\,\Phi + \beta ff.\,\mathrm{cof.}\,2\Phi + \gamma f^3.\,\mathrm{cof.}\,3\Phi + \&c. = \mathrm{R} \\ \alpha f.\,\mathrm{fin.}\,\Phi + \beta ff.\,\mathrm{fin.}\,2\Phi + \gamma f^3.\,\mathrm{fin.}\,3\Phi + \&c. = \mathfrak{R} \end{array}$$

eritque, his positis,

$$\mathrm{P} \pm \mathfrak{P}\sqrt{-1} = \mathrm{AQ} \pm \mathrm{A}\mathfrak{Q}\sqrt{-1} + \mathfrak{A}\mathrm{R} \pm \mathfrak{A}\mathfrak{R}\sqrt{-1}.$$

203. Ob signorum ambiguitatem hæ duæ oriuntur æquationes,

$$\mathrm{P} = \mathrm{AQ} + \mathfrak{A}\mathrm{R}$$
$$\mathfrak{P} = \mathrm{A}\mathfrak{Q} + \mathfrak{A}\mathfrak{R}$$

ex quibus incognitæ A & $\mathfrak{A}$ ita definiuntur, ut sit

$$\mathrm{A} = \frac{\mathrm{P}\mathfrak{R} - \mathfrak{P}\mathrm{R}}{\mathrm{Q}\mathfrak{R} - \mathfrak{Q}\mathrm{R}} \;\&\; \mathfrak{A} = \frac{\mathrm{P}\mathfrak{Q} - \mathfrak{P}\mathrm{Q}}{\mathfrak{Q}\mathrm{R} - \mathrm{Q}\mathfrak{R}}.$$

Proposita ergo fractione $\frac{M}{(pp - 2pqz.\,\mathrm{cof.}\,\Phi + qqzz)Z}$ per sequentem regulam fractio partialis ex ea oriunda $\frac{\mathrm{A} + \mathfrak{A}z}{pp + 2pqz.\,\mathrm{cof.}\,\Phi + qqzz}$ definietur. Posito $f = \frac{p}{q}$, & evolutis singulis terminis, fiat ut sequitur,

X 2 posito

LIB. I. posito $z^n = f^n \cdot \cos. n\Phi$, fit $M = P$

$\ldots\ldots z^n = f^n \cdot \sin. n\Phi$, fit $M = \mathrm{p}$

$\ldots\ldots z^n = f^n \cdot \cos. n\Phi$, fit $Z = Q$

$\ldots\ldots z^n = f^n \cdot \sin. n\Phi$, fit $Z = \mathrm{q}$

$\ldots\ldots z^n = f^n \cdot \cos. n\Phi$, fit $zZ = R$

$\ldots\ldots z^n = f^n \cdot \sin. n\Phi$, fit $zZ = \mathrm{r}$

Inventis hoc modo valoribus P, Q, R, p, q, r erit

$$A = \frac{P\mathrm{r} - \mathrm{p}R}{Q\mathrm{r} - \mathrm{q}R}, \ \& \ \mathrm{a} = \frac{\mathrm{p}Q - P\mathrm{q}}{Q\mathrm{r} - \mathrm{q}R}.$$

EXEMPLUM I.

Si fuerit proposita hæc Functio fracta $\frac{zz}{(1 - z + zz)(1 + z^4)}$ ex qua partem a denominatoris Factore $1 - z + zz$ oriundam definire oporteat, quæ sit $\frac{A + \mathrm{a}z}{1 - z + zz}$. Ac primo quidem hic Factor, cum forma generali $pp - 2pqz. \cos.\Phi + qqzz$ comparatus, dat $p = 1$, $q = 1$ & $\cos.\Phi = \frac{1}{2}$, unde fit $\Phi = 60° = \frac{\pi}{3}$. Quia itaque est $M = zz$; $Z = 1 + z^4$ & $f = 1$ erit

$$P = \cos.\frac{2}{3}\pi = -\frac{1}{2}; \quad \mathrm{p} = \frac{\sqrt{3}}{2}$$

$$Q = 1 + \cos.\frac{4}{3}\pi = \frac{1}{2}; \quad \mathrm{q} = -\frac{\sqrt{3}}{2}$$

$$R = \cos.\frac{\pi}{3} + \cos.\frac{5\pi}{3} = 1 \quad ; \quad \mathrm{r} = 0.$$

Ex his invenitur $A = -1$; & $\mathrm{a} = 0$, ideoque fractio quæsita est $\frac{-1}{1 - z + zz}$, hujusque complementum erit

$1 + z$

$\frac{1+z+zz}{1+z^4}$, cujus denominator $1+z^4$ cum habeat Factores $1+z\sqrt{2}+zz$ & $1-z\sqrt{2}+zz$, resolutio denuo suscipi potest; fit autem $\Phi=\frac{\pi}{4}$ & priori casu $f=-1$ posteriori $f=+1$. CAP. XII.

EXEMPLUM. II.

Sit igitur proposita hæc fractio resolvenda

$$\frac{1+z+zz}{(1+z\sqrt{2}+zz)(1-z\sqrt{2}+zz)}$$

& erit $M=1+z+zz$; & pro priore Factore habebitur $f=-1$; $\Phi=\frac{\pi}{4}$, & $Z=1-z\sqrt{2}+zz$, unde erit

$$\mathrm{P}=1-\cos.\frac{\pi}{4}+\cos.\frac{2\pi}{4}=\frac{\sqrt{2}-1}{\sqrt{2}}$$

$$\mathfrak{p}=-\sin.\frac{\pi}{4}+\sin.\frac{2\pi}{4}=\frac{\sqrt{2}-1}{\sqrt{2}}$$

$$\mathrm{Q}=1+\sqrt{2}.\cos.\frac{\pi}{4}+\cos.\frac{2\pi}{4}=2$$

$$\mathfrak{q}=+\sqrt{2}.\sin.\frac{\pi}{4}+\sin.\frac{2\pi}{4}=2$$

$$\mathrm{R}=-\cos.\frac{\pi}{4}-\sqrt{2}.\cos.\frac{2\pi}{4}-\cos.\frac{3\pi}{4}=0$$

$$\mathfrak{r}=-\sin.\frac{\pi}{4}-\sqrt{2}.\sin.\frac{2\pi}{4}-\sin.\frac{3\pi}{4}=-2\sqrt{2}$$

Ex his reperitur $\mathrm{Q}\mathfrak{r}-\mathfrak{q}\mathrm{R}=-4\sqrt{2}$: &

$\mathrm{A}=\frac{\sqrt{2}-1}{2\sqrt{2}}$, & $\mathfrak{A}=0$ unde ex denominatoris Factore $1+z\sqrt{2}+zz$ hæc orietur fractio partialis $\frac{(\sqrt{2}-1):2\sqrt{2}}{1+z\sqrt{2}+zz}$, alter autem Factor dabit simili modo hanc $\frac{(\sqrt{2}+1):2\sqrt{2}}{1-z\sqrt{2}+zz}$.

Hinc Functio primum proposita $\frac{zz}{(1-z+zz)(1+z^4)}$ resol-

 vitur

LIB. I. vitur in has $\frac{-1}{1-z+zz} + \frac{(\surd 2-1):2\surd 2}{1+z\surd 2+zz} + \frac{(\surd 2+1):2\surd 2}{1-z\surd 2+zz}$.

EXEMPLUM III.

Sit proposita hæc fractio resolvenda

$$\frac{1+2z+zz}{(1-\frac{8}{5}z+zz)(1+2z+3zz)}$$

Pro Factore denominatoris $1-\frac{8}{5}z+zz$ oriatur ista fractio $\frac{A+\mathfrak{A}z}{1-\frac{8}{5}z+zz}$; eritque $p=1$; $q=1$; $cos.\,\Phi=\frac{4}{5}$, unde $f=1$; $M=1+2z+zz$; $Z=1+2z+3zz$. Quia vero hic ratio Anguli Φ ad rectum non constat, Sinus & Cosinus ejus multiplorum seorsim debent investigari. Cum sit

$$cos.\,\Phi=\frac{4}{5};\ \text{erit}\ sin.\,\Phi=\frac{3}{5}$$
$$cos.\,2\Phi=\frac{7}{25};\quad sin.\,2\Phi=\frac{24}{25}$$
$$cos.\,3\Phi-\frac{-44}{125};\quad sin.\,3\Phi=\frac{117}{125};$$

hinc fit

$$P=1+2.\frac{4}{5}+\frac{7}{25}=\frac{72}{25}$$
$$\mathfrak{P}=2.\frac{3}{5}+\frac{24}{25}=\frac{54}{25}$$
$$Q=1+2.\frac{4}{5}+3.\frac{7}{25}=\frac{86}{25}$$
$$\mathfrak{Q}=2.\frac{3}{5}+3.\frac{24}{25}=\frac{102}{25}$$
$$R=\frac{4}{5}+2.\frac{7}{25}-3.\frac{44}{125}=\frac{38}{125}$$
$$\mathfrak{R}=\frac{3}{5}+2.\frac{24}{25}+3.\frac{117}{125}=\frac{666}{125}$$

ideoque $Q\mathfrak{R}-\mathfrak{Q}R=\frac{53400}{25.125}=\frac{2136}{125}$. Ergo

A =

$$A = \frac{1836}{2136} = \frac{153}{178}; \quad \mathfrak{A} = -\frac{540}{2136} = -\frac{45}{178}.$$

Quare fractio ex Factore $1 - \frac{8}{5}z + zz$ oriunda erit

$$\frac{9(17 - 5z) : 178}{1 - \frac{8}{5}z + zz}.$$

Quæramus simili modo fractionem alteri Factori respondentem; erit $p = 1$, $q = -\sqrt{3}$ & $cos.\,\Phi = \frac{1}{\sqrt{3}}$, ergo $f = -\frac{1}{\sqrt{3}}$, $M = 1 + 2z + zz$ & $Z = 1 - \frac{8}{5}z + zz$. Fiet autem, ob $cos.\,\Phi = \frac{1}{\sqrt{3}}$, $fin.\,\Phi = \frac{\sqrt{2}}{\sqrt{3}}$

$$cos.\,2\Phi = -\frac{1}{3}, \quad fin.\,2\Phi = \frac{2\sqrt{2}}{3}$$

$$cos.\,3\Phi = -\frac{5}{3\sqrt{3}}, \quad fin.\,3\Phi = \frac{\sqrt{3}}{3\sqrt{3}}$$

consequenter

$$P = 1 - \frac{2}{\sqrt{3}}\cdot\frac{1}{\sqrt{3}} + \frac{1}{3}\cdot -\frac{1}{3} = \frac{2}{9}$$

$$\mathfrak{p} = \quad -\frac{2}{\sqrt{3}}\cdot\frac{\sqrt{2}}{\sqrt{3}} + \frac{1}{3}\cdot\ \frac{2\sqrt{2}}{3} = -\frac{4\sqrt{2}}{9}$$

$$Q = 1 + \frac{8}{5\sqrt{3}}\cdot\frac{1}{\sqrt{3}} + \frac{1}{3}\cdot -\frac{1}{3} = \frac{64}{45}$$

$$\mathfrak{q} = \quad +\frac{8}{5\sqrt{3}}\cdot\frac{\sqrt{2}}{\sqrt{3}} + \frac{1}{3}\cdot\ \frac{2\sqrt{2}}{3} = \frac{34\sqrt{2}}{45}$$

$$R = -\frac{1}{\sqrt{3}}\cdot\frac{1}{\sqrt{3}} - \frac{8}{5\cdot 3}\cdot -\frac{1}{3} - \frac{1}{3\sqrt{3}}\cdot -\frac{5}{3\sqrt{3}} = \frac{4}{135}$$

$$\mathfrak{r} = -\frac{1}{\sqrt{3}}\cdot\frac{\sqrt{2}}{\sqrt{3}} - \frac{8}{5\cdot 3}\cdot\ \frac{2\sqrt{2}}{3} - \frac{1}{3\sqrt{3}}\cdot\ \frac{\sqrt{2}}{3\sqrt{3}} = -\frac{98\sqrt{2}}{135}$$

ideoque $Q\mathfrak{r} - \mathfrak{q}R = -\frac{712\sqrt{2}}{675}$; fiet ergo

$$A = \frac{100}{712} = \frac{25}{178}; \quad \mathfrak{A} = \frac{540}{712} = \frac{135}{178}.$$

Fractio

LIB. I.

Fractio ergo proposita $\frac{1+2z+zz}{(1-\frac{8}{5}z+zz)(1+2z+3zz)}$ resolvitur in $\frac{9(17-5z):178}{1-\frac{8}{5}z+zz}+\frac{5(5+27z):178}{1+2z+3zz}$.

204. Possunt autem valores litterarum R & r ex litteris Q & q definiri, cum enim sit

$$Q=\alpha+\beta f.\,cos.\,\phi+\gamma f^2.\,cos.\,2\phi+\delta f^3.\,cos.\,3\phi\ \&c.$$
$$q=\beta f.\,sin.\,\phi+\gamma f^2.\,sin.\,2\phi+\delta f^3.\,sin.\,3\phi\ \&c.$$

erit

$$Q.\,cos.\phi-q.\,sin.\phi=\alpha.\,cos.\phi+\beta f.\,cos.\,2\phi+\gamma f^2.\,cos.\,3\phi+\&c.$$

ideoque $R=f(Q.\,cos.\,\phi-q.\,sin.\,\phi)$

deinde erit

$$Q.\,sin.\phi+q.\,cos.\phi=\alpha.\,sin.\,\phi+\beta f.\,sin.\,2\phi+\gamma ff.\,sin.\,3\phi+\&c.$$

ergo $r=f(Q.\,sin.\,\phi+q.\,cos.\,\phi)$

Ex his porro fit

$$Qr-qR=(QQ+qq)f.\,sin.\,\phi$$
$$Pr-pR=(PQ+pq)f.\,sin.\,\phi+(Pq-pQ)f.\,cos.\,\phi$$

eritque consequenter

$$A=\frac{PQ+pq}{QQ+qq}+\frac{Pq-pQ}{QQ+qq}\cdot\frac{cos.\,\phi}{sin.\,\phi}$$
$$a=-\frac{Pq+pQ}{(QQ+qq)f.\,sin.\,\phi}.$$

Quare ex denominatoris Factore $pp-2pqz.\,cos.\,\phi+qqzz$ nascitur ista fractio partialis

$$\frac{(PQ+pq)f.\,sin.\,\phi+(Pq-pQ)(f.\,cos.\,\phi-z)}{(pp-2pqz.\,cos.\,\phi+qqzz)(QQ+qq)f.\,sin.\,\phi}$$

seu, ob $f=\frac{p}{q}$, hæc

$$\frac{(PQ+pq)p.\,sin.\,\phi+(Pq-pQ)(p.\,cos.\,\phi-qz)}{(pp-2pqz.\,cos.\,\phi+qqzz)(QQ+qq)p.\,sin.\,\phi}.$$

205. Oritur ergo hæc fractio partialis ex Functionis propositæ $\frac{M}{(pp-2pqz.\,cos.\,\phi+qqzz)Z}$ Factore denominatoris $pp-2pqz.\,cos.\,\phi+qqzz$, atque litteræ P, p, Q & q sequenti modo ex Functionibus M & Z inveniuntur:

posito

posito $z^n = \frac{p^n}{q^n}.\ cos.\ n\Phi$, fit $M = P$,

& $Z = Q$;

& posito $z^n = \frac{p^n}{q^n}.\ sin.\ n\Phi$, fit $M = \mathrm{p}$,

& $Z = \mathrm{q}$:

ubi notandum est Functiones M & Z, antequam hæc substitutio fiat, omnino evolvi debere, ut hujusmodi habeant formas

$$M = A + Bz + Cz^2 + Dz^3 + Ez^4 + \&c.,$$
$$\& \ Z = \alpha + \beta z + \gamma z^2 + \delta z^3 + \epsilon z^4 + \&c.;$$

eritque ideo

$$P = A + B\frac{p}{q}.cos.\Phi + C\frac{p^2}{q^2}.cos.2\Phi + D\frac{p^3}{q^3}.cos.3\Phi + \&c.$$
$$\mathrm{p} = B\frac{p}{q}.sin.\Phi + C\frac{p^2}{q^2}.sin.2\Phi + D\frac{p^3}{q^3}.sin.3\Phi + \&c.$$
$$Q = \alpha + \beta\frac{p}{q}.cos.\Phi + \gamma\frac{p^2}{q^2}.cos.2\Phi + \delta\frac{p^3}{q^3}.cos.3\Phi + \&c.$$
$$\mathrm{q} = \beta\frac{p}{q}.sin.\Phi + \gamma\frac{p^2}{q^2}.sin.2\Phi + \delta\frac{p^3}{q^3}.sin.3\Phi + \&c..$$

206. Ex præcedentibus autem intelligitur hanc resolutionem locum habere non posse, si Functio Z eundem Factorem $pp - 2pqz.cos.\Phi + qqzz$ adhuc in se complectatur; hoc enim casu in æquatione $M = \mathrm{A}Z + \mathrm{a}Zz$ facta substitutione $z^n = f^n(cos.n\Phi \pm \surd -1.\ sin.\Phi)$, ipsa quantitas Z evanesceret, nihilque propterea colligi posset. Quamobrem, si Functionis fractæ $\frac{M}{N}$ denominator habeat Factorem $(pp - 2pqz.cos.\Phi + qqzz)^2$ vel altiorem Potestatem, peculiari opus erit resolutione. Sit igitur $N = (pp - 2pqz.cos.\Phi + qqzz)^2 Z$; atque ex denominatoris Factore $(pp - 2pqz.cos.\Phi + qqzz)^2$ orientur hujusmodi duæ fractiones partiales

LIB. I.

$$\frac{A + a z}{(pp - 2pqz.\cos.\phi + qqzz)^2} + \frac{B + b z}{pp - 2pqz.\cos.\phi + qqzz},$$

ubi litteras constantes A, a, B, b determinari oportet.

207. His positis, debebit ista expressio

$$\frac{M - (A + a z) Z - (B + b z) Z (pp - 2pqz.\cos.\phi + qqzz)}{(pp - 2pqz.\cos.\phi + qqzz)^2}$$

esse Functio integra, & hanc ob rem numerator divisibilis erit per denominatorem. Primum ergo hæc expressio $M - AZ - azZ$ divisibilis esse debet per $pp - 2pqz.\cos.\phi + qqzz$; qui cum sit casus præcedens, eodem quoque modo litteræ A & a determinabuntur.

Quare, posito $z^n = \frac{p^n}{q^n}.\cos. n\phi$, fit $M = P$,

& $Z = N$:

&, posito $z^n = \frac{p^n}{q^n}.\sin. n\phi$, fit $M = p$,

& $Z = n$.

Hisque factis secundum regulam supra datam, erit

$$A = \frac{PN + pn}{N^2 + n^2} + \frac{Pn - pN}{N^2 + n^2}.\frac{\cos.\phi}{\sin.\phi}$$

$$a = -\frac{Pn + pN}{N^2 + n^2}.\frac{q}{p\sin.\phi}.$$

208. Inventis ergo hoc modo A & a, fiet $\frac{M - (A + az) Z}{pp - 2pqz.\cos.\phi + qqzz}$ Functio integra, quæ sit $= P$; atque superest ut $P - BZ - bzZ$ divisibile evadat per $pp - 2pqz.\cos.\phi + qqzz$, quæ expressio cum similis sit præcedenti, si

posito $z^n = \frac{p^n}{q^n}.\cos. n\phi$, vocetur $P = R$,

&, posito $z^n = \frac{p^n}{q^n}.\sin. n\phi$, vocetur $P = r$; erit

B =

CAP. XII.

$$\mathrm{B} = \frac{\mathrm{RN}+\mathrm{rn}}{\mathrm{N}^2+\mathrm{n}^2} + \frac{\mathrm{Rn}-\mathrm{rN}}{\mathrm{N}^2+\mathrm{n}^2}\cdot\frac{\cos.\Phi}{\sin.\Phi}$$

$$\mathrm{b} = -\frac{\mathrm{Rn}+\mathrm{rN}}{\mathrm{N}^2+\mathrm{n}^2}\cdot\frac{q}{p\sin.\Phi}.$$

209. Hinc jam generaliter concludere licet quomodo resolutio institui debeat, si denominator Functionis propositæ $\frac{M}{N}$, Factorem habeat $(pp - 2pqz.\cos.\Phi + qqzz)^k$: sit enim $N = (pp - 2pqz.\cos.\Phi + qqzz)^k Z$, ita ut hæc resolvenda sit Functio fracta

$$\frac{M}{(pp - 2pqz.\cos.\Phi + qqzz)^k Z}.$$

Præbeat ergo Factor denominatoris $(pp - 2pqz.\cos.\Phi + qqzz)^k$ has partes:

$$\frac{\mathrm{A}+\mathrm{a}z}{(pp - 2pqz.\cos.\Phi + qqzz)^k} + \frac{\mathrm{B}+\mathrm{b}z}{(pp - 2pqz.\cos.\Phi + qqzz)^{k-1}} +$$

$$\frac{\mathrm{C}+\mathrm{c}z}{(pp - 2pqz.\cos.\Phi + qqzz)^{k-2}} + \frac{\mathrm{D}+\mathrm{d}z}{(pp - 2pqz.\cos.\Phi + qqzz)^{k-3}} + \&c.$$

Jam, posito $z^n = \frac{p^n}{q^n}.\cos.n\Phi$, fit $M = \mathrm{M}$,

& $Z = \mathrm{N}$;

&, posito $z^n = \frac{p^n}{q^n}.\sin.n\Phi$, fit $M = \mathrm{m}$;

& $Z = \mathrm{n}$;

erit

$$\mathrm{A} = \frac{\mathrm{MN}+\mathrm{mn}}{\mathrm{N}^2+\mathrm{n}^2} + \frac{\mathrm{Mn}-\mathrm{mN}}{\mathrm{N}^2+\mathrm{n}^2}\cdot\frac{\cos.\Phi}{\sin.\Phi}$$

$$\mathrm{a} = -\frac{\mathrm{Mn}+\mathrm{mN}}{\mathrm{N}^2+\mathrm{n}^2}\cdot\frac{q}{p\sin.\Phi}.$$

Deinde vocetur $\frac{M - (\mathrm{A}+\mathrm{a}z)Z}{pp - 2ppz.\cos.\Phi + qqzz} = P$; atque, posito $z^n = \frac{p^n}{q^n} \cdot \cos. n\Phi$, fit $P = \mathrm{P}$,

&, posito $z^n = \frac{p^n}{q^n} \cdot \sin. n\Phi$, fit $P = \mathfrak{P}$;

erit

$$\mathrm{B} = \frac{\mathrm{PN} + \mathfrak{PN}}{\mathrm{N}^2 + \mathfrak{N}^2} + \frac{\mathrm{P}\mathfrak{N} - \mathfrak{P}\mathrm{N}}{\mathrm{N}^2 + \mathfrak{N}^2} \cdot \frac{\cos. \Phi}{\sin. \Phi}$$

$$\mathfrak{B} = -\frac{\mathrm{P}\mathfrak{N} + \mathfrak{P}\mathrm{N}}{\mathrm{N}^2 + \mathfrak{N}^2} \cdot \frac{q}{p \sin. \Phi}.$$

Tum vocetur $\frac{P - (\mathrm{B} + \mathfrak{B}z)Z}{pp - 2pqz. \cos. \Phi + qqzz} = Q$, atque

posito $z^n = \frac{p^n}{q^n} \cdot \cos. n\Phi$, fit $Q = \mathrm{Q}$,

&, posito $z^n = \frac{p^n}{q^n} \cdot \sin. n\Phi$, fit $Q = \mathfrak{Q}$;

erit

$$\mathrm{C} = \frac{\mathrm{QN} + \mathfrak{QN}}{\mathrm{N}^2 + \mathfrak{N}^2} + \frac{\mathrm{Q}\mathfrak{N} - \mathfrak{Q}\mathrm{N}}{\mathrm{N}^2 + \mathfrak{N}^2} \cdot \frac{\cos. \Phi}{\sin. \Phi}$$

$$\mathfrak{C} = -\frac{\mathrm{Q}\mathfrak{N} + \mathfrak{Q}\mathrm{N}}{\mathrm{N}^2 + \mathfrak{N}^2} \cdot \frac{q}{p \sin. \Phi}.$$

Porro vocetur $\frac{Q - (\mathrm{C} + \mathfrak{C}z)Z}{pp - 2pqz. \cos. \Phi + qqzz} = R$, atque

posito $z^n = \frac{p^n}{q^n} \cdot \cos. n\Phi$, fit $R = \mathrm{R}$,

&, posito $z^n = \frac{p^n}{q^n} \cdot \sin. n\Phi$, fit $R = \mathfrak{R}$;

erit

$$\mathrm{D} = \frac{\mathrm{RN} + \mathfrak{RN}}{\mathrm{N}^2 + \mathfrak{N}^2} + \frac{\mathrm{R}\mathfrak{N} - \mathfrak{R}\mathrm{N}}{\mathrm{N}^2 + \mathfrak{N}^2} \cdot \frac{\cos. \Phi}{\sin. \Phi}$$

$$\mathfrak{D} = -\frac{\mathrm{R}\mathfrak{N} + \mathfrak{R}\mathrm{N}}{\mathrm{N}^2 + \mathfrak{N}^2} \cdot \frac{q}{p \sin. \Phi}.$$

hocque

hocque modo progrediendum est donec ultimæ fractionis, cujus denominator est $pp - 2pqz.\cos.\Phi + qqzz$, numerator fuerit determinatus. CAP. XII.

EXEMPLUM.

Sit ista proposita Functio fracta

$$\frac{z - z^3}{(1+zz)^4(1+z^4)}$$

ex cujus denominatoris Factore $(1+zz)^4$ oriantur hæ fractiones partiales,

$$\frac{A+\mathfrak{A}z}{(1+zz)^4} + \frac{B+\mathfrak{B}z}{(1+zz)^3} + \frac{C+\mathfrak{C}z}{(1+zz)^2} + \frac{D+\mathfrak{D}z}{1+zz}.$$

Comparatione ergo instituta, erit $p=1$, $q=1$, $\cos.\Phi = 0$; ideoque $\Phi = \frac{1}{2}\pi$, porroque $M = z - z^3$ & $Z = 1 + z^4$. Hinc erit $\mathfrak{M} = 0$; $\mathfrak{m} = 2$; $\mathfrak{N} = 2$; $\mathfrak{n} = 0$, & $\sin.\Phi = 1$.

Hinc itaque invenitur

$$A = -\frac{4}{4}.0 = 0, \text{ \& } \mathfrak{A} = 1.$$

ergo $A + \mathfrak{A}z = z$; hincque $P = \frac{z - z^3 - z - z^5}{1+zz} = -z^3$, & $\mathfrak{P} = 0$, $\mathfrak{p} = 1$: unde reperitur

$$B = 0, \text{ \& } \mathfrak{B} = \frac{1}{2}.$$

Ergo $B + \mathfrak{B}z = \frac{1}{2}z$, & $Q = \frac{-z^3 - \frac{1}{2}z - \frac{1}{2}z^5}{1+zz} = -\frac{1}{2}z - \frac{1}{2}z^3$;

unde $\mathfrak{Q} = 0$ & $\mathfrak{q} = 0$, ergo
$C = 0$ & $\mathfrak{C} = 0$, hincque $R = -\frac{\frac{1}{2}z - \frac{1}{2}z^3}{1+zz} = -\frac{1}{2}z$,

ergo $\mathfrak{R} = 0$; $\mathfrak{r} = -\frac{1}{2}$; unde fit

$$D = 0 \text{ \& } \mathfrak{D} = -\frac{1}{4}.$$

LIB. I. Quamobrem fractiones quaesitae sunt hae

$$\frac{z}{(1+zz)^4}+\frac{z}{2(1+zz)^3}-\frac{z}{4(1+zz)}.$$ Reliquae vero fractionis numerator est $= S = \frac{R-(D+Dz)Z}{1+zz} = -\frac{1}{4}z + \frac{1}{4}z^3$, quae ergo erit $= \frac{-z+z^3}{4(1+z^4)}$.

210. Hac ergo methodo simul innotescit fractio complementi, quae cum inventis conjuncta producat fractionem propositam ipsam. Scilicet si fractionis

$$\frac{M}{(pp-2pqz.\,cos.\,\Phi+qqzz)^k Z}$$

inventae fuerint omnes fractiones partiales ex Factore $(pp - 2pqz.\,cos.\,\Phi + qqzz)^k$ oriundae, pro quibus formati sunt valores Functionum P, Q, R, S, T, si harum litterarum Series ulterius continuetur, erit ea, quae ultimam, qua opus est ad numeratores inveniendos, sequitur, numerator reliquae fractionis denominatorem Z habentis; nempe, si $k=1$, erit reliqua fractio $\frac{P}{Z}$; si $k=2$, erit reliqua fractio $\frac{Q}{Z}$; si $k=3$, erit ea $\frac{R}{Z}$, & ita porro. Inventa autem hac reliqua fractione denominatorem Z habente, ea per has regulas ulterius resolvi poterit.

CAPUT

CAP. XIII.

CAPUT XIII.

De Seriebus recurrentibus.

211. AD hoc Serierum genus, quas MOIVRÆUS *recurrentes* vocare solet, hic refero omnes Series quæ ex evolutione Functionis cujusque fractæ per divisionem actualem instituta nascuntur. Supra enim jam ostendimus has Series ita esse comparatas, ut quivis terminus ex aliquot præcedentibus secundum legem quandam constantem determinetur, quæ lex a denominatore Functionis fractæ pendet. Cum autem nunc Functionem quamcunque fractam in alias simpliciores resolvere docuerim, hinc Series quoque recurrens in alias simpliciores resolvetur. In hoc igitur Capite propositum est Serierum recurrentium cujusvis gradus resolutionem in simpliciores exponere.

212. Sit proposita ista Functio fracta genuina

$$\frac{a + bz + czz + dz^3 + \&c.}{1 - \alpha z - \beta zz - \gamma z^3 - \delta z^4 - \&c.}$$

quæ per divisionem evolvatur in hanc Seriem recurrentem

$$A + Bz + Cz^2 + Dz^3 + Ez^4 + Fz^5 + \&c.,$$

cujus coëfficientes quemadmodum progrediantur, supra est ostensum. Quod si jam Functio illa fracta resolvatur in fractiones suas simplices, & unaquæque in Seriem recurrentem evolvatur, manifestum est summam omnium harum Serierum ex fractionibus partialibus ortarum æqualem esse debere Seriei recurrenti.

$$A + Bz + Cz^2 + Dz^3 + Ez^4 + Fz^5 + \&c.$$

Fractiones ergo partiales, quas supra invenire docuimus, dabunt:

LIB. I. bunt Series partiales, quarum indoles ob simplicitatem facile perspicitur; omnes autem Series partiales junctim sumtæ producent Seriem recurrentem propositam; unde & hujus natura penitius cognoscetur.

213. Sint Series recurrentes ex singulis fractionibus partialibus ortæ hæ.

$$\begin{array}{l} a + bz + czz + dz^3 + ez^4 + \&c. \\ a' + b'z + c'zz + d'z^3 + e'z^4 + \&c. \\ a'' + b''z + c''zz + d''z^3 + e''z^4 + \&c. \\ a''' + b'''z + c'''zz + d'''z^3 + e'''z^4 + \&c. \\ \&c. \end{array}$$

Quoniam hæ Series junctim sumtæ æquales esse debent huic

$$A + Bz + Czz + Dz^3 + Ez^4 + \&c.,$$

necesse est ut sit

$$\begin{array}{l} A = a + a' + a'' + a''' + \&c. \\ B = b + b' + b'' + b''' + \&c. \\ C = c + c' + c'' + c''' + \&c. \\ D = d + d' + d'' + d''' + \&c. \\ \&c. \end{array}$$

Hinc, si singularum Serierum ex fractionibus partialibus ortarum definiri queant coëfficientes Potestatis z^n, horum summa dabit coëfficientem Potestatis z^n in Serie recurrente $A + Bz + Cz^2 + Dz^3 + \&c.$

214. Dubium hic suboriri posset, an, si duæ hujusmodi Series fuerint inter se æquales

$$A + Bz + Cz^2 + Dz^3 + \&c. = \mathrm{A} + \mathrm{B}z + \mathrm{C}z^2 + \mathrm{D}z^3 + \&c.,$$

necessario inde sequatur, coëfficientes similium Potestatum ipsius z inter se esse æquales; seu an sit $A = \mathrm{A}$; $B = \mathrm{B}$; $C = \mathrm{C}$;

$\mathcal{C} = C$; $\mathcal{D} = D$; &c.. Hoc autem dubium facile tolletur, si perpendamus hanc æqualitatem subsistere debere quemcunque valorem obtineat variabilis z. Sit igitur $z = 0$, atque manifestum est fore $\mathcal{A} = A$. His ergo terminis æqualibus utrinque sublatis, ac reliqua æquatione per z divisa, habebitur

CAP. XIII.

$$\mathcal{B} + \mathcal{C}z + \mathcal{D}z^2 + \&c. = B + Cz + Dz^2 + \&c.;$$

unde sequitur fore $\mathcal{B} = B$: simili autem modo ostendetur esse $\mathcal{C} = C$; $\mathcal{D} = D$, & ita porro in infinitum.

215. Contemplemur ergo Series, quæ ex fractionibus partialibus, in quas fractio quæpiam proposita resolvitur, oriuntur. Ac primo quidem patet fractionem $\frac{A}{1 - pz}$ dare Seriem $A + Apz + Ap^2z^2 + Ap^3z^3 + \&c.$, cujus terminus generalis est Ap^nz^n; hæc enim expressio vocari solet *terminus generalis*, quoniam ex ea, loco n numeros omnes successive substituendo, omnes Seriei termini nascuntur. Deinde ex fractione $\frac{A}{(1 - pz)^2}$, oritur Series $A + 2Apz + 3Ap^2z^2 + 4Ap^3z^3 + \&c.$, cujus terminus generalis est $(n + 1)Ap^nz^n$. Tum ex fractione $\frac{A}{(1 - pz)^3}$, oritur Series $A + 3Apz + 6Ap^2z^2 + 10Ap^3z^3 + \&c.$, cujus terminus generalis est $\frac{(n+1)(n+2)}{1 \cdot 2} Ap^nz^n$. Generatim autem fractio $\frac{A}{(1 - pz)^k}$ præbet Seriem hanc $A + kApz + \frac{k(k+1)}{1 \cdot 2} Ap^2z^2 + \frac{k(k+1)(k+2)}{1 \cdot 2 \cdot 3} Ap^3z^3 + \&c.$, cujus terminus generalis est $\frac{(n+1)(n+2)(n+3) \ldots\ldots (n+k-1)}{1 \cdot 2 \cdot 3 \ldots\ldots (k-1)} Ap^nz^n$. Ex ipsa autem Seriei progressione colligitur hic idem terminus $= \frac{k(k+1)(k+2) \ldots\ldots (k+n-1)}{1 \cdot 2 \cdot 3 \ldots\ldots n} Ap^nz^n$: hæc vero

 expressio illi est æqualis, id quod multiplicatione per crucem instituta patebit, fiet enim,

$$1.2.3.....n(n+1).....(n+k-1) = 1.2.3.....(k-1)k.....(k+n-1)$$

quæ est æquatio identica.

216. Quoties ergo in resolutione Functionum fractarum ad hujusmodi fractiones partiales $\frac{A}{(1-pz)^k}$ pervenitur, toties Seriei recurrentis ex illa Functione fracta ortæ $A + Bz + Cz^2 + Dz^3 +$ &c., terminus generalis assignari poterit, quippe qui erit summa terminorum generalium Serierum, quæ ex fractionibus partialibus nascuntur.

EXEMPLUM I.

Invenire terminum generalem Seriei recurrentis, quæ ex hac fractione $\frac{1-z}{1-z-2zz}$ nascitur.

Series hinc nata est $1 + 0z + 2zz + 2z^3 + 6z^4 + 10z^5 + 22z^6 + 42z^7 + 86z^8 +$ &c.. Ad coëfficientem potestatis generalis z^n inveniendum, fractio $\frac{1-z}{1-z-2zz}$ resolvatur in $\frac{\frac{2}{3}}{1+z} + \frac{\frac{1}{3}}{1-2z}$, unde oritur terminus generalis quæsitus $\left(\frac{2}{3}(-1)^n + \frac{1}{3}.2^n\right)z^n = \frac{2^n \pm 2}{3}z^n$, ubi signum $+$ valet si n sit numerus par, signum $-$ si n sit impar.

EXEMPLUM II.

Invenire terminum generalem Seriei recurrentis quæ oritur ex fractione $\frac{1-z}{1-5z+6zz}$, seu Seriei hujus $1 + 4z + 14z^2 + 46z^3 + 146z^4 + 454z^5 +$ &c.

Ob

Ob denominatorem $= (1 - 2z)(1 - 3z)$ resolvitur fractio in has $\frac{-1}{1-2z} + \frac{2}{1-3z}$, ex quibus fit terminus generalis $2 \cdot 3^n z^n - 2^n z^n = (2 \cdot 3^n - 2^n) z^n$. CAP. XIII.

EXEMPLUM III.

Invenire terminum generalem Seriei hujus $1 + 3z + 4z^2 + 7z^3 + 11z^4 + 18z^5 + 29z^6 + 47z^7 +$ &c., *quæ oritur ex evolutione fractionis* $\frac{1+2z}{1-z-zz}$.

Ob denominatoris Factores $1 - (\frac{1+\sqrt{5}}{2})z$ & $1 - (\frac{1-\sqrt{5}}{2})z$, per resolutionem prodeunt $\frac{\frac{\sqrt{5}+1}{2}}{1-(\frac{1+\sqrt{5}}{2})z} + \frac{\frac{1-\sqrt{5}}{2}}{1-(\frac{1-\sqrt{5}}{2})z}$, unde erit terminus generalis $=$

$$(\frac{1+\sqrt{5}}{2})^{n+1} z^n + (\frac{1-\sqrt{5}}{2})^{n+1} z^n.$$

EXEMPLUM IV.

Invenire terminum generalem Seriei hujus

$a + (\alpha a + b)z + (\alpha^2 a + \alpha b + \beta a)z^2 + (\alpha^3 a + \alpha^2 b + 2\alpha\beta a + \beta b)z^3 +$ &c.,

quæ oritur ex evolutione fractionis $\frac{a+bz}{1-\alpha z-\beta zz}$.

Per resolutionem oriuntur hæ duæ fractiones:

$$\frac{(a(\alpha+\sqrt{(\alpha\alpha+4\beta)})+2b):2\sqrt{(\alpha\alpha+4\beta)}}{1-(\frac{\alpha+\sqrt{(\alpha\alpha+4\beta)}}{2})z}+$$

$$\frac{(a(\sqrt{(\alpha\alpha+4\beta)}-\alpha)-2b):2\sqrt{(\alpha\alpha+4\beta)}}{1-(\frac{\alpha-\sqrt{(\alpha\alpha+4\beta)}}{2})z};$$ hinc terminus generalis erit $\frac{a(\sqrt{(\alpha\alpha+4\beta)}+\alpha)+2b}{2\sqrt{(\alpha\alpha+4\beta)}}$ $(\frac{\alpha+\sqrt{(\alpha\alpha+4\beta)}}{2})^n z^n + \frac{a(\sqrt{(\alpha\alpha+4\beta)}-\alpha)-2b}{2\sqrt{(\alpha\alpha+4\beta)}}$ $(\frac{\alpha-\sqrt{(\alpha\alpha+4\beta)}}{2})^n z^n$; ex quo omnium Serierum recurrentium, quarum quisque terminus per duos præcedentes determinatur, termini generales expedite definiri poterunt.

EXEMPLUM V.

Invenire terminum generalem hujus Seriei $1+z+2z^2+2z^3+3z^4+3z^5+4z^6+4z^7+$ &c., *qua oritur ex fractione* $\frac{1}{1-z-zz+z^3}=\frac{1}{(1-z)^2(1+z)}$.

Quanquam lex progressionis primo intuitu ita est manifesta ut explicatione non indigeat, tamen fractiones per resolutionem ortæ $\frac{\frac{1}{2}}{(1-z)^2}+\frac{\frac{1}{4}}{1-z}+\frac{\frac{1}{4}}{1+z}$ dant hunc terminum generalem $\frac{1}{2}(n+1)z^n+\frac{1}{4}z^n+\frac{1}{4}(-1)^n z^n = \frac{2n+3\pm 1}{4}z^n$, ubi signum superius valet si n fuerit numerus par, inferius si n fuerit impar.

217. Hoc pacto omnium Serierum recurrentium termini generales exhiberi possunt, quoniam omnes fractiones in hujusmodi fractiones partiales simplices resolvere licet. Quod si autem expressiones imaginarias vitare velimus, sæpenumero ad hujusmodi fractiones partiales pervenietur

$\frac{A + Bpz}{1 - 2pz.\,cos.\,\Phi + ppzz}$; $\frac{A + Bpz}{(1 - 2pz.\,cos.\,\Phi + ppzz)^2}$; &

$\frac{A + Bpz}{(1 - 2pz.\,cos.\,\Phi + ppzz)^k}$, ex quarum evolutione cujusmodi Series nascantur videndum est. Ac primo quidem, ob $cos.\,n\Phi = 2\,cos.\,\Phi.\,cos.\,(n-1)\Phi - cos.\,(n-2)\Phi$, fractio $\frac{A}{1 - 2pz.\,cos.\,\Phi + ppzz}$ evoluta dabit

$$\begin{aligned}A + 2Apz.cos.\Phi + 2Appzz.cos.2\Phi &+ 2Ap^3z^3.cos.3\Phi + 2Ap^4z^4.cos.4\Phi \\ + Appzz., &+ 2Ap^3z^3.\,cos.\,\Phi + 2Ap^4z^4.\,cos.2\Phi \\ &+ Ap^4z^4. \qquad \&c.\end{aligned}$$

cujus Seriei terminus generalis non tam facile apparet.

