ホモロジー代数非入門

alg-d

2021-09-18

ミラー配信: `https://www.youtube.com/watch?v=_vZtK_9ExpY`
このスライド: `http://alg-d.com/math/kan_extension/online2.pdf`

【定義】

R を単位的環とする．右 R-加群の図式

$$\cdots \to M_{n-1} \xrightarrow{f_{n-1}} M_n \xrightarrow{f_n} M_{n+1} \to \cdots$$

が完全列とは，各 n について $\mathrm{Im}(f_{n-1}) = \ker(f_n)$ となることをいう．

【定義】

R を単位的環とする．右 R-加群の図式

$$\cdots \to M_{n-1} \xrightarrow{f_{n-1}} M_n \xrightarrow{f_n} M_{n+1} \to \cdots$$

が完全列とは，各 n について $\mathrm{Im}(f_{n-1}) = \ker(f_n)$ となることをいう．

完全列は数学の多くの場面で出てくるから，完全列について調べようというのがホモロジー代数である (多分)．

完全列については多くのことが知られていて，特に有用な事実については「蛇の補題」「5 項補題」「9 項補題」「サラマンダーの補題」「horseshoe lemma」などが知られている．

【定理】 (蛇の補題)

次の右 R-加群の可換図式で横列は両方とも完全列とする.

$$\begin{tikzcd} & A_0 \arrow[r, "u_0"] \arrow[d, "f_0"'] & A_1 \arrow[r, "u_1"] \arrow[d, "f_1"] & A_2 \arrow[r] \arrow[d, "f_2"] & 0 \\ 0 \arrow[r] & B_0 \arrow[r, "v_0"'] & B_1 \arrow[r, "v_1"'] & B_2 & \end{tikzcd}$$

このとき完全列

$$\begin{aligned} &\ker(f_0) \xrightarrow{\overline{u_0}} \ker(f_1) \xrightarrow{\overline{u_1}} \ker(f_2) \\ &\xrightarrow{\delta} \operatorname{coker}(f_0) \xrightarrow{\overline{v_0}} \operatorname{coker}(f_1) \xrightarrow{\overline{v_1}} \operatorname{coker}(f_2) \end{aligned}$$

が存在する.

証明.

$$
\begin{tikzcd}
 & \ker(f_0) \arrow[r, "\overline{u_0}"] \arrow[d] & \ker(f_1) \arrow[r, "\overline{u_1}"] \arrow[d] & \ker(f_2) \arrow[d] & \\
 & A_0 \arrow[r, "u_0"] \arrow[d, "f_0"'] & A_1 \arrow[r, "u_1"] \arrow[d, "f_1"] & A_2 \arrow[r] \arrow[d, "f_2"] & 0 \\
0 \arrow[r] & B_0 \arrow[r, "v_0"'] \arrow[d] & B_1 \arrow[r, "v_1"'] \arrow[d] & B_2 \arrow[d] & \\
 & \mathrm{coker}(f_0) \arrow[r, "\overline{v_0}"'] & \mathrm{coker}(f_1) \arrow[r, "\overline{v_1}"'] & \mathrm{coker}(f_2) &
\end{tikzcd}
$$

□

証明.

$$
\begin{tikzcd}
 & \ker(f_0) \arrow[r, "\overline{u_0}"] \arrow[d] & \ker(f_1) \arrow[r, "\overline{u_1}"] \arrow[d] & \ker(f_2) \arrow[d, "\color{red}x"] & \\
 & A_0 \arrow[r, "u_0"] \arrow[d, "f_0"'] & A_1 \arrow[r, "u_1"] \arrow[d, "f_1"] & A_2 \arrow[r] \arrow[d, "f_2"] & 0 \\
0 \arrow[r] & B_0 \arrow[r, "v_0"'] \arrow[d] & B_1 \arrow[r, "v_1"'] \arrow[d] & B_2 \arrow[d] & \\
 & \operatorname{coker}(f_0) \arrow[r, "\overline{v_0}"'] & \operatorname{coker}(f_1) \arrow[r, "\overline{v_1}"'] & \operatorname{coker}(f_2) &
\end{tikzcd}
$$

□

証明.

$$
\begin{tikzcd}
 & \ker(f_0) \arrow[r, "\overline{u_0}"] \arrow[d] & \ker(f_1) \arrow[r, "\overline{u_1}"] \arrow[d] & \ker(f_2) \arrow[d] & \color{red}{x} \arrow[d, mapsto, red] & \\
 & A_0 \arrow[r, "u_0"] \arrow[d, "f_0"'] & A_1 \arrow[r, "u_1"] \arrow[d, "f_1"] & A_2 \arrow[rr] \arrow[d, "f_2"] & \color{red}{x} & 0 \\
0 \arrow[r] & B_0 \arrow[r, "v_0"'] \arrow[d] & B_1 \arrow[r, "v_1"'] \arrow[d] & B_2 \arrow[d] & & \\
 & \operatorname{coker}(f_0) \arrow[r, "\overline{v_0}"'] & \operatorname{coker}(f_1) \arrow[r, "\overline{v_1}"'] & \operatorname{coker}(f_2) & &
\end{tikzcd}
$$

□

証明.

$$
\begin{tikzcd}
 & \ker(f_0) \arrow[r, "\overline{u_0}"] \arrow[d] & \ker(f_1) \arrow[r, "\overline{u_1}"] \arrow[d] & \ker(f_2) \arrow[d] & \color{red}{x} \arrow[dd, mapsto, red] & \\
 & A_0 \arrow[r, "u_0"] \arrow[dd, "f_0"'] & A_1 \arrow[r, "u_1"] \arrow[dd, "f_1"] & A_2 \arrow[rr] \arrow[dd, "f_2"] & & 0 \\
 & & \color{red}{y} \arrow[rr, mapsto, red] & & \color{red}{x} & \\
0 \arrow[r] & B_0 \arrow[r, "v_0"'] \arrow[d] & B_1 \arrow[r, "v_1"'] \arrow[d] & B_2 \arrow[d] & & \\
 & \operatorname{coker}(f_0) \arrow[r, "\overline{v_0}"'] & \operatorname{coker}(f_1) \arrow[r, "\overline{v_1}"'] & \operatorname{coker}(f_2) & &
\end{tikzcd}
$$

□

証明.

$$
\begin{tikzcd}
 & \ker(f_0) \arrow[r, "\overline{u_0}"] \arrow[d] & \ker(f_1) \arrow[r, "\overline{u_1}"] \arrow[d] & \ker(f_2) \arrow[d] & \color{red}{x} \arrow[dd, mapsto, red] & \\
 & A_0 \arrow[r, "u_0"] \arrow[dd, "f_0"'] & A_1 \arrow[r, "u_1"] \arrow[dd, "f_1"] & A_2 \arrow[rr] \arrow[dd, "f_2"'] & & 0 \\
 & & \color{red}{y} \arrow[rr, mapsto, red] & & \color{red}{x} \arrow[d, mapsto, red] & \\
0 \arrow[r] & B_0 \arrow[r, "v_0"'] \arrow[d] & B_1 \arrow[r, "v_1"'] \arrow[d] & B_2 \arrow[d] & \color{red}{0} & \\
 & \operatorname{coker}(f_0) \arrow[r, "\overline{v_0}"'] & \operatorname{coker}(f_1) \arrow[r, "\overline{v_1}"'] & \operatorname{coker}(f_2) & &
\end{tikzcd}
$$

□

証明.

$$
\begin{tikzcd}
 & \ker(f_0) \arrow[r, "\overline{u_0}"] \arrow[dd] & \ker(f_1) \arrow[rr, "\overline{u_1}"] \arrow[dd] & & \ker(f_2) \arrow[dd] & & \\
 & & & & & {\color{red}x} \arrow[dd, red, mapsto] & \\
 & A_0 \arrow[r, "u_0"] \arrow[dd, "f_0"'] & A_1 \arrow[rr, "u_1"] \arrow[dd, "f_1"'] & & A_2 \arrow[rr] \arrow[dd, "f_2"'] & & 0 \\
 & & & {\color{red}y} \arrow[rr, red, mapsto] \arrow[dd, red, mapsto] & & {\color{red}x} \arrow[dd, red, mapsto] & \\
0 \arrow[r] & B_0 \arrow[r, "v_0"'] \arrow[dd] & B_1 \arrow[rr, "v_1"'] \arrow[dd] & & B_2 \arrow[dd] & & \\
 & & & {\color{red}f_1(y)} \arrow[rr, red, mapsto] & & {\color{red}0} & \\
 & \operatorname{coker}(f_0) \arrow[r, "\overline{v_0}"'] & \operatorname{coker}(f_1) \arrow[rr, "\overline{v_1}"'] & & \operatorname{coker}(f_2) & &
\end{tikzcd}
$$

□

証明.

$$
\begin{tikzcd}
 & \ker(f_0) \arrow[rr, "\overline{u_0}"] \arrow[d] & & \ker(f_1) \arrow[rr, "\overline{u_1}"] \arrow[d] & & \ker(f_2) \arrow[d] & x \arrow[dd, mapsto] & \\
 & A_0 \arrow[rr, "u_0"] \arrow[dd, "f_0"'] & & A_1 \arrow[rr, "u_1"] \arrow[dd, "f_1"'] & & A_2 \arrow[rr] \arrow[dd, "f_2"'] & & 0 \\
 & & & & y \arrow[rr, mapsto] \arrow[dd, mapsto] & & x \arrow[dd, mapsto] & \\
0 \arrow[r] & B_0 \arrow[rr, "v_0"'] \arrow[dd] & & B_1 \arrow[rr, "v_1"'] \arrow[dd] & & B_2 \arrow[dd] & & \\
 & & z \arrow[rr, mapsto] & & f_1(y) \arrow[rr, mapsto] & & 0 & \\
 & \operatorname{coker}(f_0) \arrow[rr, "\overline{v_0}"'] & & \operatorname{coker}(f_1) \arrow[rr, "\overline{v_1}"'] & & \operatorname{coker}(f_2) & &
\end{tikzcd}
$$