218. Quo igitur ad scopum perveniamus, consideremus has duas Series

$$Ppz.\,sin.\,\Phi + Pp^2z^2.\,sin.\,2\Phi + Pp^3z^3.\,sin.3\Phi + Pp^4z^4.\,sin.4\Phi + \&c.$$
$$Q + Qpz.\,cos.\Phi + Qp^2z^2.\,cos.2\Phi + Qp^3z^3.cos.3\Phi + Qp^4z^4.cos.4\Phi + \&c.$$

quæ duæ Series utique nascuntur ex evolutione fractionis, cujus denominator est $1 - 2pz.cos.\,\Phi + ppzz$. Ac prior quidem oritur ex hac fractione $\frac{Ppz.\,sin.\,\Phi}{1 - 2pz.\,cos.\,\Phi + ppzz}$, posterior vero ex hac $\frac{Q - Qpz.\,cos.\,\Phi}{1 - 2pz.\,cos.\,\Phi + ppzz}$. Addantur hæ duæ fractiones, atque summa $\frac{Q + Ppz.\,sin.\,\Phi - Qpz.\,cos.\,\Phi}{1 - 2pz.\,cos.\,\Phi + ppzz}$ dabit Seriem cujus terminus generalis erit $= (P\,sin.n\Phi + Q\,cos.n\Phi)\,p^n z^n$. Fiat autem hæc fractio propositæ $\frac{A + Bpz}{1 - 2pz.\,cos.\,\Phi + ppzz}$ æqualis, erit $Q = A$, & $P = A\,cot.\Phi + B\,cosec.\Phi$. Seriei ergo ex hac fractione $\frac{A + Bpz}{1 - 2pz.\,cos.\,\Phi + ppzz}$ ortæ terminus generalis erit $= \frac{A\,cos.\,\Phi\,sin.\,n\Phi + B\,sin.\,n\Phi + A\,sin.\,\Phi.\,cos.\,n\Phi}{sin.\,\Phi}\,p^n z^n =$ $\frac{A\,sin.\,(n+1)\Phi + B\,sin.\,n\Phi}{sin.\,\Phi}\,p^n z^n$.

LIB. I.

219. Ad terminum generalem inveniendum, si denominator fractionis fuerit Potestas, ut $(1 - 2pz.cos.\Phi + ppzz)^k$, conveniet hanc fractionem resolvi in duas etsi imaginarias

$$\frac{a}{(1-(cos.\Phi+\sqrt{-1}.sin.\Phi)pz)^k}+\frac{b}{(1-(cos.\Phi-\sqrt{-1}.sin.\Phi)pz)^k}$$

quarum simul sumtarum terminus generalis Seriei ex ipsis ortæ erit

$$\frac{(n+1)(n+2)(n+3)\ldots\ldots(n+k-1)}{1.\quad 2.\quad 3\ldots\ldots(k-1)}(cos.n\Phi+\sqrt{-1}.sin.n\Phi)\,ap^n z^n +$$
$$\frac{(n+1)(n+2)(n+3)\ldots\ldots(n+k-1)}{1.\quad 2.\quad 3\ldots\ldots(k-1)}(cos.n\Phi-\sqrt{-1}.sin.n\Phi)\,bp^n z^n.$$

Sit $a+b=f$; $a-b=\frac{g}{\sqrt{-1}}$, ut sit $a=\frac{f\sqrt{-1}+g}{2\sqrt{-1}}$ & $b=\frac{f\sqrt{-1}-g}{2\sqrt{-1}}$, eritque hæc expressio

$$\frac{(n+1)(n+2)(n+3)\ldots\ldots(n+k-1)}{1.\quad 2.\quad 3\ldots\ldots(k-1)}(f.cos.n\Phi+g.sin.n\Phi)p^n z^n$$

terminus generalis Seriei, quæ oritur ex his fractionibus

$$\frac{\frac{1}{2}f+\frac{1}{2\sqrt{-1}}g}{(1-(cos.\Phi+\sqrt{-1}.sin.\Phi)pz)^k}+\frac{\frac{1}{2}f-\frac{1}{2\sqrt{-1}}g}{(1-(cos.\Phi-\sqrt{-1}.sin.\Phi)pz)^k},$$

seu quæ oritur ex hac fractione una

$$\frac{\begin{matrix}f-kfpz.cos.\Phi+\frac{k(k-1)}{1.\,2}fp^2z^2.cos.2\Phi-\frac{k(k-1)(k-2)}{1.\,2.\,3}fp^3z^3.cos.3\Phi \\ +kgpz.sin.\Phi-\frac{k(k-1)}{1.\,2}gp^2z^2.sin.2\Phi+\frac{k(k-1)(k-2)}{1.\,2.\,3}gp^3z^3.sin.3\Phi\end{matrix}\ \&c.}{(1-2pz.cos.\Phi+ppzz)^k}$$

220. Posito ergo $k=2$, erit Seriei ex hac fractione

$$\frac{f-2pz.(f.cos.\Phi-g.sin.\Phi)+ppzz.(f.cos.2\Phi-g.sin.2\Phi)}{(1-2pz.cos.\Phi+ppzz)^2}$$

ortæ terminus generalis $=(n+1)(f.cos.n\Phi+g.sin.n\Phi)p^n z^n$.
At Seriei ex hac fractione $\frac{a}{1-2pz.cos.\Phi+ppzz}$, seu hac

$a-$

$\frac{a - 2apz.\cos.\Phi + appzz}{(1 - 2pz.\cos.\Phi + ppzz)^2}$ ortæ terminus generalis est $= \frac{a\,\text{fin.}(n+1)\Phi}{\text{fin.}\Phi} p^n z^n$. Addantur hæ fractiones invicem, ac ponatur $a + f = A$; $2a.\cos.\Phi + 2f.\cos.\Phi - 2g.\text{fin.}\Phi = -B$ & $a + f.\cos.2\Phi - g.\text{fin.}2\Phi = 0$, hinc erit $g = \frac{B + 2A\cos.\Phi}{2\,\text{fin.}\Phi}$, $a = \frac{A + B\cos.\Phi}{1 - \cos.2\Phi} = \frac{A + B\cos.\Phi}{2(\text{fin.}\Phi)^2}$ & $f = -\frac{A\cos.2\Phi - B\cos.\Phi}{2(\text{fin.}\Phi)^2}$, & $g = \frac{B\,\text{fin.}\Phi + A\,\text{fin.}2\Phi}{2(\text{fin.}\Phi)^2}$. Hanc ob rem Seriei ex hac fractione $\frac{A + Bpz}{(1 - 2pz.\cos.\Phi + ppzz)^2}$, ortæ terminus generalis est $\frac{A + B\cos.\Phi}{2(\text{fin.}\Phi)^3}\text{fin.}(n+1)\Phi.p^n z^n + (n+1)$

$$\frac{(B\,\text{fin.}\Phi.\text{fin.}n\Phi + A\,\text{fin.}2\Phi.\text{fin.}n\Phi - B\cos.\Phi.\cos.n\Phi - A\cos.2\Phi.\cos.n\Phi)}{2(\text{fin.}\Phi)^2}$$

$$p^n z^n = -\frac{(n+1)(A\cos.(n+2)\Phi + B\cos.(n+1)\Phi)}{2(\text{fin.}\Phi)^2} p^n z^n +$$

$$\frac{(A + B\cos.\Phi)\,\text{fin.}(n+1)\Phi}{2(\text{fin.}\Phi)^3} p^n z^n =$$

$$\frac{(\tfrac{1}{2}(n+3)\text{fin.}(n+1)\Phi - \tfrac{1}{2}(n+1)\text{fin.}(n+3)\Phi)}{2(\text{fin.}\Phi)^3} A p^n z^n +$$

$$\frac{(\tfrac{1}{2}(n+2)\text{fin.}n\Phi - \tfrac{1}{2}n\,\text{fin.}(n+2)\Phi)}{2(\text{fin.}\Phi)^3} B p^n z^n.$$ Est ergo iste terminus generalis quæsitus $=$

$$\frac{(n+3)\text{fin.}(n+1)\Phi - (n+1)\text{fin.}(n+3)\Phi}{4(\text{fin.}\Phi)^3} A p^n z^n +$$

$$\frac{(n+2)\text{fin.}n\Phi - n.\text{fin.}(n+2)\Phi}{4(\text{fin.}\Phi)^3} B p^n z^n$$: Seriei quæ oritur ex fractione $\frac{A + Bpz}{(1 - 2pz.\cos.\Phi + ppzz)^2}$.

221. Sit $k = 3$, eritque Seriei ex hac fractione ortæ

$$\frac{f - 3pz.(f.\cos.\Phi - g.\text{fin.}\Phi) + 3ppzz(f.\cos.2\Phi - g.\text{fin.}2\Phi) - p^3z^3(f.\cos.3\Phi - g.\text{fin.}3\Phi)}{(1 - 2pz.\cos.\Phi + ppzz)^3}$$

terminus generalis $= \frac{(n+1)(n+2)}{1 \cdot 2}(f.\cos.n\Phi + g.\text{fin.}n\Phi)p^n z^n$.

Deinde

LIB. I. Deinde Seriei ex fractione $\frac{a+bpz}{(1-2pz.\cos.\Phi+ppzz)^3}$, seu ex hac $\frac{a-2apz.\cos.\Phi+appzz+bpz-2bppzz.\cos.\Phi+bp^3z^3}{(1-2pz.\cos.\Phi+ppzz)^3}$ ortæ terminus generalis est $\frac{(n+3)\sin.(n+1)\Phi-(n+1)\sin.(n+3)\Phi}{4(\sin.\Phi)^3}ap^nz^n+\frac{(n+2)\sin.n\Phi-n\sin.(n+2)\Phi}{4(\sin.\Phi)^3}bp^nz^n$. Addantur hæ fractiones ac ponatur numerator $=A$, erit $a+f=A$, $3f.\cos.\Phi-3g.\sin.\Phi+2a.\cos.\Phi-b=0$, $3f.\cos.2\Phi-3g.\sin.2\Phi+a-2b.\cos.\Phi=0$; & $b=f.\cos.3\Phi-g.\sin.3\Phi$, hinc erit $a=\frac{f.\cos.3\Phi-g.\sin.3\Phi-3f.\cos.\Phi+3g.\sin.\Phi}{2\cos.\Phi}=2g.(\sin.\Phi)^2\tan g.\Phi-f-2f.(\sin.\Phi)^2$. Deinde reperitur $\frac{f}{g}=\frac{\sin.5\Phi-2\sin 3\Phi+\sin\Phi}{\cos.5\Phi-2\cos.3\Phi+\cos.\Phi}$ & $a+f=A=2g.(\sin.\Phi)^2\tan g.\Phi-2f(\sin.\Phi)^2$; ergo $\frac{A}{2(\sin.\Phi)^2}=\frac{g\sin.\Phi-f.\cos.\Phi}{\cos.\Phi}$; ex quibus tandem oritur $f=\frac{A(\sin.\Phi-2\sin.3\Phi+\sin.5\Phi)}{16(\sin.\Phi)^5}$, $g=\frac{A(\cos.\Phi-2\cos.3\Phi+\cos.5\Phi)}{16(\sin\Phi)^5}$, ob $16(\sin.\Phi)^5=\sin.5\Phi-5\sin.3\Phi+10\sin.\Phi$, erit $a=\frac{A(9\sin.\Phi-3\sin.3\Phi)}{16(\sin.\Phi)^5}$ & $b=\frac{A(-\sin.2\Phi+\sin.2\Phi)}{16(\sin\Phi)^5}=0$. Est autem $3\sin.\Phi-\sin.3\Phi=4(\sin.\Phi)^3$; ergo $a=\frac{3A}{4(\sin.\Phi)^2}$. Quocirca erit terminus generalis $\frac{(n+1)(n+2)}{1.\ \ 2}p^nz^n$
$A\frac{(\sin.(n+1)\Phi-2\sin.(n+3)\Phi+\sin.(n+5)\Phi)}{16(\sin.\Phi)^5}+$
$3Ap^nz^n.\frac{((n+3)\sin.(n+1)\Phi-(n+1)\sin.(n+3)\Phi)}{16(\sin.\Phi)^5}=$
$\frac{Ap^nz^n}{16(\sin.\Phi)^5}\left(\frac{(n+4)(n+5)}{1.\ \ 2}\sin.(n+1)\Phi-\frac{2(n+1)(n+5)}{1.\ \ 2}\sin.(n+3)\Phi+\frac{(n+1)(n+2)}{1.\ \ 2}\sin.(n+5)\Phi\right)$.

222. Se-

222. Seriei ergo quæ oritur ex hac fractione

$$\frac{A + Bpz}{(1 - 2pz.\,cos.\,\Phi + ppzz)^3}$$

terminus generalis erit hic

$$\frac{Ap^n z^n}{16\,(sin.\,\Phi)^5}\Big(\frac{(n+5)(n+4)}{1.\quad 2}\,sin.(n+1)\Phi - \frac{2(n+1)(n+5)}{1.\quad 2}\times$$
$$sin.(n+3)\Phi + \frac{(n+1)(n+2)}{1.\quad 2}\,sin.(n+5)\Phi\Big)$$
$$+\frac{Bp^n z^n}{16\,(sin.\,\Phi)^5}\Big(\frac{(n+4)(n+3)}{1.\quad 2}\,sin.\,n\Phi - \frac{2n(n+4)}{1.\quad 2}\,sin.(n+2)\Phi +$$
$$\frac{n(n+1)}{1.\quad 2}\,sin.(n+4)\Phi\Big).$$

Atque, ulterius progrediendo, Seriei, quæ oritur ex hac fractione

$$\frac{A + Bpz}{(1 - 2pz.\,cos.\,\Phi + ppzz)^4}$$

terminus generalis erit hic

$$+\frac{Ap^n z^n}{64\,(sin.\,\Phi)^7}\Big(\frac{(n+7)(n+6)(n+5)}{1.\quad 2.\quad 3}\,sin.(n+1)\Phi -$$
$$\frac{3(n+1)(n+7)(n+6)}{1.\quad 2.\quad 3}\,sin.(n+3)\Phi + \frac{3(n+1)(n+2)(n+7)}{1.\quad 2.\quad 3}\times$$
$$sin.(n+5)\Phi - \frac{(n+1)(n+2)(n+3)}{1.\quad 2.\quad 3}\,sin.(n+7)\Phi\Big)$$
$$+\frac{Bp^n z^n}{64\,(sin.\,\Phi)^7}\Big(\frac{(n+6)(n+5)(n+4)}{1.\quad 2.\quad 3}\,sin.\,n\Phi -$$
$$\frac{3n(n+6)(n+5)}{1.\quad 2.\quad 3}\,sin.(n+2)\Phi + \frac{3n(n+1)(n+6)}{1.\quad 2.\quad 3}$$
$$sin.(n+4)\Phi - \frac{n(n+1)(n+2)}{1.\quad 2.\quad 3}\,sin.(n+6)\Phi\Big).$$

Ex his autem expressionibus facile intelligitur, quemadmodum formæ terminorum generalium pro altioribus dignitatibus progrediantur. Ad naturam vero harum expressionum penitius inspiciendam notari convenit esse

LIB. I.

$$\sin.\Phi = \sin.\Phi$$

$$4(\sin.\Phi)^3 = 3\sin.\Phi - \sin.3\Phi$$

$$16(\sin.\Phi)^5 = 10\sin.\Phi - 5\sin.3\Phi + \sin.5\Phi$$

$$64(\sin.\Phi)^7 = 35\sin.\Phi - 21\sin.3\Phi + 7\sin.5\Phi - \sin.7\Phi$$

$$256(\sin.\Phi)^9 = 126\sin.\Phi - 84\sin.3\Phi + 36\sin.5\Phi - 9\sin.7\Phi + \sin.9\Phi.$$

&c.

223. Cum igitur hoc pacto omnes functiones fractæ in fractiones partiales reales resolvi queant, simul omnium Serierum recurrentium termini generales per expressiones reales exhiberi poterunt. Quod quo clarius appareat, exempla sequentia adjuncta sunt.

EXEMPLUM I.

Ex fractione $\frac{1}{(1-z)(1-zz)(1-z^3)} = \frac{1}{1-z-zz+z^4+z^5-z^6}$, oritur ista Series recurrens $1+z+2z^2+3z^3+4z^4+5z^5+7z^6+8z^7+10z^8+12z^9+$&c.; cujus terminus generalis desideratur. Fractio proposita secundum Factores ordinata fit $= \frac{1}{(1-z)^3(1+z)(1+z+zz)}$, quæ resolvitur in has fractiones $\frac{1}{6(1-z)^3} + \frac{1}{4(1-z)^2} + \frac{17}{72(1-z)} + \frac{1}{8(1+z)} + \frac{(2+z)}{9(1+z+zz)}$. Harum prima $\frac{1}{6(1-z)^3}$ dat terminum generalem $\frac{(n+1)(n+2)}{1.\quad 2.} . \frac{1}{6} z^n = \frac{nn+3n+2}{12} z^n$: secunda $\frac{1}{4(1-z)^2}$ dat $\frac{n+1}{4} z^n$: tertia $\frac{17}{72(1-z)}$ dat $\frac{17}{72} z^n$: quarta $\frac{1}{8(1+z)}$ dat $\frac{1}{8}(-1)^n z^n$.

Quinta

CAP. XIII.

Quinta vero $\frac{2+z}{9(1+z+zz)}$ comparata cum forma $\frac{A+Bpz}{1-2pz.\cos.\phi+ppzz}$ (218) dat $p=1$, $\phi=\frac{\varpi}{3}=60°$; $A=+\frac{2}{9}$; & $B=-\frac{1}{9}$, unde oritur terminus generalis $+\frac{2\,\text{fin.}(n+1)\phi-\text{fin.}n\phi}{9\,\text{fin.}\,\phi}(-1)^n z^n=+\frac{4\,\text{fin.}(n+1)\phi-2\,\text{fin.}n\phi}{9\sqrt{3}}$ $(-1)^n z^n=+\frac{4\,\text{fin.}(n+1)\frac{\varpi}{3}-2\,\text{fin.}\,n\frac{\varpi}{3}}{9\sqrt{3}}(-1)^n z^n$. Colligantur hæ expreſſiones omnes in unam ſummam, ac prodibit Seriei propoſitæ terminus generalis quæſitus $=(\frac{nn}{12}+\frac{n}{2}+\frac{47}{72})z^n\pm\frac{1}{8}z^n\pm\frac{4\,\text{fin.}(n+1)\frac{\varpi}{3}-2\,\text{fin.}\,n\frac{\varpi}{3}}{9\sqrt{3}}z^n$, ubi ſigna ſuperiora valent ſi n numerus par, inferiora ſin impar. Ubi notandum eſt ſi fuerit n numerus formæ $3m$ fore $\frac{4\,\text{fin.}\frac{1}{3}(n+1)\varpi-2\,\text{fin.}\frac{1}{3}n\varpi}{9\sqrt{3}}=\pm\frac{2}{9}$; ſi fuerit $n=3m+1$ erit hæc expreſſio $=\mp\frac{1}{9}$; at ſi $n=3m+2$ erit iſta expreſſio $=\mp\frac{1}{9}$, prout n fuerit numerus vel par vel impar. Ex his natura Seriei ita explicari poteſt, ut

LIB. I.

si fuerit	terminus generalis futurus sit
$n = 6m + 0$	$\left(\frac{nn}{12} + \frac{n}{2} + 1\right) z^n$
$n = 6m + 1$	$\left(\frac{nn}{12} + \frac{n}{2} + \frac{5}{12}\right) z^n$
$n = 6m + 2$	$\left(\frac{nn}{12} + \frac{n}{2} + \frac{2}{3}\right) z^n$
$n = 6m + 3$	$\left(\frac{nn}{12} + \frac{n}{2} + \frac{3}{4}\right) z^n$
$n = 6m + 4$	$\left(\frac{nn}{12} + \frac{n}{2} + \frac{2}{3}\right) z^n$
$n = 6m + 5$	$\left(\frac{nn}{12} + \frac{n}{2} + \frac{5}{12}\right) z^n$

Sic, si fuerit $n = 50$, valet forma $n = 6m + 2$, eritque terminus Seriei $= 234z^{50}$.

EXEMPLUM II.

Ex fractione $\frac{1 + z + zz}{1 - z - z^4 + z^5}$, oritur hæc Series recurrens:

$1 + 2z + 3zz + 3z^3 + 4z^4 + 5z^5 + 6z^6 + 6z^7 + 7z^8 + \&c.$, cujus terminum generalem invenire oportet. Fractio proposita ad hanc formam reducitur $\frac{1 + z + zz}{(1 - z)^2 (1 + z)(1 + zz)}$, quæ propterea resolvitur in has fractiones partiales $\frac{3}{4(1 - z)^2} + \frac{3}{8(1 - z)} + \frac{1}{8(1 + z)} - \frac{1 + z}{4(1 + zz)}$. Harum prima $\frac{3}{4(1 - z)^2}$ dat terminum generalem $\frac{3(n + 1)}{4} z^n$; secunda $\frac{3}{8(1 - z)}$ dat $\frac{3}{8} z^n$; tertia dat $\frac{1}{8}(-1)^n z^n$; & quarta $-\frac{1 + z}{4(1 + zz)}$ comparata cum forma $\frac{A + Bpz}{1 - 2pz.\cos.\Phi + ppzz}$ dat $p = 1$; $\cos.\Phi = 0$; & $\Phi = \frac{1}{2}\pi$; $A = -\frac{1}{4}$;

B =

$B = + \frac{1}{4}$, unde fit terminus generalis $= (- \frac{1}{4} \textit{fin.} \frac{1}{2} (n+1) \varpi + \frac{1}{4} \textit{fin.} \frac{1}{2} n\varpi) z^n$. Quare colligendo erit terminus generalis quæſitus $= (\frac{3}{4} n + \frac{9}{8}) z^n \pm \frac{1}{8} z^n - \frac{1}{4} (\textit{fin.} \frac{1}{2} (n+1) \varpi - \textit{fin.} \frac{1}{2} n \varpi) z^n$. Hinc

ſi fuerit	erit terminus generalis
$n = 4m + 0$	$(\frac{3}{4} n + 1) z^n$
$n = 4m + 1$	$(\frac{3}{4} n + \frac{5}{4}) z^n$
$n = 4m + 2$	$(\frac{3}{4} n + \frac{3}{2}) z^n$
$n = 4m + 3$	$(\frac{3}{4} n + \frac{3}{4}) z^n$

Ita, ſi $n = 50$, valebit $n = 4m + 2$, eritque terminus $= 39z^{50}$.

224. Propoſita ergo Serie recurrente, quoniam illa fractio unde oritur, facile cognoſcitur, ejus terminus generalis ſecundum præcepta data reperietur. Ex lege autem Seriei recurrentis, qua quiſque terminus ex præcedentibus definitur, ſtatim innoteſcit denominator fractionis, hujuſque Factores præbebunt formam termini generalis, per numeratorem enim tantum coëfficientes determinantur. Sit nempe propoſita hæc Series recurrens

$$A + Bz + Cz^2 + Dz^3 + Ez^4 + Fz^5 + \&c.,$$

cujus lex progreſſionis, qua unuſquiſque terminus ex aliquot præcedentibus determinatur, præbeat hunc fractionis denominatorem $1 - \alpha z - \beta z^2 - \gamma z^3$. Ita ut ſit $D = \alpha C + \beta B + \gamma A$; $E = \alpha D + \beta C + \gamma B$; $F = \alpha E + \beta D + \gamma C$; &c.,

LIB. I. &c., qui multiplicatores $\alpha, +\beta, +\gamma$ a MOIVRÆO *scalam relationis* constituere dicuntur. Lex ergo progressionis posita est in scala relationis, atque scala relationis statim præbet denominatorem fractionis, ex cujus resolutione proposita Series recurrens oritur.

225. Ad terminum ergo generalem, seu coëfficientem Potestatis indefinitæ z^n, inveniendum, quæri debent denominatoris $1 - \alpha z - \beta z^2 - \gamma z^3$ Factores vel simplices vel duplices, si imaginarios vitare velimus. Sint primo Factores simplices omnes inter se inæquales & reales hi $(1 - pz)(1 - qz)(1 - rz)$; atque fractio generans Seriem propositam resolvetur in $\frac{A}{1 - pz} + \frac{B}{1 - qz} + \frac{C}{1 - rz}$; unde Seriei terminus generalis erit $(Ap^n + Bq^n + Cr^n)z^n$. Si duo Factores fuerint æquales nempe $q = p$, tum terminus generalis hujusmodi erit $((An + B)p^n + Cr^n)z^n$, &, si insuper fuerit $r = q = p$, erit terminus generalis $(An^2 + Bn + C)p^n z^n$. Quod si vero denominator $1 - \alpha z - \beta z^2 - \gamma z^3$ duplicem habeat Factorem, ut sit $= (1 - pz)(1 - 2qz.cos.\Phi + qqzz)$ tum terminus generalis erit $= (Ap^n + \frac{B sin.(n+1)\Phi + C sin.n\Phi}{sin.\Phi} q^n)z^n$. Cum igitur, positis pro n successive numeris $0, 1, 2$, prodire debeant termini A, Bz, Cz^2, hinc valores litterarum A, B, C determinabuntur.

226. Sit scala relationis bimembris, seu determinetur quisque terminus per duos præcedentes, ita ut sit

$$C = \alpha B - \beta A;\ D = \alpha C - \beta B;\ E = \alpha D - \beta C, \text{ \&c.},$$

atque manifestum est Seriem hanc recurrentem, quæ sit

$$A + Bz + Cz^2 + Dz^3 + Ez^4 + \ldots\ldots\ldots + Pz^n + Qz^{n+1} + \text{\&c.},$$

oriri ex fractione cujus denominator sit $1 - \alpha z + \beta zz$. Sint hujus denominatoris Factores $(1 - pz)(1 - qz)$ erit $p + q =$

$q = \alpha$ & $pq = \mathfrak{C}$: atque Seriei terminus generalis erit $(Ap^n + Bq^n)z^n$. Hinc facto $n = 0$, erit $A = A + B$; & facto $n = 1$ erit $A = Ap + Bq$; unde fit $Aq - B = A(q - p)$ & $A = \frac{Aq - B}{q - p}$; & $B = \frac{Ap - B}{p - q}$. Inventis autem valoribus A & B, erit $P = Ap^n + Bq^n$ & $Q = Ap^{n+1} + Bq^{n+1}$. Tum vero erit $AB = \frac{BB - \alpha AB + \mathfrak{C}AA}{4\mathfrak{C} - \alpha\alpha}$.

C A P. XIII.

227. Hinc deduci poteſt modus quemvis terminum ex unico præcedente formandi, cum ad hoc per legem progreſſionis duo requirantur. Cum enim ſit

$$P = Ap^n + Bq^n \ \& \ Q = Ap.p^n + Bq.q^n$$

erit

$$Pq - Q = A(q - p)p^n \ \& \ Pp - Q = B(p - q)q^n:$$

multiplicentur hæ expreſſiones in ſe invicem; eritque

$$P^2pq - (p + q)PQ + QQ - AB(p - q)^2 p^n q^n = 0.$$

At eſt

$p + q = \alpha$; $pq = \mathfrak{C}$; $(p - q)^2 = (p + q)^2 - 4pq = \alpha\alpha - 4\mathfrak{C}$ & $p^n q^n = \mathfrak{C}^n$. Quibus ſubſtitutis habebitur

$\mathfrak{C}P^2 - \alpha PQ + QQ = (\mathfrak{C}AA - \alpha AB + BB)\mathfrak{C}^n$, ſeu $\frac{QQ - \alpha PQ + \mathfrak{C}PP}{BB - \alpha AB + \mathfrak{C}AA} = \mathfrak{C}^n$; quæ eſt inſignis proprietas Serierum recurrentium, quarum quiſque terminus per duos præcedentes determinatur. At cognito quovis termino P, erit ſequens $Q = \frac{1}{2}\alpha P + \surd((\frac{1}{4}\alpha^2 - \mathfrak{C})P^2 + (B^2 - \alpha AB + \mathfrak{C}AA)\mathfrak{C}^n)$, quæ expreſſio, etſi ſpeciem irrationalitatis præ ſe

fert,

LIB. I. fert, tamen semper est rationalis, propterea quod termini irrationales in Serie non occurrunt.

228. Ex datis porro duobus terminis contiguis quibusvis Pz^n & Qz^{n+1} commode assignari potest terminus multo magis remotus Xz^{2n}. Ponatur enim

$X = fP^2 + gPQ - hAB\mathfrak{C}^n$. Quoniam est

$P = Ap^n + Bq^n$ & $Q = Ap.p^n + Bq.q^n$ atque

$X = Ap^{2n} + Bq^{2n}$; erit ut sequitur

$$\begin{array}{rl} fP^2 = & fA^2p^{2n} + fB^2q^{2n} + 2fAB\mathfrak{C}^n \\ gPQ = & gA^2p.p^{2n} + gB^2q.q^{2n} + gAB\alpha\mathfrak{C}^n \\ -hAB\mathfrak{C}^n = & \qquad\qquad - hAB\mathfrak{C}^n \\ \hline X = & Ap^{2n} + Bq^{2n} \end{array}$$

Fiet ergo $f + gp = \frac{1}{A}$; $f + gq = \frac{1}{B}$ & $h = 2f + g\alpha$, unde $g = \frac{B - A}{AB(p - q)}$ & $f = \frac{Ap - Bq}{AB(p - q)}$. At est $B - A = \frac{\alpha A - 2B}{p - q}$; $Ap - Bq = \frac{\alpha B - 2A\mathfrak{C}}{p - q}$. Ergo $f = \frac{\alpha B - 2A\mathfrak{C}}{AB(\alpha\alpha - 4\mathfrak{C})}$ & $g = \frac{\alpha A - 2B}{AB(\alpha\alpha - 4\mathfrak{C})}$ seu $f = \frac{2A\mathfrak{C} - \alpha B}{BB - \alpha AB + \mathfrak{C}AA}$ & $g = \frac{2B - \alpha A}{BB - \alpha AB + \mathfrak{C}AA}$; ideoque $h = \frac{(4\mathfrak{C} - \alpha\alpha)A}{BB - \alpha AB + \mathfrak{C}AA}$.

Eritque ergo

$$X = \frac{(2A\mathfrak{C} - \alpha B)P^2 + (2B - \alpha A)PQ}{BB - \alpha AB + \mathfrak{C}AA} - A\mathfrak{C}^n.$$

Simili vero modo reperitur

$$X = \frac{(\alpha\mathfrak{C}A - (\alpha\alpha - 2\mathfrak{C})B)P^2 + (2B - \alpha A)Q^2}{\alpha(BB - \alpha AB + \mathfrak{C}AA)} - \frac{2B\mathfrak{C}^n}{\alpha}.$$

His conjungendis per eliminationem termini $\mathfrak{C}^n$ reperitur

$$X = \frac{(\mathfrak{C}A - \alpha B)P^2 + 2BPQ - AQQ}{BB - \alpha AB + \mathfrak{C}AA}.$$

229. Si

229. Simili modo, si statuantur termini sequentes CAP. XIII.

$A + Bz + Cz^2 + \ldots\ldots + Pz^n + Qz^{n+1} + Rz^{n+2} + \ldots\ldots + Xz^{2n} + Yz^{2n+1} + Zz^{2n+2}$,

erit

$$Z = \frac{(\beta A - \alpha B) Q^2 + 2BQR - ARR}{BB - \alpha AB + \beta AA}, \text{ \&, ob } R = \alpha Q - \beta P,$$

erit

$$Z = \frac{-\beta\beta A P^2 + 2\beta(\alpha A - B) PQ + (\alpha B - (\alpha\alpha - \beta)A)Q^2}{BB - \alpha AB + \beta AA}.$$

At est

$Z = \alpha Y - \beta X$, ergo $Y = \frac{Z + \beta X}{\alpha}$; unde fit

$Y = \frac{-\beta BP^2 + 2\beta APQ + \alpha(B - \alpha A) QQ}{BB - \alpha AB + \beta AA}$. Sic igitur porro ex X & Y definiri poterunt simili modo coëfficientes potestatum z^{4n}, & z^{4n+1}; hincque ipsarum z^{8n}, z^{8n+1}, & ita porro.

EXEMPLUM.

Sit proposita ista Series recurrens

$1 + 3z + 4z^2 + 7z^3 + 11z^4 + 18z^5 + \ldots\ldots + Pz^n + Qz^{n+1} + \&c.$,

cujus cum quilibet coëfficiens sit summa duorum præcedentium, erit denominator fractionis hanc Seriem producentis $1 - z - zz$; ideoque $\alpha = 1$; $\beta = -1$; & $A = 1$; $B = 3$; unde fit $BB - \alpha AB + \beta AA = 5$, ex quo orietur primum $Q = \frac{P + \sqrt{(5PP + 20(-1)^n)}}{2} = \frac{P + \sqrt{(5PP \pm 20)}}{2}$, ubi signum superius valet, si n sit numerus par, inferius si impar. Sic, si $n = 4$, ob $P = 11$, erit

 $Q =$

LIB. I. $Q = \frac{11 + \sqrt{(5.121 + 20)}}{2} = \frac{11 + 25}{2} = 18$. Si porro coëfficiens termini z^{2n} sit X, erit $X = \frac{-4PP + 6PQ - QQ}{5}$; ergo Potestatis z^4 coëfficiens erit $= \frac{-4.121 + 6.198 - 324}{5} = 76$. Cum autem sit $Q = \frac{P + \sqrt{(5PP \pm 20)}}{2}$ erit $QQ = \frac{3PP \pm 10 + P\sqrt{(5PP \pm 20)}}{2}$; ideoque $X = \frac{-PP \mp 2 + P\sqrt{(5PP \pm 20)}}{2}$. Ex termino ergo Seriei quocunque Pz^n, obtinentur hi $\frac{P + \sqrt{(5PP \pm 20)}}{2} z^{n+1}$, & $\frac{-PP \mp 2 + P\sqrt{(5PP \pm 20)}}{2} z^{2n}$.

230. Simili modo in Seriebus recurrentibus, quarum quilibet terminus ex tribus antecedentibus determinatur, quivis terminus ex duobus antecedentibus definiri potest. Sit enim Series hujusmodi recurrens

$$A + Bz + Cz^2 + Dz^3 + \dots\dots + Pz^n + Qz^{n+1} + Rz^{n+2} + \&c.;$$

cujus scala relationis sit $\alpha, -\beta, +\gamma$, seu quæ oriatur ex fractione cujus denominator $= 1 - \alpha z + \beta z^2 - \gamma z^3$. Quod si jam termini P, Q, R eodem modo per Factores hujus denominatoris, qui sint $(1 - pz)(1 - qz)(1 - rz)$ exprimantur, ut sit $P = \mathfrak{A}p^n + \mathfrak{B}q^n + \mathfrak{C}r^n$; $Q = \mathfrak{A}p.p^n + \mathfrak{B}q.q^n + \mathfrak{C}r.r^n$; & $R = \mathfrak{A}p^2.p^n + \mathfrak{B}q^2.q^n + \mathfrak{C}r^2.r^n$; ob $p + q + r = \alpha$; $pq + pr + qr = \beta$ & $pqr = \gamma$, reperietur hæc proportio

$$R^3 \left.\begin{matrix} -2\alpha Q \\ +\beta P \end{matrix}\right\} R^2 \left.\begin{matrix} +(\alpha\alpha + \beta)\,Q^2 \\ -(\alpha\beta + 3\gamma)\,PQ \\ +\alpha\gamma\,P^2 \end{matrix}\right\} R \begin{matrix} -(\alpha\beta - \gamma)\,Q^3 \\ +(\alpha\gamma + \beta\beta)\,PQ^2 \\ -2\beta\gamma\,P^2Q \\ +\gamma\gamma\,P^3 \end{matrix} : c^n =$$

C³

$$C^3 \left.\begin{matrix} -2\alpha B \\ +\beta A\end{matrix}\right\} C^2 \left.\begin{matrix} +(\alpha^2+\beta)B^2 \\ -(\alpha\beta+3\gamma)AB \\ +\alpha\gamma \quad A^2\end{matrix}\right\} C \begin{matrix} -(\alpha\beta-\gamma)B^3 \\ +(\alpha\gamma+\beta\beta)AB^2 \\ -\quad 2\beta\gamma A^2B \\ +\quad \gamma\gamma A^3\end{matrix} : 1.$$

CAP. XIII.

Pendet ergo inventio termini R ex duobus præcedentibus P & Q a resolutione æquationis cubicæ.

231. His de terminis generalibus Serierum recurrentium notatis, superest ut earumdem Serierum summas investigemus. Ac primo quidem manifestum est summam Seriei recurrentis in infinitum extensæ æqualem esse fractioni ex qua oritur: cujus fractionis cum denominator ex ipsa progressionis lege pateat, reliquum est ut numeratorem definiamus. Sit itaque proposita hæc Series

$$A + Bz + Cz^2 + Dz^3 + Ez^4 + Fz^5 + Gz^6 + \&c.,$$

cujus lex progressionis præbeat hunc denominatorem $1 - \alpha z + \beta z^2 - \gamma z^3 + \delta z^4$. Sumamus fractionem summæ Seriei in infinitum æqualem esse $= \frac{a + bz + cz^2 + dz^3}{1 - \alpha z + \beta z^2 - \gamma z^3 + \delta z^4}$, ex qua cum Series proposita oriri debeat, erit comparando

$$\begin{aligned} a &= A \\ b &= B - \alpha A \\ c &= C - \alpha B + \beta A \\ d &= D - \alpha C + \beta B - \gamma A: \end{aligned}$$

Hinc erit summa quæsita

$$\frac{A + (B - \alpha A)z + (C - \alpha B + \beta A)z^2 +}{1 - \alpha z + \beta z^2 - \gamma z^3 + \delta z^4}$$

$$\frac{(D - \alpha C + \beta B - \gamma A)z^3.}{1 - \alpha z + \beta z^2 - \gamma z^3 + \delta z^4}$$

232. Hinc facile intelligitur quemadmodum Seriei recurrentis summa ad datum terminum usque inveniri debeat.

 Quæ-

LIB. I. Quæratur ſcilicet Seriei modo aſſumtæ ſumma ad terminum Pz^n, atque ponatur

$$s = A + Bz + Cz^2 + Dz^3 + Ez^4 + \ldots\ldots\ldots + Pz^n;$$

quoniam hujus Seriei ſumma in infinitum conſtat, quæratur ſumma terminorum ultimum Pz^n in infinitum ſequentium, qui ſint

$$t = Qz^{n+1} + Rz^{n+2} + Sz^{n+3} + Tz^{n+4} + \&c.;$$

hæc Series per z^{n+1} diviſa dat Seriem recurrentem propoſitæ æqualem, cujus propterea ſumma erit $t =$

$$\frac{Qz^{n+1} + (R - \alpha Q)z^{n+2} + (S - \alpha R + \beta Q)z^{n+3} +}{1 - \alpha z + \beta z^2 - \gamma z^3 + \delta z^4}$$

$$\frac{(T - \alpha S + \beta R - \gamma Q)z^{n+4}.}{1 - \alpha z + \beta z^2 - \gamma z^3 + \delta z^4.}$$

Unde orietur ſumma quæſita $s =$

$$\frac{+ A + (B - \alpha A)z + (C - \alpha B + \beta A)z^2 +}{1 - \alpha z + \beta z^2 - \gamma z^3 + \delta z^4}$$

$$\frac{(D - \alpha C + \beta B - \gamma A)z^3 - Qz^{n+1} -}{1 - \alpha z + \beta z^2 - \gamma z^3 + \delta z^4}$$

$$\frac{(R - \alpha Q)z^{n+2} - (S - \alpha R + \beta Q)z^{n+3} -}{1 - \alpha z + \beta z^2 - \gamma z^3 + \delta z^4}$$

$$\frac{(T - \alpha S + \beta R - \gamma Q)z^{n+4}}{1 - \alpha z + \beta z^2 - \gamma z^3 + \delta z^4}$$

233. Quod ſi ergo ſcala relationis fuerit bimembris $\alpha, =$

α, $-\beta$; Seriei $A + Bz + Cz^2 + Dz^3 + \ldots\ldots + Pz^n$; quae oritur ex fractione $\frac{A + (B - \alpha A)z}{1 - \alpha z + \beta zz}$, summa erit

$$\frac{A + (B - \alpha A)z - Qz^{n+1} - (R - \alpha Q)z^{n+2}}{1 - \alpha z + \beta zz}.$$

At est, ex natura Seriei, $R = \alpha Q - \beta P$; unde prodibit summa

$$\frac{A + (B - \alpha A)z - Qz^{n+1} + \beta Pz^{n+2}}{1 - \alpha z + \beta zz}$$

EXEMPLUM.

Sit proposita Series $1 + 3z + 4z^2 + 7z^3 + \ldots\ldots + Pz^n$ ubi est $\alpha = 1$; $\beta = -1$; $A = 1$; $B = 3$; erit hujus summa

$$\frac{1 + 2z - Qz^{n+1} - Pz^{n+2}}{1 - z - zz}.$$

Posito vero $z = 1$; erit summa Seriei $1 + 3 + 4 + 7 + 11 + \ldots\ldots\ldots + P = P + Q - 3$. Summa ergo termini ultimi & sequentis ternario excedit summam Seriei. Quia vero est $Q = \frac{P + \sqrt{(5PP \pm 20)}}{2}$ erit summa Seriei $1 + 3 + 4 + 7 + 11 + \ldots\ldots + P = \frac{3P - 6 + \sqrt{(5PP \pm 20)}}{2}$. Ex solo ergo termino ultimo summa potest exhiberi.

CAPUT XIV.

De multiplicatione ac divisione Angulorum.

234. Sit Angulus, vel Arcus, in Circulo cujus Radius $= 1$, quicunque $= z$; ejus Sinus $= x$; Cosinus $= y$, & Tangens $= t$; erit $xx + yy = 1$ & $t = \frac{x}{y}$. Cum igitur, uti supra vidimus, tam Sinus quam Cosinus Angulorum z; $2z$; $3z$; $4z$; $5z$; &c., constituant Seriem recurrentem cujus scala relationis est $2y, -1$; primum Sinus horum Arcuum ita se habebunt:

$$\begin{aligned}
fin.\,0z &= 0\\
fin.\,1z &= x\\
fin.\,2z &= 2xy\\
fin.\,3z &= 4xy^2 - x\\
fin.\,4z &= 8xy^3 - 4xy\\
fin.\,5z &= 16xy^4 - 12xy^2 + x\\
fin.\,6z &= 32xy^5 - 32xy^3 + 6xy\\
fin.\,7z &= 64xy^6 - 80xy^4 + 24xy^2 - x\\
fin.\,8z &= 128xy^7 - 192xy^5 + 80xy^3 - 0xy
\end{aligned}$$

hinc concluditur fore

$$fin.\,nz = x\Big(2^{n-1}y^{n-1} - (n-2)\,2^{n-3}y^{n-3} + \frac{(n-3)(n-4)}{1.\;2}2^{n-5}y^{n-5} - \frac{(n-4)(n-5)(n-6)}{1.\;2.\;3}2^{n-7}y^{n-7} + \frac{(n-5)(n-6)(n-7)(n-8)}{1.\;2.\;3.\;4}2^{n-9}y^{n-9} - \&c.\Big)$$

235. Si ponamus Arcum $nz = s$; erit $fin.\,nz = fin.\,s = fin.\,(\pi - s) = fin.\,(2\pi + s) = fin.\,(3\pi - s)$ &c., hi enim

enim Sinus omnes sunt inter se æquales. Hinc obtinemus plures valores pro x, qui erunt

$$\text{fin.}\ \frac{s}{n};\ \text{fin.}\ \frac{\pi - s}{n};\ \text{fin.}\ \frac{2\pi + s}{n};\ \text{fin.}\ \frac{3\pi - s}{n};\ \text{fin.}\ \frac{4\pi + s}{n};\ \&c.,$$

qui ergo omnes æquationi inventæ æque conveniunt. Tot autem prodibunt diversi pro x valores, quot numerus n continet unitates, qui propterea erunt radices æquationis inventæ. Cavendum ergo est, ne valores æquales pro iisdem habeantur, quod fiet dum alternæ tantum expressiones assumantur. Cognitis igitur radicibus æquationis a posteriori, earum comparatio cum terminis æquationis notatu dignas præbebit proprietates. Quoniam autem ad hoc æquatio, in qua tantum x tanquam incognita insit, requiritur, pro y suus valor $\surd(1 - xx)$ substitui debet; unde duplex operatio instituenda erit, prout n fuerit vel numerus par vel impar.

236. Sit n numerus impar, quia Arcuum $-z$, $+z$, $+3z$, $+5z$; &c., differentia est $2z$, hujusque Cosinus $= 1 - 2xx$, erit progressionis Sinuum scala relationis hæc $2 - 4xx$, -1. Hinc erit

$$\begin{aligned}
\text{fin.} -z &= -x \\
\text{fin.}\ z &= x \\
\text{fin.}\ 3z &= 3x - 4x^3 \\
\text{fin.}\ 5z &= 5x - 20x^3 + 16x^5 \\
\text{fin.}\ 7z &= 7x - 56x^3 + 112x^5 - 64x^7 \\
\text{fin.}\ 9z &= 9x - 120x^3 + 432x^5 - 576x^7 + 256x^9
\end{aligned}$$

ergo

$$\text{fin.}\ nz = nx - \frac{n(nn-1)}{1.\ 2.\ 3}x^3 + \frac{n(nn-1)(nn-9)}{1.\ 2.\ 3.\ 4.\ 5}x^5 - \frac{n(nn-1)(nn-9)(nn-25)}{1.\ 2.\ 3.\ 4.\ 5.\ 6.\ 7}x^7 + \&c.,$$

si quidem n fuerit numerus impar. Hujusque æquationis radices sunt

LIB. I. sunt $fin.z$; $fin.(\frac{2\varpi}{n}+z)$; $fin.(\frac{4\varpi}{n}+z)$; $fin.(\frac{6\varpi}{n}+z)$; $fin.(\frac{8\varpi}{n}+z)$; &c., quarum numerus est n.

237. Hujus ergo æquationis

$$o = 1 - \frac{nx}{fin.nz} + \frac{n(nn-1)x^3}{1.2.3\,fin.nz} - \frac{n(nn-1)(nn-9)x^5}{1.2.3.4.5\,fin.nz} \dots \pm \frac{2^{n-1}x^n}{fin.nz},$$

(ubi signum superius valet si n unitate deficiat a multiplo quaternarii, contra inferius,) Factores sunt $(1-\frac{x}{fin.z})$ $(1-\frac{x}{fin.(\frac{2\varpi}{n}+z)})(1-\frac{x}{fin.(\frac{4\pi}{n}+z)})$ &c., ex quibus concluditur fore

$$\frac{n}{fin.nz} = \frac{1}{fin.z} + \frac{1}{fin.(\frac{2\varpi}{n}+z)} + \frac{1}{fin.(\frac{4\varpi}{n}+z)} + \frac{1}{fin.(\frac{6\varpi}{n}+z)} +$$

&c., donec habeantur n termini. Tum vero productum omnium erit

$$\mp\frac{2^{n-1}}{fin.nz} = \frac{1}{fin.z.fin.(\frac{2\varpi}{n}+z).fin.(\frac{4\varpi}{n}+z).fin.(\frac{6\varpi}{n}+z)\&c.}$$

seu $fin.nz = \mp 2^{n-1} fin.z.fin.(\frac{2\varpi}{n}+z).fin.(\frac{4\varpi}{n}+z) \times fin.(\frac{6\varpi}{n}+z)$ &c.. Et, quia terminus penultimus deest, erit

$$o = fin.z + fin.(\frac{2\varpi}{n}+z) + fin.(\frac{4\pi}{n}+z) + fin.(\frac{6\pi}{n}+z)\&c.$$

EXEMPLUM I.

Si ergo fuerit $n=3$, prodibunt hæ æqualitates $o = fin.z + fin.(120° + z) + fin.(240° + z) = fin.z + fin.(60 - z) - fin.(60 + z)$.

$\frac{3}{fin.3z}$

CAP. XIV.

$$\frac{3}{\textit{fin.}\,3z} = \frac{1}{\textit{fin.}\,z} + \frac{1}{\textit{fin.}\,(120+z)} + \frac{1}{\textit{fin.}\,(240+z)} = \frac{1}{\textit{fin.}\,z} + \frac{1}{\textit{fin.}\,(60-z)} - \frac{1}{\textit{fin.}\,(60+z)}$$

$$\textit{fin.}\,3z = -4\,\textit{fin.}\,z.\,\textit{fin.}\,(120+z)\,\textit{fin.}\,(240+z) = 4\,\textit{fin.}\,z.\,\textit{fin.}\,(60-z).\,\textit{fin.}\,(60+z).$$

Erit ergo, uti jam fupra notavimus,

$$\textit{fin.}\,(60+z) = \textit{fin.}\,z + \textit{fin.}\,(60-z),\ \&$$

$$3\,\textit{cofec.}\,3z = \textit{cofec.}\,z + \textit{cofec.}\,(60-z) - \textit{cofec.}\,(60+z).$$

EXEMPLUM II.

Ponamus effe $n = 5$, atque prodibunt hæ æquationes:

$$o = \textit{fin.}\,z + \textit{fin.}\,(\tfrac{2}{5}\pi + z) + \textit{fin.}\,(\tfrac{4}{5}\pi + z) + \textit{fin.}\,(\tfrac{6}{5}\pi + z) + \textit{fin.}\,(\tfrac{8}{5}\pi + z)$$

$$\text{feu } o = \textit{fin.}\,z + \textit{fin.}\,(\tfrac{2}{5}\pi + z) + \textit{fin.}\,(\tfrac{1}{5}\pi - z) - \textit{fin.}\,(\tfrac{1}{5}\pi + z) - \textit{fin.}\,(\tfrac{2}{5}\pi - z)$$

$$\text{feu } o = \textit{fin.}\,z + \textit{fin.}\,(\tfrac{1}{5}\pi - z) - \textit{fin.}\,(\tfrac{1}{5}\pi + z) + \textit{fin.}\,(\tfrac{2}{5}\pi + z) - \textit{fin.}\,(\tfrac{2}{5}\pi - z)$$

deinde erit

$$\frac{5}{\textit{fin.}\,5z} = \frac{1}{\textit{fin.}\,z} + \frac{1}{\textit{fin.}\,(\frac{1}{5}\pi - z)} - \frac{1}{\textit{fin.}\,(\frac{1}{5}\pi + z)} - \frac{1}{\textit{fin.}\,(\frac{2}{5}\pi - z)} + \frac{1}{\textit{fin.}\,(\frac{2}{5}\pi + z)}$$

$$\textit{fin.}\,5z = 16\,\textit{fin.}\,z.\,\textit{fin.}\,(\tfrac{1}{5}\pi - z).\,\textit{fin.}\,(\tfrac{1}{5}\pi + z) \times \textit{fin.}\,(\tfrac{2}{5}\pi - z).\,\textit{fin.}\,(\tfrac{2}{5}\pi + z)$$

EXEMPLUM III.

Hoc modo, si ponamus $n = 2m + 1$, erit

$$0 = \mathit{fin}.\,z + \mathit{fin}.\left(\frac{\pi}{n} - z\right) - \mathit{fin}.\left(\frac{\pi}{n} + z\right) - \mathit{fin}.\left(\frac{2\pi}{n} - z\right) + \mathit{fin}.\left(\frac{2\pi}{n} + z\right) + \mathit{fin}.\left(\frac{3\pi}{n} - z\right) - \mathit{fin}.\left(\frac{3\pi}{n} + z\right) - \ldots \pm \mathit{fin}.\left(\frac{m}{n}\pi - z\right) \mp \mathit{fin}.\left(\frac{m}{n}\pi + z\right)$$

ubi signa superiora valent si m sit numerus impar, inferiora si sit par. Altera æquatio erit hæc.

$$\frac{n}{\mathit{fin}.\,nz} = \frac{1}{\mathit{fin}.\,z} + \frac{1}{\mathit{fin}.\left(\frac{\pi}{n} - z\right)} - \frac{1}{\mathit{fin}.\left(\frac{\pi}{n} + z\right)} - \frac{1}{\mathit{fin}.\left(\frac{2\pi}{n} - z\right)} + \frac{1}{\mathit{fin}.\left(\frac{2\pi}{n} + z\right)} + \frac{1}{\mathit{fin}.\left(\frac{3\pi}{n} - z\right)} - \frac{1}{\mathit{fin}.\left(\frac{3\pi}{n} + z\right)} - \ldots \pm \frac{1}{\mathit{fin}.\left(\frac{m\pi}{n} - z\right)} \mp \frac{1}{\mathit{fin}.\left(\frac{m\pi}{n} + z\right)}$$

quæ ad Cosecantes commode transfertur. Tertio habetur hoc productum:

$$\mathit{fin}.\,nz = 2^{2m}\,\mathit{fin}.\,z.\,\mathit{fin}.\left(\frac{\pi}{n} - z\right).\,\mathit{fin}.\left(\frac{\pi}{n} + z\right).\,\mathit{fin}.\left(\frac{2\pi}{n} - z\right) \times \mathit{fin}.\left(\frac{2\pi}{n} + z\right).\,\mathit{fin}.\left(\frac{3\pi}{n} - z\right).\,\mathit{fin}.\left(\frac{3\pi}{n} + z\right) \ldots\ldots\ldots \times \mathit{fin}.\left(\frac{m\pi}{n} - z\right).\,\mathit{fin}.\left(\frac{m\pi}{n} + z\right).$$

238. Sit n nunc numerus par, & quoniam est $y = \sqrt{(1 - xx)}$ & $\mathit{cof}.\,2z = 1 - 2xx$, ita ut Seriei Sinuum sit scala relationis, ut ante, $2 - 4xx$, -1, erit

fin.

$\sin.\ 0z = 0$

$\sin.\ 2z = 2x\sqrt{(1-xx)}$

$\sin.\ 4z = (4x-8x^3)\sqrt{(1-xx)}$

$\sin.\ 6z = (6x-32x^3+32x^5)\sqrt{(1-xx)}$

$\sin.\ 8z = (8x-80x^3+192x^5-128x^7)\sqrt{(1-xx)}$

& generaliter

$$\sin.\ nz = \left(nx - \frac{n(nn-4)}{1.\,2.\,3}x^3 + \frac{n(nn-4)(nn-16)}{1.\,2.\,3.\,4.\,5}x^5 - \frac{n(nn-4)(nn-16)(nn-36)}{1.\,2.\,3.\,4.\,5.\,6.\,7}x^7 + \ldots\ldots \pm 2^{n-1}x^{n-1}\right)\sqrt{(1-xx)}$$

denotante n numerum quemcunque parem.

239. Ad æquationem hanc rationalem efficiendam ſumantur utrinque quadrata, ac prodibit hujuſmodi æquatio

$$(\sin.\ nz)^2 = nnxx + Px^4 + Qx^6 + \ldots - 2^{2n-2}x^{2n}$$

$$\text{ſeu } x^{2n} \ldots\ldots - \frac{nn}{2^{2n-2}}xx + \frac{1}{2^{2n-2}}(\sin.\ nz)^2 = 0$$

cujus æquationis radices erunt tam affirmativæ quam negativæ; Scilicet $\pm\sin.\ z$; $\pm\sin.\left(\frac{\pi}{n}-z\right)$; $\pm\sin.\left(\frac{2\pi}{n}+z\right)$; $\pm\sin.\left(\frac{3\pi}{n}-z\right)$; $\pm\sin.\left(\frac{4\pi}{n}+z\right)$ &c. Sumendo omnino n hujuſmodi expreſſiones. Cum igitur ultimus terminus ſit productum omnium harum radicum, extrahendo utrinque radicem quadratam erit

$$\sin.\ nz = \pm 2^{n-1}\sin.\ z.\sin.\left(\frac{\pi}{n}-z\right).\sin.\left(\frac{2\pi}{n}+z\right)\times \sin.\left(\frac{3\pi}{n}-z\right)\ldots\ldots;$$

ubi, quibus caſibus utrumvis ſignum valeat, ex caſibus particularibus erit diſpiciendum.

EXEMPLUM.

Subſtituendo autem pro n ſucceſſive numeros 2, 4, 6, &c. & eligendo n Sinus diverſos erit.

$$\sin. 2z = 2\sin. z. \sin. \left(\frac{\pi}{2} - z\right)$$

$$\sin. 4z = 8\sin. z. \sin. \left(\frac{\pi}{4} - z\right). \sin. \left(\frac{\pi}{4} + z\right). \sin. \left(\frac{\pi}{2} - z\right)$$

$$\sin. 6z = 32\sin. z. \sin. \left(\frac{\pi}{6} - z\right). \sin. \left(\frac{\pi}{6} + z\right). \sin. \left(\frac{2\pi}{6} - z\right) \times$$
$$\sin. \left(\frac{2\pi}{6} + z\right). \sin. \left(\frac{3\pi}{6} - z\right)$$

$$\sin. 8z = 128\sin. z. \sin. \left(\frac{\pi}{8} - z\right). \sin. \left(\frac{\pi}{8} + z\right). \sin. \left(\frac{2\pi}{8} - z\right) \times$$
$$\sin. \left(\frac{2\pi}{8} + z\right). \sin. \left(\frac{3\pi}{8} - z\right). \sin. \left(\frac{3\pi}{8} + z\right). \sin. \left(\frac{4\varpi}{8} - z\right)$$

240. Patet ergo fore generatim

$$\sin. nz = 2^{n-1} \sin. z. \sin. \left(\frac{\varpi}{n} - z\right). \sin. \left(\frac{\pi}{n} + z\right). \sin. \left(\frac{2\pi}{n} - z\right) \times$$
$$\sin. \left(\frac{2\pi}{n} + z\right). \sin. \left(\frac{3\pi}{n} - z\right). \sin. \left(\frac{3\pi}{n} + z\right) \ldots \sin. \left(\frac{1}{2}\pi - z\right)$$

si n fuerit numerus par. Quod si autem hæc cum superiori, ubi n erat numerus impar, comparetur tanta similitudo adesse deprehenditur, ut utramque in unam redigere liceat. Erit ergo, sive n fuerit numerus par sive impar,

$$\sin. nz = 2^{n-1} \sin. z. \sin. \left(\frac{\pi}{n} - z\right). \sin. \left(\frac{\varpi}{n} + z\right). \sin. \left(\frac{2\pi}{n} - z\right) \times$$
$$\sin. \left(\frac{2\varpi}{n} + z\right). \sin. \left(\frac{3\pi}{n} - z\right). \sin. \left(\frac{3\pi}{n} + z\right) \text{ \&c.}$$

donec tot habeantur Factores, quot numerus n continet unitates.

241. Expressiones istæ, quibus Sinus Angulorum multiplorum per Factores exponuntur, non parum utilitatis afferre possunt ad Logarithmos Sinuum Angulorum multiplorum inveniendos, itemque ad plures expressiones Sinuum per Factores, quales supra (§. 184) dedimus, reperiendas. Erit autem

ſin.

CAP. XIV.

$$\text{fin. } z = 1 \text{ fin. } z$$

$$\text{fin. } 2z = 2 \text{ fin. } z . \text{fin.} \left(\frac{\pi}{2} - z\right)$$

$$\text{fin. } 3z = 4 \text{ fin. } z . \text{fin.} \left(\frac{\pi}{3} - z\right) . \text{fin.} \left(\frac{\pi}{3} + z\right)$$

$$\text{fin. } 4z = 8 \text{ fin. } z . \text{fin.} \left(\frac{\pi}{4} - z\right) . \text{fin.} \left(\frac{\pi}{4} + z\right) . \text{fin.} \left(\frac{2\pi}{4} - z\right)$$

$$\text{fin. } 5z = 16 \text{ fin. } z . \text{fin.} \left(\frac{\pi}{5} - z\right) . \text{fin.} \left(\frac{\pi}{5} + z\right) . \text{fin.} \left(\frac{2\pi}{5} - z\right) \times \text{fin.} \left(\frac{2\pi}{5} + z\right)$$

$$\text{fin. } 6z = 32 \text{ fin. } z . \text{fin.} \left(\frac{\pi}{6} - z\right) . \text{fin.} \left(\frac{\pi}{6} + z\right) . \text{fin.} \left(\frac{2\pi}{6} - z\right) \times \text{fin.} \left(\frac{2\pi}{6} + z\right) . \text{fin.} \left(\frac{3\pi}{6} - z\right)$$

&c.

242. Cum deinde fit $\frac{\text{fin. } 2nz}{\text{fin. } nz} = 2 \text{cof. } nz$, Cofinus Angulorum multiplorum fimili modo per Factores exprimentur.

$$\text{cof. } z = 1 \text{ fin.} \left(\frac{\pi}{2} - z\right)$$

$$\text{cof. } 2z = 2 \text{ fin.} \left(\frac{\pi}{4} - z\right) . \text{fin.} \left(\frac{\pi}{4} + z\right)$$

$$\text{cof. } 3z = 4 \text{ fin.} \left(\frac{\pi}{6} - z\right) . \text{fin.} \left(\frac{\pi}{6} + z\right) . \text{fin.} \left(\frac{3\pi}{6} - z\right)$$

$$\text{cof. } 4z = 8 \text{ fin.} \left(\frac{\pi}{8} - z\right) . \text{fin.} \left(\frac{\pi}{8} + z\right) . \text{fin.} \left(\frac{3\pi}{8} - z\right) \times \text{fin.} \left(\frac{3\pi}{8} + z\right)$$

$$\text{cof. } 5z = 16 \text{ fin.} \left(\frac{\pi}{10} - z\right) . \text{fin.} \left(\frac{\pi}{10} + z\right) . \text{fin.} \left(\frac{3\pi}{10} - z\right) \times \text{fin.} \left(\frac{3\pi}{10} + z\right) . \text{fin.} \left(\frac{5\pi}{10} - z\right)$$

& generaliter

LIB. I.
$$\cos. nz = 2^{n-1} \sin.\left(\frac{\pi}{2n} - z\right).\sin.\left(\frac{\pi}{2n} + z\right).\sin.\left(\frac{3\pi}{2n} - z\right) \times \sin.\left(\frac{3\pi}{2n} + z\right).\sin.\left(\frac{5\pi}{2n} - z\right) \&c.,$$

quoad tot habeantur Factores quot numerus n continet unitates.

243. Eædem expressiones prodibunt ex consideratione Cosinuum Arcuum multiplorum, si enim fuerit $\cos. z = y$, erit ut sequitur

$$\begin{array}{lllllll}
\cos. 0z & = & 1 & & & & \\
\cos. 1z & = & y & & & & \\
\cos. 2z & = & 2yy & - 1 & & & \\
\cos. 3z & = & 4y^3 & - & 3y & & \\
\cos. 4z & = & 8y^4 & - & 8yy & + 1 & \\
\cos. 5z & = & 16y^5 & - & 20y^3 & + 5y & \\
\cos. 6z & = & 32y^6 & - & 48y^4 & + 18yy & - 1 \\
\cos. 7z & = & 64y^7 & - & 112y^5 & + 56y^3 & - 7y
\end{array}$$

& generaliter.

$$\cos. nz = 2^{n-1} y^n - \frac{n}{1} 2^{n-3} y^{n-2} + \frac{n(n-3)}{1.\ 2} 2^{n-5} y^{n-4} - \frac{n(n-4)(n-5)}{1.\ 2.\ 3} 2^{n-7} y^{n-6} + \frac{n(n-5)(n-6)(n-7)}{1.\ 2.\ 3.\ 4} 2^{n-9} y^{n-8} - \&c.,$$

cujus æquationis, cum sit $\cos. nz = \cos.(2\pi - nz) = \cos.(2\pi + nz) = \cos.(4\pi \pm nz) = \cos.(6\pi \pm nz)$ &c., erunt radices ipsius y hæ: $\cos. z$; $\cos.\left(\frac{2\pi}{n} \pm z\right)$; $\cos.\left(\frac{4\pi}{n} \pm z\right)$; $\cos.\left(\frac{6\pi}{n} \pm z\right)$ &c., quarum formularum tot diversæ sunt pro y eligendæ quot dantur; dantur autem tot, quot n continet unitates.

244. Pri-

244. Primum igitur patet, ob terminum secundum deficientem excepto casu $n=1$, fore summam harum radicum omnium $=0$. Erit ergo

$$0=\cos.z+\cos.\left(\frac{2\pi}{n}-z\right)+\cos.\left(\frac{2\pi}{n}+z\right)+\cos.\left(\frac{4\pi}{n}-z\right)+\cos.\left(\frac{4\pi}{n}+z\right)+\&c.,$$

sumendo tot terminos quot n continet unitates: Hæc autem æqualitas sponte se offert si n sit numerus par, cum quivis terminus ab alio sui negativo destruatur. Contemplemur ergo numeros impares, unitate exclusa, eritque, ob $\cos.v=-\cos.(\pi-v)$

$$0=\cos.z-\cos.\left(\frac{\pi}{3}-z\right)-\cos.\left(\frac{\pi}{3}+z\right)$$

$$0=\cos.z-\cos.\left(\frac{\pi}{5}-z\right)-\cos.\left(\frac{\pi}{5}+z\right)+\cos.\left(\frac{2\pi}{5}-z\right)+\cos.\left(\frac{2\pi}{5}+z\right)$$

$$0=\cos.z-\cos.\left(\frac{\pi}{7}-z\right)-\cos.\left(\frac{\pi}{7}+z\right)+\cos.\left(\frac{2\pi}{7}-z\right)+\cos.\left(\frac{2\pi}{7}+z\right)-\cos.\left(\frac{3\pi}{7}-z\right)-\cos.\left(\frac{3\pi}{7}+z\right)$$

& generaliter, si fuerit n numerus impar quicunque, erit

$$0=\cos.z-\cos.\left(\frac{\pi}{n}-z\right)-\cos.\left(\frac{\pi}{n}+z\right)+\cos.\left(\frac{2\pi}{n}-z\right)+\cos.\left(\frac{2\pi}{n}+z\right)-\cos.\left(\frac{3\pi}{n}-z\right)-\cos.\left(\frac{3\pi}{n}+z\right)+\cos.\left(\frac{4\pi}{n}-z\right)+\cos.\left(\frac{4\pi}{n}+z\right)-\&c.,$$

sumendo tot terminos, quot numerus n continet unitates: oportet autem n esse numerum imparem unitate majorem, uti jam monuimus.

245. Quod

Lib. I. 245. Quod ad productum ex omnibus attinet, variæ quidem prodeunt expreſſiones, prout n fuerit numerus vel impar, vel impariter par, vel pariter par: omnes autem comprehenduntur in expreſſione generali (§. 242.) inventa, ſi ſinguli Sinus in Coſinus tranſmutentur: Erit ſcilicet

$$cof.\, z = 1\, cof.\, z$$

$$cof.\, 2z = 2\, cof.(\tfrac{\pi}{4} + z).cof.(\tfrac{\pi}{4} - z)$$

$$cof.\, 3z = 4\, cof.(\tfrac{2\pi}{6} + z).cof.(\tfrac{2\pi}{6} - z).\, cof.\, z$$

$$cof.\, 4z = 8\, cof.(\tfrac{3\pi}{8} + z).cof.(\tfrac{3\pi}{8} - z).cof.(\tfrac{\pi}{8} + z)\times cof.(\tfrac{\pi}{8} - z)$$

$$cof.\, 5z = 16\, cof.(\tfrac{4\pi}{8} + z).cof.(\tfrac{4\pi}{8} - z).cof.(\tfrac{2\pi}{8} + z)\times cof.(\tfrac{2\pi}{8} - z)\, cof.\, z$$

& generaliter

$$cof.\, nz = 2^{n-1}\, cof.(\tfrac{n-1}{n}\pi + z).cof.(\tfrac{n-1}{n}\pi - z)\times cof.(\tfrac{n-3}{n}\pi + z).cof.(\tfrac{n-3}{n}\pi - z)\times cof.(\tfrac{n-5}{n}\pi + z).cof.(\tfrac{n-5}{n}\pi - z)\times cof.(\tfrac{n-7}{n}\pi + z)\ \&c.,$$

ſumtis tot Factoribus, quot numerus n continet unitates.

246. Sit n numerus impar, atque æquatio incipiatur ab unitate, erit

$0 = 1 \mp \frac{ny}{cof.\, nz} + \&c.$, ubi ſignum ſuperius valet ſi n fuerit numerus impar formæ $4m + 1$, inferius ſi $n = 4m - 1$. Hinc erit

$$+\frac{1}{cos.z}=\frac{1}{cos.z}$$

$$-\frac{3}{cos.3z}=\frac{1}{cos.z}-\frac{1}{cos.(\frac{\pi}{3}-z)}-\frac{1}{cos.(\frac{\pi}{3}+z)}$$

$$+\frac{5}{cos.5z}=\frac{1}{cos.z}-\frac{1}{cos.(\frac{\pi}{5}-z)}-\frac{1}{cos.(\frac{\pi}{5}+z)}+\frac{1}{cos.(\frac{2\pi}{5}-z)}+\frac{1}{cos.(\frac{2\pi}{5}+z)}$$

& generaliter, posito $n=2m+1$, erit

$$\frac{n}{cos.nz}=\frac{2m+1}{cos.(2m+1)z}=\frac{1}{cos.(\frac{m}{n}\pi+z)}+\frac{1}{cos.(\frac{m}{n}\pi-z)}-\frac{1}{cos.(\frac{m-1}{n}\pi+z)}-\frac{1}{cos.(\frac{m-1}{n}\pi-z)}+\frac{1}{cos.(\frac{m-2}{n}\pi+z)}+\frac{1}{cos.(\frac{m-2}{n}\pi-z)}-\frac{1}{cos.(\frac{m-3}{n}\pi+z)}-\&c.$$

sumendis tot terminis, quot n continet unitates.

247. Cum ergo sit $\frac{1}{cos.v}=sec.v.$, hinc pro Secantibus insignes proprietates deducuntur, erit nempe

$$sec.\ z=sec.z.$$

$$3sec.3z=sec.(\frac{\pi}{3}+z)+sec.(\frac{\pi}{3}-z)-sec.(\frac{0\pi}{3}+z)$$

$$5sec.5z=sec.(\frac{2\pi}{5}+z)+sec.(\frac{2\pi}{5}-z)-sec.(\frac{\pi}{5}+z)-sec.(\frac{\pi}{5}-z)+sec.(\frac{0\pi}{5}+z)$$

$$7\,sec.\,7z = sec.\left(\frac{3\pi}{7}+z\right)+sec.\left(\frac{3\pi}{7}-z\right)-sec.\left(\frac{2\pi}{7}+z\right)-$$
$$sec.\left(\frac{2\pi}{7}-z\right)+sec.\left(\frac{\varpi}{7}+z\right)+sec.\left(\frac{\varpi}{7}-z\right)-$$
$$sec.\left(\frac{0\varpi}{7}+z\right)$$

& generaliter, posito $n = 2m+1$, erit

$$n.sec.\,nz = sec.\left(\frac{m}{n}\pi+z\right)+sec.\left(\frac{m}{n}\varpi-z\right)-$$
$$sec.\left(\frac{m-1}{n}\pi+z\right)-sec.\left(\frac{m-1}{n}\varpi-z\right)+$$
$$sec.\left(\frac{m-2}{n}\pi+z\right)+sec.\left(\frac{m-2}{n}\varpi-z\right)-$$
$$sec.\left(\frac{m-3}{n}\pi+z\right)-sec.\left(\frac{m-3}{n}\varpi-z\right)+$$
$$sec.\left(\frac{m-4}{n}\pi+z\right)+\ldots\ldots\ldots\ldots\pm sec.\,z.$$

248. Pro Cosecantibus autem erit ex §. 237.

$$cosec.\,z = cosec.\,z$$
$$3\,cosec.\,3z = cosec.\,z + cosec.\left(\frac{\pi}{3}-z\right)-cosec.\left(\frac{\pi}{3}+z\right)$$
$$5\,cosec.\,5z = cosec.\,z + cosec.\left(\frac{\pi}{5}-z\right)-cosec.\left(\frac{\pi}{5}+z\right)-$$
$$cosec.\left(\frac{2\pi}{5}-z\right)+cosec.\left(\frac{2\pi}{5}+z\right)$$
$$7\,cosec.\,7z = cosec.\,z + cosec.\left(\frac{\pi}{7}-z\right)-cosec.\left(\frac{\pi}{7}+z\right)-$$
$$cosec.\left(\frac{2\pi}{7}-z\right)+cosec.\left(\frac{2\pi}{7}+z\right)+$$
$$cosec.\left(\frac{3\pi}{7}-z\right)-cosec.\left(\frac{3\pi}{7}+z\right).$$

& generaliter, ponendo $n = 2m+1$, erit

n. cosec.

$$m.\,cosec.\,nz = cosec.\,z + cosec.\left(\frac{\pi}{n} - z\right) - cosec.\left(\frac{\pi}{n} + z\right) - cosec.\left(\frac{2\pi}{n} - z\right) + cosec.\left(\frac{2\pi}{n} + z\right) + cosec.\left(\frac{3\pi}{n} - z\right) - cosec.\left(\frac{3\pi}{n} + z\right) - \ldots\ldots\ldots\ldots \mp cosec.\left(\frac{m\pi}{n} - z\right) \pm cosec.\left(\frac{m\pi}{n} + z\right)$$

ubi ſigna ſuperiora valent ſi m fuerit numerus par, inferiora ſi m ſit impar.

249. Cum ſit, uti ſupra vidimus, $cof.\,nz + \sqrt{-1}.\,fin.\,nz = (cof.\,z \pm \sqrt{-1}.\,fin.\,z)^n$, erit $cof.\,nz = \frac{(cof.\,z + \sqrt{-1}.\,fin.\,z)^n + (cof.\,z - \sqrt{-1}.\,fin.\,z)^n}{2}$, & $fin.\,nz = \frac{(cof.\,z + \sqrt{-1}.\,fin.\,z)^n - (cof.\,z - \sqrt{-1}.\,fin.\,z)^n}{2\sqrt{-1}}$, erit

$$tang.\,nz = \frac{(cof.\,z + \sqrt{-1}.\,fin.\,z)^n - (cof.\,z - \sqrt{-1}.\,fin.\,z)^n}{(cof.\,z + \sqrt{-1}.\,fin.\,z)^n \sqrt{-1} + (cof.\,z - \sqrt{-1}.\,fin.\,z)^n \sqrt{-1}}.$$

Ponamus $tang.\,z = \frac{fin.\,z}{cof.\,z} = t$, erit $tang.\,nz = \frac{(1 + t\sqrt{-1})^n - (1 - t\sqrt{-1})^n}{(1 + t\sqrt{-1})^n \sqrt{-1} + (1 - t\sqrt{-1})^n \sqrt{-1}}$, unde oriuntur Tangentes Angulorum multiplorum ſequentes

$$tang.\,z = t$$

$$tang.\,2z = \frac{2t}{1 - tt}$$

$$tang.\,3z = \frac{3t - t^3}{1 - 3tt}$$

$$tang.\,4z = \frac{4t - 4t^3}{1 - 6tt + t^4}$$

$$tang.\,5z = \frac{5t - 10t^3 + t^5}{1 - 10tt + 5t^4}$$

& generaliter

$$tang. nz = \frac{nt - \frac{n(n-1)(n-2)}{1.\ 2.\ 3} t^3 + \frac{n(n-1)(n-2)(n-3)(n-4)}{1.\ 2.\ 3.\ 4.\ 5} t^5 - \&c.}{1 - \frac{n(n-1)}{1.\ 2} tt + \frac{n(n-1)(n-2)(n-3)}{1.\ 2.\ 3.\ 4} t^4 - \&c.}.$$

Cum jam sit $tang.\ nz = tang.(\pi + nz) = tang.(2\pi + nz) = tang.(3\pi + nz)$ &c., erunt valores ipsius t, seu radices æquationis, hæ, $tang. z$; $tang.(\frac{\pi}{n} + z)$; $tang.(\frac{2\pi}{n} + z)$; $tang.(\frac{3\pi}{n} + z)$; &c., quarum numerus est n.

250. Quod si æquatio ab unitate incipiat, erit

$$o = 1 - \frac{nt}{tang.\ nz} - \frac{n(n-1)tt}{1.\ 2} + \frac{n(n-1)(n-2)t^3}{1.\ 2.\ 3\ tang.\ nz} + \&c..$$

Ex comparatione ergo coëfficientium cum radicibus, erit

$$n.cot.\ nz = cot.\ z + cot.(\frac{\pi}{n} + z) + cot.(\frac{2\pi}{n} + z) + cot.(\frac{3\pi}{n} + z) + cot.(\frac{4\pi}{n} + z) + \ldots\ldots + cot.(\frac{n-1}{n}\pi + z)$$

deinde erit summa quadratorum harum Cotangentium omnium $= \frac{nn}{(sin.\ nz)^2} - n$, similique modo ulteriores Potestates possunt definiri. Ponendo autem loco n numeros definitos, erit.

$$cot.\ z = cot.\ z$$

$$2cot.\ 2z = cot.\ z + cot.(\frac{\pi}{2} + z)$$

$$3cot.\ 3z = cot.\ z + cot.(\frac{\pi}{3} + z) + cot.(\frac{2\pi}{3} + z)$$

$$4cot.\ 4z = cot.\ z + cot.(\frac{\pi}{4} + z) + cot.(\frac{2\pi}{4} + z) + cot.(\frac{3\pi}{4} + z)$$

5 cot.

$$5\cot.5z = \cot.z + \cot.\left(\frac{\pi}{5}+z\right) + \cot.\left(\frac{2\pi}{5}+z\right) + \cot.\left(\frac{3\pi}{5}+z\right) + \cot.\left(\frac{4\pi}{5}+z\right).$$

CAP. XIV.

251. Quia vero eſt $\cot.v = -\cot.(\pi - v)$, erit

$$\cot.z = \cot.z$$

$$2\cot.2z = \cot.z - \cot.\left(\frac{\pi}{2}-z\right)$$

$$3\cot.3z = \cot.z - \cot.\left(\frac{\pi}{3}-z\right) + \cot.\left(\frac{\pi}{3}+z\right)$$

$$4\cot.4z = \cot.z - \cot.\left(\frac{\pi}{4}-z\right) + \cot.\left(\frac{\pi}{4}+z\right) - \cot.\left(\frac{2\pi}{4}-z\right)$$

$$5\cot.5z = \cot.z - \cot.\left(\frac{\pi}{5}-z\right) + \cot.\left(\frac{\pi}{5}+z\right) - \cot.\left(\frac{2\pi}{5}-z\right) + \cot.\left(\frac{2\pi}{5}+z\right)$$

& generaliter

$$n.\cot.nz = \cot.z - \cot.\left(\frac{\pi}{n}-z\right) + \cot.\left(\frac{\pi}{n}+z\right) - \cot.\left(\frac{2\pi}{n}-z\right) + \cot.\left(\frac{2\pi}{n}+z\right) - \cot.\left(\frac{3\pi}{n}-z\right) + \cot.\left(\frac{3\pi}{n}+z\right) - \text{\&c.}$$

donec tot habeantur termini, quot numerus n continet unitates.

252. Incipiamus æquationem inventam a Poteſtate ſumma; ubi primum diſtingendi ſunt caſus, quibus n eſt vel numerus par, vel impar. Sit n numerus impar, $n = 2m + 1$ erit

$$t - tang.\,z = 0$$

$$t^3 - 3tt.\,tang.\,3z - 3t + tang.\,3z = 0$$

$$t^5 - 5t^4.\,tang.\,5z - 10t^3 + 10tt.\,tang.\,5z + 5t - tang.\,5z = 0$$

& generaliter

$$t^n - nt^{n-1}\,tang.\,nz - \ldots\ldots\ldots\ldots \mp tang.\,nz = 0$$

ubi ſignum ſuperius $-$ valet, ſi m ſit numerus par, inferius $+$ ſi m ſit numerus impar. Erit ergo ex coëfficiente ſecundi termini

$$tang.\;z = tang.\,z$$

$$3tang.\,3z = tang.\,z + tang.\,\left(\frac{\pi}{3} + z\right) + tang.\,\left(\frac{2\pi}{3} + z\right)$$

$$5tang.\,5z = tang.\,z + tang.\,\left(\frac{\pi}{5} + z\right) + tang.\,\left(\frac{2\pi}{5} + z\right) + tang.\,\left(\frac{3\pi}{5} + z\right) + tang.\,\left(\frac{4\pi}{5} + z\right).$$

&c.

253. Cum igitur ſit $tang.\,v = -\,tang.\,(\pi - v)$, Anguli recto majores ad Angulos recto minores reducuntur, eritque

$$tang.\;z = tang.\,z$$

$$3tang.\,3z = tang.\,z - tang.\,\left(\frac{\pi}{3} - z\right) + tang.\,\left(\frac{\pi}{3} + z\right)$$

$$5tang.\,5z = tang.\,z - tang.\,\left(\frac{\pi}{5} - z\right) + tang.\,\left(\frac{\pi}{5} + z\right) - tang.\,\left(\frac{2\pi}{5} - z\right) + tang.\,\left(\frac{2\pi}{5} + z\right)$$

$$7tang.\,7z = tang.\,z - tang.\,\left(\frac{\pi}{7} - z\right) + tang.\,\left(\frac{\pi}{7} + z\right) - tang.\,\left(\frac{2\pi}{7} - z\right) + tang.\,\left(\frac{2\pi}{7} + z\right) - tang.\,\left(\frac{3\pi}{7} - z\right) + tang.\,\left(\frac{3\pi}{7} + z\right)$$

& gene-

& generaliter, si $n = 2m + 1$, erit

$$n.tang.nz = tang.z - tang.(\frac{\pi}{n} - z) + tang.(\frac{\pi}{n} + z) - tang.(\frac{2\pi}{n} - z) + tang.(\frac{2\pi}{n} + z) - tang.(\frac{3\pi}{n} - z) + \dots\dots\dots - tang.(\frac{m\pi}{n} - z) + tang.(\frac{m\pi}{n} + z).$$

254. Tum vero productum ex his Tangentibus omnibus erit $= tang.\,nz$, propterea quod per signorum negativorum numerum alternatim parem & imparem, superior signorum ambiguitas tollitur. Sic erit

$$tang.\ z = tang.z$$
$$tang.3z = tang.z.\,tang.(\frac{\pi}{3} - z).\,tang.(\frac{\pi}{3} + z)$$
$$tang.5z = tang.z.\,tang.(\frac{\pi}{5} - z).\,tang.(\frac{\pi}{5} + z).tang.(\frac{2\pi}{5} - z)\times tang.(\frac{2\pi}{5} + z)$$

& generaliter, si $n = 2m + 1$, erit

$$tang.nz = tang.z.\,tang.(\frac{\pi}{n} - z).tang.(\frac{\pi}{n} + z).tang.(\frac{2\pi}{n} - z)\times tang.(\frac{2\pi}{n} + z).tang.(\frac{3\pi}{n} - z)\dots\dots\dots\times tang.(\frac{m\pi}{n} - z).tang.(\frac{m\pi}{n} + z).$$

255. Sit jam n numerus par, atque, incipiendo a Potestate summa, erit

$$tt + 2t.\,cot.\,2z - 1 = 0$$
$$t^4 + 4t^3.cot.\,4z - 6tt - 4t.cot.\,4z + 1 = 0.$$

&

& generaliter, si $n = 2m$, erit

$$t^n + nt^{n-1} \cot. nz - \ldots \ldots \ldots \ldots \mp 1 = 0$$

ubi signum superius — valet si m sit numerus impar, inferius + si m sit par. Comparando ergo radices cum coëfficiente secundi termini, erit

$$-2 \cot. 2z = \tang. z + \tang. \left(\frac{\pi}{2} + z\right)$$

$$-4 \cot. 4z = \tang. z + \tang. \left(\frac{\pi}{4} + z\right) + \tang. \left(\frac{2\pi}{4} + z\right) + \tang. \left(\frac{3\pi}{4} + z\right)$$

$$-6 \cot. 6z = \tang. z + \tang. \left(\frac{\pi}{6} + z\right) + \tang. \left(\frac{2\pi}{6} + z\right) + \tang. \left(\frac{3\pi}{6} + z\right) + \tang. \left(\frac{4\pi}{6} + z\right) + \tang. \left(\frac{5\pi}{6} + z\right).$$

&c.