□

証明.

$$
\begin{tikzcd}
 & \ker(f_0) \arrow[r, "\overline{u_0}"] \arrow[d] & \ker(f_1) \arrow[r, "\overline{u_1}"] \arrow[d] & \ker(f_2) \arrow[d] & \\
 & A_0 \arrow[r, "u_0"] \arrow[d, "f_0"'] & A_1 \arrow[r, "u_1"] \arrow[d, "f_1"'] & A_2 \arrow[r] \arrow[d, "f_2"'] & 0 \\
0 \arrow[r] & B_0 \arrow[r, "v_0"'] \arrow[d] & B_1 \arrow[r, "v_1"'] \arrow[d] & B_2 \arrow[d] & \\
 & \operatorname{coker}(f_0) \arrow[r, "\overline{v_0}"'] & \operatorname{coker}(f_1) \arrow[r, "\overline{v_1}"'] & \operatorname{coker}(f_2) &
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & & x \arrow[d, mapsto] \\
 & y \arrow[r, mapsto] \arrow[d, mapsto] & x \arrow[d, mapsto] \\
z \arrow[r, mapsto] \arrow[d, mapsto] & f_1(y) \arrow[r, mapsto] & 0 \\
\delta(x) & &
\end{tikzcd}
$$

□

これをなるべく一般の圏へ一般化したい.

(1) 完全列をどのように定義するか?
(2) 元を取らずにどのように証明するか?

これをなるべく一般の圏へ一般化したい．

(1) 完全列をどのように定義するか?
(2) 元を取らずにどのように証明するか?

これに関してよく知られているのはアーベル圏で，$\ker$ や Im が定義できるから完全列を考えることができる．

これをなるべく一般の圏へ一般化したい．

(1) 完全列をどのように定義するか?
(2) 元を取らずにどのように証明するか?

これに関してよく知られているのはアーベル圏で，ker や Im が定義できるから完全列を考えることができる．

Q. どうやって証明をするか？

- よくあるのは Mitchell の埋込定理

これをなるべく一般の圏へ一般化したい．

(1) 完全列をどのように定義するか?
(2) 元を取らずにどのように証明するか?

これに関してよく知られているのはアーベル圏で，$\ker$ や Im が定義できるから完全列を考えることができる．

Q. どうやって証明をするか？

- よくあるのは Mitchell の埋込定理
- 「元を取る」を generalized element を使って正当化する方法 (圏論の基礎)

これをなるべく一般の圏へ一般化したい．

(1) 完全列をどのように定義するか?
(2) 元を取らずにどのように証明するか?

これに関して よく知られているのはアーベル圏で，ker や Im が定義できるから完全列を考えることができる．

Q. どうやって証明をするか？

- よくあるのは Mitchell の埋込定理
- 「元を取る」を generalized element を使って正当化する方法 (圏論の基礎)
- ker とかの普遍性を駆使して示す (圏論の技法)

これをなるべく一般の圏へ一般化したい．

(1) 完全列をどのように定義するか?
(2) 元を取らずにどのように証明するか?

これに関してよく知られているのはアーベル圏で，ker や Im が定義できるから完全列を考えることができる．

Q. どうやって証明をするか？

- よくあるのは Mitchell の埋込定理
- 「元を取る」を generalized element を使って正当化する方法 (圏論の基礎)
- ker とかの普遍性を駆使して示す (圏論の技法)
  $\longrightarrow$ これはアーベル圏でなくてもよくて，Quillen 完全圏というのが知られている．

ここでは A. Jafari による弱完全圏 (Weakly Exact Category) を説明する.

ここでは A. Jafari による弱完全圏 (Weakly Exact Category) を説明する．

Q. アーベル圏，Quillen 完全圏ではダメなの？

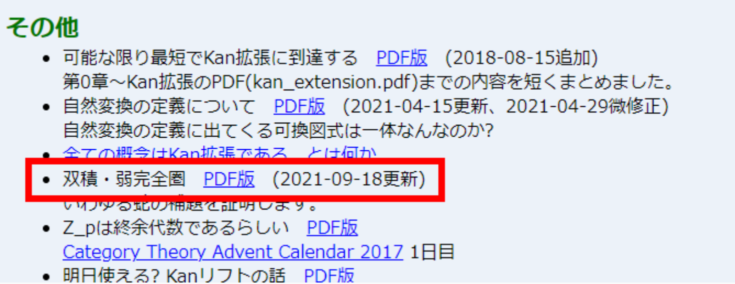
ここでは A. Jafari による弱完全圏 (Weakly Exact Category) を説明する．

Q. アーベル圏，Quillen 完全圏ではダメなの？
$\longrightarrow$ これらは **Ab**-豊穣圏になってしまうので，**Ab**-豊穣圏にならない圏 (例えば **Grp**) に適用できない．弱完全圏はそうではないので **Grp** 等にも適用できる．

参考文献

- A. Jafari, Weakly Exact Categories and the Snake Lemma, https://arxiv.org/abs/0901.2372

今回やる内容の詳しい証明は『双積・弱完全圏』の PDF を参照
http://alg-d.com/math/kan_extension/



準備

まず $\mathrm{Im}(f)$ は一般の圏で定義できる．

[1] k がモノ射だから，このような h は一意であり，また $h \circ g = g'$ となる．

準備

まず $\mathrm{Im}(f)$ は一般の圏で定義できる．

【定義】

圏 C の射 $f\colon a \to b$ の像 (image) とは，モノ射 $\mathrm{Im}(f)\colon x \to b$ であって次の条件を満たすものをいう．

(1) ある射 $g\colon a \to x$ が存在して $f = \mathrm{Im}(f) \circ g$ となる．

(2) モノ射 $k\colon x' \to b$ と射 $g'\colon a \to x'$ が存在して $f = k \circ g'$ となるならば，射 $h\colon x \to x'$ が存在して $\mathrm{Im}(f) = k \circ h$ となる [1]．

$$
\begin{tikzcd}
 & x \arrow[rd, "\mathrm{Im}(f)"] \arrow[dd, dashed, "h"] & \\
a \arrow[ru, "g"] \arrow[rd, "g'"'] & & b \\
 & x' \arrow[ru, "k"'] &
\end{tikzcd}
$$

[1] k がモノ射だから，このような h は一意であり，また $h \circ g = g'$ となる．

準備

次に $\ker(f), \operatorname{coker}(f)$ を定義するため零射を導入する．

準備

次に $\ker(f), \operatorname{coker}(f)$ を定義するため零射を導入する．

【定義】

$f\colon a \to b$ を圏 C の射とする．

(1) f が定数射
$\iff$ 任意の $c \in C$ と $g, h\colon c \to a$ に対して $f \circ g = f \circ h$．

(2) f が余定数射 $\iff$ C^{op} において f が定数射．

(3) f が零射 $\iff$ f が定数射かつ余定数射．

次に $\ker(f), \operatorname{coker}(f)$ を定義するため零射を導入する．

【定義】

$f\colon a \to b$ を圏 C の射とする．

(1) f が定数射
$\iff$ 任意の $c \in C$ と $g, h\colon c \to a$ に対して $f \circ g = f \circ h$．

(2) f が余定数射 $\iff$ C^{op} において f が定数射．

(3) f が零射 $\iff$ f が定数射かつ余定数射．

【定義】

圏 C が零射を持つ
$\iff$ 任意の $a, b \in C$ に対して零射 $a \to b$ が存在．

準備

【命題】

零射 $a \to b$ と零射 $b \to a$ が存在すれば，零射 $a \to b$ は一意． $\square$

従って圏 C が零射を持つとき「a から b への零射」は一意である．
そこでそれを 0_{ab} もしくは単に 0 で表す．

【命題】

零射 $a \to b$ と零射 $b \to a$ が存在すれば，零射 $a \to b$ は一意． $\square$

従って圏 C が零射を持つとき「a から b への零射」は一意である．そこでそれを 0_{ab} もしくは単に 0 で表す．

【定義】

零射を持つ圏 C において，射 $f\colon a \to b$ の核とは，f と 0_{ab} の equalizer のことをいう．また余核とは f と 0_{ab} の coequalizer のことをいう．

準備

即ち，$f\colon a \to b$ の核とは射 $\ker(f)\colon x \to a$ であって

(1) $f \circ \ker(f) = 0_{xb}$ である．

(2) $g\colon c \to a$ が $f \circ g = 0_{cb}$ を満たすとき，$h\colon c \to x$ が一意に存在して $\ker(f) \circ h = g$ となる．

$$\begin{tikzcd} x \arrow[r, "\ker(f)"] & a \arrow[r, "f"] & b \\ c \arrow[u, dashed, "h"] \arrow[ur, "g"'] & & \end{tikzcd}$$

ここでは射 k が f の核であることを $k \cong \ker(f)$ で表す．
また射 k が f の余核であることを $k \cong \mathrm{coker}(f)$ で表す．

【命題】

$a \xrightarrow{f} b \xrightarrow{g} c$ で f はエピ射とする．このとき

$$g \text{ の余核が存在する} \iff g \circ f \text{ の余核が存在する}$$

であり，これらは一致する (即ち $\mathrm{coker}(g) \cong \mathrm{coker}(g \circ f)$)．

証明.

普遍性から容易に分かる． $\square$

準備

【命題】

$f\colon a \to b$ の余核 $\mathrm{coker}(f)\colon b \to c$ が存在して
更にその核 $\ker(\mathrm{coker}(f))\colon d \to b$ も存在するとき
余核 $\mathrm{coker}(\ker(\mathrm{coker}(f)))$ も存在して

$$\mathrm{coker}(\ker(\mathrm{coker}(f))) \cong \mathrm{coker}(f).$$

証明.

普遍性から容易に分かる. $\square$

準備

零射を構成するよく知られた方法は，零対象を使うものである．

準備

零射を構成するよく知られた方法は，零対象を使うものである．

【定義】

対象 $x \in C$ が零対象 $\iff$ x が始対象かつ終対象．

零対象は記号では 0 で表すことが多いので，ここでは以下そのようにする．

準備

零射を構成するよく知られた方法は，零対象を使うものである．

【定義】

対象 $x \in C$ が零対象 $\iff$ x が始対象かつ終対象．

零対象は記号では 0 で表すことが多いので，ここでは以下そのようにする．

【命題】

零対象を持つ圏は零射を持つ．

証明.

$a, b \in C$ に対して $0_{ab} := (a \xrightarrow{!} 0 \xrightarrow{!} b)$ とすれば 0_{ab} は零射である． $\square$

【定義】

零射を持つ圏 C の射 f が正規モノ射 (normal monomorphism)
$\Longleftrightarrow$ ある射 g により $f \cong \ker(g)$ と書ける.

正規モノ射はモノ射である. (逆は不成立)

【定義】

零射を持つ圏 C の射 f が正規モノ射 (normal monomorphism)
$\Longleftrightarrow$ ある射 g により $f \cong \ker(g)$ と書ける.

正規モノ射はモノ射である. (逆は不成立)

【定義】

零射を持つ圏 C が正規圏 (normal category)
$\Longleftrightarrow$ C の任意のモノ射が正規モノ射である.

準備

【定義】

零射を持つ圏 C の射 f が正規モノ射 (normal monomorphism)
$\iff$ ある射 g により $f \cong \ker(g)$ と書ける．

正規モノ射はモノ射である．(逆は不成立)

【定義】

零射を持つ圏 C が正規圏 (normal category)
$\iff$ C の任意のモノ射が正規モノ射である．

【命題】

C を余核を持つ正規圏とする．射 f に対して，$\ker(\mathrm{coker}(f))$ が存在するならば $\ker(\mathrm{coker}(f))$ が f の像になる．

準備

【命題】

C を余核を持つ正規圏とする．射 f に対して，$\ker(\mathrm{coker}(f))$ が存在するならば $\ker(\mathrm{coker}(f))$ が f の像になる．

証明．

$$\begin{tikzcd}
 & d \arrow[dr, "\ker(\mathrm{coker}(f))"] & & \\
a \arrow[rr, "f"] & & b \arrow[rr, "\mathrm{coker}(f)"] & & c
\end{tikzcd}$$

【命題】

C を余核を持つ正規圏とする．射 f に対して，$\ker(\mathrm{coker}(f))$ が存在するならば $\ker(\mathrm{coker}(f))$ が f の像になる．

証明.

まず $\mathrm{coker}(f) \circ f = 0$ だから，$\ker(\mathrm{coker}(f))$ の普遍性により次の h を得る．$\ker(\mathrm{coker}(f))$ が f の像であることを示す．

$$\begin{tikzcd} & d \arrow[dr, "\ker(\mathrm{coker}(f))"] & & \\ a \arrow[ur, dashed, "h"] \arrow[rr, "f"] & & b \arrow[r, "\mathrm{coker}(f)"] & c \end{tikzcd}$$

準備

【命題】

C を余核を持つ正規圏とする．射 f に対して，$\ker(\mathrm{coker}(f))$ が存在するならば $\ker(\mathrm{coker}(f))$ が f の像になる．

証明.

そこでモノ射 $k\colon x \to b$ と射 $g\colon a \to x$ を使って
$f = (a \xrightarrow{g} x \xrightarrow{k} b)$ と書けたとする．

$$
\begin{tikzcd}
 & d \arrow[dr, "\ker(\mathrm{coker}(f))"] & & \\
a \arrow[ur, "h"] \arrow[rr, "f"] \arrow[dr, dashed, "g"'] & & b \arrow[r, "\mathrm{coker}(f)"] & c \\
 & x \arrow[ur, dashed, "k"'] & &
\end{tikzcd}
$$

準備

【命題】

C を余核を持つ正規圏とする．射 f に対して，$\ker(\mathrm{coker}(f))$ が存在するならば $\ker(\mathrm{coker}(f))$ が f の像になる．

証明．

モノ射 k の余核を取る．
k が正規モノ射だから $\ker(\mathrm{coker}(k)) \cong k$ である．

$$
\begin{tikzcd}
 & d \arrow[dr, "\ker(\mathrm{coker}(f))"] & & \\
a \arrow[ur, "h"] \arrow[rr, "f"] \arrow[dr, "g"'] & & b \arrow[r, "\mathrm{coker}(f)"] \arrow[dr, dashed, "\mathrm{coker}(k)"'] & c \\
 & x \arrow[ur, "k"'] & & u
\end{tikzcd}
$$

【命題】

C を余核を持つ正規圏とする．射 f に対して，$\ker(\mathrm{coker}(f))$ が存在するならば $\ker(\mathrm{coker}(f))$ が f の像になる．

証明.

$\mathrm{coker}(k) \circ f = \mathrm{coker}(k) \circ k \circ g = 0$ だから

$$\begin{tikzcd}
 & d \arrow[dr, "\ker(\mathrm{coker}(f))"] & & \\
a \arrow[ur, "h"] \arrow[rr, "f"] \arrow[dr, "g"'] & & b \arrow[r, "\mathrm{coker}(f)"] \arrow[dr, "\mathrm{coker}(k)"'] & c \\
 & x \arrow[ur, "k"'] & & u
\end{tikzcd}$$

準備

【命題】

C を余核を持つ正規圏とする．射 f に対して，$\ker(\mathrm{coker}(f))$ が存在するならば $\ker(\mathrm{coker}(f))$ が f の像になる．

証明．

$\mathrm{coker}(k) \circ f = \mathrm{coker}(k) \circ k \circ g = 0$ だから

$$
\begin{tikzcd}
 & d \arrow[dr, "\ker(\mathrm{coker}(f))"] & & \\
a \arrow[ur, "h"] \arrow[rr, "f"] \arrow[dr, "g"'] & & b \arrow[r, "\mathrm{coker}(f)"] \arrow[dr, "\mathrm{coker}(k)"'] & c \\
 & x \arrow[ur, "k"'] & & u
\end{tikzcd}
$$

準備

【命題】

C を余核を持つ正規圏とする．射 f に対して，$\ker(\mathrm{coker}(f))$ が存在するならば $\ker(\mathrm{coker}(f))$ が f の像になる．

証明．

$\mathrm{coker}(k) \circ f = \mathrm{coker}(k) \circ k \circ g = 0$ だから

$$
\begin{tikzcd}
 & d \arrow[dr, "\ker(\mathrm{coker}(f))"] & & \\
a \arrow[ur, "h"] \arrow[rr, "f"] \arrow[dr, "g"'] & & b \arrow[r, "\mathrm{coker}(f)"] \arrow[ddr, "\mathrm{coker}(k)"'] & c \\
 & x \arrow[ur, "k"'] & & \\
 & & & u
\end{tikzcd}
$$

準備

【命題】

C を余核を持つ正規圏とする．射 f に対して，$\ker(\mathrm{coker}(f))$ が存在するならば $\ker(\mathrm{coker}(f))$ が f の像になる．

証明.

$\mathrm{coker}(k) \circ f = \mathrm{coker}(k) \circ k \circ g = 0$ だから
$\mathrm{coker}(f)$ の普遍性により $p\colon c \to u$ を得る．

$$
\begin{tikzcd}
 & d \arrow[dr, "\ker(\mathrm{coker}(f))"] & & \\
a \arrow[ur, "h"] \arrow[rr, "f"] \arrow[dr, "g"'] & & b \arrow[r, "\mathrm{coker}(f)"] \arrow[dr, "\mathrm{coker}(k)"'] & c \arrow[d, dashed, "p"] \\
 & x \arrow[ur, "k"'] & & u
\end{tikzcd}
$$

【命題】

C を余核を持つ正規圏とする．射 f に対して，$\mathrm{ker}(\mathrm{coker}(f))$ が存在するならば $\mathrm{ker}(\mathrm{coker}(f))$ が f の像になる．

証明.

このとき $\mathrm{coker}(k) \circ \mathrm{ker}(\mathrm{coker}(f)) = 0$ だから

$$
\begin{tikzcd}
 & d \arrow[dr, "\mathrm{ker}(\mathrm{coker}(f))"] & & \\
a \arrow[ur, "h"] \arrow[rr, "f"] \arrow[dr, "g"'] & & b \arrow[r, "\mathrm{coker}(f)"] \arrow[dr, "\mathrm{coker}(k)"'] & c \arrow[d, "p"] \\
 & x \arrow[ur, "k"'] & & u
\end{tikzcd}
$$

【命題】

C を余核を持つ正規圏とする．射 f に対して，$\ker(\mathrm{coker}(f))$ が存在するならば $\ker(\mathrm{coker}(f))$ が f の像になる．

証明.