256. Cum sit $\tang. v = - \tang. (\pi - v)$, sequentes formabuntur æquationes

$$2 \cot. 2z = - \tang. z + \tang. \left(\frac{\pi}{2} - z\right)$$

$$4 \cot. 4z = - \tang. z + \tang. \left(\frac{\pi}{4} - z\right) - \tang. \left(\frac{\pi}{4} + z\right) + \tang. \left(\frac{2\pi}{4} - z\right)$$

$$6 \cot. 6z = - \tang. z + \tang. \left(\frac{\pi}{6} - z\right) - \tang. \left(\frac{\pi}{6} + z\right) + \tang. \left(\frac{2\pi}{6} - z\right) - \tang. \left(\frac{2\pi}{6} + z\right) + \tang. \left(\frac{3\pi}{6} - z\right)$$

&

& generaliter, si $n = 2m$, erit

$$n.cot.nz = -tang.z + tang.\left(\frac{\varpi}{n} - z\right) - tang.\left(\frac{\varpi}{n} + z\right) + tang.\left(\frac{2\varpi}{n} - z\right) - tang.\left(\frac{2\varpi}{n} + z\right) + tang.\left(\frac{3\varpi}{n} - z\right) - tang.\left(\frac{3\varpi}{n} + z\right) + \ldots\ldots + tang.\left(\frac{m\varpi}{n} - z\right).$$

257. Per has formas iterum ambiguitas producti ex omnibus radicibus destruitur; eritque idcirco

$$1 = tang.z.\,tang.\left(\frac{\varpi}{2} - z\right)$$

$$1 = tang.z.\,tang.\left(\frac{\varpi}{4} - z\right).tang.\left(\frac{\varpi}{4} + z\right).tang.\left(\frac{2\varpi}{4} - z\right)$$

$$1 = tang.z.\,tang.\left(\frac{\varpi}{6} - z\right).tang.\left(\frac{\varpi}{6} + z\right).tang.\left(\frac{2\varpi}{6} - z\right) \times tang.\left(\frac{2\varpi}{6} + z\right).tang.\left(\frac{3\varpi}{6} - z\right).$$

&c.

Harum vero æquationum ratio statim sponte in oculos incurrit, cum perpetuo bini Anguli reperiantur, quorum alter est alterius complementum ad rectum. Hujusmodi ergo binorum Angulorum Tangentes productum dant $= 1$; ideoque omnium productum unitati debet esse æquale.

258. Quoniam Sinus & Cosinus Angulorum progressionem arithmeticam constituentium Seriem recurrentem præbent, per Caput præcedens summa hujusmodi Sinuum & Cosinuum quotcunque exhiberi poterit. Sint Anguli in arithmetica progressione

$$a,\ a+b,\ a+2b,\ a+3b,\ a+4b,\ a+5b,\ \&c.$$

& quæratur primo summa Sinuum horum Angulorum in infinitum progredientium; ponatur ergo

$s = \sin. a + \sin. (a+b) + \sin. (a+2b) + \sin. (a+3b) + \&c.$

& quia hæc Series eſt recurrens, cujus ſcala relationis eſt $2\cos. b, -1$, orietur hæc Series ex evolutione fractionis, cujus denominator eſt $1 - 2z. \cos. b + zz$, poſito $z = 1$. Ipſa vero fractio erit $= \frac{\sin. a + z(\sin. (a+b) - 2\sin. a.\cos. b)}{1 - 2z.\cos. b + zz}$, quare, facto $z = 1$, erit $s = \frac{\sin. a + \sin. (a+b) - 2\sin. a.\cos. b}{2 - 2\cos. b} = \frac{\sin. a - \sin. (a-b)}{2(1-\cos. b)}$, ob $2\sin. a.\cos. b = \sin. (a+b) + \sin. (a-b)$. Cum autem ſit $\sin. f - \sin. g = 2\cos. \frac{f+g}{2}.\sin. \frac{f-g}{2}$, erit $\sin. a - \sin. (a-b) = 2\cos. (a - \frac{1}{2}b).\sin. \frac{1}{2}b$: & $1 - \cos. b = 2(\sin. \frac{1}{2}b)^2$, unde erit $s = \frac{\cos. (a - \frac{1}{2}b)}{2\sin. \frac{1}{2}b}$.

259. Hinc itaque ſumma quotcunque Sinuum, quorum Arcus in arithmetica progreſſione incedunt, aſſignari poterit; quæratur nempe ſumma hujus progreſſionis

$$\sin. a + \sin. (a+b) + \sin. (a+2b) + \sin. (a+3b) + \dots\dots + \sin. (a+nb).$$

Quia ſumma hujus progreſſionis in infinitum continuatæ eſt $\frac{\cos. (a - \frac{1}{2}b)}{2\sin. \frac{1}{2}b}$, conſiderentur termini ultimum ſequentes in infinitum hi

$$\sin. (a+(n+1)b) + \sin. (a+(n+2)b) + \sin. (a+(n+3)b) + \&c.$$

quia horum Sinuum ſumma eſt $= \frac{\cos. (a+(n+\frac{1}{2})b)}{2\sin. \frac{1}{2}b}$; ſi hæc a priori ſubtrahatur, remanebit ſumma quæſita. Scilicet, ſi fuerit

$$s = \sin. a + \sin. (a+b) + \sin. (a+2b) + \dots\dots + \sin. (a+nb),$$

erit $s = \frac{\cos. (a - \frac{1}{2}b) - \cos. (a + (n+\frac{1}{2})b)}{2\sin. \frac{1}{2}b} = \frac{\sin. (a + \frac{1}{2}nb)\sin. \frac{1}{2}(n+1)b}{\sin. \frac{1}{2}b}$.

260. Pari

260. Pari modo, si consideretur summa Cosinuum, atque ponatur CAP. XIV.

$$s = \cos. a + \cos.(a+b) + \cos.(a+2b) + \cos.(a+3b) + \text{\&c.}$$

in infinitum, erit $s = \frac{\cos. a + z(\cos.(a+b) - 2\cos.a.\cos.b)}{1 - 2z.\cos.b + zz}$, posito $z = 1$. Quare, ob $2\cos.a.\cos.b = \cos.(a-b) + \cos.(a+b)$, fiet $s = \frac{\cos.a - \cos.(a-b)}{2(1-\cos.b)}$. At est $\cos.f - \cos.g = 2\sin.\frac{f+g}{2} \times \sin.\frac{g-f}{2}$; unde erit $\cos.a - \cos.(a-b) = -2\sin.(a - \frac{1}{2}b) \times \sin.\frac{1}{2}b$, & ob $1 - \cos.b = 2(\sin.\frac{1}{2}b)^2$, erit $s = -\frac{\sin.(a-\frac{1}{2}b)}{2\sin.\frac{1}{2}b}$. Quare, cum simili modo sit hujus Seriei

$$\cos.(a+(n+1)b) + \cos.(a+(n+2)b) + \cos.(a+(n+3)b) + \text{\&c.}$$

summa $= -\frac{\sin.(a+(n+\frac{1}{2})b)}{2\sin.\frac{1}{2}b}$, si hæc ab illa subtrahatur, relinquetur summa hujus Seriei

$$s = \cos.a + \cos.(a+b) + \cos.(a+2b) + \cos.(a+3b) + \ldots\ldots + \cos.(a+nb):$$

eritque $s = -\frac{\sin.(a-\frac{1}{2}b) + \sin.(a+(n+\frac{1}{2})b)}{2\sin.\frac{1}{2}b} = \frac{\cos.(a+\frac{1}{2}nb).\sin.\frac{1}{2}(n+1)b}{\sin.\frac{1}{2}b}$.

261. Plurimæ aliæ quæstiones circa Sinus & Tangentes ex principiis allatis resolvi possent; cujusmodi sunt, si quadrata, altioresve Potestates Sinuum, Tangentiumve summari deberent, verum quia hæc ex reliquis æquationum superiorum coëfficientibus similiter derivantur, iis hic diutius non immoror. Quod autem ad has postremas summationes attinet, notandum est quamcunque Sinuum Cosinuumque Potestatem per singulos Sinus Cosinusve explicari posse, quod, ut clarius perspiciatur, breviter exponamus.

Lib. I. 262. Ad hoc expediendum juvabit ex præcedentibus hæc Lemmata depromſiſſe

$$2\sin.a.\sin.z = \cos.(a-z) - \cos.(a+z)$$
$$2\cos.a\sin.z = \sin.(a+z) - \sin.(a-z)$$
$$2\sin.a.\cos.z = \sin.(a+z) + \sin.(a-z)$$
$$2\cos.a.\cos.z = \cos.(a-z) + \cos.(a+z)$$

Hinc igitur primum Poteſtates Sinuum reperiuntur

$$\sin.z = \sin.z$$
$$2(\sin.z)^2 = 1 - \cos.2z$$
$$4(\sin.z)^3 = 3\sin.z - \sin.3z$$
$$8(\sin.z)^4 = 3 - 4\cos.2z + \cos.4z$$
$$16(\sin.z)^5 = 10\sin.z - 5\sin.3z + \sin.5z$$
$$32(\sin.z)^6 = 10 - 15\cos.2z + 6\cos.4z - \cos.6z$$
$$64(\sin.z)^7 = 35\sin.z - 21\sin.3z + 7\sin.5z - \sin.7z$$
$$128(\sin.z)^8 = 35 - 56\cos.2z + 28\cos.4z - 8\cos.6z + \cos.8z$$
$$256(\sin.z)^9 = 126\sin.z - 84\sin.3z + 36\sin.5z - 9\sin.7z + \sin.9z$$

&c.

Lex, qua hi coëfficientes progrediuntur, ex unciis Binomii elevati intelligitur, niſi quod numerus abſolutus in Poteſtatibus paribus ſemiſſis tantum ſit ejus, quem unciæ præbent.

263. Pari modo Poteſtates Coſinuum definientur

$$\cos.z = \cos.z$$
$$2(\cos.z)^2 = 1 + \cos.2z$$
$$4(\cos.z)^3 = 3\cos.z + \cos.3z$$
$$8(\cos.z)^4 = 3 + 4\cos.2z + \cos.4z$$
$$16(\cos.z)^5 = 10\cos.z + 5\cos.3z + \cos.5z$$
$$32(\cos.z)^6 = 10 + 15\cos.2z + 6\cos.4z + \cos.6z$$
$$64(\cos.z)^7 = 35\cos.z + 21\cos.3z + 7\cos.5z + \cos.7z$$

&c.

Hic ratione legis progreſſionis eadem ſunt monenda quæ circa Sinus notavimus.

CAPUT XV.

De Seriebus ex evolutione Factorum ortis.

264. Sit propositum productum ex Factoribus, numero sive finitis sive infinitis, constans hujusmodi

$$(1+\alpha z)(1+\beta z)(1+\gamma z)(1+\delta z)(1+\epsilon z)(1+\zeta z)\ \&c.,$$

quod, si per multiplicationem actualem evolvatur, det

$$1+Az+Bz^2+Cz^3+Dz^4+Ez^5+Fz^6+\&c.,$$

atque manifestum est coëfficientes A, B, C, D, E, &c., ita formari ex numeris α, β, γ, δ, ϵ, ζ, &c., ut sit

$A = \alpha+\beta+\gamma+\delta+\epsilon+\zeta+$ &c. $=$ summæ singulorum
$B =$ summæ Factorum ex binis diversis
$C =$ summæ Factorum ex ternis diversis
$D =$ summæ Factorum ex quaternis diversis
$E =$ summæ Factorum ex quinis diversis
&c.

donec perveniatur ad productum ex omnibus.

265. Quod si ergo ponatur $z=1$, productum hoc

$$(1+\alpha)(1+\beta)(1+\gamma)(1+\delta)(1+\epsilon)\ \&c.$$

æquabitur unitati cum Serie numerorum omnium, qui ex his α, β, γ, δ, ϵ, &c., vel sumendis singulis, vel duobus pluribusve diversis in se multiplicandis, nascuntur. Atque si idem numerus duobus pluribusve modis resultare queat, etiam idem bis pluriesve in hac numerorum Serie occurret.

266. Si ponatur $z=-1$, productum hoc

 (1 —

$$(1-\alpha)(1-\beta)(1-\gamma)(1-\delta)(1-\epsilon) \text{ \&c.}$$

æquabitur unitati cum Serie numerorum omnium, qui ex his $\alpha, \beta, \gamma, \delta, \epsilon, \xi$, &c. vel ſumendis ſingulis, vel duobus pluribuſve diverſis in ſe multiplicandis, naſcuntur; ut ante quidem, verum hoc diſcrimine, ut ii numeri, qui vel ex ſingulis, vel ternis, vel quinis, vel numero imparibus naſcuntur, ſint negativi, illi vero, qui vel ex binis, vel quaternis, vel ſenis vel numero paribus reſultant, ſint affirmativi.

267. Scribantur pro $\alpha, \beta, \gamma, \delta$, &c., numeri primi omnes 2, 3, 5, 7, 11, 13, &c., atque hoc productum

$$(1+2)(1+3)(1+5)(1+7)(1+11)(1+13) \text{ \&c.} = P$$

æquabitur unitati, cum Serie omnium numerorum vel primorum ipſorum, vel ex primis diverſis per multiplicationem ortorum. Erit ergo

$$P = 1+2+3+5+6+7+10+11+13+14+15+17+ \text{ \&c.},$$

in qua Serie omnes occurrunt numeri naturales, exceptis Poteſtatibus, iiſque qui per quamvis Poteſtatem ſunt diviſibiles. Deſunt ſcilicet numeri 4, 8, 9, 12, 16, 18 &c., quoniam ſunt vel Poteſtates, ut 4, 8, 9, 16, &c., vel per Poteſtates diviſibiles ut 12, 18, &c.

268. Simili modo res ſe habebit, ſi pro $\alpha, \beta, \gamma, \delta$, &c.. Poteſtates quæcunque numerorum primorum ſubſtituantur. Scilicet ſi ponamus

$$P = \left(1+\frac{1}{2^n}\right)\left(1+\frac{1}{3^n}\right)\left(1+\frac{1}{5^n}\right)\left(1+\frac{1}{7^n}\right)\left(1+\frac{1}{11^n}\right) \text{ \&c.},$$

erit enim multiplicatione inſtituta:

$$P = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{6^n} + \frac{1}{7^n} + \frac{1}{10^n} + \frac{1}{11^n} + \text{ \&c.}$$

in

in quibus fractionibus omnes occurrunt numeri præter illos qui vel ipsi sunt Potestates, vel per Potestatem quampiam divisibiles. Cum enim omnes numeri integri sint vel primi vel ex primis per multiplicationem compositi, hic ii tantum numeri excludentur, in quorum formationem idem numerus primus bis vel pluries ingreditur. CAP. XV.

269. Si numeri $\alpha, \beta, \gamma, \delta,$ &c., negative capiantur, ut ante (*266.*) fecimus, atque ponatur

$$P = \left(1 - \frac{1}{2^n}\right)\left(1 - \frac{1}{3^n}\right)\left(1 - \frac{1}{5^n}\right)\left(1 - \frac{1}{7^n}\right)\left(1 - \frac{1}{11^n}\right) \text{ \&c., erit}$$

$$P = 1 - \frac{1}{2^n} - \frac{1}{3^n} - \frac{1}{5^n} + \frac{1}{6^n} - \frac{1}{7^n} + \frac{1}{10^n} - \frac{1}{11^n} - \frac{1}{13^n} + \frac{1}{15^n} - \text{\&c.},$$

ubi iterum, ut ante, omnes occurrunt numeri præter Potestates ac divisibiles per Potestates. Verum ipsi numeri primi, & qui ex ternis, quinis, numerove imparibus constant, signum habent præfixum —, qui autem ex binis, vel quaternis, vel senis, vel numero paribus formantur, signum habent +. Sic in hac Serie occurret terminus $\frac{1}{30^n}$, quia est $30 = 2 \cdot 3 \cdot 5$, neque adeo Potestatem complectitur, habebit vero hic terminus $\frac{1}{30^n}$ signum —, quia 30 est productum ex tribus numeris primis.

270. Consideremus jam hanc expressionem

$$\frac{1}{(1-\alpha z)(1-\beta z)(1-\gamma z)(1-\delta z)(1-\varepsilon z) \text{ \&c.}}$$

quæ per divisionem actualem evoluta præbeat hanc Seriem:

1 +

$$1 + Az + Bz^2 + Cz^3 + Dz^4 + Ez^5 + Fz^6 + \&c.$$

atque manifestum est coëfficientes A, B, C, D, E, &c., sequenti modo ex numeris α, β, γ, δ, ϵ, &c., componi, ut sit

A = summæ singulorum
B = summæ Factorum ex binis
C = summæ Factorum ex ternis
D = summæ Factorum ex quaternis
&c.
} non exclusis Factoribus iisdem.

271. Posito ergo $z = 1$, ista expressio

$$\frac{1}{(1-\alpha)(1-\beta)(1-\gamma)(1-\delta)(1-\epsilon)\ \&c.}$$

æquabitur unitati cum Serie numerorum omnium, qui ex his α, β, γ, δ, ϵ, ξ, &c., vel sumendis singulis, vel duobus pluribusve in se multiplicandis, oriuntur, non exclusis æqualibus. Hoc ergo differt ista numerorum Series ab illa, quæ (§. 265.) prodiit, quod ibi Factores tantum diversi sumi debebant, hic autem idem Factor bis pluriesve occurrere possit. Hic scilicet omnes numeri occurrunt, qui per multiplicationem ex his α, β, γ, δ, &c., provenire possunt.

272. Hanc ob rem Series semper ex terminorum numero infinito constat, sive Factorum numerus fuerit infinitus, sive finitus. Sic erit

$$\frac{1}{1-\frac{1}{2}} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \&c.,$$

ubi omnes numeri adsunt, qui ex binario solo per multiplicationem oriuntur; seu omnes binarii Potestates. Deinde erit

$$\frac{1}{(1-\frac{1}{2})(1-\frac{1}{3})} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \frac{1}{9} + \frac{1}{12} + \frac{1}{16} + \frac{1}{18} + \&c.,$$

ubi

ubi alii numeri non occurrunt, nisi qui ex his duobus 2 & 3 per multiplicationem originem trahunt; seu qui alios Divisores præter 2 & 3 non habent.

273. Si igitur pro $\alpha, \beta, \gamma, \delta$, &c., unitas per singulos omnes numeros primos scribatur, ac ponatur

$$P = \frac{1}{(1-\frac{1}{2})(1-\frac{1}{3})(1-\frac{1}{5})(1-\frac{1}{7})(1-\frac{1}{11})(1-\frac{1}{13})\ \&c.},$$

fiet

$$P = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \&c.,$$

ubi omnes numeri tam primi, quam qui ex primis per multiplicationem nascuntur, occurrunt. Cum autem omnes numeri vel sint ipsi primi, vel ex primis per multiplicationem oriundi, manifestum est, hic omnes omnino numeros integros in denominatoribus adesse debere.

274. Idem evenit, si numerorum primorum Potestates quæcunque accipiantur: si enim ponatur

$$P = \frac{1}{(1-\frac{1}{2^n})(1-\frac{1}{3^n})(1-\frac{1}{5^n})(1-\frac{1}{7^n})(1-\frac{1}{11^n})\&c.},$$

fiet

$$P = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \frac{1}{5^n} + \frac{1}{6^n} + \frac{1}{7^n} + \frac{1}{8^n} + \&c.,$$

ubi omnes numeri naturales nullo excepto occurrunt. Quod si autem in Factoribus ubique signum + statuatur, ut sit

$$P = \frac{1}{(1+\frac{1}{2^n})(1+\frac{1}{3^n})(1+\frac{1}{5^n})(1+\frac{1}{7^n})(1+\frac{1}{11^n})\ \&c.},$$

erit

$$P = 1 - \frac{1}{2^n} - \frac{1}{3^n} + \frac{1}{4^n} - \frac{1}{5^n} + \frac{1}{6^n} - \frac{1}{7^n} - \frac{1}{8^n} + \frac{1}{9^n} + \frac{1}{10^n} - \&c.,$$

ubi numeri primi habent ſignum —; qui ſunt producti ex duobus primis, ſive iiſdem ſive diverſis, ſignum habent +; & generatim, quorum numerorum numerus Factorum primorum eſt par, ſignum habent +, qui autem ex Factoribus primis numero imparibus conſtant, habent ſignum —. Sic terminus $\frac{1}{240^n}$, ob $240 = 2.2.2.2.3.5$, habebit ſignum +, cujus legis ratio percipitur ex §. 270, ſi ponatur $z = -1$.

275. Si hæc cum ſuperioribus conferantur, naſcentur binæ Series quarum productum unitati æquatur. Sit enim

$$P = \frac{1}{(1-\frac{1}{2^n})(1-\frac{1}{3^n})(1-\frac{1}{5^n})(1-\frac{1}{7^n})(1-\frac{1}{11^n})\ \&c.},$$

&

$$Q = (1-\frac{1}{2^n})(1-\frac{1}{3^n})(1-\frac{1}{5^n})(1-\frac{1}{7^n})(1-\frac{1}{11^n})\ \&c.,$$

erit

$$P = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \frac{1}{5^n} + \frac{1}{6^n} + \frac{1}{7^n} + \frac{1}{8^n} + \&c.,$$

$$Q = 1 - \frac{1}{2^n} - \frac{1}{3^n} - \frac{1}{5^n} + \frac{1}{6^n} - \frac{1}{7^n} + \frac{1}{10^n} - \frac{1}{11^n}\ \&c.$$

(269.), atque manifeſtum eſt fore $PQ = 1$.

276. Sin autem ponatur

$$P = \frac{1}{(1+\frac{1}{2^n})(1+\frac{1}{3^n})(1+\frac{1}{5^n})(1+\frac{1}{7^n})(1+\frac{1}{11^n})\ \&c.},$$

&

$Q =$

$$Q = (1 + \frac{1}{2^n})(1 + \frac{1}{3^n})(1 + \frac{1}{5^n})(1 + \frac{1}{7^n})(1 + \frac{1}{11^n})\ \&c.,$$ CAP. XV.

erit

$$P = 1 - \frac{1}{2^n} - \frac{1}{3^n} + \frac{1}{4^n} - \frac{1}{5^n} + \frac{1}{6^n} - \frac{1}{7^n} - \frac{1}{8^n} + \frac{1}{9^n} + \&c.$$

$$Q = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{6^n} + \frac{1}{7^n} + \frac{1}{10^n} + \frac{1}{11^n} + \&c.$$

ſimilique modo habebitur $PQ = 1$. Cognita ergo alterius Seriei ſumma, ſimul alterius innoteſcet.

277. Viciſſim porro ex cognitis ſummis harum Serierum, aſſignari poterunt valores Factorum infinitorum. Sit nimirum

$$M = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \frac{1}{5^n} + \frac{1}{6^n} + \frac{1}{7^n} + \&c.$$

$$N = 1 + \frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \frac{1}{4^{2n}} + \frac{1}{5^{2n}} + \frac{1}{6^{2n}} + \frac{1}{7^{2n}} + \&c.;$$

eritque

$$M = \frac{1}{(1 - \frac{1}{2^n})(1 - \frac{1}{3^n})(1 - \frac{1}{5^n})(1 - \frac{1}{7^n})(1 - \frac{1}{11^n})\ \&c.}$$

$$N = \frac{1}{(1 - \frac{1}{2^{2n}})(1 - \frac{1}{3^{2n}})(1 - \frac{1}{5^{2n}})(1 - \frac{1}{7^{2n}})(1 - \frac{1}{11^{2n}})\ \&c.}$$

Hinc per diviſionem naſcitur

$$\frac{M}{N} = (1 + \frac{1}{2^n})(1 + \frac{1}{3^n})(1 + \frac{1}{5^n})(1 + \frac{1}{7^n})(1 + \frac{1}{11^n})\ \&c.$$

 denique

denique vero erit

$$\frac{MM}{N} = \frac{2^n+1}{2^n-1} \cdot \frac{3^n+1}{3^n-1} \cdot \frac{5^n+1}{5^n-1} \cdot \frac{7^n+1}{7^n-1} \cdot \frac{11^n+1}{11^n-1} \cdot \&c.$$

Ex cognitis ergo M & N, præter valores horum productorum, summæ harum Serierum habebuntur

$$\frac{1}{M} = 1 - \frac{1}{2^n} - \frac{1}{3^n} - \frac{1}{5^n} + \frac{1}{6^n} - \frac{1}{7^n} + \frac{1}{10^n} - \frac{1}{11^n} - \&c.$$

$$\frac{1}{N} = 1 - \frac{1}{2^{2n}} - \frac{1}{3^{2n}} - \frac{1}{5^{2n}} + \frac{1}{6^{2n}} - \frac{1}{7^{2n}} + \frac{1}{10^{2n}} - \frac{1}{11^{2n}} - \&c.$$

$$\frac{M}{N} = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{6^n} + \frac{1}{7^n} + \frac{1}{10^n} + \frac{1}{11^n} + \&c.$$

$$\frac{N}{M} = 1 - \frac{1}{2^n} - \frac{1}{3^n} + \frac{1}{4^n} - \frac{1}{5^n} + \frac{1}{6^n} - \frac{1}{7^n} - \frac{1}{8^n} + \frac{1}{9^n} + \frac{1}{10^n} - \&c.,$$

ex quarum combinatione multæ aliæ deduci possunt.

EXEMPLUM I.

Sit $n = 1$, &, quoniam supra demonstravimus esse, $l\frac{1}{1-x} = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \frac{x^6}{6} + \&c.$, erit, posito $x = 1$, $l\frac{1}{1-1} = l\infty = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \&c.$ At Logarithmus numeri infinite magni ∞ ipse est infinite magnus, ex quo erit

$M =$

$M = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \&c. = \infty.$ CAP. XV.

Hinc ob $\frac{1}{M} = \frac{1}{\infty} = 0$, fiet

$$0 = 1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{5} + \frac{1}{6} - \frac{1}{7} + \frac{1}{10} - \frac{1}{11} - \frac{1}{13} + \frac{1}{14} + \frac{1}{15} \&c..$$

Tum vero in productis habebitur

$$M = \infty = \frac{1}{(1-\frac{1}{2})(1-\frac{1}{3})(1-\frac{1}{5})(1-\frac{1}{7})(1-\frac{1}{11})\&c.},$$

unde fit

$$\infty = \frac{2}{1}\cdot\frac{3}{2}\cdot\frac{5}{4}\cdot\frac{7}{6}\cdot\frac{11}{10}\cdot\frac{13}{12}\cdot\frac{17}{16}\cdot\frac{19}{18}\cdot \&c.,$$

&

$$0 = \frac{1}{2}\cdot\frac{2}{3}\cdot\frac{4}{5}\cdot\frac{6}{7}\cdot\frac{10}{11}\cdot\frac{12}{13}\cdot\frac{16}{17}\cdot\frac{18}{19}\cdot \&c..$$

Deinde per summationem Serierum supra traditam erit

$N = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2} + \&c. = \frac{\pi\pi}{6}$, hinc obtinentur istæ summæ Serierum

$$\frac{6}{\pi\pi} = 1 - \frac{1}{2^2} - \frac{1}{3^2} - \frac{1}{5^2} + \frac{1}{6^2} - \frac{1}{7^2} + \frac{1}{10^2} - \frac{1}{11^2} - \&c.$$

$$\infty = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{10} + \frac{1}{11} + \&c.$$

$$0 = 1 - \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} - \frac{1}{7} - \frac{1}{8} + \frac{1}{9} + \frac{1}{10} - \frac{1}{11} \&c.$$

Denique pro Factoribus orietur

Lib. I. $\frac{\pi\pi}{6} = \frac{2^2}{2^2-1} \cdot \frac{3^2}{3^2-1} \cdot \frac{5^2}{5^2-1} \cdot \frac{7^2}{7^2-1} \cdot \frac{11^2}{11^2-1} \cdot$ &c.,

seu

$$\frac{\pi\pi}{6} = \frac{4}{3} \cdot \frac{9}{8} \cdot \frac{25}{24} \cdot \frac{49}{48} \cdot \frac{121}{120} \cdot \frac{169}{168} \cdot \text{\&c.}$$

&, ob $\frac{M}{N} = \infty$ seu $\frac{N}{M} = 0$, habebitur

$$\infty = \frac{3}{2} \cdot \frac{4}{3} \cdot \frac{6}{5} \cdot \frac{8}{7} \cdot \frac{12}{11} \cdot \frac{14}{13} \cdot \frac{18}{17} \cdot \frac{20}{19} \cdot \text{\&c.},$$

seu

$$0 = \frac{2}{3} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{7}{8} \cdot \frac{11}{12} \cdot \frac{13}{14} \cdot \frac{17}{18} \cdot \frac{19}{20} \cdot \text{\&c.},$$

atque

$$\infty = \frac{3}{1} \cdot \frac{4}{2} \cdot \frac{6}{4} \cdot \frac{8}{6} \cdot \frac{12}{10} \cdot \frac{14}{12} \cdot \frac{18}{16} \cdot \frac{20}{18} \cdot \text{\&c.},$$

seu

$$0 = \frac{1}{3} \cdot \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{6}{7} \cdot \frac{8}{9} \cdot \frac{9}{10} \cdot \text{\&c.};$$

quarum fractionum (excepta prima) numeratores unitate deficiunt a denominatoribus, summæ autem ex numeratoribus & denominatoribus cujusque fractionis constanter præbent numeros primos, 3, 5, 7, 11, 13, 17, 19, &c.

Exemplum II.

Sit $n = 2$, eritque ex superioribus

$$M = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2} + \text{\&c.} = \frac{\pi\pi}{6}$$

$$N = 1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \frac{1}{6^4} + \frac{1}{7^4} + \text{\&c.} = \frac{\pi^4}{90}$$

Hinc primo istæ Series summantur

$\frac{6}{\pi\pi}$

$$\frac{6}{\pi\pi} = 1 - \frac{1}{2^2} - \frac{1}{3^2} - \frac{1}{5^2} + \frac{1}{6^2} - \frac{1}{7^2} + \frac{1}{10^2} - \frac{1}{11^2} - \&c.$$

$$\frac{90}{\pi^4} = 1 - \frac{1}{2^4} - \frac{1}{3^4} - \frac{1}{5^4} + \frac{1}{6^4} - \frac{1}{7^4} + \frac{1}{10^4} - \frac{1}{11^4} - \&c.$$

$$\frac{15}{\pi^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2} + \frac{1}{10^2} + \frac{1}{11^2} + \&c.$$

$$\frac{\pi\pi}{15} = 1 - \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \frac{1}{5^2} + \frac{1}{6^2} - \frac{1}{7^2} - \frac{1}{8^2} + \frac{1}{9^2} + \frac{1}{10^2} \ \&c.,$$

Deinde valores ſequentium productorum innoteſcunt

$$\frac{\pi\pi}{6} = \frac{2^2}{2^2-1} \cdot \frac{3^2}{3^2-1} \cdot \frac{5^2}{5^2-1} \cdot \frac{7^2}{7^2-1} \cdot \frac{11^2}{11^2-1} \cdot \&c.$$

$$\frac{\pi^4}{90} = \frac{2^4}{2^4-1} \cdot \frac{3^4}{3^4-1} \cdot \frac{5^4}{5^4-1} \cdot \frac{7^4}{7^4-1} \cdot \frac{11^4}{11^4-1} \cdot \&c.$$

$$\frac{15}{\pi\pi} = \frac{2^2+1}{2^2} \cdot \frac{3^2+1}{3^2} \cdot \frac{5^2+1}{5^2} \cdot \frac{7^2+1}{7^2} \cdot \frac{11^2+1}{11^2} \cdot \&c.,$$

ſeu

$$\frac{\pi\pi}{15} = \frac{4}{5} \cdot \frac{9}{10} \cdot \frac{25}{26} \cdot \frac{49}{50} \cdot \frac{121}{122} \cdot \frac{169}{170} \cdot \&c.,$$

&

$$\frac{5}{2} = \frac{2^2+1}{2^2-1} \cdot \frac{3^2+1}{3^2-1} \cdot \frac{5^2+1}{5^2-1} \cdot \frac{7^2+1}{7^2-1} \cdot \frac{11^2+1}{11^2-1} \cdot \&c.;$$

ſive

$$\frac{5}{2} = \frac{5}{3} \cdot \frac{5}{4} \cdot \frac{13}{12} \cdot \frac{25}{24} \cdot \frac{61}{60} \cdot \frac{85}{84} \cdot \ \&c.,$$

vel

$$\frac{3}{2} = \frac{5}{4} \cdot \frac{13}{12} \cdot \frac{25}{24} \cdot \frac{61}{60} \cdot \frac{85}{84} \cdot \&c. .$$

In his fractionibus numeratores unitate ſuperant denominatores, ſimul vero ſumti præbent quadrata numerorum primorum 3^2, 5^2, 7^2, 11^2, &c.

EXEM-

EXEMPLUM III.

Quia ex superioribus valores ipsius M tantum si n sit numerus par, assignare licet, ponamus $n = 4$, eritque

$$M = 1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \frac{1}{6^4} + \&c. = \frac{\pi^4}{90}$$

$$N = 1 + \frac{1}{2^8} + \frac{1}{3^8} + \frac{1}{4^8} + \frac{1}{5^8} + \frac{1}{6^8} + \&c. = \frac{\pi^8}{9450}$$

Hinc primæ sequentes Series summantur

$$\frac{90}{\pi^4} = 1 - \frac{1}{2^4} - \frac{1}{3^4} - \frac{1}{5^4} + \frac{1}{6^4} - \frac{1}{7^4} + \frac{1}{10^4} - \frac{1}{11^4} \ \&c.$$

$$\frac{9450}{\pi^8} = 1 - \frac{1}{2^8} - \frac{1}{3^8} - \frac{1}{5^8} + \frac{1}{6^8} - \frac{1}{7^8} + \frac{1}{10^8} - \frac{1}{11^8} \ \&c.$$

$$\frac{105}{\pi^4} = 1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{6^4} + \frac{1}{7^4} + \frac{1}{10^4} + \frac{1}{11^4} \ \&c.$$

$$\frac{\pi^4}{105} = 1 - \frac{1}{2^4} - \frac{1}{3^4} + \frac{1}{4^4} - \frac{1}{5^4} + \frac{1}{6^4} - \frac{1}{7^4} - \frac{1}{8^4} + \frac{1}{9^4} \ \&c..$$

Deinde etiam valores sequentium productorum obtinentur

$$\frac{\pi^4}{90} = \frac{2^4}{2^4-1} \cdot \frac{3^4}{3^4-1} \cdot \frac{5^4}{5^4-1} \cdot \frac{7^4}{7^4-1} \cdot \frac{11^4}{11^4-1} \cdot \&c.$$

$$\frac{\pi^8}{9450} = \frac{2^8}{2^8-1} \cdot \frac{3^8}{3^8-1} \cdot \frac{5^8}{5^8-1} \cdot \frac{7^8}{7^8-1} \cdot \frac{11^8}{11^8-1} \cdot \&c.$$

$$\frac{105}{\pi^4} = \frac{2^4+1}{2^4} \cdot \frac{3^4+1}{3^4} \cdot \frac{5^4+1}{5^4} \cdot \frac{7^4+1}{7^4} \cdot \frac{11^4+1}{11^4} \cdot \&c.$$

&

$$\frac{7}{6} = \frac{2^4+1}{2^4-1} \cdot \frac{3^4+1}{3^4-1} \cdot \frac{5^4+1}{5^4-1} \cdot \frac{7^4+1}{7^4-1} \cdot \frac{11^4+1}{11^4-1} \cdot \&c.$$

seu

$$\frac{35}{34} = \frac{41}{40} \cdot \frac{313}{312} \cdot \frac{1201}{1200} \cdot \frac{7321}{7320} \cdot \&c.,$$

in

in his Factoribus numeratores unitate superant denominatores, simul vero sumti præbent bi-quadrata numerorum primorum imparium 3, 5, 7, 11, &c. CAP. XV.

278. Quoniam hic summam Seriei

$$M = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \frac{1}{5^n} + \frac{1}{6^n} + \&c.$$

ad Factores reduximus, ad Logarithmos commode progredi licebit. Nam, cum sit

$$M = \frac{1}{(1 - \frac{1}{2^n})(1 - \frac{1}{3^n})(1 - \frac{1}{5^n})(1 - \frac{1}{7^n})(1 - \frac{1}{11^n})\ \&c.},$$

erit

$$lM = -l(1 - \frac{1}{2^n}) - l(1 - \frac{3}{3^n}) - l(1 - \frac{1}{5^n}) - l(1 - \frac{1}{7^n}) - \&c.$$

Hinc, sumendis Logarithmis hyperbolicis, erit

$$\begin{aligned} lM = &+ 1(\frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{7^n} + \frac{1}{11^n} + \&c.) \\ &+ \frac{1}{2}(\frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \frac{1}{5^{2n}} + \frac{1}{7^{2n}} + \frac{1}{11^{2n}} + \&c.) \\ &+ \frac{1}{3}(\frac{1}{2^{3n}} + \frac{1}{3^{3n}} + \frac{1}{5^{3n}} + \frac{1}{7^{3n}} + \frac{1}{11^{3n}} + \&c.) \\ &+ \frac{1}{4}(\frac{1}{2^{4n}} + \frac{1}{3^{4n}} + \frac{1}{5^{4n}} + \frac{1}{7^{4n}} + \frac{1}{11^{4n}} + \&c.) \\ &\&c. \end{aligned}$$

Quod si insuper ponamus

$$N = 1 + \frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \frac{1}{4^{2n}} + \frac{1}{5^{2n}} + \frac{1}{6^{2n}} + \&c.,$$

ut sit

$$N = \frac{1}{(1-\frac{1}{2^{2n}})(1-\frac{1}{3^{2n}})(1-\frac{1}{5^{2n}})(1-\frac{1}{7^{2n}})(1-\frac{1}{11^{2n}})\ \&c.},$$

fiet, Logarithmis hyperbolicis sumendis,

$$\begin{aligned} lN = &+ 1\left(\frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \frac{1}{5^{2n}} + \frac{1}{7^{2n}} + \frac{1}{11^{2n}} + \&c.\right) \\ &+ \frac{1}{2}\left(\frac{1}{2^{4n}} + \frac{1}{3^{4n}} + \frac{1}{5^{4n}} + \frac{1}{7^{4n}} + \frac{1}{11^{4n}} + \&c.\right) \\ &+ \frac{1}{3}\left(\frac{1}{2^{6n}} + \frac{1}{3^{6n}} + \frac{1}{5^{6n}} + \frac{1}{7^{6n}} + \frac{1}{11^{6n}} + \&c.\right) \\ &+ \frac{1}{4}\left(\frac{1}{2^{8n}} + \frac{1}{3^{8n}} + \frac{1}{5^{8n}} + \frac{1}{7^{8n}} + \frac{1}{11^{8n}} + \&c.\right) \\ &\&c. \end{aligned}$$

Ex his conjunctis fiet $lM - \frac{1}{2}lN =$

$$\begin{aligned} &+ 1\left(\frac{1}{2^{n}} + \frac{1}{3^{n}} + \frac{1}{5^{n}} + \frac{1}{7^{n}} + \frac{1}{11^{n}} + \&c.\right) \\ &+ \frac{1}{3}\left(\frac{1}{2^{3n}} + \frac{1}{3^{3n}} + \frac{1}{5^{3n}} + \frac{1}{7^{3n}} + \frac{1}{11^{3n}} + \&c.\right) \\ &+ \frac{1}{5}\left(\frac{1}{2^{5n}} + \frac{1}{3^{5n}} + \frac{1}{5^{5n}} + \frac{1}{7^{5n}} + \frac{1}{11^{5n}} + \&c.\right) \\ &+ \frac{1}{7}\left(\frac{1}{2^{7n}} + \frac{1}{3^{7n}} + \frac{1}{5^{7n}} + \frac{1}{7^{7n}} + \frac{1}{11^{7n}} + \&c.\right) \\ &\&c. \end{aligned}$$

279. Si

279. Si $n = 1$ erit $M = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \&c. = l\infty$, & $N = \frac{\pi\pi}{6}$; hincque erit $l.l\infty - \frac{1}{2} l \frac{\pi\pi}{6} =$

$$+ 1 \left(\frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \&c. \right)$$
$$+ \frac{1}{3} \left(\frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{5^3} + \frac{1}{7^3} + \frac{1}{11^3} + \&c. \right)$$
$$+ \frac{1}{5} \left(\frac{1}{2^5} + \frac{1}{3^5} + \frac{1}{5^5} + \frac{1}{7^5} + \frac{1}{11^5} + \&c. \right)$$
$$+ \frac{1}{7} \left(\frac{1}{2^7} + \frac{1}{3^7} + \frac{1}{5^7} + \frac{1}{7^7} + \frac{1}{11^7} + \&c. \right)$$
&c.