このとき $\mathrm{coker}(k) \circ \ker(\mathrm{coker}(f)) = 0$ だから

$$\begin{tikzcd}
& d \arrow[dr, "\ker(\mathrm{coker}(f))"] & & \\
a \arrow[ur, "h"] \arrow[rr, "f"] \arrow[dr, "g"'] & & b \arrow[r, "\mathrm{coker}(f)"] \arrow[dr, "\mathrm{coker}(k)"'] & c \arrow[d, "p"] \\
& x \arrow[ur, "k"'] & & u
\end{tikzcd}$$

【命題】

C を余核を持つ正規圏とする．射 f に対して，$\ker(\mathrm{coker}(f))$ が存在するならば $\ker(\mathrm{coker}(f))$ が f の像になる．

証明.

このとき $\mathrm{coker}(k) \circ \ker(\mathrm{coker}(f)) = 0$ だから

$$\begin{tikzcd}
& d \arrow[dr, "\ker(\mathrm{coker}(f))"] & & \\
a \arrow[ur, "h"] \arrow[rr, "f"] \arrow[dr, "g"'] & & b \arrow[r, "\mathrm{coker}(f)"] \arrow[ddr, "\mathrm{coker}(k)"'] & c \arrow[dd, "p"] \\
& x \arrow[ur, "k"'] & & \\
& & & u
\end{tikzcd}$$

【命題】

C を余核を持つ正規圏とする．射 f に対して，$\ker(\mathrm{coker}(f))$ が存在するならば $\ker(\mathrm{coker}(f))$ が f の像になる．

証明．

このとき $\mathrm{coker}(k) \circ \ker(\mathrm{coker}(f)) = 0$ だから
$k \cong \ker(\mathrm{coker}(k))$ の普遍性により $d \to x$ が得られる．

$$
\begin{tikzcd}
 & d \arrow[dd, dashed] \arrow[dr, "\ker(\mathrm{coker}(f))"] & & \\
a \arrow[ur, "h"] \arrow[dr, "g"'] & & b \arrow[r, "\mathrm{coker}(f)"] \arrow[dr, "\mathrm{coker}(k)"'] & c \arrow[d, "p"] \\
 & x \arrow[ur, "k"'] & & u
\end{tikzcd}
$$

□

【定義】

圏 C がアーベル圏とは次の条件を満たすことをいう．

(1) C は零対象を持つ．

(2) C は直積と余直積を持つ．

(3) C は核と余核を持つ．

(4) C は正規かつ余正規である．

今さっき示した命題より，アーベル圏では $\ker(\operatorname{coker}(f)) \cong \operatorname{Im}(f)$ である．

弱完全圏

弱完全圏

【定義】

C を零対象を持つ圏とする．$D \subset \mathrm{Mor}(C)$ が「deflation の集まり」とは以下の条件を満たすことをいう．

(1) D は同型射と一意な射 $!\colon a \to 0$ を全て含む．

(2) 任意の $f \in D$ は核 $\ker(f)$ を持ち，$f \cong \mathrm{coker}(\ker(f))$．

(3) D は合成で閉じている．

(4) $a \xrightarrow{f} b \xrightarrow{g} c$ で $f, g \circ f \in D$ ならば $g \in D$ である．

このとき D の元を deflation と呼び，deflation の核となる射を inflation と呼ぶ．

弱完全圏

【定義】

$D \subset \mathrm{Mor}(C)$ を deflation の集まりとする．
$a \xrightarrow{f} b \xrightarrow{g} c$ が短完全列 $\Longleftrightarrow g \in D$ かつ $f \cong \ker(g)$．

ここでは inflation を $\rightarrowtail$，deflation を $\twoheadrightarrow$ で表す．

例えば短完全列は $a \overset{f}{\rightarrowtail} b \overset{g}{\twoheadrightarrow} c$ のようになる．

【定義】

零対象を持つ圏 C が弱完全圏とは，次の条件 (9 項補題 (上) と呼ぶ) を満たす deflation の集まり $D \subset \mathrm{Mor}(C)$ が与えられていることをいう．

次の可換図式で縦列が全て短完全列のとき
中央と下の横列が短完全列ならば上の横列も短完全列となる．

$$
\begin{tikzcd}
a_0 \arrow[r] \arrow[d, tail] & a_1 \arrow[r] \arrow[d, tail] & a_2 \arrow[d, tail] \\
b_0 \arrow[r, tail] \arrow[d, -latex] & b_1 \arrow[r, -latex] \arrow[d, -latex] & b_2 \arrow[d, -latex] \\
c_0 \arrow[r, tail] & c_1 \arrow[r, -latex] & c_2
\end{tikzcd}
$$

弱完全圏

【例】

C をアーベル圏とするとき $D = \{f \in \mathrm{Mor}(C) \mid f$ はエピ射 $\}$ により弱完全圏となる．この場合 inflation とはモノ射のことであり

$$a \xrightarrow{f} b \xrightarrow{g} c \text{ が (弱完全圏の意味で) 短完全列}$$
$$\iff 0 \to a \xrightarrow{f} b \xrightarrow{g} c \to 0 \text{ が (通常の意味で) 完全列}$$

となる．(後述) □

【例】

群の圏 $\mathbf{Grp}$ は $D = \{f \in \mathrm{Mor}(\mathbf{Grp}) \mid f$ はエピ射 $\}$ により弱完全圏となる．(後述) □

弱完全圏

以下，弱完全圏 C が与えられているものとする．

弱完全圏

以下，弱完全圏 C が与えられているものとする．

【命題】

inflation は正規モノ射 (従ってモノ射) であり，deflation は正規エピ射 (従ってエピ射) である． $\square$

弱完全圏

【命題】 (9 項補題 (下))

次の可換図式で縦列が全て短完全列のとき，上と中央の横列が短完全列ならば下の縦列も短完全列となる．

$$
\begin{tikzcd}
a_0 \arrow[r, tail, red, "u_0"] \arrow[d, tail, red, "f_0"'] & a_1 \arrow[r, cyan, "u_1"] \arrow[d, tail, red, "f_1"] & a_2 \arrow[d, tail, red, "f_2"] \\
b_0 \arrow[r, tail, red, "v_0"] \arrow[d, cyan, "g_0"'] & b_1 \arrow[r, cyan, "v_1"] \arrow[d, cyan, "g_1"] & b_2 \arrow[d, cyan, "g_2"] \\
c_0 \arrow[r, "w_0"'] & c_1 \arrow[r, "w_1"'] & c_2
\end{tikzcd}
$$

弱完全圏

証明.

まず弱完全圏の定義より w_1 は deflation である.

$$
\begin{tikzcd}
& a_0 \arrow[r, tail, "u_0"] \arrow[d, tail, "f_0"'] & a_1 \arrow[r, "u_1"] \arrow[d, tail, "f_1"] & a_2 \arrow[d, tail, "f_2"] \\
& b_0 \arrow[r, tail, "v_0"] \arrow[ddl, bend right, "g_0"'] & b_1 \arrow[r, "v_1"] \arrow[d, "g_1"] & b_2 \arrow[d, "g_2"] \\
& & c_1 \arrow[r, "w_1"'] & c_2 \\
c_0 \arrow[uurr, bend right, "w_0"'] & & &
\end{tikzcd}
$$

弱完全圏

証明.

そこで $\ker(w_1)$ を考える.

$$\begin{tikzcd}
 & a_0 \arrow[r, tail, "u_0"] \arrow[d, tail, "f_0"'] & a_1 \arrow[r, "u_1"] \arrow[d, tail, "f_1"] & a_2 \arrow[d, tail, "f_2"] \\
 & b_0 \arrow[r, tail, "v_0"] \arrow[ddl, bend right, "g_0"'] & b_1 \arrow[r, "v_1"] \arrow[d, "g_1"] & b_2 \arrow[d, "g_2"] \\
 & c_0' \arrow[r, tail, dashed, "\ker(w_1)"'] & c_1 \arrow[r, "w_1"'] & c_2 \\
c_0 \arrow[urr, bend right, "w_0"'] & & &
\end{tikzcd}$$

証明.

$w_1 \circ g_1 \circ v_0 = 0$ である．

$$
\begin{tikzcd}
& a_0 \arrow[r, tail, "u_0"] \arrow[d, tail, "f_0"'] & a_1 \arrow[r, "u_1"] \arrow[d, tail, "f_1"] & a_2 \arrow[d, tail, "f_2"] \\
& b_0 \arrow[r, tail, "v_0"] \arrow[ddl, bend right, "g_0"'] & b_1 \arrow[r, "v_1"] \arrow[d, "g_1"] & b_2 \arrow[d, "g_2"] \\
& c_0' \arrow[r, tail, "\ker(w_1)"'] & c_1 \arrow[r, "w_1"'] & c_2 \\
c_0 \arrow[uurr, bend right, "w_0"'] & & &
\end{tikzcd}
$$

証明.

$w_1 \circ g_1 \circ v_0 = 0$ である.

$$
\begin{tikzcd}
 & a_0 \arrow[r, tail, "u_0"] \arrow[d, tail, "f_0"'] & a_1 \arrow[r, "u_1"] \arrow[d, tail, "f_1"] & a_2 \arrow[d, tail, "f_2"] \\
 & b_0 \arrow[r, tail, "v_0"] \arrow[dddl, bend right, "g_0"'] \arrow[ddrr, thick] & b_1 \arrow[r, "v_1"] \arrow[d, "g_1"] & b_2 \arrow[d, "g_2"] \\
 & c_0' \arrow[r, tail, "\ker(w_1)"'] & c_1 \arrow[r, "w_1"'] & c_2 \\
c_0 \arrow[urr, bend right, "w_0"'] & & &
\end{tikzcd}
$$

証明.

$w_1 \circ g_1 \circ v_0 = 0$ である．

$$
\begin{tikzcd}
 & a_0 \arrow[r, tail, "u_0"] \arrow[d, tail, "f_0"'] & a_1 \arrow[r, "u_1"] \arrow[d, tail, "f_1"] & a_2 \arrow[d, tail, "f_2"] \\
 & b_0 \arrow[r, tail, "v_0"] \arrow[ddl, bend right, "g_0"'] & b_1 \arrow[r, "v_1"] \arrow[d, "g_1"] & b_2 \arrow[d, "g_2"] \\
 & c'_0 \arrow[r, tail, "\ker(w_1)"'] & c_1 \arrow[r, "w_1"'] & c_2 \\
c_0 \arrow[uurr, bend right, "w_0"'] & & &
\end{tikzcd}
$$

弱完全圏

証明.

よって $\ker(w_1)$ の普遍性から h を得る．

$$\begin{tikzcd}
& a_0 \arrow[r, tail, "u_0"] \arrow[d, tail, "f_0"'] & a_1 \arrow[r, "u_1"] \arrow[d, tail, "f_1"] & a_2 \arrow[d, tail, "f_2"] \\
& b_0 \arrow[r, tail, "v_0"] \arrow[d, dashed, "h"] \arrow[ddl, bend right, "g_0"'] & b_1 \arrow[r, "v_1"] \arrow[d, "g_1"] & b_2 \arrow[d, "g_2"] \\
& c'_0 \arrow[r, tail, "\ker(w_1)"'] & c_1 \arrow[r, "w_1"'] & c_2 \\
c_0 \arrow[uurr, bend right, "w_0"'] & & &
\end{tikzcd}$$

証明.

9 項補題 (上) により左の縦列は短完全列，よって h は deflation.

$$
\begin{tikzcd}
 & a_0 \arrow[r, tail, "u_0"] \arrow[d, tail, "f_0"'] & a_1 \arrow[r, "u_1"] \arrow[d, tail, "f_1"] & a_2 \arrow[d, tail, "f_2"] \\
 & b_0 \arrow[r, tail, "v_0"] \arrow[d, "h"] \arrow[ddl, bend right, "g_0"'] & b_1 \arrow[r, "v_1"] \arrow[d, "g_1"] & b_2 \arrow[d, "g_2"] \\
 & c_0' \arrow[r, tail, "\ker(w_1)"'] & c_1 \arrow[r, "w_1"'] & c_2 \\
c_0 \arrow[uurr, bend right, "w_0"'] & & &
\end{tikzcd}
$$

証明.

$g_0 \cong \mathrm{coker}(f_0) \cong h$ より同型 k が存在する．

$$
\begin{tikzcd}
 & a_0 \arrow[r, tail, "u_0"] \arrow[d, tail, "f_0"'] & a_1 \arrow[r, "u_1"] \arrow[d, tail, "f_1"] & a_2 \arrow[d, tail, "f_2"] \\
 & b_0 \arrow[r, tail, "v_0"] \arrow[d, "h"] \arrow[ddl, bend right, "g_0"'] & b_1 \arrow[r, "v_1"] \arrow[d, "g_1"] & b_2 \arrow[d, "g_2"] \\
 & c_0' \arrow[r, tail, "\ker(w_1)"'] & c_1 \arrow[r, "w_1"'] & c_2 \\
c_0 \arrow[ur, dashed, "\cong", "k"'] \arrow[urr, bend right, "w_0"'] & & &
\end{tikzcd}
$$

弱完全圏

証明.

故に $w_0 \cong \ker(w_1)$ である．

$$
\begin{tikzcd}
 & a_0 \arrow[r, tail, "u_0"] \arrow[d, tail, "f_0"'] & a_1 \arrow[r, "u_1"] \arrow[d, tail, "f_1"] & a_2 \arrow[d, tail, "f_2"] \\
 & b_0 \arrow[r, tail, "v_0"] \arrow[d, "h"] \arrow[ddl, bend right, "g_0"'] & b_1 \arrow[r, "v_1"] \arrow[d, "g_1"] & b_2 \arrow[d, "g_2"] \\
 & c_0' \arrow[r, tail, "\ker(w_1)"'] & c_1 \arrow[r, "w_1"'] & c_2 \\
c_0 \arrow[ur, "\sim", "k"'] \arrow[urr, tail, bend right, "w_0"'] & & &
\end{tikzcd}
$$

□

【命題】

$a \xrightarrow{f} b \xrightarrow{g} c$ で g と $g \circ f$ が inflation ならば f も inflation である．

弱完全圏

証明.

次の図式で f が inflation であることを示す.

$$\begin{tikzcd} a \arrow[r, "f"] \arrow[d, tail, "\mathrm{id}"'] & b \arrow[d, tail, "g"] \\ a \arrow[r, tail, "g\circ f"] & c \end{tikzcd}$$

証明.

まず次のように余核を取る.

$$\begin{tikzcd}
a \arrow[r, "f"] \arrow[d, tail, "\mathrm{id}"'] & b \arrow[d, tail, "g"'] & \\
a \arrow[r, tail, "g\circ f"] \arrow[d] & c \arrow[r, "\mathrm{coker}(g\circ f)"] \arrow[d, "\mathrm{coker}(g)"] & s \\
0 \arrow[r, tail] & t \arrow[r, "\mathrm{id}"'] & t
\end{tikzcd}$$

弱完全圏

証明.

$\operatorname{coker}(g \circ f)$ の普遍性から h が得られる.

$$
\begin{tikzcd}
a \arrow[r, "f"] \arrow[d, tail, "\mathrm{id}"'] & b \arrow[d, tail, "g"'] & \\
a \arrow[r, tail, "g\circ f"] \arrow[d] & c \arrow[r, "\operatorname{coker}(g\circ f)"] \arrow[d, "\operatorname{coker}(g)"] & s \arrow[d, dashed, "h"] \\
0 \arrow[r, tail] & t \arrow[r, "\mathrm{id}"'] & t
\end{tikzcd}
$$

弱完全圏

証明.

弱完全圏の定義より h は deflation である.

$$
\begin{tikzcd}
a \arrow[r, "f"] \arrow[d, tail, "\mathrm{id}"'] & b \arrow[d, tail, "g"'] & \\
a \arrow[r, tail, "g\circ f"] \arrow[d] & c \arrow[r, "\mathrm{coker}(g\circ f)"] \arrow[d, "\mathrm{coker}(g)"] & s \arrow[d, "h"] \\
0 \arrow[r, tail] & t \arrow[r, "\mathrm{id}"'] & t
\end{tikzcd}
$$

弱完全圏

証明.

そこで $\ker(h)$ を考える.

$$
\begin{tikzcd}
a \arrow[r, "f"] \arrow[d, tail, "\mathrm{id}"'] & b \arrow[d, tail, "g"'] & & u \arrow[d, tail, dashed, "\ker(h)"] \\
a \arrow[r, tail, "g\circ f"] \arrow[d] & c \arrow[rr, "\mathrm{coker}(g\circ f)"] \arrow[d, "\mathrm{coker}(g)"] & & s \arrow[d, "h"] \\
0 \arrow[r, tail] & t \arrow[rr, "\mathrm{id}"'] & & t
\end{tikzcd}
$$

弱完全圏

証明.

$\ker(h)$ の普遍性から k が得られる.

$$
\begin{tikzcd}
a \arrow[r, "f"] \arrow[d, tail, "\mathrm{id}"'] & b \arrow[r, dashed, "k"] \arrow[d, tail, "g"'] & u \arrow[d, tail, "\ker(h)"] \\
a \arrow[r, tail, "g \circ f"] \arrow[d] & c \arrow[r, "\mathrm{coker}(g \circ f)"] \arrow[d, "\mathrm{coker}(g)"] & s \arrow[d, "h"] \\
0 \arrow[r, tail] & t \arrow[r, "\mathrm{id}"'] & t
\end{tikzcd}
$$

証明.

9 項補題 (上) により上の横列は短完全列，よって f は inflation．

$$
\begin{tikzcd}
a \arrow[r, "f"] \arrow[d, tail, "\mathrm{id}"'] & b \arrow[r, dashed, "k"] \arrow[d, tail, "g"'] & u \arrow[d, tail, "\ker(h)"] \\
a \arrow[r, tail, "g \circ f"] \arrow[d, two heads] & c \arrow[r, two heads, "\mathrm{coker}(g \circ f)"] \arrow[d, two heads, "\mathrm{coker}(g)"] & s \arrow[d, two heads, "h"] \\
0 \arrow[r, tail] & t \arrow[r, two heads, "\mathrm{id}"'] & t
\end{tikzcd}
$$

□

弱完全圏

【定義】

$\mathcal{C}$ の射 $f\colon a \to b$ が admissible
$\iff$ ある $x \in \mathcal{C}$ と deflation $f'\colon a \to x$，inflation $f''\colon x \to b$ が存在して $f = f'' \circ f'$ となる．

$$
\begin{tikzcd}
a \arrow[rr, "f"] \arrow[dr, two heads, "f'"'] & & b \\
& x \arrow[ur, tail, "f''"'] &
\end{tikzcd}
$$

アーベル圏では全ての射が admissible である (epi-mono factorization)