Verum hæ Series, præter primam, non solum summas habent finitas, sed etiam cunctæ simul sumtæ summam efficiunt finitam, eamque satis parvam: unde necesse est ut Seriei primæ $\frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \&c.$, summa sit infinite magna, quantitate scilicet satis parva deficiet a Logarithmo hyperbolico Seriei $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \&c.$

280. Sit $n = 2$; erit $M = \frac{\pi\pi}{6}$ & $N = \frac{\pi^4}{90}$: unde fit

$$2l\pi - l6 = 1 \left(\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{11^2} + \&c. \right)$$
$$+ \frac{1}{2} \left(\frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \frac{1}{11^4} + \&c. \right)$$
$$+ \frac{1}{3} \left(\frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{5^6} + \frac{1}{7^6} + \frac{1}{11^6} + \&c. \right)$$
&c.

$$4l\pi - l90 = +1\left(\frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \frac{1}{11^4} + \&c.\right)$$
$$+\frac{1}{2}\left(\frac{1}{2^8} + \frac{1}{3^8} + \frac{1}{5^8} + \frac{1}{7^8} + \frac{1}{11^8} + \&c.\right)$$
$$+\frac{1}{3}\left(\frac{1}{2^{12}} + \frac{1}{3^{12}} + \frac{1}{5^{12}} + \frac{1}{7^{12}} + \frac{1}{11^{12}} + \&c.\right)$$
&c.

$$\frac{1}{2}l\frac{5}{2} = 1\left(\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{11^2} + \&c.\right)$$
$$+\frac{1}{3}\left(\frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{5^6} + \frac{1}{7^6} + \frac{1}{11^6} + \&c.\right)$$
$$+\frac{1}{5}\left(\frac{1}{2^{10}} + \frac{1}{3^{10}} + \frac{1}{5^{10}} + \frac{1}{7^{10}} + \frac{1}{11^{10}} + \&c.\right)$$
&c.

281. Quanquam lex, qua numeri primi progrediuntur, non constat, tamen harum Serierum altiorum Potestatum summæ non difficulter proxime assignari poterunt. Sit enim hæc Series

$$M = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \frac{1}{5^n} + \frac{1}{6^n} + \frac{1}{7^n} + \&c.,$$
&
$$S = \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{7^n} + \frac{1}{11^n} + \frac{1}{13^n} + \&c.$$
erit
$$S = M - 1 - \frac{1}{4^n} - \frac{1}{6^n} - \frac{1}{8^n} - \frac{1}{9^n} - \frac{1}{10^n} - \&c.,$$
& ob
$$\frac{M}{2^n} = \frac{1}{2^n} + \frac{1}{4^n} + \frac{1}{6^n} + \frac{1}{8^n} + \frac{1}{10^n} + \frac{1}{12^n} + \&c.,$$
erit
$$S = M - \frac{M}{2^n} - 1 + \frac{1}{2^n} - \frac{1}{9^n} - \frac{1}{15^n} - \frac{1}{21^n} - \&c.,$$
seu

S =

Cap. XV.

$$S = (M - 1)\left(1 - \frac{1}{2^n}\right) - \frac{1}{9^n} - \frac{1}{15^n} - \frac{1}{21^n} - \frac{1}{25^n} - \frac{1}{27^n} - \&c.,$$

& ob

$$M\left(1 - \frac{1}{2^n}\right)\frac{1}{3^n} = \frac{1}{3^n} + \frac{1}{9^n} + \frac{1}{15^n} + \frac{1}{21^n} + \&c.,$$

erit

$$S = (M - 1)\left(1 - \frac{1}{2^n}\right)\left(1 - \frac{1}{3^n}\right) + \frac{1}{6^n} - \frac{1}{25^n} - \frac{1}{35^n} - \frac{1}{45^n} - \&c..$$

Hinc, ob datam ſummam M, valor ipſius S commode invenitur, ſi quidem n fuerit numerus mediocriter magnus.

282. Inventis autem ſummis altiorum Poteſtatum, etiam ſummæ Poteſtatum minorum ex formulis inventis exhiberi poſſunt. Atque hac methodo ſequentes prodierunt ſummæ Seriei

$$\frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{7^n} + \frac{1}{11^n} + \frac{1}{13^n} + \frac{1}{17^n} + \&c.,$$

ſi ſit	erit ſumma Seriei
$n = 2$;	0, 452247420041222
$n = 4$;	0, 076993139764252
$n = 6$;	0, 017070086850639
$n = 8$;	0, 004061405366515
$n = 10$;	0, 000993603573633
$n = 12$;	0, 000246026470033
$n = 14$;	0, 000061244396725
$n = 16$;	0, 000015282026219
$n = 18$;	0, 000003817278702

$n = 20;$	0, 000000953961123
$n = 22;$	0, 000000238450446
$n = 24;$	0, 000000059608184
$n = 26;$	0, 000000014901555
$n = 28;$	0, 000000003725333
$n = 30;$	0, 000000000931323
$n = 32;$	0, 000000000232830
$n = 34;$	0, 000000000058207
$n = 36;$	0, 000000000014551

reliquæ ſummæ parium Poteſtatum in ratione quadrupla decreſcunt.

283. Hæc autem Seriei $1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} +$ &c., in productum infinitum converſio etiam directe inſtitui poteſt hoc modo : ſit

$$A = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \frac{1}{5^n} + \frac{1}{6^n} + \frac{1}{7^n} + \frac{1}{8^n} + \text{\&c., ſubtrahe}$$

$$\frac{1}{2^n} A = \frac{1}{2^n} + \frac{1}{4^n} + \frac{1}{6^n} + \frac{1}{8^n} + \text{\&c.,}$$

erit

$$\left(1 - \frac{1}{2^n}\right) A = 1 + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{7^n} + \frac{1}{9^n} + \frac{1}{11^n} + \text{\&c.}$$

$= B$: ſic ſublati ſunt omnes termini per 2 diviſibiles,

$$\text{ſubtr. } \frac{1}{3^n} B = \frac{1}{3^n} + \frac{1}{9^n} + \frac{1}{15^n} + \frac{1}{21^n} + \text{\&c.,}$$

erit

$$\left(1 - \frac{1}{3^n}\right) B = 1 + \frac{1}{5^n} + \frac{1}{7^n} + \frac{1}{11^n} + \frac{1}{13^n} + \text{\&c.} = C:$$

ſic inſuper ſublati ſunt omnes termini per 3 diviſibiles,

ſubtr.

subtr. $\frac{1}{5^n} C = \frac{1}{5^n} + \frac{1}{25^n} + \frac{1}{35^n} + \frac{1}{55^n} + \&c.,$

erit

$$\left(1 - \frac{1}{5^n}\right) C = 1 + \frac{1}{7^n} + \frac{1}{11^n} + \frac{1}{13^n} + \frac{1}{17^n} + \&c.,$$

sic sublati etiam sunt omnes termini per 5 divisibiles. Pari modo tolluntur termini divisibiles per 7, 11, reliquosque numeros primos; manifestum autem est sublatis omnibus terminis, qui per numeros primos divisibiles sint, solam unitatem relinqui. Quare pro B, C, D, E, &c., valoribus restitutis tandem orietur

$$A\left(1 - \frac{1}{2^n}\right)\left(1 - \frac{1}{3^n}\right)\left(1 - \frac{1}{5^n}\right)\left(1 - \frac{1}{7^n}\right)\left(1 - \frac{1}{11^n}\right) \&c. = 1,$$

unde Seriei propositæ summa erit =

$$A = \frac{1}{\left(1 - \frac{1}{2^n}\right)\left(1 - \frac{1}{3^n}\right)\left(1 - \frac{1}{5^n}\right)\left(1 - \frac{1}{7^n}\right)\left(1 - \frac{1}{11^n}\right) \&c.},$$

seu

$$A = \frac{2^n}{2^n - 1} \cdot \frac{3^n}{3^n - 1} \cdot \frac{5^n}{5^n - 1} \cdot \frac{7^n}{7^n - 1} \cdot \frac{11^n}{11^n - 1} \cdot \&c.$$

284. Hæc methodus jam commode adhiberi poterit ad alias Series, quarum summas supra invenimus, in producta infinita convertendas. Invenimus autem supra (175.) summas harum Serierum

$$1 - \frac{1}{3^n} + \frac{1}{5^n} - \frac{1}{7^n} + \frac{1}{9^n} - \frac{1}{11^n} + \frac{1}{13^n} - \&c.,$$

si n fuerit numerus impar, summa enim est $= N\pi^n$ & valores ipsius N loco citato dedimus. Notandum autem est

cum

LIB. I. cum hic tantum numeri impares occurrunt, eos qui ſint formæ $4m+1$ habere ſignum $+$, reliquos formæ $4m-1$ ſignum $-$. Sit igitur

$$A = 1 - \frac{1}{3^n} + \frac{1}{5^n} - \frac{1}{7^n} + \frac{1}{9^n} - \frac{1}{11^n} + \frac{1}{13^n} - \frac{1}{15^n} + \text{\&c.}$$

$$\frac{1}{3^n}A = \frac{1}{3^n} - \frac{1}{9^n} + \frac{1}{15^n} - \frac{1}{21^n} + \frac{1}{27^n} - \text{\&c.}, \text{ addatur, erit}$$

$$\left(1 + \frac{1}{3^n}\right)A = 1 + \frac{1}{5^n} - \frac{1}{7^n} - \frac{1}{11^n} + \frac{1}{13^n} + \frac{1}{17^n} \text{ \&c.}, = B$$

$$\frac{1}{5^n}B = \frac{1}{5^n} + \frac{1}{25^n} - \frac{1}{35^n} - \frac{1}{55^n} \text{ \&c.}, \text{ ſubtrahatur, erit}$$

$$\left(1 - \frac{1}{5^n}\right)B = 1 - \frac{1}{7^n} - \frac{1}{11^n} + \frac{1}{13^n} + \frac{1}{17^n} - \text{\&c.} = C,$$

ubi jam numeri per 3 & 5 diviſibiles deſunt,

$$\frac{1}{7^n}C = \frac{1}{7^n} - \frac{1}{49^n} - \frac{1}{77^n} + \text{\&c.}, \text{ addatur, erit}$$

$$\left(1 + \frac{1}{7^n}\right)C = 1 - \frac{1}{11^n} + \frac{1}{13^n} + \frac{1}{17^n} - \text{\&c.} = D,$$

ſic etiam numeri per 7 diviſibiles ſunt ſublati

$$\frac{1}{11^n}D = \frac{1}{11^n} - \frac{1}{121^n} + \text{\&c.}, \text{ addatur, erit}$$

$$\left(1 + \frac{1}{11^n}\right)D = 1 + \frac{1}{13^n} + \frac{1}{17^n} - \text{\&c.} = E$$

ſic numeri per 11 diviſibiles quoque ſunt ſublati. Auferendis autem

autem hoc modo reliquis numeris omnibus per reliquos numeros primos divisibilibus, tandem prodibit CAP. XIV.

$$A(1+\frac{1}{3^n})(1-\frac{1}{5^n})(1+\frac{1}{7^n})(1+\frac{1}{11^n})(1-\frac{1}{13^n}) \text{ \&c.} = 1,$$

seu

$$A = \frac{3^n}{3^n+1} \cdot \frac{5^n}{5^n-1} \cdot \frac{7^n}{7^n+1} \cdot \frac{11^n}{11^n+1} \cdot \frac{13^n}{13^n-1} \times \frac{17^n}{17^n-1} \cdot \text{\&c.},$$

ubi in numeratoribus occurrunt Potestates omnium numerorum primorum, quæ in denominatoribus insunt unitate sive auctæ sive minutæ, prout numeri primi fuerint formæ $4m-1$, vel $4m+1$.

285. Posito ergo $n=1$, ob $A=\frac{\pi}{4}$, erit

$$\frac{\pi}{4} = \frac{3}{4} \cdot \frac{5}{4} \cdot \frac{7}{8} \cdot \frac{11}{12} \cdot \frac{13}{12} \cdot \frac{17}{16} \cdot \frac{19}{20} \cdot \frac{23}{24} \cdot \text{\&c.};$$

supra autem invenimus esse

$$\frac{\pi\pi}{6} = \frac{4}{3} \cdot \frac{3^2}{2.4} \cdot \frac{5^2}{4.6} \cdot \frac{7^2}{6.8} \cdot \frac{11^2}{10.12} \cdot \frac{13^2}{12.14} \cdot \frac{17^2}{16.18} \cdot \frac{19^2}{18.20} \cdot \text{\&c.};$$

Dividatur secunda per primam & orietur

$$\frac{2\pi}{3} = \frac{4}{3} \cdot \frac{3}{2} \cdot \frac{5}{6} \cdot \frac{7}{6} \cdot \frac{11}{10} \cdot \frac{13}{14} \cdot \frac{17}{18} \cdot \frac{19}{18} \cdot \frac{23}{22} \cdot \text{\&c.},$$

seu

$$\frac{\pi}{2} = \frac{2}{3} \cdot \frac{5}{6} \cdot \frac{7}{6} \cdot \frac{11}{10} \cdot \frac{13}{14} \cdot \frac{17}{18} \cdot \frac{19}{18} \cdot \frac{23}{22} \cdot \text{\&c.},$$

ubi numeri primi constituunt numeratores, denominatores vero sunt numeri impariter pares, unitate differentes a nume-

LIB. I. ratoribus. Quod si hæc denuo per primam $\frac{\pi}{4}$ dividatur, erit

$$2 = \frac{4}{2} \cdot \frac{4}{6} \cdot \frac{8}{6} \cdot \frac{12}{10} \cdot \frac{12}{14} \cdot \frac{16}{18} \cdot \frac{20}{18} \cdot \frac{24}{22} \cdot \&c.,$$

seu

$$2 = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdot \frac{8}{9} \cdot \frac{10}{9} \cdot \frac{12}{11} \cdot \&c.,$$

quæ fractiones oriuntur ex numeris primis imparibus 3, 5, 7, 11, 13, 17, &c., quemque in duas partes unitate differentes dispescendo, & partes pares pro numeratoribus, impares pro denominatoribus sumendo.

286. Si hæ expressiones cum *Wallisiana* comparentur

$$\frac{\pi}{2} = \frac{2.\ 2.\ 4.\ 4.\ 6.\ 6.\ 8.\ 8.\ 10.\ 10.\ 12}{1.\ 3.\ 3.\ 5.\ 5.\ 7.\ 7.\ 9.\ 9.\ 11.\ 11}\ \&c.,$$

seu

$$\frac{4}{\pi} = \frac{3.\ 3}{2.\ 4} \cdot \frac{5.\ 5}{4.\ 6} \cdot \frac{7.\ 7}{6.\ 8} \cdot \frac{9.\ 9}{8.\ 10} \cdot \frac{11.\ 11}{10.\ 12} \cdot \&c.,$$

cum sit

$$\frac{\pi\pi}{8} = \frac{3.\ 3}{2.\ 4} \cdot \frac{5.\ 5}{4.\ 6} \cdot \frac{7.\ 7}{6.\ 8} \cdot \frac{11.\ 11}{10.\ 12} \cdot \frac{13.\ 13}{12.\ 14} \cdot \&c.,$$

illa per hanc divisa dabit

$$\frac{32}{\pi^3} = \frac{9.\ 9}{8.10} \cdot \frac{15.\ 15}{14.\ 16} \cdot \frac{21.\ 21}{20.\ 22} \cdot \frac{25.\ 25}{24.\ 26} \cdot \&c.,$$

ubi in numeratoribus occurrunt omnes numeri impares non primi.

287. Sit jam $n = 3$ erit $A = \frac{\pi^3}{32}$, unde fit

$$\frac{\pi^3}{32} = \frac{3^3}{3^3+1} \cdot \frac{5^3}{5^3-1} \cdot \frac{7^3}{7^3+1} \cdot \frac{11^3}{11^3+1} \cdot \frac{13^3}{13^3-1} \cdot \frac{17^3}{17^3-1} \cdot$$

&c.. At ex Serie

$$\frac{\pi^6}{945} = 1 + \frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{4^6} + \frac{1}{5^6} + \&c.,$$

fit

$\frac{\pi^6}{945}$

$$\frac{\pi^6}{945} = \frac{2^6}{2^6-1} \cdot \frac{3^6}{3^6-1} \cdot \frac{5^6}{5^6-1} \cdot \frac{7^6}{7^6-1} \cdot \frac{11^6}{11^6-1} \cdot \frac{13^6}{13^6-1} \cdot \&c.,\ \text{seu}$$

$$\frac{\pi^6}{960} = \frac{3^6}{3^6-1} \cdot \frac{5^6}{5^6-1} \cdot \frac{7^6}{7^6-1} \cdot \frac{11^6}{11^6-1} \cdot \frac{13^6}{13^6-1} \cdot \&c.,$$

quæ per primam divisa dabit

$$\frac{\pi^3}{30} = \frac{3^3}{3^3-1} \cdot \frac{5^3}{5^3+1} \cdot \frac{7^3}{7^3-1} \cdot \frac{11^3}{11^3-1} \cdot \frac{13^3}{13^3+1} \cdot \frac{17^3}{17^3+1} \cdot \&c.;$$

hæc vero denuo per primam divisa dabit

$$\frac{16}{15} = \frac{3^3+1}{3^3-1} \cdot \frac{5^3-1}{5^3+1} \cdot \frac{7^3+1}{7^3-1} \cdot \frac{11^3+1}{11^3-1} \cdot \frac{13^3-1}{13^3+1} \cdot \frac{17^3-1}{17^3+1} \cdot \&c.,\ \text{seu}$$

$$\frac{16}{15} = \frac{14}{13} \cdot \frac{62}{63} \cdot \frac{172}{171} \cdot \frac{666}{665} \cdot \frac{1098}{1099} \cdot \&c.;$$

quæ fractiones formantur ex cubis numerorum primorum imparium, quemque in duas partes unitate differentes dispescendo, ac partes pares pro numeratoribus, impares pro denominatoribus sumendo.

288. Ex his expressionibus denuo novæ Series formari possunt, in quibus omnes numeri naturales denominatores constituunt. Cum enim sit

$$\frac{\pi}{4} = \frac{3}{3+1} \cdot \frac{5}{5-1} \cdot \frac{7}{7+1} \cdot \frac{11}{11+1} \cdot \frac{13}{13-1} \cdot \&c.;$$

erit

$$\frac{\pi}{6} = \frac{1}{(1+\frac{1}{2})(1+\frac{1}{3})(1-\frac{1}{5})(1+\frac{1}{7})(1+\frac{1}{11})(1-\frac{1}{13})\&c.},$$

unde per evolutionem hæc Series nascetur

$$\frac{\pi}{6} = 1 - \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} - \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10}\ \&c.,$$

 ubi

LIB. I. ubi ratio ſignorum ita eſt comparata, ut binarius habeat —; numeri primi formæ $4m-1$ ſignum —; & numeri primi formæ $4m+1$ ſignum +; numeri autem compoſiti ea habent ſigna, quæ ipſis ratione multiplicationis ex primis conveniunt. Sic patebit ſignum fractionis $\frac{1}{60}$, ob $60 =$
— — — +
2 . 2 . 3 . 5, quod erit —. Simili modo porro erit

$$\frac{\pi}{2} = \frac{1}{(1-\frac{1}{2})(1+\frac{1}{3})(1-\frac{1}{5})(1+\frac{1}{7})(1+\frac{1}{11})(1-\frac{1}{13})\ \&c.},$$

unde orietur hæc Series

$$\frac{\pi}{2} = 1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{1}{6} - \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10}\ \&c.,$$

ubi binarius habet ſignum +; numeri primi formæ $4m-1$ ſignum —; numeri primi formæ $4m+1$ ſignum +; & numerus quiſque compoſitus id habet ſignum, quod ipſi ratione compoſitionis ex primis convenit, ſecundum regulas multiplicationis.

289. Cum deinde ſit

$$\frac{\pi}{2} = \frac{1}{(1-\frac{1}{3})(1+\frac{1}{5})(1-\frac{1}{7})(1-\frac{1}{11})(1+\frac{1}{13})\ \&c.},$$

erit per evolutionem

$$\frac{\pi}{2} = 1 + \frac{1}{3} - \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \frac{1}{13} - \frac{1}{15}\ \&c.,$$

ubi tantum numeri impares occurrunt, ſigna autem ita ſunt comparata, ut numeri primi formæ $4m-1$ ſignum habeant +; numeri primi formæ $4m+1$ ſignum —; unde ſimul numerorum compoſitorum ſigna definiuntur. Binæ porro Series hinc formari poſſunt, ubi omnes numeri occurrunt, erit ſcilicet

$\pi =$

$$\pi = \frac{1}{(1-\frac{1}{2})(1-\frac{1}{3})(1+\frac{1}{5})(1-\frac{1}{7})(1-\frac{1}{11})(1+\frac{1}{13})\ \&c.,}$$ CAP. XV.

unde per evolutionem oritur

$$\pi = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} - \frac{1}{10}\ \&c.,$$

ubi binarius ſignum habet +; numeri primi formæ $4m-1$ ſignum +; numeri vero primi formæ $4m+1$ ſignum —. Tum vero etiam erit

$$\frac{\pi}{3} = \frac{1}{(1+\frac{1}{2})(1-\frac{1}{3})(1+\frac{1}{5})(1-\frac{1}{7})(1-\frac{1}{11})(1+\frac{1}{13})\ \&c.,}$$

unde per evolutionem oritur

$$\frac{\pi}{3} = 1 - \frac{1}{2} + \frac{1}{3} + \frac{1}{4} - \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} + \frac{1}{10}\ \&c.,$$

ubi binarius habet ſignum —, numeri primi formæ $4m-1$ ſignum +, & numeri primi formæ $4m+1$ ſignum —.

290. Poſſunt hinc etiam innumerabiles aliæ ſignorum conditiones exhiberi, ita ut Seriei

$$1,\ \frac{1}{2},\ \frac{1}{3},\ \frac{1}{4},\ \frac{1}{5},\ \frac{1}{6},\ \frac{1}{7},\ \frac{1}{8},\ \&c.,$$

ſumma aſſignari queat. Cum ſcilicet ſit

$$\frac{\pi}{2} = \frac{1}{(1-\frac{1}{2})(1+\frac{1}{3})(1-\frac{1}{5})(1+\frac{1}{7})(1+\frac{1}{11})\ \&c..}$$

 Multi-

Multiplicetur hæc expressio per $\frac{1+\frac{1}{3}}{1-\frac{1}{3}} = 2$, erit

$$\pi = \frac{1}{(1-\frac{1}{2})(1-\frac{1}{3})(1-\frac{1}{5})(1+\frac{1}{7})(1+\frac{1}{11})\ \&c.},$$

&

$$\pi = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} - \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} - \frac{1}{11}\ \&c.,$$

ubi binarius signum habet $+$; ternarius $+$; reliqui numeri primi omnes formæ $4m-1$ signum $-$; at numeri primi formæ $4m+1$ signum $+$; & unde pro numeris compositis ratio signorum intelligitur. Simili modo, cum sit

$$\pi = \frac{1}{(1-\frac{1}{2})(1-\frac{1}{3})(1+\frac{1}{5})(1-\frac{1}{7})(1-\frac{1}{11})\ \&c.},$$

multiplicetur per $\frac{1+\frac{1}{5}}{1-\frac{1}{5}} = \frac{3}{2}$, erit

$$\frac{3\pi}{2} = \frac{1}{(1-\frac{1}{2})(1-\frac{1}{3})(1-\frac{1}{5})(1-\frac{1}{7})(1-\frac{1}{11})(1+\frac{1}{13})(1+\frac{1}{17})\ \&c.},$$

unde per evolutionem oritur

$$\frac{3\pi}{2} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \&c.,$$

ubi binarius habet signum $+$; numeri primi formæ $4m-1$ signum $+$; & numeri primi formæ $4m+1$, præter quinarium, signum $-$.

291. Pos-

191. Possunt etiam innumerabiles hujusmodi Series exhiberi, quarum summa sit $= 0$. Cum enim sit CAP. XV.

$$0 = \frac{2}{3} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{7}{8} \cdot \frac{11}{12} \cdot \frac{13}{14} \cdot \frac{17}{18} \cdot \&c.,$$

erit

$$0 = \frac{1}{(1+\frac{1}{2})(1+\frac{1}{3})(1+\frac{1}{5})(1+\frac{1}{7})(1+\frac{1}{11})(1+\frac{1}{13})\ \&c.},$$

unde, ut supra vidimus, oritur

$$0 = 1 - \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} - \frac{1}{7} - \frac{1}{8} + \frac{1}{9} + \frac{1}{10}\ \&c.,$$

ubi omnes numeri primi signum habent $-$; compositorumque numerorum signa regulam multiplicationis sequuntur. Multiplicemus autem illam expressionem per $\frac{1+\frac{1}{2}}{1-\frac{1}{2}} = 3$, erit pariter

$$0 = \frac{1}{(1-\frac{1}{2})(1+\frac{1}{3})(1+\frac{1}{5})(1+\frac{1}{7})(1+\frac{1}{11})(1+\frac{1}{13})\ \&c.}$$

unde per evolutionem nascitur

$$0 = 1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} - \frac{1}{6} - \frac{1}{7} + \frac{1}{8} + \frac{1}{9} - \frac{1}{10}\ \&c.,$$

ubi binarius habet signum $+$; reliqui numeri primi omnes signum $-$. Simili modo quoque erit

$0 =$

$$o = \frac{1}{(1+\frac{1}{2})(1-\frac{1}{3})(1-\frac{1}{5})(1+\frac{1}{7})(1+\frac{1}{11})(1+\frac{1}{13}) \&c.},$$

unde oritur ista Series

$$o = 1 - \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{1}{6} - \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} \&c.,$$

ubi omnes numeri primi, præter 3 & 5, habent signum —. In genere autem notandum est, quoties omnes numeri primi, exceptis tantum aliquibus, habeant signum —, summam Seriei fore = o. Contra autem quoties omnes numeri primi, exceptis tantum aliquibus, habeant signum +, tum summam Seriei fore infinite magnam.

292. Supra etiam (176.) summam dedimus Seriei

$$A = 1 - \frac{1}{2^n} + \frac{1}{4^n} - \frac{1}{5^n} + \frac{1}{7^n} - \frac{1}{8^n} + \frac{1}{10^n} - \frac{1}{11^n} + \frac{1}{13^n} \&c.,$$

si fuerit n numerus impar: Erit ergo

$$\frac{1}{2^n}A = \frac{1}{2^n} - \frac{1}{4^n} + \frac{1}{8^n} - \frac{1}{10^n} + \frac{1}{14^n} - \&c.,$$

quæ addita dat

$$B = (1 + \frac{1}{2^n})A = 1 - \frac{1}{5^n} + \frac{1}{7^n} - \frac{1}{11^n} + \frac{1}{13^n} - \frac{1}{17^n} + \frac{1}{19^n} - \frac{1}{23^n} + \frac{1}{25^n} - \&c.$$

$$\frac{1}{5^n}B = \frac{1}{5^n} - \frac{1}{25^n} + \frac{1}{35^n} - \frac{1}{55^n} \&c.,$$ addatur,

erit

$C =$

$$C = \left(1 + \frac{1}{5^n}\right)B = 1 + \frac{1}{7^n} - \frac{1}{11^n} + \frac{1}{13^n} - \frac{1}{17^n} + \frac{1}{19^n} - \frac{1}{23^n} \text{ \&c.}$$

$$\frac{1}{7^n}C = \frac{1}{7^n} + \frac{1}{49^n} - \frac{1}{77^n} + \text{\&c.}, \text{ subtrahatur},$$

erit

$$D = \left(1 - \frac{1}{7^n}\right)C = 1 - \frac{1}{11^n} + \frac{1}{13^n} - \frac{1}{17^n} + \frac{1}{19^n} - \text{\&c.};$$

Ex his tandem fiet

$$A\left(1+\frac{1}{2^n}\right)\left(1+\frac{1}{5^n}\right)\left(1-\frac{1}{7^n}\right)\left(1+\frac{1}{11^n}\right)\left(1-\frac{1}{13^n}\right)\text{\&c.} = 1,$$

ubi numeri primi unitate excedentes multipla ſenarii habent ſignum $-$, deficientes autem ſignum $+$. Eritque

$$A = \frac{2^n}{2^n+1} \cdot \frac{5^n}{5^n+1} \cdot \frac{7^n}{7^n-1} \cdot \frac{11^n}{11^n+1} \cdot \frac{13^n}{13^n-1} \cdot \text{\&c.}$$

293. Conſideremus caſum $n = 1$, quo $A = \frac{\pi}{3\sqrt{3}}$; eritque

$$\frac{\pi}{3\sqrt{3}} = \frac{2}{3} \cdot \frac{5}{6} \cdot \frac{7}{6} \cdot \frac{11}{12} \cdot \frac{13}{12} \cdot \frac{17}{18} \cdot \frac{19}{18} \cdot \text{\&c.},$$

ubi in numeratoribus poſt 3 occurrunt omnes numeri primi, denominatores vero a numeratoribus unitate diſcrepant, ſuntque omnes per 6 diviſibiles. Cum jam ſit

$$\frac{\pi\pi}{6} = \frac{4}{3} \cdot \frac{9}{8} \cdot \frac{5.5}{4.6} \cdot \frac{7.7}{6.8} \cdot \frac{11.11}{10.12} \cdot \frac{13.13}{12.14} \text{ \&c.},$$

erit, hac expreſſione per illam diviſa,

$$\frac{\pi\sqrt{3}}{2} = \frac{9}{4} \cdot \frac{5}{4} \cdot \frac{7}{8} \cdot \frac{11}{10} \cdot \frac{13}{14} \cdot \frac{17}{16} \cdot \frac{19}{20} \cdot \text{\&c.},$$

LIB. I. ubi denominatores non sunt per 6 divisibiles. Vel erit

$$\frac{\pi}{2\sqrt{3}} = \frac{5}{6}\cdot\frac{7}{6}\cdot\frac{11}{12}\cdot\frac{13}{12}\cdot\frac{17}{18}\cdot\frac{19}{18}\cdot\frac{23}{24}\cdot \text{\&c.}$$

$$\frac{2\pi}{3\sqrt{3}} = \frac{5}{4}\cdot\frac{7}{8}\cdot\frac{11}{10}\cdot\frac{13}{14}\cdot\frac{17}{16}\cdot\frac{19}{20}\cdot\frac{23}{22}\cdot \text{\&c.},$$

quarum hæc per illam divisa dat

$$\frac{4}{3} = \frac{6}{4}\cdot\frac{6}{8}\cdot\frac{12}{10}\cdot\frac{12}{14}\cdot\frac{18}{16}\cdot\frac{18}{20}\cdot\frac{24}{22}\cdot \text{\&c.},$$

seu

$$\frac{4}{3} = \frac{3}{2}\cdot\frac{3}{4}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdot\frac{9}{8}\cdot\frac{9}{10}\cdot\frac{12}{11}\cdot \text{\&c.},$$

ubi singulæ fractiones ex numeris primis 5, 7, 11, &c., formantur, singulos numeros primos in duas partes unitate differentes dispescendo, & partes per 3 divisibiles constanter pro numeratoribus sumendo.

294. Quoniam vero supra vidimus esse

$$\frac{\pi}{4} = \frac{3}{4}\cdot\frac{5}{4}\cdot\frac{7}{8}\cdot\frac{11}{12}\cdot\frac{13}{12}\cdot\frac{17}{16}\cdot \text{\&c.}$$

seu

$$\frac{\pi}{3} = \frac{5}{4}\cdot\frac{7}{8}\cdot\frac{11}{12}\cdot\frac{13}{12}\cdot\frac{17}{16}\cdot\frac{19}{20}\cdot \text{\&c.},$$

si superiores $\frac{\pi}{2\sqrt{3}}$ & $\frac{2\pi}{3\sqrt{3}}$ per hanc dividantur, orietur

$$\frac{\sqrt{3}}{2} = \frac{2}{3}\cdot\frac{4}{3}\cdot\frac{8}{9}\cdot\frac{10}{9}\cdot\frac{14}{15}\cdot\frac{16}{15}\cdot \text{\&c.}$$

$$\frac{2}{\sqrt{3}} = \frac{6}{5}\cdot\frac{6}{7}\cdot\frac{12}{11}\cdot\frac{18}{19}\cdot\frac{24}{23}\cdot\frac{30}{29}\cdot \text{\&c.},$$

In priori expressione fractiones formantur ex numeris primis formæ $12m + 6 \pm 1$, in posteriore ex numeris primis formæ $12m \pm 1$, singulos in duas partes unitate discrepantes dispescendo, & partes pares pro numeratoribus, impares vero pro denominatoribus sumendo.

295. Contemplemur adhuc Seriem supra inventam (179), quæ ita progrediebatur

$$\frac{\pi}{2\sqrt{2}} = 1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \frac{1}{13} - \frac{1}{15} + \text{\&c.} = A, \text{ erit}$$

$$\frac{1}{3}A = \frac{1}{3} + \frac{1}{9} - \frac{1}{15} - \frac{1}{21} + \frac{1}{27} + \frac{1}{33} - \text{\&c.: subtrahatur}$$

$$\left(1 - \frac{1}{3}\right)A = 1 - \frac{1}{5} - \frac{1}{7} + \frac{1}{11} - \frac{1}{13} + \frac{1}{17} + \frac{1}{19} - \text{\&c.} = B$$

$$\frac{1}{5}B = \frac{1}{5} - \frac{1}{25} - \frac{1}{35} + \frac{1}{55} - \text{\&c.: addatur, erit}$$

$$\left(1 + \frac{1}{5}\right)B = 1 - \frac{1}{7} + \frac{1}{11} - \frac{1}{13} + \frac{1}{17}\ \text{\&c.} = C:$$

ficque progrediendo tandem pervenietur ad

$$\frac{\pi}{2\sqrt{2}}\left(1 - \frac{1}{3}\right)\left(1 + \frac{1}{5}\right)\left(1 + \frac{1}{7}\right)\left(1 - \frac{1}{11}\right)\left(1 + \frac{1}{13}\right)\left(1 - \frac{1}{17}\right)\left(1 - \frac{1}{19}\right)\ \text{\&c.} = 1,$$

ubi figna ita fe habent, ut numerorum primorum formæ $8m + 1$, vel $8m + 3$, figna fint $-$; numerorum primorum vero formæ $8m + 5$, vel $8m + 7$, figna fint $+$. Hinc itaque erit

$$\frac{\pi}{2\sqrt{2}} = \frac{3}{2} \cdot \frac{5}{6} \cdot \frac{7}{8} \cdot \frac{11}{10} \cdot \frac{13}{14} \cdot \frac{17}{16} \cdot \frac{19}{18} \cdot \frac{23}{24} \cdot \text{\&c.},$$

ubi omnes denominatores vel divifibiles funt per 8, vel tantum funt numeri impariter pares. Cum igitur fit

$$\frac{\pi}{4} = \frac{3}{4} \cdot \frac{5}{4} \cdot \frac{7}{8} \cdot \frac{11}{12} \cdot \frac{13}{12} \cdot \frac{17}{16} \cdot \frac{19}{20} \cdot \frac{23}{24} \cdot \text{\&c.}$$

$$\frac{\pi}{2} = \frac{3}{2} \cdot \frac{5}{6} \cdot \frac{7}{6} \cdot \frac{11}{10} \cdot \frac{13}{14} \cdot \frac{17}{18} \cdot \frac{19}{18} \cdot \frac{23}{22} \cdot \text{\&c.},$$

&

$$\frac{\pi\pi}{8} = \frac{3.3}{2.4} \cdot \frac{5.5}{4.6} \cdot \frac{7.7}{6.8} \cdot \frac{11.11}{10.12} \cdot \frac{13.13}{12.14} \cdot \&c.,$$

erit

$$\frac{\pi}{2\sqrt{2}} = \frac{3}{4} \cdot \frac{5}{4} \cdot \frac{7}{6} \cdot \frac{11}{12} \cdot \frac{13}{12} \cdot \frac{17}{18} \cdot \frac{19}{20} \cdot \frac{23}{22} \cdot \&c.,$$

ubi nulli denominatores per 8 divisibiles occurrunt, pariter pares vero adsunt, quoties unitate differunt a numeratoribus. Prima vero per ultimam divisa dat

$$1 = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdot \frac{9}{8} \cdot \frac{10}{9} \cdot \frac{11}{12} \cdot \&c.,$$

quæ fractiones formantur ex numeris primis, singulos in duas partes unitate discrepantes dispescendo, & partes pares, (nisi sint pariter pares) pro numeratoribus sumendo.

296. Simili modo reliquæ Series, quas supra pro expressione arcuum circularium invenimus (179. *& seqq.*) in Factores transformari possunt, qui ex numeris primis constituantur. Sicque multæ aliæ insignes proprietates tam hujusmodi Factorum, quam Serierum infinitarum erui poterunt. Quoniam vero præcipuas hic jam commemoravi, pluribus evolvendis hic non immorabor. Sed ad aliud huic affine argumentum procedam. Quemadmodum scilicet in hoc Capite numeri, quatenus per multiplicationem oriuntur, sunt considerati, ita in sequenti generatio numerorum per additionem perpendetur.

CAPUT

CAPUT XVI.

De Partitione numerorum.

297. PRoposita sit ista expressio

$$(1+x^{\alpha}z)(1+x^{\beta}z)(1+x^{\gamma}z)(1+x^{\delta}z)(1+x^{\varepsilon}z)\ \&c.,$$

quæ cujusmodi induat formam, si per multiplicationem evolvatur, inquiramus. Ponamus prodire

$$1+Pz+Qz^2+Rz^3+Sz^4+\&c.,$$

atque manifestum est P fore summam Potestatum

$x^{\alpha}+x^{\beta}+x^{\gamma}+x^{\delta}+x^{\varepsilon}+$ &c.. Deinde Q est summa Factorum ex binis Potestatibus diversis, seu Q erit aggregatum plurium Potestatum ipsius x, quarum Exponentes sunt summæ duorum terminorum diversorum hujus Seriei

$$\alpha,\ \beta,\ \gamma,\ \delta,\ \varepsilon,\ \zeta,\ \eta,\ \&c.$$

Simili modo R erit aggregatum Potestatum ipsius x, quarum Exponentes sunt summæ trium terminorum diversorum. Atque S erit aggregatum Potestatum ipsius x, quarum Exponentes sunt summæ quatuor terminorum diversorum ejusdem Seriei, α, β, γ, δ, ε, &c., & ita porro.

298. Singulæ hæ Potestates ipsius x, quæ in valoribus literarum P, Q, R, S, &c., insunt, unitatem pro coëfficiente habebunt, si quidem earum Exponentes unico modo ex

LIB. I. α, β, γ, δ, &c., formari queant: sin autem ejusdem Potestatis Exponens pluribus modis possit esse summa duorum, trium, pluriumve terminorum Seriei α, β, γ, δ, ϵ, &c., tum etiam Potestas illa coëfficientem habebit, qui unitatem toties in se complectatur. Sic, si in valore ipsius Q reperiatur Nx^n, indicio hoc erit numerum n esse N diversis modis summam duorum terminorum diversorum Seriei α, β, γ, &c. Atque si in evolutione Factorum propositorum occurrat terminus $Nx^n z^m$, ejus coëfficiens N indicabit quot variis modis numerus n possit esse summa m terminorum diversorum Seriei α, β, γ, δ, ϵ, ζ, &c.

299. Quod si ergo productum propositum

$$(1 + x^\alpha z)(1 + x^\beta z)(1 + x^\gamma z)(1 + x^\delta z) \text{ \&c.}$$

per multiplicationem veram evolvatur, ex expressione resultante statim apparebit, quot variis modis datus numerus possit esse summa tot terminorum diversorum Seriei α, β, γ, δ, ϵ, ζ, &c., quot quis voluerit. Scilicet, si quæratur quot variis modis numerus n possit esse summa m terminorum illius Seriei diversorum, in expressione evoluta quæri debet terminus $x^n z^m$, ejusque coëfficiens indicabit numerum quæsitum.

300. Quo hæc fiant planiora, sit propositum hoc productum ex Factoribus constans infinitis

$$(1 + xz)(1 + x^2 z)(1 + x^3 z)(1 + x^4 z)(1 + x^5 z) \text{ \&c.},$$

quod per multiplicationem actualem evolutum dat

$$
\begin{aligned}
&1+z\,(x+x^2+x^3+x^4+x^5+x^6+x^7+x^8+x^9+\&c.)\\
&+z^2(x^3+x^4+2x^5+2x^6+3x^7+3x^8+4x^9+4x^{10}+5x^{11}+\&c.)\\
&+z^3(x^6+x^7+2x^8+3x^9+4x^{10}+5x^{11}+7x^{12}+8x^{13}+10x^{14}+\&c.)\\
&+z^4(x^{10}+x^{11}+2x^{12}+3x^{13}+5x^{14}+6x^{15}+9x^{16}+11x^{17}+15x^{18}+\&c.)\\
&+z^5(x^{15}+x^{16}+2x^{17}+3x^{18}+5x^{19}+7x^{20}+10x^{21}+13x^{22}+18x^{23}+\&c.)\\
&+z^6(x^{21}+x^{22}+2x^{23}+3x^{24}+5x^{25}+7x^{26}+11x^{27}+14x^{28}+20x^{29}+\&c.)\\
&+z^7(x^{28}+x^{29}+2x^{30}+3x^{31}+5x^{32}+7x^{33}+11x^{34}+15x^{35}+21x^{36}+\&c.)\\
&+z^8(x^{36}+x^{37}+2x^{38}+3x^{39}+5x^{40}+7x^{41}+11x^{42}+15x^{43}+22x^{44}+\&c.)\\
&\qquad\qquad\&c.
\end{aligned}
$$

Ex his ergo Seriebus ſtatim definire licet quot variis modis propoſitus numerus ex dato terminorum diverſorum hujus Seriei 1, 2, 3, 4, 5, 6, 7, 8, &c., numero oriri queat. Sic, ſi quæratur quot variis modis numerus 35 poſſit eſſe ſumma ſeptem terminorum diverſorum Seriei 1, 2, 3, 4, 5, 6, 7, &c., quæratur in Serie z^7 multiplicante Poteſtas x^{35}, ejuſque coëfficiens 15 indicabit numerum propoſitum 35 quindecim variis modis eſſe ſummam ſeptem terminorum Seriei 1, 2, 3, 4, 5, 6, 7, 8, &c.