【命題】

admissible な射 $f\colon a \to b$ を $f = (\, a \xrightarrow{f'} x \overset{f''}{\rightarrowtail} b \,)$ と分解したとき，次の同型が成り立つ．

$$\ker(f) \cong \ker(f'), \quad \mathrm{coker}(\ker(f)) \cong f',$$
$$\mathrm{coker}(f) \cong \mathrm{coker}(f''), \quad \ker(\mathrm{coker}(f)) \cong f''.$$

$$\begin{tikzcd}
\bullet \arrow[r, tail, "\ker(f) \cong \ker(f')"] & a \arrow[rr, "f"] \arrow[dr, "f'"'] & & b \arrow[r, "\mathrm{coker}(f) \cong \mathrm{coker}(f'')"] & \bullet \\
& & x \arrow[ur, tail, "f''"'] & &
\end{tikzcd}$$

弱完全圏

【命題】

admissible な射 $f\colon a \to b$ を $f = \left(a \xrightarrow{f'} x \overset{f''}{\rightarrowtail} b \right)$ と分解したとき，次の同型が成り立つ．

$$
\begin{aligned}
&\ker(f) \cong \ker(f'), \quad \operatorname{coker}(\ker(f)) \cong f', \\
&\operatorname{coker}(f) \cong \operatorname{coker}(f''), \quad \ker(\operatorname{coker}(f)) \cong f''.
\end{aligned}
$$

$$
\begin{tikzcd}
\bullet \arrow[r, tail, "\ker(f)\cong\ker(f')"] & a \arrow[rr, "f"] \arrow[dr, "f'"'] & & b \arrow[r, "\operatorname{coker}(f)\cong\operatorname{coker}(f'')"] & \bullet \\
& & x \arrow[ur, tail, "f''"'] & &
\end{tikzcd}
$$

証明.

f' がエピ射だから $\operatorname{coker}(f) \cong \operatorname{coker}(f'')$ となり，よって $\ker(\operatorname{coker}(f)) \cong f''$ が分かる．同様にして $\ker(f) \cong \ker(f')$，$\operatorname{coker}(\ker(f)) \cong f'$． □

従って f が admissible のとき

$$\mathrm{coker}(\ker(f)) \cong f', \quad \ker(\mathrm{coker}(f)) \cong f''$$

より $f = (\ a \xrightarrow{f'} x \overset{f''}{\rightarrowtail} b\ )$ という分解の仕方は同型を除いて一意であることが分かる．

そこで以下，そのような分解を $f = (\ a \xrightarrow{f^{\mathrm{d}}} [f] \overset{f^{\mathrm{i}}}{\rightarrowtail} b\ )$ で表す．

弱完全圏

【定義】

射の有限列 $a_0 \xrightarrow{f_0} a_1 \xrightarrow{f_1} \cdots \xrightarrow{f_n} a_{n+1}$ が完全列とは，$0 \leq k \leq n$ に対して f_k が admissible であり，かつ $0 \leq k < n$ に対して

$$[f_k] \overset{f_k^{\mathrm{i}}}{\rightarrowtail} a_{k+1} \overset{f_{k+1}^{\mathrm{d}}}{\twoheadrightarrow} [f_{k+1}]$$

が短完全列になることをいう．

$$\begin{tikzcd}
\cdots & a_k \arrow[rr, "f_k"] \arrow[dr, "f_k^{\mathrm{d}}"'] & & a_{k+1} \arrow[rr, "f_{k+1}"] \arrow[dr, "f_{k+1}^{\mathrm{d}}"'] & & a_{k+2} & \cdots \\
& & {[f_k]} \arrow[ur, tail, "f_k^{\mathrm{i}}"'] & & {[f_{k+1}]} \arrow[ur, tail, "f_{k+1}^{\mathrm{i}}"'] & &
\end{tikzcd}$$

弱完全圏

【命題】

$0 \to a \xrightarrow{f} b \xrightarrow{g} c$ が完全列 $\Longleftrightarrow$ g が admissible で $\ker(g) \cong f$.

弱完全圏

【命題】

$0 \to a \xrightarrow{f} b \xrightarrow{g} c$ が完全列 $\iff$ g が admissible で $\ker(g) \cong f$.

証明.

($\Longrightarrow$) 完全列の定義より g は admissible である．また完全列の定義より次の図式を得る．

$$\begin{tikzcd}
0 \arrow[rr] \arrow[dr] & & a \arrow[rr, "f"] \arrow[dr, "f^{\mathrm{d}}"'] & & b \arrow[rr, "g"] \arrow[dr, "g^{\mathrm{d}}"'] & & c \\
& 0 \arrow[ur, tail, "!"'] & & {[f]} \arrow[ur, tail, "f^{\mathrm{i}}"'] & & {[g]} \arrow[ur, tail, "g^{\mathrm{i}}"'] &
\end{tikzcd}$$

ここで $0 \overset{!}{\rightarrowtail} a \xrightarrow{f^{\mathrm{d}}} [f]$ と $[f] \overset{f^{\mathrm{i}}}{\rightarrowtail} b \xrightarrow{g^{\mathrm{d}}} z$ は短完全列である．よって $f^{\mathrm{d}} \cong \mathrm{coker}(!)$ は同型射である．従って $\ker(g^{\mathrm{d}}) \cong f$ が分かる．故に $\ker(g) \cong \ker(g^{\mathrm{d}}) \cong f$ である． $\square$

弱完全圏

証明.

$(\Longleftarrow)$ $f \cong \ker(g) \cong \ker(g^{\mathrm{d}})$ だから f は inflation である．よって

$$\begin{tikzcd}
0 \arrow[rr] \arrow[dr] & & a \arrow[rr, "f"] \arrow[dr, "\mathrm{id}"'] & & b \arrow[rr, "g"] \arrow[dr, "g^{\mathrm{d}}"'] & & c \\
& 0 \arrow[ur, tail, "!"'] & & a \arrow[ur, tail, "f"'] & & {[g]} \arrow[ur, tail, "g^{\mathrm{i}}"'] &
\end{tikzcd}$$

が得られるから $0 \to a \xrightarrow{f} b \xrightarrow{g} c$ は完全列である． $\square$

弱完全圏

【命題】

$a \xrightarrow{f} b \xrightarrow{g} c \to 0$ が完全列
$\iff f$ が admissible で $\mathrm{coker}(f) \cong g$ となる．

証明．

先の命題と同様． $\square$

弱完全圏

【命題】

$0 \to a \xrightarrow{f} b \xrightarrow{g} c \to 0$ が完全列 $\iff a \xrightarrow{f} b \xrightarrow{g} c$ が短完全列.

弱完全圏

【命題】

$0 \to a \xrightarrow{f} b \xrightarrow{g} c \to 0$ が完全列 $\iff a \xrightarrow{f} b \xrightarrow{g} c$ が短完全列.

証明.

$(\Longrightarrow)$ さっき示したように g は $\mathrm{deflation}$ であり，$\ker(g) \cong f$ である.

弱完全圏

【命題】

$0 \to a \xrightarrow{f} b \xrightarrow{g} c \to 0$ が完全列 $\iff$ $a \xrightarrow{f} b \xrightarrow{g} c$ が短完全列．

証明．

$(\Longrightarrow)$ さっき示したように g は deflation であり，$\ker(g) \cong f$ である．

$(\Longleftarrow)$ 次の図式により完全列の条件が成り立つ．

$$
\begin{tikzcd}
0 \arrow[rr] \arrow[dr] & & a \arrow[rr, "f"] \arrow[dr, "\mathrm{id}_a"'] & & b \arrow[rr, "g"] \arrow[dr, "g"'] & & c \arrow[rr] \arrow[dr] & & 0 \\
& 0 \arrow[ur, tail] & & a \arrow[ur, tail, "f"'] & & c \arrow[ur, tail, "\mathrm{id}_c"'] & & 0 \arrow[ur, tail] &
\end{tikzcd}
$$

$\square$

【補題】 (蛇の補題 (弱))

次の可換図式で横列は両方とも短完全列とする．

$$\begin{tikzcd} a_0 \arrow[r, tail, "u_0"] \arrow[d, "f_0"'] & a_1 \arrow[r, two heads, "u_1"] \arrow[d, "f_1"] & a_2 \arrow[d, "f_2"] \\ b_0 \arrow[r, tail, "v_0"'] & b_1 \arrow[r, two heads, "v_1"'] & b_2 \end{tikzcd}$$

また f_i は admissible で f_i の核と余核をそれぞれ $\ker(f_i)\colon d_i \to a_i$，$\operatorname{coker}(f_i)\colon b_i \to c_i$ とする．このとき完全列

$$d_0 \xrightarrow{\overline{u_0}} d_1 \xrightarrow{\overline{u_1}} d_2 \xrightarrow{\delta} c_0 \xrightarrow{\overline{v_0}} c_1 \xrightarrow{\overline{v_1}} c_2$$

が存在し，更に $\overline{u_0}$ は inflation で $\overline{v_1}$ は deflation である．

【定理】 (蛇の補題)

次の可換図式で横列は両方とも完全列とする．

$$\begin{tikzcd} & a_0 \arrow[r, "u_0"] \arrow[d, "f_0"'] & a_1 \arrow[r, "u_1"] \arrow[d, "f_1"] & a_2 \arrow[r] \arrow[d, "f_2"] & 0 \\ 0 \arrow[r] & b_0 \arrow[r, "v_0"'] & b_1 \arrow[r, "v_1"'] & b_2 & \end{tikzcd}$$

また f_i は admissible で f_i の核と余核をそれぞれ
$\ker(f_i)\colon d_i \to a_i$，$\operatorname{coker}(f_i)\colon b_i \to c_i$ とする．このとき完全列

$$d_0 \xrightarrow{\overline{u_0}} d_1 \xrightarrow{\overline{u_1}} d_2 \xrightarrow{\delta} c_0 \xrightarrow{\overline{v_0}} c_1 \xrightarrow{\overline{v_1}} c_2$$

が存在する．

【例】

アーベル圏における次の可換図式で横列は両方とも完全列とする．

$$\begin{tikzcd} & a_0 \arrow[r, "u_0"] \arrow[d, "f_0"'] & a_1 \arrow[r, "u_1"] \arrow[d, "f_1"] & a_2 \arrow[r] \arrow[d, "f_2"] & 0 \\ 0 \arrow[r] & b_0 \arrow[r, "v_0"'] & b_1 \arrow[r, "v_1"'] & b_2 & \end{tikzcd}$$

f_i の核，余核をそれぞれ $\ker(f_i)\colon d_i \to a_i$，$\operatorname{coker}(f_i)\colon b_i \to c_i$ とする．このとき完全列

$$d_0 \xrightarrow{\overline{u_0}} d_1 \xrightarrow{\overline{u_1}} d_2 \xrightarrow{\delta} c_0 \xrightarrow{\overline{v_0}} c_1 \xrightarrow{\overline{v_1}} c_2$$

が存在する． □

【例】

Grp における次の可換図式で横列は両方とも完全列とする．

$$\begin{tikzcd}
 & a_0 \arrow[r, "u_0"] \arrow[d, "f_0"'] & a_1 \arrow[r, "u_1"] \arrow[d, "f_1"] & a_2 \arrow[r] \arrow[d, "f_2"] & 0 \\
0 \arrow[r] & b_0 \arrow[r, "v_0"'] & b_1 \arrow[r, "v_1"'] & b_2 &
\end{tikzcd}$$

また f_i は admissible で f_i の核，余核をそれぞれ $\ker(f_i)\colon d_i \to a_i$，$\mathrm{coker}(f_i)\colon b_i \to c_i$ とする．このとき完全列

$$d_0 \xrightarrow{\overline{u_0}} d_1 \xrightarrow{\overline{u_1}} d_2 \xrightarrow{\delta} c_0 \xrightarrow{\overline{v_0}} c_1 \xrightarrow{\overline{v_1}} c_2$$

が存在する． □

弱完全圏

【補題】 (蛇の補題 (弱))

次の可換図式で横列は両方とも短完全列とする．

$$\begin{tikzcd} a_0 \arrow[r, tail, "u_0"] \arrow[d, "f_0"'] & a_1 \arrow[r, two heads, "u_1"] \arrow[d, "f_1"] & a_2 \arrow[d, "f_2"] \\ b_0 \arrow[r, tail, "v_0"'] & b_1 \arrow[r, two heads, "v_1"'] & b_2 \end{tikzcd}$$

また f_i は admissible で f_i の核と余核をそれぞれ $\ker(f_i)\colon d_i \to a_i$，$\mathrm{coker}(f_i)\colon b_i \to c_i$ とする．このとき完全列

$$d_0 \xrightarrow{\overline{u_0}} d_1 \xrightarrow{\overline{u_1}} d_2 \xrightarrow{\delta} c_0 \xrightarrow{\overline{v_0}} c_1 \xrightarrow{\overline{v_1}} c_2$$

が存在し，更に $\overline{u_0}$ は inflation で $\overline{v_1}$ は deflation である．

弱完全圏

証明.

これから以下の射を定義し，完全列となっていることを示す．

$$
\begin{tikzcd}
d_0 \arrow[rr, tail, "\overline{u_0}"] \arrow[dr, "\mathrm{id}"'] & & d_1 \arrow[rr, "\overline{u_1}"] \arrow[dr, "\overline{u_1}'"'] & & d_2 \arrow[rr, "\delta"] \arrow[dr, "\delta'"'] & & c_0 \\
& d_0 \arrow[ur, tail, "\overline{u_0}"'] & & z_0 \arrow[ur, tail, "\ker(\delta')"'] & & z_1 \arrow[ur, tail, "\ker(\overline{v_0}')"'] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
c_0 \arrow[rr, "\overline{v_0}"] \arrow[dr, "\overline{v_0}'"'] & & c_1 \arrow[rr, "\overline{v_1}"] \arrow[dr, "\overline{v_1}"'] & & c_2 \\
& z_2 \arrow[ur, tail, "\ker(\overline{v_1})"'] & & c_2 \arrow[ur, tail, "\mathrm{id}"'] &
\end{tikzcd}
$$

弱完全圏

スペースがないのでここに字を書きます

$$
\begin{tikzcd}
a_0 \arrow[r, rightarrowtail, "u_0"] \arrow[d, "f_0^{\mathrm{d}}"'] & a_1 \arrow[r, "u_1"] \arrow[d, "f_1^{\mathrm{d}}"'] & a_2 \arrow[d, "f_2^{\mathrm{d}}"] \\
{[f_0]} \arrow[d, rightarrowtail, "f_0^{\mathrm{i}}"'] & {[f_1]} \arrow[d, rightarrowtail, "f_1^{\mathrm{i}}"'] & {[f_2]} \arrow[d, rightarrowtail, "f_2^{\mathrm{i}}"] \\
b_0 \arrow[r, rightarrowtail, "v_0"] & b_1 \arrow[r, "v_1"] & b_2
\end{tikzcd}
$$

f_i の核，余核

$$
\begin{tikzcd}[row sep=large, column sep=huge]
d_0 \arrow[d, tail, dashed, "\ker(f_0)"'] & d_1 \arrow[d, tail, dashed, "\ker(f_1)"] & d_2 \arrow[d, tail, dashed, "\ker(f_2)"] \\
a_0 \arrow[r, tail, "u_0"] \arrow[d, "f_0^{\mathrm{d}}"'] & a_1 \arrow[r, "u_1"] \arrow[d, "f_1^{\mathrm{d}}"'] & a_2 \arrow[d, "f_2^{\mathrm{d}}"] \\
{[f_0]} \arrow[d, tail, "f_0^{\mathrm{i}}"'] & {[f_1]} \arrow[d, tail, "f_1^{\mathrm{i}}"'] & {[f_2]} \arrow[d, tail, "f_2^{\mathrm{i}}"] \\
b_0 \arrow[r, tail, "v_0"] \arrow[d, dashed, "\mathrm{coker}(f_0)"'] & b_1 \arrow[r, "v_1"] \arrow[d, dashed, "\mathrm{coker}(f_1)"'] & b_2 \arrow[d, dashed, "\mathrm{coker}(f_2)"] \\
c_0 & c_1 & c_2
\end{tikzcd}
$$

普遍性から得られる射

$$
\begin{tikzcd}
d_0 \arrow[d, tail, "\mathrm{ker}(f_0)"'] & d_1 \arrow[d, tail, "\mathrm{ker}(f_1)"] & d_2 \arrow[d, tail, "\mathrm{ker}(f_2)"] \\
a_0 \arrow[r, tail, "u_0"] \arrow[d, "f_0^{\mathrm{d}}"'] & a_1 \arrow[r, "u_1"] \arrow[d, "f_1^{\mathrm{d}}"'] & a_2 \arrow[d, "f_2^{\mathrm{d}}"] \\
{[f_0]} \arrow[d, tail, "f_0^{\mathrm{i}}"'] & {[f_1]} \arrow[r, dashed, "w_1"'] \arrow[d, tail, "f_1^{\mathrm{i}}"'] & {[f_2]} \arrow[d, tail, "f_2^{\mathrm{i}}"] \\
b_0 \arrow[r, tail, "v_0"] \arrow[d, "\mathrm{coker}(f_0)"'] & b_1 \arrow[r, "v_1"] \arrow[d, "\mathrm{coker}(f_1)"'] & b_2 \arrow[d, "\mathrm{coker}(f_2)"] \\
c_0 & c_1 \arrow[r, dashed, "\overline{v_1}"'] & c_2
\end{tikzcd}
$$

$w_1, \overline{v_1}$ の核

$$
\begin{tikzcd}
d_0 \arrow[d, tail, "\ker(f_0)"'] & & d_1 \arrow[d, tail, "\ker(f_1)"] & d_2 \arrow[d, tail, "\ker(f_2)"] \\
a_0 \arrow[rr, tail, "u_0"] \arrow[d, "f_0^{\mathrm{d}}"'] & & a_1 \arrow[r, "u_1"] \arrow[d, "f_1^{\mathrm{d}}"'] & a_2 \arrow[d, "f_2^{\mathrm{d}}"] \\
{[f_0]} \arrow[d, tail, "f_0^{\mathrm{i}}"'] & s \arrow[r, tail, dashed, "\ker(w_1)"] & {[f_1]} \arrow[r, "w_1"'] \arrow[d, tail, "f_1^{\mathrm{i}}"'] & {[f_2]} \arrow[d, tail, "f_2^{\mathrm{i}}"] \\
b_0 \arrow[rr, tail, "v_0"] \arrow[d, "\operatorname{coker}(f_0)"'] & & b_1 \arrow[r, "v_1"] \arrow[d, "\operatorname{coker}(f_1)"'] & b_2 \arrow[d, "\operatorname{coker}(f_2)"] \\
c_0 & z_2 \arrow[r, tail, dashed, "\ker(\overline{v_1})"'] & c_1 \arrow[r, "\overline{v_1}"'] & c_2
\end{tikzcd}
$$