301. Quod ſi autem ponatur $z = 1$, & ſimiles Poteſtates ipſius x in unam ſummam conjiciantur, ſeu, quod eodem redit, ſi evolvatur hæc expreſſio infinita

$$(1+x)(1+x^2)(1+x^3)(1+x^4)(1+x^5)(1+x^6)\&c.,$$

quo facto orietur hæc Series

$$1+x+x^2+2x^3+2x^4+3x^5+4x^6+5x^7+6x^8+\&c.,$$

ubi quivis coëfficiens indicat, quot variis modis Exponens Poteſtatis ipſius x conjunctæ ex terminis diverſis Seriei 1, 2, 3, 4, 5, 6, 7, &c., per additionem emergere poſſit. Sic apparet numerum 8 ſex modis per additionem diverſorum numerorum produci, qui ſunt

$8 = 8$	$8 = 5 + 3$
$8 = 7 + 1$	$8 = 5 + 2 + 1$
$8 = 6 + 2$	$8 = 4 + 3 + 1$

ubi

 ubi notandum est numerum propositum ipsum simul computari debere, quia numerus terminorum non definitur, ideoque unitas inde non excluditur.

302. Hinc igitur intelligitur, quomodo quisque numerus per additionem diversorum numerorum producatur. Conditio autem diversitatis omittetur, si Factores illos in denominatorem transponamus. Sit igitur proposita hæc expressio

$$\frac{1}{(1-x^{\alpha}z)(1-x^{\beta}z)(1-x^{\gamma}z)(1-x^{\delta}z)(1-x^{\epsilon}z)\,\&c.,}$$

quæ per divisionem evoluta det

$$1 + Pz + Qz^2 + Rz^3 + Sz^4 + \&c..$$

Atque manifestum est fore P aggregatum Potestatum ipsius x, quarum Exponentes contineantur in hac Serie

$$\alpha,\ \beta,\ \gamma,\ \delta,\ \epsilon,\ \zeta,\ \eta,\ \&c.,$$

Deinde Q erit aggregatum Potestatum ipsius x, quarum Exponentes sint summæ duorum terminorum hujus Seriei, sive eorundem sive diversorum. Tum erit R summa Potestatum ipsius x, quarum Exponentes ex additione trium terminorum illius Seriei oriantur; & S summa Potestatum, quarum Exponentes ex additione quatuor terminorum in illa Serie contentorum formantur, & ita porro.

303. Si igitur tota expressio per singulos terminos explicetur, & termini similes conjunctim exprimantur, intelligetur quot variis modis propositus numerus n per additionem m terminorum, sive diversorum sive non diversorum, Seriei α, β, γ, δ, ϵ, ζ, &c., produci queat. Quæratur scilicet in expressione evoluta terminus $x^n z^m$, ejusque coëfficiens, qui sit N, ita ut totus terminus sit $= Nx^n z^m$, atque coëfficiens N indicabit quot variis modis numerus n per additionem m terminorum

minorum in Serie $\alpha, \beta, \gamma, \delta, \epsilon$, &c., contentorum produci queat. Hoc igitur pacto quæstio priori, quam ante sumus contemplati, similis resolvetur. CAP. XVI.

304. Accommodemus hæc ad casum inprimis notatu dignum, sitque proposita hæc expressio

$$\frac{1}{(1-xz)(1-x^2z)(1-x^3z)(1-x^4z)(1-x^5z)\,\&c.},$$

quæ per divisionem evoluta dabit

$$\begin{aligned}
&1+z\,(x+x^2+x^3+x^4+x^5+x^6+x^7+x^8+x^9+\&c.)\\
&+z^2(x^2+x^3+2x^4+2x^5+3x^6+3x^7+4x^8+4x^9+5x^{10}+\&c.)\\
&+z^3(x^3+x^4+2x^5+3x^6+4x^7+5x^8+7x^9+8x^{10}+10x^{11}+\&c.)\\
&+z^4(x^4+x^5+2x^6+3x^7+5x^8+6x^9+9x^{10}+11x^{11}+15x^{12}+\&c.)\\
&+z^5(x^5+x^6+2x^7+3x^8+5x^9+7x^{10}+10x^{11}+13x^{12}+18x^{13}+\&c.)\\
&+z^6(x^6+x^7+2x^8+3x^9+5x^{10}+7x^{11}+11x^{12}+14x^{13}+20x^{14}+\&c.)\\
&+z^7(x^7+x^8+2x^9+3x^{10}+5x^{11}+7x^{12}+11x^{13}+15x^{14}+21x^{15}+\&c.)\\
&+z^8(x^8+x^9+2x^{10}+3x^{11}+5x^{12}+7x^{13}+11x^{14}+15x^{15}+22x^{16}+\&c.)\\
&\qquad\&c.,
\end{aligned}$$

Ex his ergo Seriebus statim definire licet quot variis modis propositus numerus per additionem ex dato terminorum hujus Seriei 1, 2, 3, 4, 5, 6, 7, &c., numero produci queat. Sic, si quæratur quot variis modis numerus 13 oriri possit per additionem quinque numerorum integrorum, spectari debebit terminus $x^{13}z^5$, cujus coëfficiens 18 indicat numerum propositum 13 ex quinque numerorum additione octodecim modis oriri posse.

305. Si ponatur $z = 1$, atque similes Potestates ipsius x conjunctim exprimantur, hæc expressio

$$\frac{1}{(1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)(1-x^6)\,\&c.},$$

evolvetur in hanc Seriem

 $1 + x + 2x^2 + 3x^3 + 5x^4 + 7x^5 + 11x^6 + 15x^7 + 22x^8$ &c.;

in qua quilibet coëfficiens indicat, quot variis modis Exponens Poteſtatis adjunctæ per additionem produci queat ex numeris integris, ſive æqualibus ſive inæqualibus. Scilicet ex termino $11x^6$ cognoſcitur numerum 6 undecim modis per additionem numerorum integrorum produci poſſe, qui ſunt

$6 = 6$	$6 = 3 + 1 + 1 + 1$
$6 = 5 + 1$	$6 = 2 + 2 + 2$
$6 = 4 + 2$	$6 = 2 + 2 + 1 + 1$
$6 = 4 + 1 + 1$	$6 = 2 + 1 + 1 + 1 + 1$
$6 = 3 + 3$	$6 = 1 + 1 + 1 + 1 + 1 + 1$
$6 = 3 + 2 + 1$	

ubi quoque notari debet, ipſum numerum propoſitum, cum in Serie numerorum 1, 2, 3, 4, 5, 6, &c., propoſita contineatur, unum modum præbere.

306. His in genere expoſitis, diligentius inquiramus in modum hanc compoſitionum multitudinem inveniendi. Ac primo quidem conſideremus eam ex numeris integris compoſitionem, in qua numeri tantum diverſi admittuntur, quam prius commemoravimus. Sit igitur in hunc finem propoſita hæc expreſſio

$$Z = (1 + xz)(1 + x^2z)(1 + x^3z)(1 + x^4z)(1 + x^5z) \text{ \&c.},$$

quæ evoluta & ſecundum Poteſtates ipſius z digeſta præbeat

$$Z = 1 + Pz + Qz^2 + Rz^3 + Sz^4 + Tz^5 + \text{\&c.};$$

ubi methodus deſideratur has ipſius x Functiones P, Q, R, S, T, &c., expedite inveniendi, hoc enim pacto quæſtioni propoſitæ convenientiſſime ſatisfiet.

307. Patet autem, ſi loco z ponatur xz, prodire

(1+

$(1+x^2z)(1+x^3z)(1+x^4z)(1+x^5z)$ &c. $= \frac{Z}{1+xz}$
ergo, posito xz loco z, valor producti, qui erat Z, abibit in $\frac{Z}{1+xz}$; sicque, cum sit

$$Z = 1 + Pz + Qz^2 + Rz^3 + Sz^4 + \&c.,$$

erit

$$\frac{Z}{1+xz} = 1 + Pxz + Qx^2z^2 + Rx^3z^3 + Sx^4z^4 + \&c.,$$

multiplicetur ergo actu per $1+xz$, atque prodibit

$$\begin{aligned} Z = 1 &+ Pxz + Qx^2z^2 + Rx^3z^3 + Sx^4z^4 + \&c. \\ &+ xz + Px^2z^2 + Qx^3z^3 + Rx^4z^4 + \&c., \end{aligned}$$

qui valor ipsius Z cum superiori comparatus dabit

$$P = \frac{x}{1-x};\ Q = \frac{Px^2}{1-x^2};\ R = \frac{Qx^3}{1-x^3};\ S = \frac{Rx^4}{1-x^4}\ \&c.,$$

Sequentes ergo pro P, Q, R, S, &c., obtinentur valores

$$P = \frac{x}{1-x}$$

$$Q = \frac{x^3}{(1-x)(1-x^2)}$$

$$R = \frac{x^6}{(1-x)(1-x^2)(1-x^3)}$$

$$S = \frac{x^{10}}{(1-x)(1-x^2)(1-x^3)(1-x^4)}$$

$$T = \frac{x^{15}}{(1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)}$$

&c.

308. Sic igitur seorsim unamquamque Seriem Potestatum ipsius x exhibere possumus, ex qua definire licet, quot variis modis propositus numerus ex dato partium integrarum numero per additionem formari possit. Manifestum autem porro est has singulas Series esse recurrentes, quia ex evolutione Functionis fractæ ipsius x nascuntur. Prima scilicet expressio

 $P =$

LIB. I. $n = \frac{x}{1-x}$, dat Seriem geometricam

$$x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + \&c.,$$

ex qua quidem manifestum est quemvis numerum semel in Serie numerorum integrorum contineri.

309. Expressio secunda $\frac{x^3}{(1-x)(1-xx)}$, dat hanc Seriem

$$x^3 + x^4 + 2x^5 + 2x^6 + 3x^7 + 3x^8 + 4x^9 + 4x^{10} + \&c.,$$

in qua cujusvis termini coëfficiens indicat quot modis Exponens ipsius x in duas partes inæquales dispertiri possit. Sic terminus $4x^9$ indicat, numerum 9 quatuor modis in duas partes inæquales secari posse. Quod si hanc Seriem per x^3 dividamus, prodibit Series, quam præbet ista fractio $\frac{1}{(1-x)(1-x^2)}$, quæ erit

$$1 + x + 2x^2 + 2x^3 + 3x^4 + 3x^5 + 4x^6 + 4x^7 + \&c.,$$

cujus terminus generalis sit $= Nx^n$; atque ex genesi hujus Seriei intelligitur coëfficientem N indicare, quot variis modis Exponens n ex numeris 1 & 2 per additionem nasci queat. Cum igitur prioris Seriei terminus generalis sit $= Nx^{n+3}$, deducitur hinc istud theorema.

Quot variis modis numerus n *per additionem ex numeris* 1 *&* 2 *produci potest, totidem variis modis numerus* n + 3 *in duas partes inæquales secari poterit.*

310. Expressio tertia $\frac{x^6}{(1-x)(1-x^2)(1-x^3)}$ in Seriem evoluta dabit

$$x^6 + x^7 + 2x^8 + 3x^9 + 4x^{10} + 5x^{11} + 7x^{12} + 8x^{13} + \&c.,$$

in qua cujusvis termini coëfficiens indicat quot variis modis Exponens Potestatis x adjunctæ in tres partes inæquales dispertirii tiri possit. Quod si autem hæc fractio $\frac{1}{(1-x)(1-x^2)(1-x^3)}$ evolvatur, prodibit hæc Series

$$1 + x + 2x^2 + 3x^3 + 4x^4 + 5x^5 + 7x^6 + 8x^7 + \&c.,$$

cujus terminus generalis si ponatur $= Nx^n$, coëfficiens N indicabit quot variis modis numerus n ex numeris 1, 2, 3, per additionem produci possit. Cum igitur prioris Seriei terminus generalis sit Nx^{n+6}, sequetur hinc istud theorema.

Quot variis modis numerus n *per additionem ex numeris* 1, 2, 3, *produci potest*, *totidem variis modis numerus* n + 6 *in tres partes inæquales secari poterit.*

311. Expressio quarta $\frac{x^{10}}{(1-x)(1-x^2)(1-x^3)(1-x^4)}$ in Seriem recurrentem evoluta dabit

$$x^{10} + x^{11} + 2x^{12} + 3x^{13} + 5x^{14} + 6x^{15} + 9x^{16} + \&c.,$$

in qua cujusvis termini coëfficiens indicabit quot variis modis Exponens Potestatis x adjunctæ in quatuor partes inæquales dispertiri possit. Quod si autem hæc expressio

$\frac{1}{(1-x)(1-x^2)(1-x^3)(1-x^4)}$ evolvatur, prodibit superior Series per x^{10} divisa, nempe

$$1 + x + 2x^2 + 3x^3 + 5x^4 + 6x^5 + 9x^6 + 11x^7 + \&c.,$$

cujus terminum generalem ponamus $= Nx^n$; atque hinc patebit coëfficientem N indicare, quot variis modis numerus n per additionem oriri possit ex his quatuor numeris 1, 2, 3, 4. Cum igitur prioris Seriei terminus generalis futurus sit $= Nx^{n+10}$, deducitur hoc theorema.

LIB. I. *Quot variis modis numerus* n *per additionem produci potest ex numeris* 1, 2, 3, 4, *totidem variis modis numerus* n + 10 *in quatuor partes inæquales secari poterit.*

312. Generaliter ergo, si hæc expressio

$$\frac{1}{(1-x)(1-x^2)(1-x^3)\ldots\ldots(1-x^m)}$$

in Seriem evolvatur, ejusque terminus generalis fuerit $= Nx^n$, coëfficiens N indicabit, quot variis modis numerus n per additionem produci possit ex his numeris 1, 2, 3, 4 .... m. Quod si autem hæc expressio

$$\frac{x^{\frac{m(m+1)}{2}}}{(1-x)(1-x^2)(1-x^3)\ldots\ldots(1-x^m)}$$

in Seriem evolvatur, erit ejus terminus generalis $=$ $Nx^{n+\frac{m(m+1)}{2}}$: atque hic coëfficiens N indicat quot variis modis numerus $n+\frac{m(m+1)}{1.\ 2}$ in m partes inæquales secari possit, unde hoc habetur theorema.

Quot variis modis numerus n *per additionem produci potest ex numeris* 1, 2, 3, 4 .... m, *totidem modis numerus* $n+\frac{m(m+1)}{1.\ 2}$ *in* m *partes inæquales secari poterit.*

313. Ex posita partitione numerorum in partes inæquales, perpendamus quoque partitionem in partes, ubi æqualitas partium non excluditur; quæ partitio ex hac expressione originem habet

$$Z=\frac{1}{(1-xz)(1-x^2z)(1-x^3z)(1-x^4z)(1-x^5z)\ \&c.}$$

Ponamus evolutione per divisionem instituta prodire

Z =

$$Z = 1 + Pz + Qz^2 + Rz^3 + Sz^4 + Tz^5 + \&c.$$

Perſpicuum autem eſt, ſi loco z ponatur xz, prodire

$$\frac{1}{(1-x^2z)(1-x^3z)(1-x^4z)(1-x^5z)\&c.} = (1-xz)Z.$$

Facta ergo in Serie evoluta eadem mutatione, fiet

$$(1-xz)Z = 1 + Pxz + Qx^2z^2 + Rx^3z^3 + Sx^4z^4 + \&c.$$

Multiplicetur ergo ſuperior Series pariter per $(1-xz)$, eritque

$$\begin{aligned}(1-xz)Z = 1 &+ Pz + Qz^2 + Rz^3 + Sz^4 + \&c.\\ &- xz - Pxz^2 - Qxz^3 - Rxz^4 - \&c.\end{aligned}$$

Comparatione ergo inſtituta orietur

$$P = \frac{x}{1-x};\ Q = \frac{Px}{1-x^2};\ R = \frac{Qx}{1-x^3};\ S = \frac{Rx}{1-x^4}\ \&c.,$$

unde pro P, Q, R, S, &c., ſequentes valores proveniunt.

$$P = \frac{x}{1-x}$$

$$Q = \frac{x^2}{(1-x)(1-x^2)}$$

$$R = \frac{x^3}{(1-x)(1-x^2)(1-x^3)}$$

$$S = \frac{x^4}{(1-x)(1-x^2)(1-x^3)(1-x^4)}$$

&c.

314. Expreſſiones iſtæ a ſuperioribus aliter non diſcrepant, niſi quod numeratores hic minores habeant Exponentes quam caſu præcedente. Atque hanc ob rem Series, quæ per evolutionem naſcuntur, ratione coëfficientium omnino convenient, quæ convenientia jam ex comparatione (§. §. 300. & 304.) perſpi-

LIB. I. perſpicitur, nunc vero demum ejus ratio intelligitur. Hinc ergo omnino ſimilia theoremata conſequentur, quæ ſunt.

Quot variis modis numerus n *per additionem produci poteſt ex numeris* 1, 2, *totidem modis numerus* n + 2 *in duas partes diſpertiri poterit.*

Quot variis modis numerus n *per additionem produci poteſt ex numeris* 1, 2, 3, *totidem modis numerus* n + 3 *in tres partes diſpertiri poterit.*

Quot variis modis numerus n *per additionem produci poteſt ex numeris* 1, 2, 3, 4, *totidem modis numerus* n + 4 *in quatuor partes diſpertiri poterit.*

Atque generaliter habebitur hoc theorema:

Quot variis modis numerus n *per additionem produci poteſt ex numeris* 1, 2, 3, ... m, *totidem modis numerus* n + m *in* m *partes diſpertiri poterit.*

315. Sive ergo quæratur quot modis datus numerus in *m* partes inæquales, ſive in *m* partes, æqualibus non excluſis, diſpertiri poſſit, utraque quæſtio reſolvetur ſi cognoſcatur quot modis quiſque numerus per additionem produci poſſit ex numeris 1, 2, 3, 4 .... *m*, quemadmodum hoc patebit ex ſequentibus theorematis, quæ ex ſuperioribus ſunt derivata.

Numerus n *tot modis in* m *partes inæquales diſpertiri poteſt, quot modis numerus* $n - \frac{m(m+1)}{2}$ *per additionem produci poteſt ex numeris* 1, 2, 3, 4, ... m.

Numerus n, *tot modis in* m *partes ſive æquales ſive inæquales diſpertiri poteſt quot modis numerus* n — m *per additionem produci poteſt ex numeris* 1, 2, 3, ... m.

Hinc porro ſequuntur hæc theoremata.

Numerus n *totidem modis in* m *partes inæquales ſecari poteſt, quot modis numerus* $n - \frac{m(m-1)}{2}$ *in* m *partes, ſive æquales ſive inæquales, diſpertitur.*

Numerus n *totidem modis in* m *partes, ſive inæquales ſive æquales, ſecari*

secari potest, quot modis numerus $n + \frac{m(m-1)}{2}$ *in* m *partes inæquales dispertiri potest.*

316. Per formationem autem Serierum recurrentium inveniri poterit, quot variis modis datus numerus *n* per additionem produci possit ex numeris 1, 2, 3, ... *m*. Ad hoc enim inveniendum evolvi debebit fractio

$$\frac{1}{(1-x)(1-x^2)(1-x^3)\ldots\ldots\ldots(1-x^m)}$$

atque Series recurrens continuari debebit usque ad terminum Nx^n, cujus coëfficiens N indicabit, quot modis numerus *n* per additionem produci possit ex numeris 1, 2, 3, 4, ... *m*. At vero hic solvendi modus non parum habebit difficultatis, si numeri *m* & *n* sint modice magni; scala enim relationis, quam præbet denominator per multiplicationem evolutus, ex pluribus terminis constat, unde operosum erit Seriem ad plures terminos continuare.

317. Hæc autem disquisitio minus erit molesta, si casus simpliciores primum expediantur, ex his enim facile erit ad casus magis compositos progredi. Sit Seriei, quæ ex hac fractione oritur,

$$\frac{1}{(1-x)(1-x^2)(1-x^3)\ldots\ldots\ldots(1-x^m)}$$

terminus generalis $= Nx^n$; at Seriei ex hac forma

$$\frac{x^m}{(1-x)(1-x^2)(1-x^3)\ldots\ldots\ldots(1-x^m)}$$

ortæ terminus generalis sit Mx^n, ubi coëfficiens M indicabit quot variis modis numerus $n - m$ per additionem produci

LIB. I. possit ex numeris 1, 2, 3, ... m. Subtrahatur posterior expressio a priori, ac remanebit

$$\frac{1}{(1-x)(1-x^2)(1-x^3)\,.\,.\,.\,.\,.\,.\,.\,.\,(1-x^{m-1})}$$

atque manifestum est Seriei hinc ortæ terminum generalem futurum esse $(N-M)x^n$; quare coëfficiens $N-M$ indicabit quot variis modis numerus n per additionem produci possit ex numeris 1, 2, 3, ... $(m-1)$.

318. Hinc ergo sequentem regulam nanciscimur.

Sit L numerus modorum, quibus numerus n per additionem produci potest ex numeris 1, 2, 3, ... $(m-1)$.

Sit M numerus modorum, quibus numerus $n-m$ per additionem produci potest ex numeris 1, 2, 3, ... m.

Sitque N numerus modorum, quibus numerus n per additionem produci potest ex numeris 1, 2, 3, ... m.

His positis, erit, ut vidimus, $L=N-M$; ideoque $N=L+M$. Quod si ergo jam invenerimus quot variis modis numeri n & $n-m$ per additionem produci queant, ille ex numeris 1, 2, 3, ... $(m-1)$ hic vero ex numeris 1, 2, 3, ... m; hinc addendo cognoscemus, quot variis modis numerus n per additionem produci queat ex numeris 1, 2, 3, ... m. Ope hujus theorematis a casibus simplicioribus, qui nihil habent difficultatis, continuo ad magis compositos progredi licebit, hocque modo tabula hic annexa est computata, * cujus usus ita se habet.

Si quæratur quot variis modis numerus 50 in 7 partes inæquales dispertiri possit; sumatur in prima columna verticali numerus $50-\frac{7\cdot 8}{2}=22$, in horizontali autem suprema numerus romanus VII; atque numerus in angulo positus 522 indicabit modorum numerum quæsitum.

Sin autem quæratur, quot variis modis numerus 50 in 7 partes, sive æquales sive inæquales, dispertiri possit, in prima columna

* *Vide Tab. pag.* 275.

columna verticali sumatur numerus $50 - 7 = 43$, cui in columna 7^{ma} respondebit numerus quæsitus 8946. CAP. XVI.

319. Series hujus tabulæ verticales, etsi sunt recurrentes, tamen ingentem habent connexionem cum numeris naturalibus, trigonalibus, pyramidalibus, & sequentibus, quam paucis exponere operæ pretium erit. Quoniam enim ex fractione $\frac{1}{(1-x)(1-xx)}$ oritur Series $1+x+2x^2+2x^3+3x^4+3x^5+$ &c., ac proinde ex fractione $\frac{x}{(1-x)(1-xx)}$ hæc $x+x^2+2x^3+2x^4+3x^5+3x^6+$ &c.. Si duæ hæ Series addantur, nascitur ista

$$1+2x+3x^2+4x^3\ 5x^4+6x^5+7x^6+\text{ \&c.},$$

quæ per divisionem oritur ex fractione $\frac{1+x}{(1-x)(1-xx)} = \frac{1}{(1-x)^2}$; unde patet Seriei postremæ terminos numericos Seriem numerorum naturalium constituere. Hinc ex Serie tabulæ secunda addendo binos terminos proveniet Series numerorum naturalium, posito $x = 1$.

$$\begin{array}{l} 1+1+2+2+3+3+4+4+5+5+6+6+\text{\&c.} \\ 1+2+3+4+5+6+7+8+9+10+11+12+\text{\&c.} \end{array}$$

Vicissim ergo ex Serie numerorum naturalium superior invenitur, subtrahendo quemque terminum Seriei superioris a termino inferioris sequente.

320. Series verticalis tertia oritur ex fractione $\frac{1}{(1-x)(1-xx)(1-x^3)}$. Cum autem sit $\frac{1}{(1-x)^3} = \frac{(1+x)(1+x+xx)}{(1-x)(1-xx)(1-x^3)}$, manifestum est, si primo Seriei illius terni termini addantur, tum bini hujus novæ Seriei denuo addantur, prodire debere numeros trigonales, id quod ex schemate sequente apparebit

 1 + 1 + 2 + 3 + 4 + 5 + 7 + 8 + 10 + 12 + 14 + 16 + 19 &c.
——— 1 + 2 + 4 + 6 + 9 + 12 + 16 + 20 + 25 + 30 + 36 + 42 + 49 &c.
1 + 3 + 6 + 10 + 15 + 21 + 28 + 36 + 45 + 55 + 66 + 78 + 91 &c.

Viciſſim autem apparet quomodo ex Serie trigonalium erui debeat Series ſuperior.

321. Simili modo, quia Series quarta oritur ex fractione

$$\frac{1}{(1-x)(1-xx)(1-x^3)(1-x^4)}, \text{ erit } \frac{(1+x)(1+x+xx)(1+x+xx+x^3)}{(1-x)(1-xx)(1-x^3)(1-x^4)} = \frac{1}{(1-x)^4}.$$

Si in Serie quarta primum quaterni termini addantur, tum in Serie reſultante terni, denique in hac bini, prodibit Series numerorum pyramidalium uti ex ſequenti calculo videre licet.

1 + 1 + 2 + 3 + 5 + 6 + 9 + 11 + 15 + 18 + 23 + 27 + &c.
1 + 2 + 4 + 7 + 11 + 16 + 23 + 31 + 41 + 53 + 67 + 83 + &c.
1 + 3 + 7 + 13 + 22 + 34 + 50 + 70 + 95 + 125 + 161 + 203 + &c.
1 + 4 + 10 + 20 + 35 + 56 + 84 + 120 + 165 + 220 + 286 + 364 + &c.

Simili autem modo Series quinta deducet ad numeros pyramidales ſecundi ordinis, ſexta ad tertii ordinis, & ita porro.

322. Viciſſim igitur ex numeris figuratis illæ ipſæ Series, quæ in tabulis occurrunt, formari poterunt, per operationes, quæ ex inſpectione calculi ſequentis ſponte elucebunt.

1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + &c.
1 + 1 + 2 + 2 + 3 + 3 + 4 + 4 + 5 + 5 + &c. II

1 + 3 + 6 + 10 + 15 + 21 + 28 + 36 + 45 + 55 + &c.
1 + 2 + 4 + 6 + 9 + 12 + 16 + 20 + 25 + 30 + &c.
1 + 1 + 2 + 3 + 4 + 5 + 7 + 8 + 10 + 12 + &c. III

1 + 4 + 10 + 20 + 35 + 56 + 84 + 120 + 165 + 220 + &c.
1 + 3 + 7 + 13 + 22 + 34 + 50 + 70 + 95 + 125 + &c.
1 + 2 + 4 + 7 + 11 + 16 + 23 + 31 + 41 + 53 + &c.
1 + 1 + 2 + 3 + 5 + 6 + 9 + 11 + 15 + 18 + &c. IV

1 + 5 + 15 + 35 + 70 + 126 + 210 + 330 + 495 + 715 + &c.
1 + 4 + 11 + 24 + 46 + 80 + 130 + 200 + 295 + 420 + &c.
1 + 3 + 7 + 14 + 25 + 41 + 64 + 95 + 136 + 189 + &c.
1 + 2 + 4 + 7 + 12 + 18 + 27 + 38 + 53 + 71 + &c.
1 + 1 + 2 + 3 + 5 + 7 + 10 + 13 + 18 + 23 + &c. V
&c.

In his ordinibus primæ Series ſunt numeri figurati, unde ſubtrahendo quemvis terminum Seriei ſecundæ a termino primæ ſequente formatur Series ſecunda. Tum Seriei tertiæ bini termini conjunctim ſubtrahantur a termino ſequente Seriei ſecundæ, ſicque oritur Series tertia; hocque modo ſubtrahendo ulterius ſummam trium, quatuor, & ita porro terminorum a termino ſuperioris Seriei ſequente, formabuntur reliquæ Series donec perveniatur ad Seriem, quæ incipit ab 1 + 1 + 2 &c., hæcque erit Series in tabula exhibita.

323. Series verticales tabulæ omnes ſimiliter incipiunt, continuoque plures habent terminos communes; ex quo intelligitur in infinitum has Series inter ſe fore congruentes. Prodibit autem Series, quæ oritur ex hac fractione

$$\frac{1}{(1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)(1-x^6)(1-x^7)\&c.},$$

quæ cum ſit recurrens, primum denominator ſpectari debet, ut

 hinc

LIB. I. hinc ſcala relationis habeatur. Quod ſi autem Factores denominatoris continuo in ſe multiplicentur, prodibit

$$1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26} - x^{35} - x^{40} + x^{51} + \&c.,$$

quæ Series ſi attentius conſideretur, aliæ Poteſtates ipſius x adeſſe non deprehenduntur, niſi quarum Exponentes contineantur in hac formula $\frac{3nn \pm n}{2}$; atque, ſi n ſit numerus impar, Poteſtates erunt negativæ; affirmativæ autem ſi n fuerit numerus par.

324. Cum igitur ſcala relationis ſit

$$+1, +1, 0, 0, -1, 0, -1, 0, 0, 0, 0, +1, 0, 0, +1, 0, 0, \&c.,$$

Series recurrens ex evolutione fractionis

$$\frac{1}{(1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)(1-x^6((1-x^7)\&c.},$$

oriunda erit hæc

$$1 + x + 2x^2 + 3x^3 + 5x^4 + 7x^5 + 11x^6 + 15x^7 + 22x^8 + 30x^9 + 42x^{10} + 56x^{11} + 77x^{12} + 101x^{13} + 135x^{14} + 176x^{15} + 231x^{16} + 297x^{17} + 385x^{18} + 490x^{19} + 627x^{20} + 792x^{21} + 1002x^{22} + 1250x^{23} + 1570x^{24} \&c..$$

In hac ergo Serie coëfficiens quiſque indicat, quot variis modis Exponens ipſius x per additionem ex numeris integris oriri queat. Sic numerus 7 quindecim modis per additionem oriri poteſt.

$7=7$	$7=4+2+1$	$7=3+1+1+1+1$
$7=6+1$	$7=4+1+1+1$	$7=2+2+2+1$
$7=5+2$	$7=3+3+1$	$7=2+2+1+1+1$
$7=5+1+1$	$7=3+2+2$	$7=2+1+1+1+1+1$
$7=4+3$	$7=3+2+1+1$	$7=1+1+1+1+1+1+1$

325. Quod

325. Quod si autem hoc productum

$$(1+x)(1+x^2)(1+x^3)(1+x^4)(1+x^5)(1+x^6)\ \&c.,$$

evolvatur, sequens prodibit Series

$$1+x+x^2+2x^3+2x^4+3x^5+4x^6+5x^7+6x^8+8x^9+10x^{10}+\ \&c.,$$

in qua quisque coëfficiens indicat, quot variis modis Exponens ipsius x per additionem numerorum inæqualium oriri possit, Sic numerus 9 octo variis modis per additionem ex numeris inæqualibus formari potest.

$9 = 9$	$9 = 6 + 2 + 1$
$9 = 8 + 1$	$9 = 5 + 4$
$9 = 7 + 2$	$9 = 5 + 3 + 1$
$9 = 6 + 3$	$9 = 4 + 3 + 2$

326. Ut comparationem inter has formas instituamus, sit

$$P = (1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)(1-x^6)\ \&c.,$$

&

$$Q = (1+x)(1+x^2)(1+x^3)(1+x^4)(1+x^5)(1+x^6)\ \&c.,$$

erit

$$PQ = (1-x^2)(1-x^4)(1-x^6)(1-x^8)(1-x^{10})(1-x^{12})\ \&c.,$$

qui Factores cum omnes in P contineantur, dividatur P per PQ, erit $\frac{1}{Q} = (1-x)(1-x^3)(1-x^5)(1-x^7)(1-x^9)\ \&c.,$

ideoque

$$Q = \frac{1}{(1-x)(1-x^3)(1-x^5)(1-x^7)(1-x^9)\ \&c.},$$

quæ fractio si evolvatur, prodibit Series, in qua quisque coëfficiens indicabit, quot variis modis Exponens ipsius x, per additionem ex numeris imparibus produci possit. Cum igitur hæc expressio æqualis sit illi, quam in §. præcedente contemplati sumus, sequitur hinc istud theorema.

Quot

Quot modis datus numerus per additionem formari potest ex omnibus numeris integris inter se inaequalibus; totidem modis idem numerus formari poterit per additionem ex numeris tantum imparibus, sive aequalibus sive inaequalibus.

327. Cum igitur, ut ante vidimus, sit

$$P = 1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26} - x^{35} - x^{40} + \&c.,$$ erit, scribendo xx loco x,

$$PQ = 1 - x^2 - x^4 + x^{10} + x^{14} - x^{24} - x^{30} + x^{44} + x^{52} - \&c.,$$

Quocirca erit hanc per illam dividendo

$$Q = \frac{1 - x^2 - x^4 + x^{10} + x^{14} - x^{24} - x^{30} + \&c.}{1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26} - \&c.}.$$

Erit ergo Series Q pariter recurrens, atque ex Serie $\frac{1}{P}$ oritur, hanc per $1 - x^2 - x^4 + x^{10} + x^{14} - x^{24}$ &c., multiplicando. Nempe, cum sit ex (324), $\frac{1}{P} = 1 + x + 2x^2 + 3x^3 + 5x^4 + 7x^5 + 11x^6 + 15x^7 + 22x^8 + 30x^9 + \&c.,$ si is multiplicetur per

$$1 - x^2 - x^4 + x^{10} + x^{14} - \&c.,$$

fiet

$$\begin{aligned} &1 + x + 2x^2 + 3x^3 + 5x^4 + 7x^5 + 11x^6 + 15x^7 + 22x^8 + 30x^9 + \&c.\\ &\quad - x^2 - x^3 - 2x^4 - 3x^5 - 5x^6 - 7x^7 - 11x^8 - 15x^9 - \&c.\\ &\qquad\qquad - x^4 - x^5 - 2x^6 - 3x^7 - 5x^8 - 7x^9 - \&c. \end{aligned}$$

aut

$$1 + x + x^2 + 2x^3 + 2x^4 + 3x^5 + 4x^6 + 5x^7 + 6x^8 + 8x^9 + \&c.$$

$= Q$ Hinc ergo, si formatio numerorum per additionem numerorum, sive aequalium sive inaequalium constet, deducetur formatio numerorum per additionem numerorum inaequalium, hincque porro formatio numerorum per additionem numerorum imparium tantum.

328. Restant in hoc genere casus quidam memorabiles, quorum evolutio non omni utilitate carebit in numerorum natura cognoscenda. Consideretur nempe haec expressio

$(1 + x)$

$(1+x)(1+x^2)(1+x^4)(1+x^8)(1+x^{16})(1+x^{32})$ &c., CAP. XVI.

in qua Exponentes ipsius x in ratione dupla progrediuntur. Hæc expressio si evolvatur, reperietur quidem hæc Series

$$1+x+x^2+x^3+x^4+x^5+x^6+x^7+x^8+\&c.;$$

quoniam vero dubium esse potest, utrum hæc Series in infinitum hac lege geometrica progrediatur, hanc ipsam Seriem investigemus. Sit igitur

$$P=(1+x)(1+x^2)(1+x^4)(1+x^8)(1+x^{16})\ \&c.;$$

ac ponatur Series per evolutionem oriunda

$$P=1+\alpha x+\beta x^2+\gamma x^3+\delta x^4+\epsilon x^5+\zeta x^6+\eta x^7+\theta x^8+\&c.,$$

Patet autem si loco x scribatur $x\,x$, tum prodire productum $(1+xx)(1+x^4)(1+x^8)(1+x^{16})(1+x^{32})\&c. = \frac{P}{1+x}$: facta ergo in Serie eadem substitutione erit

$$\frac{P}{1+x}=1+\alpha x^2+\beta x^4+\gamma x^6+\delta x^8+\epsilon x^{10}+\zeta x^{12}+\&c.,$$

multiplicetur ergo per $1+x$, eritque

$$P=1+x+\alpha x^2+\alpha x^3+\beta x^4+\beta x^5+\gamma x^6+\gamma x^7+\delta x^8+\delta x^9+\&c.,$$

qui valor ipsius P si cum superiori comparetur, habebitur

$$\alpha=1;\ \beta=\alpha;\ \gamma=\alpha;\ \delta=\beta;\ \epsilon=\beta;\ \zeta=\gamma;\ \eta=\gamma;\ \&c.,$$

erunt ergo omnes coëfficientes $=1$, ideoque productum propositum P evolutum dabit Seriem geometricam

$$1+x+x^2+x^3+x^4+x^5+x^6+x^7+\&c.,$$

329. Cum igitur hic omnes ipsius x Potestates, singulæque semel occurrant, ex forma producti $(1+x)(1+x^2)(1+x^4)$ &c., sequitur, omnem numerum integrum ex terminis progressionis

LIB. I. geometricæ duplæ 1, 2, 4, 8, 16, 32, &c., diversis per additionem formari posse, hocque unico modo. Nota est hæc proprietas in praxi ponderandi, si enim habeantur pondera 1, 2, 4, 8, 16, 32, &c., librarum; his solis ponderibus omnia onera ponderari poterunt, nisi partes libræ requirant. Sic his decem ponderibus, nempe 1 ℔, 2 ℔, 4 ℔, 8 ℔, 16 ℔, 32 ℔, 64 ℔, 128 ℔, 256 ℔, 512 ℔, omnia pondera usque ad 1024 ℔, librari possunt, & si unum pondus 1024 ℔, addatur omnibus oneribus usque ad 2048 ℔, ponderandis sufficient.

330. Ostendi autem insuper solet in praxi ponderandi paucioribus ponderibus, quæ scilicet in ratione geometrica tripla progrediantur, nempe 1, 3, 9, 27, 81, &c, librarum pariter omnia onera ponderari posse, nisi opus sit fractionibus. In hac autem praxi pondera non solum uni lanci, sed ambabus, uti necessitas exigit, imponi debent. Nititur ergo ista praxis hoc fundamento, quod ex terminis progressionis geometricæ triplæ 1, 3, 9, 27, 81, &c., diversis semper sumendis per additionem ac subtractionem omnes omnino numeri produci queant; erit scilicet.

$1 = 1$	$5 = 9 - 3 - 1$	$9 = 9$
$2 = 3 - 1$	$6 = 9 - 3$	$10 = 9 + 1$
$3 = 3$	$7 = 9 - 3 + 1$	$11 = 9 + 3 - 1$
$4 = 3 + 1$	$8 = 9 - 1$	$12 = 9 + 3$

&c.

331. Ad hanc veritatem ostendendam considero hoc productum infinitum

$$(x^{-1}+1+x^{1})(x^{-3}+1+x^{3})(x^{-9}+1+x^{9})(x^{-27}+1+x^{27})\,\&c. = P,$$

quod evolutum alias non dabit Potestates ipsius x, nisi quarum Exponentes formari possint ex numeris 1, 3, 9, 27, 81, &c., sive

TABULA

ad paginam 275 Tom. I.

	I	II	III	IV	V	VI	VII	VIII	IX	X	XI
1	1	1	1	1	1	1	1	1	1	1	1
2	1	2	2	2	2	2	2	2	2	2	2
3	1	2	3	3	3	3	3	3	3	3	3
4	1	3	4	5	5	5	5	5	5	5	5
5	1	3	5	6	7	7	7	7	7	7	7
6	1	4	7	9	10	11	11	11	11	11	11
7	1	4	8	11	13	14	15	15	15	15	15
8	1	5	10	15	18	20	21	22	22	22	22
9	1	5	12	18	23	26	28	29	30	30	30
10	1	6	14	23	30	35	38	40	41	42	42
11	1	6	16	27	37	44	49	52	54	55	56
12	1	7	19	34	47	58	65	70	73	75	76
13	1	7	21	39	57	71	82	89	94	97	99
14	1	8	24	47	70	90	105	116	123	128	131
15	1	8	27	54	84	110	131	146	157	164	169
16	1	9	30	64	101	126	164	186	201	[illegible]	[illegible]

five addendo five fubtrahendo: num vero omnes Poteftates prodeant, fingulæque femel, fic exploro. Sit CAP. XVI.

$$P = \&c. + cx^{-3} + bx^{-2} + ax^{-1} + 1 + \alpha x^1 + \beta x^2 + \gamma x^3 + \delta x^4 + \epsilon x^5 + \&c.,$$

manifeftum vero eft, fi x^3 loco x fcribatur, tum prodire

$$\frac{P}{x^{-1} + 1 + x^1} = bx^{-6} + ax^{-3} + 1 + \alpha x^3 + \beta x^6 + \gamma x^9 + \&c.:$$

Hinc igitur reperitur $P =$ &c.

$$+ ax^{-4} + ax^{-3} + ax^{-2} + x^{-1} + 1 + x + \alpha x^2 + \alpha x^3 + \alpha x^4 + \beta x^5 + \beta x^6 + \beta x^7 + \&c.,$$

quæ expreffio cum affumta comparata dabit

$\alpha = 1$; $\beta = \alpha$; $\gamma = \alpha$; $\delta = \alpha$; $\epsilon = \beta$; $\zeta = \beta$; &c., & $a = 1$, $b = a$, $c = a$, $d = a$, $e = b$, &c..

Hinc itaque erit

$$P = 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + \&c. + x^{-1} + x^{-2} + x^{-3} + x^{-4} + x^{-5} + x^{-6} + x^{-7} + \&c.,$$

unde patet omnes ipfius x Poteftates, tam affirmativas quam negativas, hic occurrere, atque adeo omnes numeros ex terminis progreffionis geometricæ triplæ, vel addendo vel fubtrahendo, formari poffe; & unumquemque numerum unico tantum modo.

LIB. I.

CAPUT XVII.

De usu Serierum recurrentium in radicibus æquationum indagandis.

332. INdicavit *Vir Celeb. Daniel* BERNOULLI insignem usum Serierum recurrentium in investigandis radicibus æquationum cujusvis gradus, in *Comment. Acad. Petropol. Tomo III.*, ubi ostendit, quemadmodum cujusque æquationis algebraicæ, quotcunque fuerit dimensionum, valores radicum veris proximi ope Serierum recurrentium assignari queant. Quæ inventio, cum sæpenumero maximam afferat utilitatem, eam hic diligentius explicare constitui, ut intelligatur, quibus casibus adhiberi possit. Interdum enim præter expectationem evenit, ut nulla æquationis radix ope hujus methodi cognosci queat. Quocirca, ut vis hujus methodi clarius perspiciatur, ex proprietatibus Serierum recurrentium totum fundamentum, quo nititur, contemplemur.

333. Quoniam omnis Series recurrens ex evolutione cujusdam fractionis rationalis oritur, sit ista fractio

$$= \frac{a + bz + cz^2 + dz^3 + ez^4 + \&c.}{1 - \alpha z - \beta z^2 - \gamma z^3 - \delta z^4 - \&c.},$$

unde oriatur sequens Series recurrens

$$A + Bz + Cz^2 + Dz^3 + Ez^4 + Fz^5 + \&c.,$$

cujus coëfficientes A, B, C, D, &c., ita determinantur ut sit

$A = a$

$$A = \alpha$$
$$B = \alpha A + \beta$$
$$C = \alpha B + \beta A + \gamma$$
$$D = \alpha C + \beta B + \gamma A + \delta$$
$$E = \alpha D + \beta C + \gamma B + \delta A + \epsilon$$
&c.

Terminus autem generalis, ſeu coëfficiens Poteſtatis z^n, invenitur ex reſolutione fractionis propoſitæ in fractiones ſimplices, quarum denominatores ſint Factores denominatoris $1 - \alpha z - \beta zz - \gamma z^3 -$ &c., uti (*Cap. XIII.*) eſt oſtenſum.

334. Forma autem termini generalis potiſſimum pendet ab indole Factorum ſimplicium denominatoris, utrum ſint reales an imaginarii, & utrum ſint inter ſe inæquales & eorum bini pluresve æquales. Quos varios caſus ut ordine percurramus, ponamus primum omnes denominatoris Factores ſimplices cum reales eſſe tum inter ſe inæquales. Sint ergo Factores ſimplices denominatoris omnes $(1-pz)(1-qz)(1-rz)(1-sz)$ &c., ex quibus fractio propoſita in ſequentes fractiones ſimplices reſolvatur $\frac{A}{1-pz} + \frac{B}{1-qz} + \frac{C}{1-rz} + \frac{D}{1-sz} +$ &c.. Quibus cognitis erit Seriei recurrentis terminus generalis $=$ $z^n(Ap^n + Bq^n + Cr^n + Ds^n +$ &c.$)$, quem ſtatuamus $= Pz^n$; ſit ſcilicet P coëfficiens Poteſtatis z^n, ſequentiumque Q, R, &c., ita ut Series recurrens fiat
$$A + Bz + Cz^2 + Dz^3 + \ldots\ldots\ldots + Pz^n + Qz^{n+1} + Rz^{n+2} +$$
&c.

335. Ponamus jam n eſſe numerum maximum, ſeu Seriem recurrentem ad plurimos terminos eſſe continuatam; quoniam numerorum inæqualium Poteſtates eo magis fiunt inæquales, quo fuerint altiores; tanta erit diverſitas in Poteſtatibus

LIB. I. Ap^n, Bq^n, Cr^n, &c., ut ea, quæ oritur ex maximo numerorum p, q, r, &c., reliquas magnitudine longe ſuperet, præ eaque reliquæ penitus evaneſcant, ſi n fuerit numerus plane infinite magnus. Cum igitur numeri p, q, r, &c., ſint inter ſe inæquales, ponamus inter eos p eſſe maximum; ac propterea, ſi n ſit numerus infinitus, fiet $P = Ap^n$; ſin autem n ſit numerus vehementer magnus erit tantum proxime $P = Ap^n$. Simili vero modo erit $Q = Ap^{n+1}$, ideoque $\frac{Q}{P} = p$. Unde patet, ſi Series recurrens jam longe fuerit producta, coëfficientem cujuſque termini per præcedentem diviſum proxime eſſe exhibiturum valorem maximæ litteræ p.

336. Si igitur in fractione propoſita

$$\frac{a + bz + cz^2 + dz^3 + \&c.}{1 - \alpha z - \beta z^2 - \gamma z^3 - \delta z^4 - \&c.}$$

denominator habeat omnes Factores ſimplices reales & inter ſe inæquales, ex Serie recurrente inde orta cognoſci poterit unus Factor ſimplex, is ſcilicet $1 - pz$, in quo littera p omnium maximum habet valorem. Neque in hoc negotio coëfficientes numeratoris a, b, c, d. &c., in computum ingrediuntur, ſed quicunque ii ſtatuantur, tamen denique idem verus valor litteræ maximæ p invenitur. Verus quidem valor ipſius p tum demum innoteſcit, quando Series in infinitum fuerit continuata; interim tamen ſi jam plures ejus termini fuerint formati, eo propius valor ipſius p cognoſcetur, quo major fuerit terminorum numerus, & quo magis littera iſta p excedat reliquas q, r, s, &c.: perinde vero eſt utrum hæc maxima littera p fuerit ſigno + an ſigno — affecta, quoniam ejus Poteſtates æque increſcunt.

337. Quemadmodum nunc hæc inveſtigatio ad inventionem radicum æquationis cujuſvis algebraicæ accommodari poſſit,

fit, ſatis eſt perſpicuum. Ex Factoribus enim denominatoris $1 - \alpha z - \beta zz - \gamma z^3 - \delta z^4 -$ &c., cognitis facile aſſignantur radices æquationis hujus CAP. XVII.

$$1 - \alpha z - \beta z^2 - \gamma z^3 - \delta z^4 - \&c. = 0,$$

ita ut, ſi Factor fuerit $1 - pz$, hujus æquationis radix una futura ſit $z = \frac{1}{p}$. Cum igitur ex Serie recurrente reperiatur maximus numerus p, indidem obtinebitur minima radix æquationis $1 - \alpha z - \beta z^2 - \gamma z^3 -$ &c. $= 0$. Vel, ſi ponatur $z = \frac{1}{x}$ ut prodeat hæc æquatio

$$x^m - \alpha x^{m-1} - \beta x^{m-2} - \gamma x^{m-3} - \&c. = 0,$$

ejuſdem methodi ope eruitur maxima hujus æquationis radix $x = p$.

338. Si igitur proponatur æquatio hæc

$$x^m - \alpha x^{m-1} - \beta x^{m-2} - \gamma x^{m-3} - \&c. = 0,$$

quæ omnes radices habeat reales & inter ſe inæquales, harum radicum maxima ſequenti modo reperietur. Formetur ex coëfficientibus hujus æquationis fractio

$$\frac{a + bz + cz^2 + dz^3 + \&c.}{1 - \alpha z - \beta z^2 - \gamma z^3 - \delta z^4 - \&c.}$$

Hincque formetur Series recurrens, aſſumendo pro arbitrio numeratorem, ſeu, quod eòdem redit, aſſumendo pro libitu terminos initiales; quæ ſit

$$A + Bz + Cz^2 + Dz^3 + \ldots\ldots\ldots + Pz^n + Qz^{n+1}$$

dabitque fractio $\frac{Q}{P}$ valorem radicis maximæ x pro æquatione propoſita, eo propius, quo major fuerit numerus n.

EXEM-

EXEMPLUM I.

Sit proposita ista æquatio $xx - 3x - 1 = 0$, *cujus maximam radicem inveniri oporteat.*

Formetur fractio $\frac{a + bz}{1 - 3z - zz}$, unde positis duobus primis terminis 1, 2, orietur ista Series recurrens

1, 2, 7, 23, 76, 251, 829, 2738, &c.,

erit ergo $\frac{2738}{829}$ proxime æqualis radici æquationis propositæ maximæ. Valor autem hujus fractionis in partibus decimalibus expressus est

3, 3027744

æquationis vero radix maxima est $= \frac{3 + \sqrt{13}}{2} =$

3, 3027756,

quæ inventam superat tantum una parte millionesima. Ceterum notandum est fractiones $\frac{Q}{P}$ alternatim vera radice esse majores & minores.

EXEMPLUM II.

Proposita sit ista æquatio $3x - 4x^3 = \frac{1}{2}$ *cujus radices exhibent Sinus trium Arcuum, quorum triplorum Sinus est* $= \frac{1}{2}$.

Æquatione perducta ad hanc formam $0 = 1 - 6x * + 8x^3$, quæratur hujus, ut in numeris integris maneamus, radix minima, ita ut non opus sit pro x ponere $\frac{1}{z}$. Formetur ergo hæc fractio

$$\frac{a + bx + cxx}{1 - 6x * + 8x^3}$$

ex qua sumendis pro lubitu tribus terminis initialibus 0, 0, 1, quia

quia hoc modo calculus facillime expeditur, orietur hæc Series recurrens, omittendis potestatibus ipsius x quia tantum coëfficientibus opus est, CAP. XVII.

0; 0; 1; 6; 36; 208; 1200; 6912; 39808; 229248.

Erit ergo proxime æquationis radix minima $= \frac{39808}{229248} = \frac{311}{1791} = 0,1736515$, quæ propterea esse deberet Sinus anguli $10°$; hic autem ex tabulis est 0, 1736482, quem superat radix inventa parte $\frac{33}{10000000}$. Facilius autem hæc eadem radix inveniri potest ponendo $x = \frac{1}{2}y$, ut prodeat æquatio $1 - 3y * + y^3 = 0$, ex qua simili modo tractata oritur Series

0, 0, 1, 3, 9, 26, 75, 216, 622, 1791, 5157 &c.,

erit ergo proxime æquationis radix minima $y = \frac{1791}{5157} = \frac{199}{573} = 0,3472949$, unde fit $x = \frac{1}{2}y = 0,1736479$, qui valor decies propius accedit quam præcedens.

EXEMPLUM III.

Si desideretur ejusdem æquationis propositæ $0 = 1 - 6x * + 8x^3$, *radix maxima.*

Ponatur $x = \frac{y}{2}$, eritque $y^3 * - 3y + 1 = 0$. Cujus æquationis radix maxima reperietur per Seriem recurrentem cujus scala relationis est 0, 3, -1, unde ergo oritur, sumtis tribus terminis initialibus pro arbitrio,

1, 1, 1, 2, 2, 5, 4, 13, 7, 35, 8, 98, -11, &c.,

in qua Serie cum ad terminos negativos perveniatur, id indicio est maximam radicem esse negativam, est enim $x = -$

 sin.

LIB. I. $fin. 70° = -0,9396926$. Quare hujus ratio in terminis initialibus est habenda, hoc modo

$$1-2+4-7+14-25+49-89+172-316+605-\&c.,$$

ex qua erit $y = \frac{-605}{316}$ & $x = \frac{-605}{632} = -0,957$, quæ a veritate vehementer abludit.

339. Ratio hujus dissensus potissimum est, quod æquationis propositæ radices sint $fin. 10°$, $fin. 50°$, & $-fin. 70°$, quarum binæ maximæ tam parum a se invicem discrepant, ut in Potestatibus, ad quas Seriem continuavimus, secunda radix $fin. 50°$ adhuc notabilem teneat rationem ad radicem maximam, ideoque præ ea non evanescant. Hincque etiam saltus pendet, quod alternatim valores inventi fiant nimis magni & nimis parvi: Sic, sumendo

$$y = \frac{-316}{172}, \text{ fit } x = \frac{-158}{172} = \frac{-79}{86} = -0,918.$$

Nam, quoniam Potestates radicis maximæ alternatim fiunt affirmativæ & negativæ, alternatim quoque Potestates secundæ radicis adduntur & tolluntur: quamobrem, quo hæc discrepantia fiat insensibilis, Series vehementer ulterius debet continuari.

340. Aliud vero remedium huic incommodo afferri potest, transmutando æquationem ope idoneæ substitutionis in aliam formam, cujus radices sibi non amplius sint tam vicinæ. Sic, si in æquatione $0 = 1 - 6x + 8x^3$ cujus radices sunt $-fin. 70°$, $+fin. 50°$, $+fin. 10°$, ponatur $x = y - 1$, æquationis $0 = 8y^3 - 24yy + 18y - 1$ radices erunt $1 - fin. 70°$; $1 + fin. 50°$; $1 + fin. 10°$; ideoque ejus radix minima erit $1 - fin. 70°$, cum tamen hæc $fin. 70°$ esset radix maxima æquationis præcedentis; atque $1 + fin. 50°$ nunc est radix maxima, cum $fin. 50°$ ante esset media. Atque hoc modo quævis radix per substitutionem in maximam minimamve radicem novæ æquationis transmutari, ideoque per methodum hic traditam inveniri poterit.

poterit. Quia præterea in hoc exemplo radix $1 - fin. 70°$ multo minor est, quam binæ reliquæ, etiam facile per Seriem recurrentem proxime cognoscetur. CAP. XVII.

EXEMPLUM IV.

Invenire radicem minimam æquationis $0 = 8y^3 - 24yy + 18y - 1$, *quæ ab unitate subtracta relinquet Sinum anguli* $70°$.

Ponatur $y = \frac{1}{2}z$, ut sit $0 = z^3 - 6zz + 9z - 1$, cujus radix minima invenietur per Seriem recurrentem, cujus scala relationis est $9, -6, +1$, pro radice autem maxima invenienda scala relationis sumi deberet $6, -9, +1$. Pro minima ergo formetur hæc Series

$1, 1, 1, 4, 31, 256, 2122; 17593; 145861;$ &c.,

erit ergo proxime $z = \frac{17593}{145861} = 0,12061483$ & $y = 0,06030741$, atque $fin. 70° = 1 - y = 0,93969258$, quæ a veritate ne in ultima quidem figura discrepat. Ex hoc ergo exemplo intelligitur quantam utilitatem idonea transformatio æquationis ope substitutionis ad inventionem radicum afferat, & quod hoc pacto methodus tradita non solum ad maximas minimasve radices adstringatur, sed etiam omnes radices exhibere queat.

341. Cognita ergo jam quacunque æquationis propositæ radice proxime, ita ut, verbi gratia, numerus k quam minime a quapiam radice differat, ponatur $x - k = y$ seu $x = y + k$, hocque modo prodibit æquatio, cujus radix minima erit $= x - k$, quæ igitur si per Series recurrentes indagetur, quod facillime fiet, quia hæc radix multo minor erit, quam ceteræ, si ea ad k addatur habebitur radix vera ipsius x, pro æquatione proposita. Hoc vero artificium tam late patet, ut etiamsi æquatio contineat radices imaginarias, usum suum retineat.

342. Imprimis autem sine hoc artificio radix cognosci nequit,

LIB. I. quit, cui datur alia æqualis sed signo contrario affecta. Scilicet, si æquatio cujus maxima radix p, eadem radicem habeat $-p$, tum, etiamsi Series recurrens in infinitum continuetur, tamen radix hæc p nunquam obtinebitur. Sit, ut hoc exemplo illustremus, proposita æquatio $x^3 - x^2 - 5x + 5 = 0$, cujus maxima radix est $\sqrt{5}$, præter quam vero inest quoque $-\sqrt{5}$. Si igitur modo ante præscripto, pro radice maxima invenienda, utamur, atque Seriem recurrentem formemus ex scala relationis $1, +5, -5$, quæ erit

$$1, 2, 3, 8, 13, 38, 63, 188, 313, 938, 1563, \text{ \&c.},$$

ubi ad nullam rationem constantem pervenitur. Termini vero alterni rationem æquabilem induunt, quorum si quisque per præcedentem dividatur, reperietur quadratum maximæ radicis, sic enim est proxime $5 = \frac{1563}{313} = \frac{938}{188} = \frac{313}{63}$. Quoties ergo termini tantum alterni sese ad rationem constantem componunt, toties quadratum radicis quæsitæ proxime obtinetur. Ipsa autem radix $x = \sqrt{5}$ invenitur ponendo $x = y + 2$ unde fit $1 - 3y - 5yy - y^3 = 0$, cujus radix minima cognoscetur ex Serie

$$1, 1, 1, 9, 33, 145, 609, 2585, 10945, \text{ \&c.},$$

erit enim proxime $= \frac{2585}{10945} = 0,2361$, at $2,2361$ est proxime $= \sqrt{5}$, quæ est radix maxima æquationis.

343. Quanquam numerator fractionis, ex qua Series recurrens formatur, a nostro arbitrio pendet, tamen idonea ejus constitutio plurimum confert, ut valor radicis cito vero proxime exhibeatur. Cum enim assumtis, ut supra, Factoribus denominatoris (334.), sit terminus generalis Seriei recurrentis $=$

$z^n (Ap^n + Bq^n + Cr^n + \text{ \&c. })$, isti coëfficientes A, B, C, &c., per numeratorem fractionis determinantur; unde fieri potest, ut A sive magnum sive parvum valorem obtineat: priori casu radix maxima p cito reperitur, posteriore vero tarde. Quin etiam numerator ita accipi potest ut A prorsus evanescat,

quo,

quo casu, etiamsi Series in infinitum continuetur, tamen nunquam radicem maximam p præbebit. Hoc autem evenit si numerator ita accipiatur, ut ipse eundem habeat Factorem $1 - pz$, sic enim ex computo penitus tolletur. Sic, si proponatur æquatio $x^3 - 6xx + 10x - 3 = 0$, cujus maxima radix est $= 3$, indeque formetur fractio

$$\frac{1 - 3z}{1 - 6z + 10z^2 - 3z^3}$$

ut Seriei recurrentis sit scala relationis $6, -10, +3$

$$1 + 3, 8, 21, 55, 144, 377, \&c.,$$

cujus termini prorsus non convergunt ad rationem, $1 : 3$. Eadem enim Series oritur ex fractione $\frac{1}{1 - 3z + zz}$, ac propterea maximam radicem æquationis $x^2 - 3x + 1 = 0$ exhibet.

344. Quin etiam numerator ita assumi potest, ut per Seriem recurrentem quævis radix æquationis reperiatur, quod fiet si numerator fuerit productum ex omnibus Factoribus denominatoris præter eum, cui respondet radix quam velimus. Sic, si in priori exemplo sumatur numerator $1 - 3z + zz$, fractio $\frac{1 - 3z + zz}{1 - 6z + 10z^2 - 3z^3}$, dabit hanc Seriem recurrentem $1, 3, 9, 27, 81, 243$, &c., quæ, cum sit geometrica, statim monstrat radicem $x = 3$. Fractio enim illa æqualis est huic simplici $\frac{1}{1 - 3z}$. Hinc apparet, si termini initiales, quos pro lubitu assumere licet, ita accipiantur, ut progressionem geometricam constituant, cujus Exponens æquetur uni radici æquationis, tum totam Seriem recurrentem fore geometricam, ideoque eam ipsam radicem esse exhibituram, etiamsi neque sit maxima neque minima.

345. Ne igitur, dum quærimus radicem vel maximam vel minimam, præter expectationem nobis alia radix per Seriem recurrentem exhibeatur, ejusmodi numerator debet eligi, qui

LIB. I. cum denominatore nullum Factorem habeat communem, quod fiet si pro numeratore unitas accipiatur, unde terminus primus Seriei erit $= 1$, ex quo solo secundum scalam relationis sequentes omnes definiantur. Hocque modo semper certe radix æquationis vel maxima vel minima, prout fuerit propositum, eruetur. Sic, proposita æquatione $y^3 * - 3y + 1 = 0$, cujus radix maxima desideratur, ex scala relationis $0, +3, -1$ incipiendo ab unitate sequens oritur Series recurrens

$$1 - 0 + 3 - 1 + 9 - 6 + 28 - 27 + 90 - 109 + 297 - 517 + 1000 - 1848 + 3517 - 6544 + \&c.,$$

quæ manifesto ad rationem constantem convergit, ostenditque radicem maximam esse negativam, atque proxime $y = \frac{-6544}{3517} = -1,860676$, quæ esse debebat $= -1,86793852$. Ratio autem supra est allata, cur tam lente ad verum valorem appropinquetur, propterea quod altera radix non multo sit minor maxima, simulque sit affirmativa.

346. His probe perpensis, quæ cum in genere tum ad exempla allata monuimus, summa utilitas hujus methodi ad investigandas æquationum radices luculenter perspicietur; artificia vero, quibus operatio contrahi, eoque promtior reddi queat, satis quoque sunt indicata; ita ut nihil insuper addendum esset, nisi casus, quibus æquatio vel radices habet æquales vel imaginarias, evolvendi superessent. Ponamus ergo denominatorem fractionis

$$\frac{a + bz + cz^2 + dz^3 + \&c.}{1 - \alpha z - \beta z^2 - \gamma z^3 - \delta z^4 - \&c.}$$

habere Factorem $(1 - pz)^2$, reliquis Factoribus existentibus $1 - qz$, $1 - rz$, &c.. Seriei ergo recurrentis hinc natæ terminus generalis erit $= z^n((n+1)Ap^n + Bp^n + Cq^n + \&c.)$, quæ cujusmodi valorem sit adeptura, si n fuerit numerus vehemen-

vehementer magnus, duo casus sunt distinguendi, alter quo p est numerus major reliquis q, r, &c., alter quo p non præbet radicem maximam. Casu priori, quo p simul est radix maxima, ob coëfficientem $(n+1)$ reliqui termini $Bp^n + Cq^n$ &c., non tam cito præ eo evanescent, quam ante: sin autem q fuerit $> p$, tum quoque tarde terminus $(n+1)Ap^n$ præ Bq^n evanescet, ideoque investigatio radicis maximæ admodum evadet molesta. CAP. XVII.

EXEMPLUM I.

Sit proposita æquatio $x - 3xx + 4 = 0$, *cujus maxima radix* 2 *bis occurrit.*

Quæratur ergo maxima radix hæc modo ante exposito per evolutionem fractionis

$$\frac{1}{1 - 3z * + 4z^3},$$

quæ dabit hanc Seriem recurrentem

$$1, 3, 9, 23, 57, 135, 313, 711, 1593, \&c.,$$

ubi quidem quivis terminus per præcedentem divisus dat quotum binario majorem. Cujus ratio ex termino generali facillime patet, rejectis enim in eo terminis Bp^n, Cq^n &c., erit terminus potestati z^n respondens $= (n+1)Ap^n + Bp^n$, sequens $= (n+2)Ap^{n+1} + Bp^{n+1}$, qui per illum divisus dat $\frac{(n+2)A + B}{(n+1)A + B}p > p$, nisi n jam in infinitum excreverit.

EXEM-

EXEMPLUM II.

Sit jam proposita aequatio $x^3 - xx - 5x - 3 = 0$, *cujus maxima radix* $= 3$, *reliquae duae aequales* $= -1$, & quaeratur maxima radix ope Seriei recurrentis, cujus scala relationis est $1, +5, +3$; unde oritur

$$1, 1, 6, 14, 47, 135, 412, 1228, \&c.,$$

quae ideo satis cito valorem 3 exhibet, quod Potestates minoris radicis -1, etiamsi multiplicentur per $n+1$, tamen mox prae Potestatibus ipsius 3 evanescant.

EXEMPLUM III.

Sin autem proponeretur aequatio $x^3 + xx - 8x - 12 = 0$, cujus radices sunt $3, -2, -2$, multo tardius maxima sese prodet. Orietur enim haec Series

$$1, -1, 9, -5, 65, 3, 457, 347, 3345, 4915, \&c.,$$

quae adhuc longissime continuari deberet, antequam pateret, radicem inde oriundam esse $= 3$.

347. Simili modo si tres Factores essent aequales, ita ut denominatoris Factor unus esset $(1 - pz)^3$, reliqui $1 - qz$, $1 - rz$, &c., Seriei recurrentis terminus generalis erit $=$
$$z^n\left(\frac{(n+1)(n+2)}{1 \cdot 2}Ap^n + (n+1)Bp^n + Cp^n + Dq^n + Cr^n \&c.\right)$$
Si ergo p fuerit maxima radix, atque n fuerit numerus tantus, ut Potestates q^n, r^n &c. prae p^n evanescant, tum ex Serie recurrente orietur radix $=$
$$\frac{\frac{1}{2}(n+2)(n+3)A + (n+2)B + C}{\frac{1}{2}(n+1)(n+2)A + (n+1)B + C}p,$$
quae, nisi sit n numerus maximus & quasi infinitus, verum ipsius

sius p valorem indicabit. Erit autem iste radicis valor $= p +$ CAP. XVII.

$$\frac{(n+2)A+B}{\frac{1}{2}(n+1)(n+2)A+(n+1)B+C}p.$$

Quod si autem p non fuerit radix maxima, tum inventio maximæ multo magis adhuc impedietur; unde sequitur æquationes, quæ contineant radices æquales, hac methodo per Series recurrentes multo difficilius resolvi, quam si omnes radices essent inter se inæquales.

348. Videamus nunc quomodo Series recurrens in infinitum continuata debeat esse comparata, quando denominator fractionis habet Factores imaginarios. Sint igitur fractionis

$$\frac{a+bz+cz^2+dz^3+\&c.}{1-\alpha z-\beta z^2-\gamma z^3-\delta z^4-\&c.}$$

Factores denominatoris reales $1-qz$, $1-rz$, &c., insuperque Factor trinomialis $1-2pz.\,cos.\,\Phi+ppzz$ continens duos Factores simplices imaginarios. Quod si ergo Series recurrens ex illa fractione orta fuerit

$$A+Bz+Cz^2+Dz^3+\ldots\ldots\ldots\ldots+Pz^n+Qz^{n+1},$$

erit, per ea quæ supra exposuimus, coëfficiens $P=$

$\frac{A.\,sin.\,(n+1)\Phi+B.\,sin.\,n\Phi}{sin.\,\Phi}p^n+Cq^n+Dr^n+\&c.$. Si igitur numerus p minor fuerit, quam unus ceterorum q, r, &c., ita ut maxima radix æquationis

$$x^m-\alpha x^{m-1}-\beta x^{m-2}-\gamma x^{m-3}-\&c.=0,$$

sit realis, tum ea per Series recurrentes æque reperietur, ac si nullæ radices inessent imaginariæ.

349. Inventio ergo maximæ radicis realis per radices imaginarias non perturbabitur, si hæ ita fuerint comparatæ, ut binarum, quæ Factorem realem componunt, productum non sit

LIB. I. majus quadrato radicis maximæ. Sin autem binæ ejusmodi insint radices imaginariæ, ut earum productum adæquet vel adeo superet quadratum maximæ radicis realis, tum investigatio ante exposita nihil declarabit, propterea quod Potestas p^n, præ simili Potestate radicis maximæ nunquam evanescit, etiamsi Series in infinitum continuetur. Cujus exempla illustrationis causa hic adjicere visum est.

EXEMPLUM I.

Sit proposita æquatio $x^3 - 2x - 4 = 0$, *cujus radicem maximam investigari oporteat.*

Resolvitur hæc æquatio in duos Factores $(x-2)(xx+2x+2)$; unde unam habet radicem realem 2 & duas reliquas imaginarias, quarum productum est 2, minus quam quadratum radicis realis. Quam ob rem ea per modum hactenus traditum cognosci poterit. Formetur ergo Series recurrens ex scala relationis $0, +2, +4$, quæ erit

$$1, 0, 2, 4, 4, 16, 24, 48, 112, 192, 416, 832, \&c.,$$

unde satis luculenter radix realis 2 cognosci potest.

EXEMPLUM II.

Proposita sit æquatio $x^3 - 4xx + 8x - 8 = 0$, *cujus radix una realis est* 2, *binarum imaginariarum productum vero* $= 4$, *ideoque æquale quadrato radicis realis* 2.

Quæramus ergo radicem per Seriem recurrentem, quod quo facilius fieri queat, ponamus $x = 2y$, ut habeatur $y^3 - 2yy + 2y - 1 = 0$, unde formetur Series recurrens

$$1, 2, 2, 1, 0, 0, 1, 2, 2, 1, 0, 0, 1, 2, 2, 1, \&c.,$$

in qua cum iidem termini perpetuo revertantur, nihil inde aliud

aliud colligi poteſt, niſi radicem maximam vel non eſſe realem, vel dari imaginarias, quarum productum æquale ſit aut ſuperet quadratum radicis realis. CAP. XVII.

EXEMPLUM III.

Sit jam propoſita æquatio $x^3 - 3xx + 4x - 2 = 0$, *cujus radix realis eſt* 1, *imaginariarum vero productum* = 2.

Formetur ergo ex ſcala relationis 3, − 4, + 2, Series

1, 3, 5, 5, 1, − 7, − 15, − 15, −, + 1, 33, 65, 65, 1, &c.,

in qua cum termini modo fiant affirmativi, modo negativi, radix realis 1 inde nullo modo cognoſci poterit. Hujuſmodi vero revolutiones ſemper oſtendunt radicem, quam Series præbere debebat, eſſe imaginariam; hic enim radices imaginariæ poteſtate ſunt majores quam realis 1.

350. Sit igitur in fractione generali productum binarum radicum imaginariarum pp majus quam ullius radicis realis quadratum, ita ut præ p^n reliquæ poteſtates q^n, r^n, &c., evaneſcant ſi n ſit numerus infinitus. Hoc ergo caſu fiet $P = \frac{A.\,ſin.(n+1)\phi + B.\,ſin.\,n\phi}{ſin.\,\phi}\,p^n$, & $Q = \frac{A.\,ſin.(n+2)\phi + B.\,ſin.(n+1)\phi}{ſin.\,\phi}\,p^{n+1}$ ideoque $\frac{Q}{P} = \frac{A.\,ſin.(n+2)\phi + B.\,ſin.(n+1)\phi}{A.\,ſin.(n+1)\phi + B.\,ſin.\,n\phi}\,p$. Quæ expreſſio nunquam valorem conſtantem induet, etiamſi n ſit numerus infinitus. Sinus enim Angulorum perpetuo maxime manent mutabiles, ita ut mox ſint affirmativi mox negativi.

351. Interim tamen ſi fractiones ſequentes $\frac{R}{Q}$, $\frac{S}{R}$ ſimili modo ſumantur, indeque litteræ A & B eliminentur, ſimul numerus n ex calculo egredietur; reperietur enim $Ppp + R = 2Qp.\,coſ.\,\phi$, unde fit $coſ.\,\phi = \frac{Ppp + R}{2Qp}$; ſimiliter vero erit

 $coſ.\,\phi =$

 $cos. \Phi = \frac{Qpp+S}{2Rp}$, ex quorum duorum valorum comparatione fit $p = \sqrt{\frac{RR-QS}{QQ-PR}}$, atque $cos. \Phi = \frac{QR-PS}{2\sqrt{(Q^2-PR)(R^2-QS)}}$. Quam ob rem si Series recurrens jam eo usque fuerit continuata, ut præ p^n reliquarum radicum Potestates evanescant, tum hoc modo Factor trinomialis $1 - 2pz.cos.\Phi + ppzz$ poterit inveniri.

352. Quoniam iste calculus non satis exercitatis molestiam creare posset, eum totum hic apponam. Ex valore ipsius $\frac{Q}{P}$ invento oritur $AP.p.sin.(n+2)\Phi + BPp.sin.(n+1)\Phi = AQ.sin.(n+1)\Phi + BQ.sin.n\Phi$, unde fit $\frac{A}{B} = \frac{Q.sin.n\Phi - Pp.sin.(n+1)\Phi}{Pp.sin.(n+2)\Phi - Q.sin.(n+1)\Phi}$. Pari ratione erit $\frac{A}{B} = \frac{R.sin.(n+1)\Phi - Qp.sin.(n+2)\Phi}{Qp.sin.(n+3)\Phi - R.sin.(n+2)\Phi}$; æquatis his duobus valoribus fiet

$$0 = QQp.sin.n\Phi.sin.(n+3)\Phi - QR.sin.n\Phi.sin.(n+2)\Phi - PQpp.sin.(n+1)\Phi.sin.(n+3)\Phi - QQp.sin.(n+1)\Phi.sin.(n+2)\Phi + QR.sin.(n+1)\Phi.sin.(n+1)\Phi + PQpp.sin.(n+1)\Phi.sin.(n+2)\Phi.$$

Cum autem sit $sin.a.sin.b = \frac{1}{2}.cos.(a-b) - \frac{1}{2}.cos.(a+b)$ fiet $0 = \frac{1}{2}QQp.(cos.3\Phi - cos.\Phi) + \frac{1}{2}QR(1-cos.2\Phi) + \frac{1}{2}PQpp.(1-cos.2\Phi)$ quæ per $\frac{1}{2}Q$ divisa dat $(Ppp.+R)(1-cos.2\Phi) = Qp.(cos.\Phi - cos.3\Phi)$. At est $cos.\Phi = cos.2\Phi.cos.\Phi + sin.2\Phi.sin.\Phi$ & $cos.3\Phi = cos.2\Phi.cos.\Phi - sin.2\Phi.sin.\Phi$ unde $cos.\Phi - cos.3\Phi = 2sin.2\Phi.sin.\Phi = 4sin.\Phi^2 \times cos.\Phi$ & $1 - cos.\Phi = 2sin.\Phi^2$, ex quo erit $Ppp + R = 2Qp.cos.\Phi$, & $cos.\Phi = \frac{Ppp+R}{2Qp}$, atque $cos.\Phi = \frac{Qpp+S}{2Rp}$: unde superiores

superiores valores prodeunt, scilicet $p = \sqrt{\frac{RR - QS}{QQ - PR}}$ & $cos.\phi = \frac{QR - PS}{2\sqrt{(Q^2 - PR)(RR - QS)}}$.

353. Si denominator fractionis, ex qua Series recurrens formatur, plures habeat Factores trinomiales inter se æquales, tum, spectata forma termini generalis supra data, patebit inventionem radicum multo magis fieri incertam. Interim tamen si una quæcunque radix realis jam proxime fuerit detecta, tum æquationis transformatione semper valor ejusdem radicis multo propior eruetur. Ponatur enim x æqualis valori illi jam detecto $+ y$, atque novæ æquationis quæratur minima radix pro y, quæ addita ad illum valorem præbebit verum ipsius x valorem.

EXEMPLUM.

Sit proposita ista æquatio $x^3 - 3xx + 5x - 4 = 0$, *cujus unam radicem fere esse* $= 1$ *inde constat, quod, posito* $x = 1$, *prodit* $x^3 - 3xx + 5x - 4 = -1$.

Ponatur ergo $x = 1 + y$, fietque $1 - 2y - y^3 = 0$, unde pro radice minima invenienda formetur Series recurrens, cujus scala relationis $2, 0, +1$, quæ erit

$$1, 2, 4, 9, 20, 44, 97, 214, 472, 1041, 2296, \&c.,$$

unde radix minima ipsius y erit proxime $\frac{1041}{2296} = 0,453397$, ita ut sit $x = 1,453397$, qui valor tam prope vix alia methodo æque facile obtineri poterit.

354. Quod si autem Series quæcunque recurrens tandem tam prope ad progressionem geometricam convergat, tum ex ipsa lege progressionis statim facile cognosci poterit, cujusnam æquationis radix sit futura quotus qui ex divisione unius termini per præcedentem oritur. Sint

termini Seriei recurrentis a principio jam longiſſime remoti, ita ut cum progreſſione geometrica confundantur; ſitque $T = \alpha S + \beta R + \gamma Q + \delta P$, ſeu ſcala relationis α, $+\beta$, $+\gamma + \delta$. Ponatur valor fractionis $\frac{Q}{P} = x$; erit $\frac{R}{P} = xx$; $\frac{S}{P} = x^3$ & $\frac{T}{P} = x^4$, qui in ſuperiori æquatione ſubſtituti dabunt

$$x^4 = \alpha x^3 + \beta x^2 + \gamma x + \delta.$$

unde patet quotum $\frac{Q}{P}$ tandem præbere radicem unam æquationis inventæ. Hoc vero & præcedens methodus indicat, præterea vero docet fractionem $\frac{Q}{P}$ dare maximam æquationis radicem.

355. Poteſt quoque hæc methodus inveſtigandarum radicum ſæpenumero utiliter adhiberi, ſi æquatio ſit infinita. Ad quod oſtendendum propoſita ſit æquatio $\frac{1}{2} = z - \frac{z^3}{6} + \frac{z^5}{120} - \frac{z^7}{5040} +$ &c., cujus radix minima z exhibet Arcum $30°$, ſeu Semiperipheriæ Circuli ſextantem. Perducatur ergo æquatio ad hanc formam

$$1 - 2z + \frac{z^3}{3} - \frac{z^5}{60} + \frac{z^7}{2520} - \&c. = 0.$$

Hinc ergo formetur Series recurrens, cujus ſcala relationis eſt infinita, ſcilicet

$$2,\ 0,\ -\frac{1}{3},\ 0,\ +\frac{1}{60},\ 0,\ -\frac{1}{2520},\ 0\ \&c.,$$

eritque Series recurrens

$$1,\ 2,\ 4,\ \frac{23}{3},\ \frac{44}{3},\ \frac{1681}{60},\ \frac{2408}{45},\ \&c.,$$

erit

erit ergo proxime $z = \frac{1681.45}{2408.60} = \frac{1681.3}{2408.4} = \frac{5043}{9632} = 0,52356$: CAP. XVII.
At ex proportione Peripheriæ ad Diametrum cognita debebat esse $z = 0,523598$, ita ut radix inventa tantum parte $\frac{3}{100000}$ a vero discrepet. Hoc autem in hac æquatione commode usu venit, quod ejus omnes radices sint reales, atque a minima reliquæ satis notabiliter discrepent. Quæ conditio cum rarissime in æquationibus infinitis locum habeat, huic methodo ad eas resolvendas parum usus relinquitur.

CAPUT XVIII.

De fractionibus continuis.

356. Quoniam in præcedentibus Capitibus plura, cum de Seriebus infinitis, tum de productis ex infinitis Factoribus conflatis disserui, non incongruum fore visum est, si etiam nonnulla de tertio quodam expressionum infinitarum genere addidero, quod continuis fractionibus vel divisionibus continetur. Quanquam enim hoc genus parum adhuc est excultum, tamen non dubitamus, quin ex eo amplissimus usus in analysin infinitorum aliquando sit redundaturus. Exhibui enim jam aliquoties ejusmodi specimina, quibus hæc expectatio non parum probabilis redditur. Imprimis vero ad ipsam Arithmeticam & Algebram communem non contemnenda subsidia affert ista speculatio, quæ hoc Capite breviter indicare atque exponere constitui.

357. Fractionem autem continuam voco ejusmodi fractionem, cujus denominator constat ex numero integro cum fractione, cujus denominator denuo est aggregatum ex integro & fractione, quæ porro simili modo sit comparata, sive ista affectio in infinitum progrediatur sive alicubi sistatur. Hujusmodi ergo fractio continua erit sequens expressio

a +

$$a+\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{d+\cfrac{1}{e+\cfrac{1}{f+\&c.}}}}} \quad \text{vel} \quad a+\cfrac{\alpha}{b+\cfrac{\beta}{c+\cfrac{\gamma}{d+\cfrac{\delta}{e+\cfrac{\epsilon}{f+\&c.}}}}},$$

in quarum forma priori omnes fractionum numeratores ſunt unitates, quam potiſſimum hic contemplabor, in altera vero forma ſunt numeratores numeri quicunque.

358. Expoſita ergo fractionum harum continuarum forma, primum videndum eſt, quemadmodum earum ſignificatio conſueto more expreſſa inveniri queat. Quæ ut facilius inveniri poſſit, progrediamur per gradus, abrumpendo illas fractiones primo in prima, tum in ſecunda, poſt in tertia & ita porro fractione; quo facto patebit fore

$$a = a$$

$$a+\cfrac{1}{b} = \frac{ab+1}{b}$$

$$a+\cfrac{1}{b+\cfrac{1}{c}} = \frac{abc+a+c}{bc+1}$$

$$a+\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{d}}} = \frac{abcd+ab+ad+cd+1}{bcd+b+d}$$

$$a+\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{d+\cfrac{1}{e}}}} = \frac{abcde+abe+ade+cde+abc+a+c+e}{bcde+be+de+bc+1}$$

&c.

359. Etſi in his fractionibus ordinariis non facile lex, ſecundum quam numerator ac denominator ex litteris a, b, c, d, &c., componantur, perſpicitur, tamen attendenti ſtatim patebit, quemadmodum quælibet fractio ex præcedentibus formari queat. Quilibet enim numerator eſt aggregatum ex numeratore ultimo per novam litteram multiplicato, & ex numeratore

meratore penultimo ſimplici: eademque lex in denominatoribus obſervatur. Scriptis ergo ordine litteris a, b, c, d, &c., ex iis fractiones inventæ facile formabuntur hoc modo CAP. XVIII.

$$\overset{a}{\frac{1}{0}};\ \overset{b}{\frac{a}{1}};\ \overset{c}{\frac{ab+1}{b}};\ \overset{d}{\frac{abc+a+c}{bc+1}};\ \overset{e}{\frac{abcd+ab+ad+cd+1}{bcd+b+d}}$$

ubi quilibet numerator invenitur, ſi præcedentium ultimus per indicem ſupra ſcriptum multiplicetur atque ad productum antepenultimus addatur; quæ eadem lex pro denominatoribus valet. Quo autem hac lege ab ipſo initio uti liceat, præfixi fractionem $\frac{1}{0}$ quæ, etiamſi e fractione continua non oriatur, tamen progreſſionis legem clariorem efficit. Quælibet autem fractio exhibet valorem fractionis continuæ uſque ad eam litteram, quæ antecedenti imminet, incluſive continuata.

360. Simili modo altera fractionum continuarum forma

$$a+\cfrac{\alpha}{b+\cfrac{\beta}{c+\cfrac{\gamma}{d+\cfrac{\delta}{e+\cfrac{\epsilon}{f+\&c.}}}}},$$

dabit, prout aliis aliiſque locis abrumpitur, ſequentes valores

$$a = a$$

$$a+\frac{\alpha}{b} = \frac{ab+\alpha}{b}$$

$$a+\cfrac{\alpha}{b+\cfrac{\beta}{c}} = \frac{abc+\beta a+\alpha c}{bc+\beta}$$

$$a+\cfrac{\alpha}{b+\cfrac{\beta}{c+\cfrac{\gamma}{d}}} = \frac{abcd+\beta ad+\alpha cd+\gamma ab+\alpha\gamma}{bcd+\beta d+\gamma b}$$

&c.,

 quarum

Lib. I. quarum fractionum quæque ex binis præcedentibus sequentem in modum invenietur

$$\begin{array}{ccccc} a & b & c & d & e \\ \frac{1}{0}; & \frac{a}{1}; & \frac{ab+\alpha}{b}; & \frac{abc+\beta a+\alpha c}{bc+\beta}; & \frac{abcd+\beta ad+\alpha cd+\gamma ab+\alpha\gamma}{bcd+\beta d+\gamma b} \\ \alpha & \beta & \gamma & \delta & \varepsilon \end{array}$$

361. Fractionibus scilicet formandis supra inscribantur indices a, b, c, d, &c., infra autem subscribantur indices α, β, γ, δ, &c.. Prima fractio iterum constituatur $\frac{1}{0}$, secunda $\frac{a}{1}$, tum sequentium quævis formabitur si antecedentium ultimæ numerator per indicem supra scriptum, penultimæ vero numerator per indicem infra scriptum multiplicetur & ambo producta addantur, aggregatum erit numerator fractionis sequentis: simili modo ejus denominator erit aggregatum ex ultimo denominatore per indicem supra scriptum, & ex penultimo denominatore per indicem infra scriptum multiplicatis. Quælibet vero fractio hoc modo inventa præbebit valorem fractionis continuæ ad eum usque denominatorem, qui fractioni antecedenti est inscriptus, continuatæ inclusive.

362. Quod si ergo hæ fractiones eousque continuentur quoad fractio continua indices suppeditet, tum ultima fractio verum dabit valorem fractionis continuæ. Præcedentes fractiones vero continuo propius ad hunc valorem accedent, ideoque perquam idoneam appropinquationem suggerent. Ponamus enim verum valorem fractionis continuæ

$$a+\cfrac{\alpha}{b+\cfrac{\beta}{c+\cfrac{\gamma}{d+\cfrac{\varepsilon}{e+\&c.,}}}} \quad \text{esse} = x$$

atque manifestum est fractionem primam $\frac{1}{0}$ esse majorem quam

quam x; secunda vero $\frac{a}{1}$ minor erit quam x; tertia $a + \frac{\alpha}{b}$ CAP. XVIII.
iterum vero valore erit major; quarta denuo minor, atque ita porro hæ fractiones alternatim erunt majores & minores quam x. Porro autem perspicuum est quamlibet fractionem propius accedere ad verum valorem x quam ulla præcedentium; unde hoc pacto citissime & commodissime valor ipsius x proxime obtinetur; etiamsi fractio continua in infinitum progrediatur, dummodo numeratores α, β, γ, δ, &c., non nimis crescant; sin autem omnes isti numeratores fuerint unitates, tum appropinquatio nulli incommodo est obnoxia.

363. Quo ratio hujus appropinquationis ad verum fractionis continuæ valorem melius percipiatur, consideremus fractionum inventarum differentias. Ac, prima quidem $\frac{1}{0}$ prætermissa, differentia inter secundam ac tertiam est $= \frac{\alpha}{b}$; quarta a tertia subtracta relinquit $\frac{\alpha\beta}{b(bc+\beta)}$; quarta a quinta subtracta relinquit $\frac{\alpha\beta\gamma}{(bc+\beta)(bcd+\beta d+\gamma)}$, &c.. Hinc exprimetur valor fractionis continuæ per Seriem terminorum consuetam hoc modo, ut sit

$$x = a + \frac{\alpha}{b} - \frac{\alpha\beta}{b(bc+\beta)} + \frac{\alpha\beta\gamma}{(bc+\beta)(bcd+\beta d+\gamma b)} - \text{\&c.},$$

quæ Series toties abrumpitur quoties fractio continua non in infinitum progreditur.

364. Modum ergo invenimus fractionem continuam quamcunque in Seriem terminorum, quorum signa alternantur, convertendi, si quidem prima littera a evanescat. Si enim fuerit

$$x = \cfrac{\alpha}{b + \cfrac{\beta}{c + \cfrac{\gamma}{d + \cfrac{\delta}{e + \cfrac{\varepsilon}{f + \&c.,}}}}}$$

erit per ea, quæ modo invenimus,

$$x = \frac{\alpha}{b} - \frac{\alpha\beta}{b(bc+\beta)} + \frac{\alpha\beta\gamma}{(bc+\beta)(bcd+\beta d+\gamma b)} - \frac{\alpha\beta\gamma\delta}{(bcd+\beta d+\gamma b)(bcde+\beta de+\gamma be+\delta bc+\beta\delta)} + \&c..$$

Unde, si α, β, γ, δ, &c. fuerint numeri non crescentes, uti omnes unitates, denominatores vero a, b, c, d, &c. numeri integri quicunque affirmativi, valor fractionis continuæ exprimetur per Seriem terminorum maxime convergentem.

365. His probe consideratis, poterit vicissim Series quæcunque terminorum alternantium in fractionem continuam converti, seu fractio continua inveniri cujus valor æqualis sit summæ Seriei propositæ. Sit enim proposita hæc Series

$$x = A - B + C - D + E - F + \&c.,$$

erit, singulis terminis cum Serie ex fractione continua orta comparandis

$$A = \frac{\alpha}{b}; \qquad \text{hincque } \alpha = Ab,$$

$$\frac{B}{A} = \frac{\beta}{bc+\beta}; \qquad \text{unde fit } \beta = \frac{Bbc}{A-B}$$

$$\frac{C}{B} = \frac{\gamma b}{bcd+\beta d+\gamma b}; \qquad \gamma = \frac{Cd(bc+\beta)}{b(B-C)}$$

$$\frac{D}{C} = \frac{\delta(bc+\beta)}{bcde+\beta de+\gamma be+\delta bc+\beta\delta}; \qquad \delta = \frac{De(bcd+\beta d+\gamma b)}{(bc+\beta)(C-D)}$$

&c..

At, cum sit $\beta = \frac{Bbc}{A-B}$, erit $bc+\beta = \frac{Abc}{A-B}$; unde

$\gamma =$

$\gamma = \frac{ACcd}{(A-B)(B-C)}$. Porro fit $bcd + \mathfrak{C}d + \gamma b =$ $(bc+\mathfrak{C})d+\gamma b = \frac{Abcd}{A-B} + \frac{ACbcd}{(A-B)(B-C)} = \frac{ABbcd}{(A-B)(B-C)}$, unde erit $\frac{bcd+\mathfrak{C}d+\gamma b}{bc+\mathfrak{C}} = \frac{Bd}{B-C}$ & $\delta = \frac{BDde}{(B-C)(C-D)}$; fimili modo reperietur $\varepsilon = \frac{CEef}{(C-D)(D-E)}$ & ita porro.

366. Quo ifta lex clarius appareat, ponamus effe

$$P = b$$
$$Q = bc + \mathfrak{C}$$
$$R = bcd + \mathfrak{C}d + \gamma b$$
$$S = bcde + \mathfrak{C}de + \gamma be + \delta bc + \mathfrak{C}\delta$$
$$T = bcdef + \&c.$$
$$V = bcdefg + \&c.,$$

erit ex lege harum expreffionum

$$Q = Pc + \mathfrak{C}$$
$$R = Qd + \gamma P$$
$$S = Re + \delta Q$$
$$T = Sf + \varepsilon R$$
$$V = Tg + \zeta S$$
$$\&c..$$

Cum igitur his adhibendis litteris fit

$$x = \frac{\alpha}{P} - \frac{\alpha\mathfrak{C}}{PQ} + \frac{\alpha\mathfrak{C}\gamma}{QR} - \frac{\alpha\mathfrak{C}\gamma\delta}{RS} + \frac{\alpha\mathfrak{C}\gamma\delta\varepsilon}{ST} - \&c.,$$

367. Quoniam ergo ponimus effe

$$x = A - B + C - D + E - F + \&c.,$$

erit

$$A = \frac{\alpha}{P};\ \alpha = AP$$

LIB. I.

$$\frac{B}{A} = \frac{\beta}{Q};\quad \beta = \frac{BQ}{A}$$
$$\frac{C}{B} = \frac{\gamma P}{R};\quad \gamma = \frac{CR}{BP}$$
$$\frac{D}{C} = \frac{\delta Q}{S};\quad \delta = \frac{DS}{CQ}$$
$$\frac{E}{D} = \frac{\varepsilon R}{T};\quad \varepsilon = \frac{ET}{DR}$$
&c. &c.

Porro vero differentiis ſumendis habebitur

$$A - B = \frac{\alpha(Q - \beta)}{PQ} = \frac{\alpha c}{Q} = \frac{APc}{Q}$$
$$B - C = \frac{\alpha\beta(R - \gamma P)}{PQR} = \frac{\alpha\beta d}{PR} = \frac{BQd}{R}$$
$$C - D = \frac{\alpha\beta\gamma(S - \delta Q)}{QRS} = \frac{\alpha\beta\gamma e}{QS} = \frac{CRe}{S}$$
$$D - E = \frac{\alpha\beta\gamma\delta(T - \varepsilon R)}{RST} = \frac{\alpha\beta\gamma\delta f}{RT} = \frac{DSf}{T},$$
&c. &c. &c.

Si bini igitur in ſe invicem ducantur, fiet

$$(A - B)(B - C) = ABcd.\frac{P}{R};\quad \&\quad \frac{R}{P} = \frac{ABcd}{(A - B)(B - C)}$$
$$(B - C)(C - D) = BCde.\frac{Q}{S};\quad \&\quad \frac{S}{Q} = \frac{BCed}{(B - C)(C - D)}$$
$$(C - D)(D - E) = CDef.\frac{R}{T};\quad \&\quad \frac{T}{R} = \frac{CDef}{(C - D)(D - E)}$$
&c..

Unde, cum ſit $P = b$; $Q = \frac{\alpha c}{A - B} = \frac{Abc}{A - B}$, erit

$$\alpha = Ab$$
$$\beta = \frac{Bbc}{A - B}$$
$$\gamma = \frac{ACcd}{(A - B)(B - C)}$$

$\delta =$

$$\delta = \frac{BDde}{(B-C)(C-D)}$$
$$\varepsilon = \frac{CEef}{(C-D)(D-E)}$$
&c..

368. Inventis ergo valoribus numeratorum α, β, γ, δ, &c., denominatores b, c, d, e, &c., arbitrio nostro relinquuntur: ita autem eos assumi convenit, ut, cum ipsi sint numeri integri, tum valores integros pro α, β, γ, δ, &c., exhibeant. Hoc vero pendet quoque a natura numerorum A, B, C, &c., utrum sint integri an fracti. Ponamus esse numeros integros, atque quæsito satisfiet statuendo

$b = 1$		$\alpha = A$
$c = A - C$		$\beta = B$
$d = B - C$	unde fit	$\gamma = AC$
$e = C - D$		$\delta = BD$
$f = D - E$		$\varepsilon = CE$
&c.		&c.

Quocirca, si fuerit,

$$x = A - B + C - D + E - F + \&c.,$$

idem ipsius x valor per fractionem continuam ita exprimi poterit, ut sit

$$x = \cfrac{A}{1 + \cfrac{B}{A - B + \cfrac{AC}{B - C + \cfrac{BD}{C - D + \cfrac{CE}{D - E + \&c.}}}}}$$

369. Sin autem omnes termini Seriei sint numeri fracti, ita ut fuerit

$$x = \frac{1}{A} - \frac{1}{B} + \frac{1}{C} - \frac{1}{D} + \frac{1}{E} - \&c.,$$

habebuntur pro α, β, γ, δ, &c., sequentes valores

$\alpha =$

Lib. I.

$$\alpha = \frac{b}{A};\ \beta = \frac{Abc}{B-A};\ \gamma = \frac{B^2cd}{(B-A)(C-B)};$$

$$\delta = \frac{C^2de}{(C-B)(D-C)};\ \varepsilon = \frac{D^2ef}{(D-C)(E-D)};\ \&c..$$

Ponatur ergo ut ſequitur

$b = A;$	$\alpha = 1$
$c = B - A;$	$\beta = AA$
$d = C - B;$	$\gamma = BB$
$e = D - C;$	$\delta = CC$

&c.,

eritque per fractionem continuam

$$x = \cfrac{1}{A + \cfrac{AA}{B - A + \cfrac{BB}{C - B + \cfrac{CC}{D - C + \&c.}}}},$$

Exemplum I.

Transformetur hæc Series infinita

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \&c.,$$

in fractionem continuam.

Erit ergo $A = 1$, $B = 2$, $C = 3$, $D = 4$, &c., atque, cum Seriei propoſitæ valor ſit $= l2$, erit

$$l2 = \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{4}{1 + \cfrac{9}{1 + \cfrac{16}{1 + \cfrac{25}{1 + \&c.}}}}}}$$

Exemplum II.

Transformetur hæc Series infinita

$$\frac{\pi}{4}$$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \&c.;$$

ubi π denotat peripheriam circuli, cujus diameter $= 1$, in fractionem continuam.

Substitutis loco A, B, C, D, &c., numeris 1, 3, 5, 7, &c., orietur

$$\frac{\pi}{4} = \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{9}{2 + \cfrac{25}{2 + \cfrac{49}{2 + \&c.}}}}},$$

hincque, invertendo fractionem, erit

$$\frac{4}{\pi} = 1 + \cfrac{1}{2 + \cfrac{9}{2 + \cfrac{25}{2 + \cfrac{49}{2 + \&c.}}}},$$

quæ est expressio, quam BROUNCKERUS primum pro quadratura circuli protulit.

EXEMPLUM III.

Sit proposita ista Series infinita

$$x = \frac{1}{m} - \frac{1}{m+n} + \frac{1}{m+2n} - \frac{1}{m+3n} + \&c.;$$

quæ, ob $A = m$; $B = m + n$; $C = m + 2n$; &c., in hanc fractionem continuam mutatur

$$x = \cfrac{1}{m + \cfrac{mm}{n + \cfrac{(m+n)^2}{n + \cfrac{(m+2n)^2}{n + \cfrac{(m+3n)^2}{n + \&c.}}}}}$$

ex qua fit, invertendo,

$$\frac{1}{x} - m = \cfrac{mm}{n + \cfrac{(m+n)^2}{n + \cfrac{(m+2n)^2}{n + \cfrac{(m+3n)^2}{n + \&c.}}}}$$

EXEMPLUM IV.

Quoniam, supra §. 178., invenimus esse

$$\frac{\pi \operatorname{cos.} \frac{m\pi}{n}}{n \operatorname{sin.} \frac{m\pi}{n}} = \frac{1}{m} - \frac{1}{n-m} + \frac{1}{n+m} - \frac{1}{2n-m} + \frac{1}{2n+m} - \&c.,$$

erit, pro fractione continuanda, $A = m$; $B = n - m$; $C = n + m$; $D = 2n - m$; &c., unde fiet

$$\frac{\pi \operatorname{cos.} \frac{m\pi}{n}}{n \operatorname{sin.} \frac{m\pi}{n}} = \cfrac{1}{m + \cfrac{mm}{n - 2m + \cfrac{(n-m)^2}{2m + \cfrac{(n+m)^2}{n - 2m + \cfrac{(2n-m)^2}{2m + \cfrac{(2n+m)^2}{n - 2m + \&c.}}}}}}$$

370. Si Series proposita per continuos Factores progrediatur, ut sit

$$x = \frac{1}{A} - \frac{1}{AB} + \frac{1}{ABC} - \frac{1}{ABCD} + \frac{1}{ABCDE} - \&c.;$$

tum prodibunt sequentes determinationes

$$\alpha = \frac{b}{A};\ \beta = \frac{bc}{B-1};\ \gamma = \frac{Bcd}{(B-1)(C-1)};$$

$$\delta = \frac{Cde}{(C-1)(D-1)};\ \varepsilon = \frac{Def}{(D-1)(E-1)};\ \&c.;$$

fiat ergo ut sequitur,

$$b = A;$$

$$b = A; \qquad \alpha = 1$$
$$c = B - 1; \qquad \beta = A$$
$$d = C - 1; \qquad \gamma = B$$
$$e = D - 1; \qquad \delta = C$$
$$f = E - 1; \qquad \varepsilon = D$$

&c.;

unde consequenter fiet

$$x = \cfrac{1}{A + \cfrac{A}{B - 1 + \cfrac{B}{C - 1 + \cfrac{C}{D - 1 + \cfrac{D}{E - 1 + \&c.}}}}}$$

EXEMPLUM I.

Quoniam, posito e numero cujus Logarithmus est $= 1$; supra invenimus esse

$$\frac{1}{e} = 1 - \frac{1}{1} + \frac{1}{1.2} - \frac{1}{1.2.3} + \frac{1}{1.2.3.4} - \&c.,$$

seu

$$1 - \frac{1}{e} = \frac{1}{1} - \frac{1}{1.2} + \frac{1}{1.2.3} - \frac{1}{1.2.3.4} + \&c.,$$

hæc Series in fractionem continuam convertetur ponendo $A = 1$, $B = 2$, $C = 3$, $D = 4$, &c.: quo ergo facto habebitur

$$1 - \frac{1}{e} = \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{2}{2 + \cfrac{3}{3 + \cfrac{4}{4 + \cfrac{5}{5 + \&c.}}}}}},$$

unde, asymmetria initio rejecta, erit

$$\frac{1}{e - 1} = \cfrac{1}{1 + \cfrac{2}{2 + \cfrac{3}{3 + \cfrac{4}{4 + \cfrac{5}{5 + \&c.}}}}}$$

EXEMPLUM II.

Invenimus quoque arcus, qui radio æqualis sumitur, cosinum esse $= 1 - \frac{1}{2} + \frac{1}{2.12} - \frac{1}{2.12.30} + \frac{1}{2.12.30.56} -$ &c.. Si ergo fiat $A = 1$, $B = 2$, $C = 12$, $D = 30$; $E = 56$, &c., atque Cosinus arcus qui radio æquatur, ponatur $= x$; erit

$$x = \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{2}{11 + \cfrac{12}{29 + \cfrac{30}{55 + \&c.}}}}},$$

seu

$$\frac{1}{x} - 1 = \cfrac{1}{1 + \cfrac{2}{11 + \cfrac{12}{29 + \cfrac{30}{55 + \&c.}}}}$$

371. Sit Series insuper cum geometrica conjuncta, scilicet

$$x = A - Bz + Cz^2 - Dz^3 + Ez^4 - Fz^5 + \&c.;$$

erit

$$\alpha = Ab;\ \beta = \frac{Bbcz}{A - Bz};\ \gamma = \frac{ACcdz}{(A - Bz)(B - Cz)};$$
$$\delta = \frac{BDdez}{(B - Cz)(C - Dz)};\ \varepsilon = \frac{CEefz}{(C - Dz)(D - Ez)};\ \&c.:$$

Ponatur nunc

$$\begin{aligned} b &= 1; & \alpha &= A \\ c &= A - Bz; & \beta &= Bz \\ d &= B - Cz; & \gamma &= ACz \\ e &= C - Dz; & \delta &= BDz, \end{aligned}$$

unde fiet

$x =$

$$x = \cfrac{A}{1 + \cfrac{Bz}{A - Bz + \cfrac{ACz}{B - Cz + \cfrac{BDz}{C - Dz + \&c.}}}}$$

372. Quo autem hoc negotium generalius abſolvamus, ponamus eſſe

$$x = \frac{A}{L} - \frac{By}{Mz} + \frac{Cy^2}{Nz^2} + \frac{Dy^3}{Oz^3} - \frac{Ey^4}{Pz^4} + \&c.,$$

fietque, comparatione inſtituta,

$$\alpha = \frac{Ab}{L};\ \beta = \frac{BLbcy}{AMz - BLy};\ \gamma = \frac{ACM^2 cdyz}{(AMz - BLy)(BNz - CMy)};$$

$$\delta = \frac{BDN^2 deyz}{(BNz - CMy)(COz - DNy)};\ \&c.,$$

ſtatuantur valores b. c. d, &c., ſequenti modo

$$\begin{array}{lll} b = L; & \text{erit} & \alpha = A \\ c = AMz - BLy; & & \beta = BLLy \\ d = BNz - CMy; & & \gamma = ACM^2yz \\ e = COz - DNy; & & \delta = BDN^2yz \\ f = DPz - EOy; & & \varepsilon = CEO^2yz \\ \&c. & & \&c. \end{array}$$

unde Series propoſita per ſequentem fractionem continuam exprimetur

$$x = \cfrac{A}{L + \cfrac{BLLy}{AMz - BLy + \cfrac{ACMMyz}{BNz - CMy + \cfrac{BDNNyz}{COz - DNy + \&c.}}}}$$

373. Habeat denique Series propoſita hujuſmodi formam

$$x = \frac{A}{L} - \frac{ABy}{LMz} + \frac{ABCy^2}{LMNz^2} - \frac{ABCDy^3}{LMNOz^3} + \&c.,$$

atque ſequentes valores prodibunt

LIB. I.

$$\alpha = \frac{Ab}{L};\ \beta = \frac{Bbcy}{Mz - By};\ \gamma = \frac{CMcdyz}{(Mz - By)(Nz - Cy)};$$

$$\delta = \frac{DNdeyz}{(Nz - Cy)(Oz - Dy)};\ \varepsilon = \frac{EOefyz}{(Oz - Dy)(Pz - Ey)};$$

&c.,

ad valores ergo integros inveniendos fiat

$b = Lz;$	erit	$\alpha = Az$
$c = Mz - By;$		$\beta = BLyz$
$d = Nz - Cy;$		$\gamma = CMyz$
$e = Oz - Dy;$		$\delta = DNyz$
$f = Pz - Ey;$		$\varepsilon = EOyz$
&c.		&c.

Unde valor Seriei propositæ ita exprimetur, ut sit

$$x = \cfrac{Az}{Lz + \cfrac{BLyz}{Mz - By + \cfrac{CMyz}{Nz - Cy + \cfrac{DNyz}{Oz - Dy + \&c.}}}}.$$

Vel, ut lex progressionis statim a principio fiat manifesta, erit

$$\frac{Az}{x} - Ay = Lz - Ay + \cfrac{BLyz}{Mz - By + \cfrac{CMyz}{Nz - Cy + \cfrac{DNyz}{Oz - Dy + \&c.}}}$$

374. Hoc modo innumerabiles inveniri poterunt fractiones continuæ in infinitum progredientes, quarum valor verus exhiberi queat. Cum enim, ex supra traditis, infinitæ Series, quarum summæ constent, ad hoc negotium accommodari queant, unaquæque transformari poterit in fractionem continuam, cujus adeo valor summæ illius Seriei est æqualis. Exempla, quæ jam hic sunt allata, sufficiunt ad hunc usum ostendendum: verumtamen optandum esset, ut methodus detegeretur, cujus beneficio, si proposita fuerit fractio continua quæcunque, ejus valor immediate inveniri posset. Quanquam enim fractio continua

tinua transmutari potest in Seriem infinitam, cujus summa per methodos cognitas investigari queat, tamen plerumque istæ Series tantopere fiunt intricatæ, ut earum summa, etiamsi sit satis simplex, vix ac ne vix quidem obtineri possit. CAP. XVIII.

375. Quo autem clarius perspiciatur, dari ejusmodi fractiones continuas, quarum valor aliunde facile assignari queat, etiamsi ex Seriebus infinitis, in quas convertuntur, nihil admodum colligere liceat, consideremus hanc fractionem continuam

$$x = \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \&c.}}}},$$

cujus omnes denominatores sunt inter se æquales; si enim hinc modo supra exposito, fractiones formemus

$$\overset{0,}{\frac{1}{0}}, \overset{2,}{\frac{0}{1}}, \overset{2,}{\frac{1}{2}}, \overset{2,}{\frac{2}{5}}, \overset{2,}{\frac{5}{12}}, \overset{2,}{\frac{12}{29}}, \overset{2}{\frac{29}{70}}, \&c. :$$

Hinc autem porro oritur hæc Series

$$x = 0 + \frac{1}{2} - \frac{1}{2.5} + \frac{1}{5.12} - \frac{1}{12.29} + \frac{1}{29.70} - \&c.,$$

vel, si bini termini conjungantur, erit

$$x = \frac{2}{1.5} + \frac{2}{5.29} + \frac{2}{29.169} + \&c.,$$

vel

$$x = \frac{1}{2} - \frac{2}{2.12} - \frac{2}{12.70} - \&c..$$

Quin etiam, cum sit

$$x = \frac{1}{4} - \frac{1}{2.2.5} + \frac{1}{2.5.12} - \frac{1}{2.12.29} + \&c.$$

+

LIB. I.

$$+\frac{1}{4}-\frac{1}{2.2.5}+\frac{1}{2.5.12}-\frac{1}{2.12.29}+\&c.,$$

erit

$$x=\frac{1}{4}+\frac{1}{1.5}-\frac{1}{2.12}+\frac{1}{5.29}-\frac{1}{12.70}\ \&c.;$$

quæ Series etiamsi vehementer convergant, tamen vera earum summa ex earum forma colligi nequit.

376. Pro hujusmodi autem fractionibus continuis, in quibus denominatores omnes vel sunt æquales, vel iidem revertuntur; ita ut ea fractio, si ab initio aliquot terminis truncetur, toti adhuc sit æqualis, facilis habetur modus earum summas explorandi. In exemplo enim proposito, cum sit

$$x=\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\&c.}}}}$$

erit $x=\frac{1}{2+x}$, ideoque $xx+2x=1$ & $x+1=\sqrt{2}$; ita ut valor hujus fractionis continuæ sit $=\sqrt{2}-1$. Fractiones vero ex fractione continua ante erutæ, continuo propius ad hunc valorem accedunt, idque tam cito, ut vix promptior modus ad valorem hunc irrationalem per numeros rationales proxime exprimendum, inveniri queat. Est enim $\sqrt{2}-1$ tam prope $=\frac{29}{70}$, ut error sit insensibilis: namque, radicem extrahendo, erit

$$\sqrt{2}-1=0,41421356236,$$

atque

$$\frac{29}{70}=0,41428571428,$$

ita ut error tantum in partibus centesimis millesimis consistat.

377. Quemadmodum ergo fractiones continuæ commodissimum suppeditant modum ad valorem $\sqrt{2}$ appropinquandi, ita indidem

indidem facillima via aperitur ad radices aliorum numerorum proxime investigandas. Ponamus hunc in finem CAP. XVIII.

$$x = \cfrac{1}{a + \cfrac{1}{a + \cfrac{1}{a + \cfrac{1}{a + \cfrac{1}{a + \&c.}}}}},$$

erit $x = \frac{1}{a+x}$ & $xx + ax = 1$, unde fit $x = -\frac{1}{2}a + \surd(1 + \frac{1}{4}aa) = \frac{\surd(aa+4) - a}{2}$. Hæc ergo fractio continua inserviet valori radicis quadratæ ex numero $aa + 4$ inveniendo. Hincque adeo substituendo loco a successive numeros 1, 2, 3, 4, &c., reperientur $\surd 5$; $\surd 2$; $\surd 13$; $\surd 5$; $\surd 29$; $\surd 10$; $\surd 53$; &c., perductis scilicet his radicibus ad formam simplicissimam: erit ergo

$$\begin{array}{l} 1, \quad 1, \quad 1, \quad 1, \quad 1, \quad 1, \\ \frac{0}{1}, \frac{1}{1}, \frac{1}{2}, \frac{2}{3}, \frac{3}{5}, \frac{5}{8}, \&c. = \frac{\surd 5 - 1}{2} \\ 2, \quad 2, \quad 2, \quad 2, \quad 2, \quad 2 \\ \frac{0}{1}, \frac{1}{2}, \frac{2}{5}, \frac{5}{12}, \frac{12}{29}, \frac{29}{70}, \&c. = \surd 2 - 1 \\ 3, \quad 3, \quad 3, \quad 3, \quad 3, \quad 3 \\ \frac{0}{1}, \frac{1}{3}, \frac{3}{10}, \frac{10}{33}, \frac{33}{109}, \frac{109}{360}, \&c. = \frac{\surd 13 - 3}{2} \\ 4, \quad 4, \quad 4, \quad 4, \quad 4, \quad 4 \\ \frac{0}{1}, \frac{1}{4}, \frac{4}{17}, \frac{17}{72}, \frac{72}{305}, \frac{305}{1292}, \&c. = \surd 5 - 2 \\ \&c., \end{array}$$

notandum autem eo promptiorem esse approximationem, quo major fuerit numerus a: sic in ultimo exemplo erit $\surd 5 = 2 \frac{305}{1292}$, ut error minor sit quam $\frac{1}{1292 \cdot 5473}$, ubi 5473 est denominator sequentis fractionis $\frac{1292}{5473}$.

LIB. I. 378. Hoc vero modo aliorum numerorum radices exhiberi nequeunt, nisi qui sint summa duorum quadratorum. Ut igitur hæc approximatio ad alios numeros extendatur, ponamus esse

$$x = \cfrac{1}{a + \cfrac{1}{b + \cfrac{1}{a + \cfrac{1}{b + \cfrac{1}{a + \cfrac{1}{b + \&c.}}}}}},$$

erit

$$x = \cfrac{1}{a + \cfrac{1}{b + x}} = \frac{b + x}{ab + 1 + ax}; \text{ ideoque } axx + abx = b,$$

&

$$x = -\frac{1}{2}b \pm \sqrt{\left(\frac{1}{4}bb + \frac{b}{a}\right)} = \frac{-ab + \sqrt{(aabb + 4ab)}}{2a}.$$

Unde jam omnium numerorum radices inveniri poterunt. Sit, verbi gratia, $a = 2$, $b = 7$; erit $x = \frac{-14 + \sqrt{14 \cdot 18}}{4} = \frac{-7 + 3\sqrt{7}}{2}$; at valorem ipsius x proxime exhibebunt sequentes fractiones

$$\begin{matrix} 2, & 7, & 2, & 7, & 2, & 7 & \\ \frac{0}{1}, & \frac{1}{2}, & \frac{7}{15}, & \frac{15}{32}, & \frac{112}{239}, & \frac{239}{510}, & \&c., \end{matrix}$$

Erit ergo proxime $\frac{-7 + 3\sqrt{7}}{2} = \frac{239}{510}$ & $\sqrt{7} = \frac{2024}{765} =$ 2,6457516; at revera est $\sqrt{7} = 2,64575131$; ita ut error minor sit quam $\frac{3}{10000000}$.

379. Progrediamur autem ulterius ponendo

$$x = \cfrac{1}{a + \cfrac{1}{b + \cfrac{1}{c + \cfrac{1}{a + \cfrac{1}{b + \cfrac{1}{c + \cfrac{1}{a + \&c.}}}}}}},$$

erit

$$x = \cfrac{1}{a + \cfrac{1}{b + \cfrac{1}{c + x}}} = \cfrac{1}{a + \cfrac{c + x}{bx + bc + 1}} = \frac{bx + bc + 1}{(ab + 1)x + abc + a + c},$$

unde $(ab + 1)xx + (abc + a - b + c)x = bc + 1$ atque

$x =$

$$x = \frac{-abc - a + b - c + \sqrt{((abc + a + b + c)^2 + 4)}}{2(ab + 1)};$$

ubi quantitas post signum radicale posita iterum est summa duorum quadratorum, neque ergo hæc forma radicibus ex aliis numeris extrahendis inservit, nisi ad quos prima forma jam sufficeerat. Simili modo si quatuor litteræ a, b, c, d, continuo repetitæ denominatores fractionis continuæ constituant, tum ea plus non inserviet quam secunda, quæ duas tantum litteras continebat, & ita porro.

380. Cum igitur fractiones continuæ tam utiliter ad extractionem radicis quadratæ adhiberi queant, simul inservient æquationibus quadratis resolvendis; quod quidem ex ipso calculo est manifestum, dum x per æquationem quadraticam affectam determinatur. Potest autem vicissim facile cujusque æquationis quadratæ radix per fractionem continuam hoc modo exprimi. Sit proposita ista æquatio

$$xx = ax + b;$$

ex qua, cum sit $x = a + \frac{b}{x}$, substituatur in ultimo termino loco x valor idem jam inventus, eritque

$$x = a + \cfrac{b}{a + \cfrac{b}{x}},$$

simili ergo modo procedendo, erit per fractionem continuam infinitam

$$x = a + \cfrac{b}{a + \cfrac{b}{a + \cfrac{b}{a + \&c.}}},$$

quæ autem, cum numeratores b non sint unitates, non tam commode adhiberi potest.

381. Ut autem usus in arithmetica ostendatur, primum notandum est omnem fractionem ordinariam in fractionem con-

 tinuam

LIB. I. tinuam converti posse. Sit enim proposita fractio $x = \frac{A}{B}$; in qua sit $A > B$; dividatur A per B, sitque quotus $= a$ & residuum C; tum per hoc residuum C dividatur præcedens divisor B, prodeatque quotus b & relinquatur residuum D, per quod denuo præcedens divisor C dividatur; sicque hæc operatio, quæ vulgo ad maximum communem divisorem numerorum A & B investigandum usurpari solet, continuetur, donec ipsa finiatur; sequenti modo

$$\begin{array}{l} B\,)\,A\,(\,a \\ \quad C\,)\,B\,(\,b \\ \qquad D\,)\,C\,(\,c \\ \qquad\quad E\,)\,D\,(\,d \\ \qquad\qquad F \text{ \&c.}, \end{array}$$

eritque per naturam divisionis

$$A = aB + C;\ \text{unde}\ \frac{A}{B} = a + \frac{C}{B};$$

$$B = bC + D; \qquad \frac{B}{C} = b + \frac{D}{C}; \quad \frac{C}{B} = \cfrac{1}{b + \cfrac{D}{C}}$$

$$C = cD + E; \qquad \frac{C}{D} = c + \frac{E}{D}; \quad \frac{D}{C} = \cfrac{1}{c + \cfrac{E}{D}}$$

$$D = dE + F; \qquad \frac{D}{E} = d + \frac{F}{E}; \quad \frac{E}{D} = \cfrac{1}{d + \cfrac{F}{E}}$$

&c. &c. &c.

hinc, sequentes valores in præcedentibus substituendo, erit

$$x = \frac{A}{B} = a + \frac{C}{B} = a + \cfrac{1}{b + \cfrac{D}{C}} = a + \cfrac{1}{b + \cfrac{1}{c + \cfrac{E}{D}}},$$

unde tandem x per meros quotos inventos $a, b, c, d,$ &c. sequentem in modum exprimetur, ut sit

$x =$

$$x = a + \cfrac{1}{b + \cfrac{1}{c + \cfrac{1}{d + \cfrac{1}{e + \cfrac{1}{f + \&c.}}}}}$$

EXEMPLUM I.

Sit propoſita iſta fractio $\frac{1461}{59}$, quæ ſequenti modo in fractionem continuam tranſmutabitur, cujus omnes numeratores erunt unitates. Inſtituatur ſcilicet eadem operatio, qua maximus communis diviſor numerorum 59 & 1461 quæri ſolet.

```
59) 1461 (24
    118
    ---
     281
     236
     ---
      45) 59 (1
          45
          --
          14) 45 (3
              42
              --
               3) 14 (4
                  12
                  --
                   2) 3 (1
                      2
                      -
                      1) 2 (2
                         2
                         -
                         0
```

Hinc ergo ex quotis fiet

$$\frac{1461}{59} = 24 + \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{4 + \cfrac{1}{1 + \cfrac{1}{2}}}}}.$$

EXEMPLUM II.

Fractiones quoque decimales eodem modo tranſmutari poterunt; ſit enim propoſita

$$\sqrt{2} = 1,41421356 = \frac{141421356}{100000000},$$

unde hæc operatio inſtituatur

100000000	141421356	1
82842712	100000000	2
17157288	41421356	2
14213560	34314576	2
2943728	7106780	2
2438648	5887456	2
505080	1219324	2
418728	1010160	2
&c.	209364	

Ex qua operatione jam patet omnes denominatores esse 2, atque adeo esse

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{\&c.}}}}},$$

cujus expressionis ratio jam ex superioribus patet.

EXEMPLUM III.

Imprimis vero etiam hic attentione dignus est numerus e, cujus logarithmus est $= 1$, qui est $e = 2,718281828459$, unde oritur $\frac{e-1}{2} = 0,8591409142295$, quæ fractio decimalis, si superiori modo tractetur, dabit quotos sequentes

8591409142295	10000000000000	1
8451545146224	8591409142295	6
139863996071	1408590857704	10
139312557916	1398639960710	14
551438155	9950896994	18
550224488	9925886790	22
1213667	25010204	
	&c.	

si iste calculus exactius adhuc, assumto valore ipsius e, ulterius continuetur, tum prodibunt isti quoti

1, 6, 10, 14, 18, 22, 26, 30, 34, &c.,

qui, demto primo, progressionem arithmeticam constituunt, unde patet fore

$e - 1$

$$\frac{e-1}{2} = \cfrac{1}{1+\cfrac{1}{6+\cfrac{1}{10+\cfrac{1}{14+\cfrac{1}{18+\cfrac{1}{22+\cfrac{1}{\&c.}}}}}}},$$

cujus fractionis ratio ex calculo infinitesimali dari potest.

382. Cum igitur ex hujusmodi expressionibus fractiones erui queant, quæ quam citissime ad verum valorem expressionis deducant, hæc methodus adhiberi poterit ad fractiones decimales per ordinarias fractiones, quæ ad ipsas proxime accedant, exprimendas. Quin etiam, si fractio fuerit proposita cujus numerator & denominator sint numeri valde magni, fractiones ex minoribus numeris constantes inveniri poterunt quæ, etiamsi propositæ non sint penitus æquales, tamen ab ea quam minime discrepent. Hincque problema a WALLISIO olim tractatum facile resolvi potest, quo quæruntur fractiones minoribus numeris expressæ, quæ tam prope exhauriant valorem fractionis cujuspiam in numeris majoribus propositæ, quantum fieri poterit numeris non majoribus. Fractiones autem nostra hac methodo ortæ tam prope ad valorem fractionis continuæ, ex qua eliciuntur, accedunt, ut nullæ numeris non majoribus constantes dentur quæ propius accedant.

EXEMPLUM I.

Exprimatur ratio diametri ad peripheriam numeris tam exiguis, ut accuratior exhiberi nequeat, nisi numeri majores adhibeantur. Si fractio decimalis cognita

$$3,1415926535 \ \&c.,$$

modo exposito per divisionem continuam evolvatur, reperientur sequentes quoti

$$3, \quad 7, \quad 15, \quad 1, \quad 292, \quad 1, \quad 1, \quad \&c.,$$

ex quibus sequentes fractiones formabuntur,

$$\frac{1}{0}, \ \frac{3}{1}, \ \frac{22}{7}, \ \frac{333}{106}, \ \frac{355}{113}, \ \frac{103993}{33102}, \ \&c.,$$

secunda fractio jam ostendit esse diametrum ad peripheriam ut

1 : 3,

1 : 3, neque certe numeris non majoribus accuratius dari poterit. Tertia fractio dat rationem *Archimedeam* 7 : 22, at quinta *Metianam* quæ ad verum tam prope accedit, ut error minor ſit parte $\frac{1}{113.33102}$. Ceterum hæ fractiones alternatim vero ſunt majores minoreſque.

EXEMPLUM II.

Exprimatur ratio diei ad annum ſolarem medium in numeris minimis proxime. Cum annus iſte ſit $365^d, 5^h, 48', 55''$, continebit in fractione annus unus $365\frac{20935}{86400}$ dies. Tantum ergo opus eſt ut hæc fractio evolvatur, quæ dabit ſequentes quotos

$$4, \; 7, \; 1, \; 6, \; 1, \; 2, \; 2, \; 4$$

unde iſtæ eliciuntur fractiones

$$\frac{0}{1}, \; \frac{1}{4}, \; \frac{7}{29}, \; \frac{8}{33}, \; \frac{55}{227}, \; \frac{63}{260}, \; \frac{181}{747}, \; \&c..$$

Horæ ergo cum minutis primis & ſecundis, quæ ſupra 365 dies adſunt, quatuor annis unum diem circiter faciunt, unde calendarium *Julianum* originem habet. Exactius autem 33 annis 8 dies implentur, vel 747 annis 181 dies; unde ſequitur quadringentis annis abundare 97 dies. Quare, cum hoc intervallo calendarium *Julianum* inſerat 100 dies, *Gregorianus* quaternis ſeculis tres annos biſſextiles in communes convertit.

FINIS TOMI PRIMI.