普遍性から得られる射に 9 項補題 (上)

$$
\begin{tikzcd}
d_0 \arrow[d, tail, "\ker(f_0)"'] & & d_1 \arrow[d, tail, "\ker(f_1)"] & d_2 \arrow[d, tail, "\ker(f_2)"] \\
a_0 \arrow[rr, tail, "u_0"] \arrow[d, "f_0^{\mathrm{d}}"'] & & a_1 \arrow[r, "u_1"] \arrow[d, "f_1^{\mathrm{d}}"'] & a_2 \arrow[d, "f_2^{\mathrm{d}}"] \\
{[f_0]} \arrow[d, tail, "f_0^{\mathrm{i}}"'] & s \arrow[dl, tail, dashed, bend right, "i_0"] \arrow[r, tail, "\ker(w_1)"] & {[f_1]} \arrow[r, "w_1"'] \arrow[d, tail, "f_1^{\mathrm{i}}"'] & {[f_2]} \arrow[d, tail, "f_2^{\mathrm{i}}"] \\
b_0 \arrow[rr, tail, "v_0"] \arrow[d, "\operatorname{coker}(f_0)"'] \arrow[dr, dashed, bend right, "p_0"] & & b_1 \arrow[r, "v_1"] \arrow[d, "\operatorname{coker}(f_1)"'] & b_2 \arrow[d, "\operatorname{coker}(f_2)"] \\
c_0 & z_2 \arrow[r, tail, "\ker(\overline{v_1})"'] & c_1 \arrow[r, "\overline{v_1}"'] & c_2
\end{tikzcd}
$$

$\ker(\overline{v_1})$ の普遍性から得られる射

$$
\begin{tikzcd}
d_0 \arrow[d, tail, "\ker(f_0)"'] & & d_1 \arrow[d, tail, "\ker(f_1)"] & d_2 \arrow[d, tail, "\ker(f_2)"] \\
a_0 \arrow[rr, tail, "u_0"] \arrow[d, "f_0^{\mathrm{d}}"'] & & a_1 \arrow[r, "u_1"] \arrow[d, "f_1^{\mathrm{d}}"'] & a_2 \arrow[d, "f_2^{\mathrm{d}}"] \\
{[f_0]} \arrow[d, tail, "f_0^{\mathrm{i}}"'] & s \arrow[r, tail, "\ker(w_1)"] \arrow[dl, tail, bend right, "i_0"] & {[f_1]} \arrow[r, "w_1"'] \arrow[d, tail, "f_1^{\mathrm{i}}"'] & {[f_2]} \arrow[d, tail, "f_2^{\mathrm{i}}"] \\
b_0 \arrow[rr, tail, "v_0"] \arrow[d, "\mathrm{coker}(f_0)"'] \arrow[dr, bend left, "p_0"] & & b_1 \arrow[r, "v_1"] \arrow[d, "\mathrm{coker}(f_1)"'] & b_2 \arrow[d, "\mathrm{coker}(f_2)"] \\
c_0 \arrow[r, dashed, "\overline{v_0}'"'] & z_2 \arrow[r, tail, "\ker(\overline{v_1})"'] & c_1 \arrow[r, "\overline{v_1}"'] & c_2
\end{tikzcd}
$$

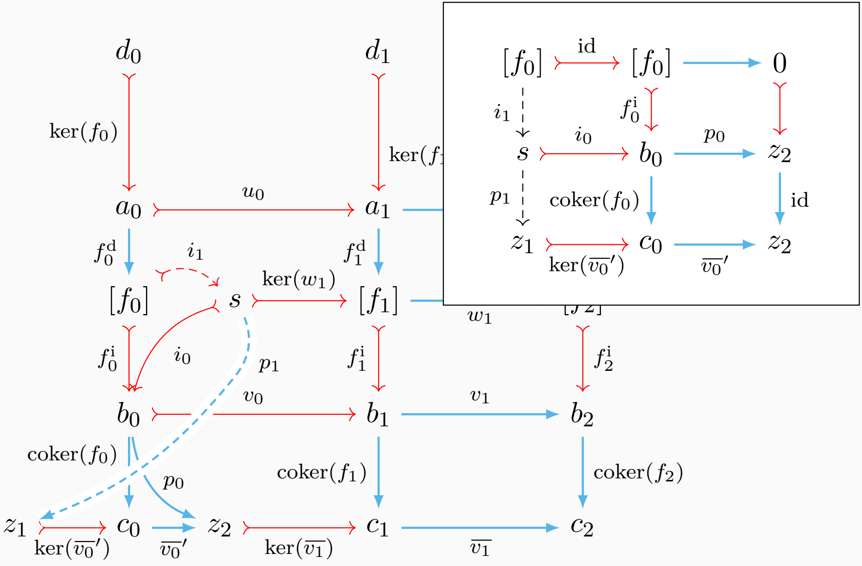
$\overline{v_0}'$ の核

$$
\begin{tikzcd}
 & d_0 \arrow[d, tail, "\ker(f_0)"'] & & d_1 \arrow[d, tail, "\ker(f_1)"] & d_2 \arrow[d, tail, "\ker(f_2)"] \\
 & a_0 \arrow[rr, tail, "u_0"] \arrow[d, "f_0^{\mathrm{d}}"'] & & a_1 \arrow[r, "u_1"] \arrow[d, "f_1^{\mathrm{d}}"'] & a_2 \arrow[d, "f_2^{\mathrm{d}}"] \\
 & {[f_0]} \arrow[d, tail, "f_0^{\mathrm{i}}"'] & s \arrow[r, tail, "\ker(w_1)"] \arrow[dl, tail, bend right, "i_0"] & {[f_1]} \arrow[r, "w_1"'] \arrow[d, tail, "f_1^{\mathrm{i}}"'] & {[f_2]} \arrow[d, tail, "f_2^{\mathrm{i}}"] \\
 & b_0 \arrow[rr, tail, "v_0"] \arrow[d, "\mathrm{coker}(f_0)"'] \arrow[dr, bend right, "p_0"] & & b_1 \arrow[r, "v_1"] \arrow[d, "\mathrm{coker}(f_1)"'] & b_2 \arrow[d, "\mathrm{coker}(f_2)"] \\
z_1 \arrow[r, tail, dashed, "\ker(\overline{v_0}')"'] & c_0 \arrow[r, "\overline{v_0}'"'] & z_2 \arrow[r, tail, "\ker(\overline{v_1})"'] & c_1 \arrow[r, "\overline{v_1}"'] & c_2
\end{tikzcd}
$$

弱完全圏

$$
\begin{tikzcd}
 & d_0 \arrow[d, tail, "\ker(f_0)"'] & & d_1 \arrow[d, tail, "\ker(f_1)"] & d_2 \arrow[d, tail, "\ker(f_2)"] \\
 & a_0 \arrow[rr, tail, "u_0"] \arrow[d, "f_0^{\mathrm{d}}"'] & & a_1 \arrow[r, "u_1"] \arrow[d, "f_1^{\mathrm{d}}"'] & a_2 \arrow[d, "f_2^{\mathrm{d}}"] \\
 & {[f_0]} \arrow[r, tail, dashed, bend left, "i_1"] \arrow[d, tail, "f_0^{\mathrm{i}}"'] & s \arrow[r, tail, "\ker(w_1)"] \arrow[dl, tail, bend right, "i_0"] \arrow[ddll, dashed, bend left, "p_1"] & {[f_1]} \arrow[r, "w_1"'] \arrow[d, tail, "f_1^{\mathrm{i}}"'] & {[f_2]} \arrow[d, tail, "f_2^{\mathrm{i}}"] \\
 & b_0 \arrow[rr, tail, "v_0"] \arrow[d, "\mathrm{coker}(f_0)"'] \arrow[dr, bend left, "p_0"] & & b_1 \arrow[r, "v_1"] \arrow[d, "\mathrm{coker}(f_1)"'] & b_2 \arrow[d, "\mathrm{coker}(f_2)"] \\
z_1 \arrow[r, tail, "\ker(\overline{v_0}')"'] & c_0 \arrow[r, "\overline{v_0}'"'] & z_2 \arrow[r, tail, "\ker(\overline{v_1})"'] & c_1 \arrow[r, "\overline{v_1}"'] & c_2
\end{tikzcd}
$$

弱完全圏


d_0
d_1
$\ker(f_0)$
a_0
u_0
a_1
f_0^{d}
f_1^{d}
i_1
$[f_0]$
s
$\ker(w_1)$
$[f_1]$
w_1
f_0^{i}
i_0
p_1
f_1^{i}
f_2^{i}
v_0
v_1
b_0
b_1
b_2
$\mathrm{coker}(f_0)$
p_0
$\mathrm{coker}(f_1)$
$\mathrm{coker}(f_2)$
z_1
c_0
z_2
c_1
c_2
$\ker(\overline{v_0}')$
$\overline{v_0}'$
$\ker(\overline{v_1})$
$\overline{v_1}$
$[f_0]$
id
$[f_0]$
0
i_1
f_0^{i}
s
i_0
b_0
p_0
z_2
p_1
$\mathrm{coker}(f_0)$
id
z_1
c_0
z_2
$\ker(\overline{v_0}')$
$\overline{v_0}'$

弱完全圏

$$\begin{tikzcd}
 & d_0 \arrow[d, tail, "\ker(f_0)"'] & & d_1 \arrow[d, tail, "\ker(f_1)"] & d_2 \arrow[d, tail, "\ker(f_2)"] \\
 & a_0 \arrow[rr, tail, "u_0"] \arrow[d, "f_0^{\mathrm{d}}"'] & & a_1 \arrow[r, "u_1"] \arrow[d, "f_1^{\mathrm{d}}"'] & a_2 \arrow[d, "f_2^{\mathrm{d}}"] \\
 & {[f_0]} \arrow[r, tail, dashed, bend left, "i_1"] \arrow[d, tail, "f_0^{\mathrm{i}}"'] & s \arrow[r, tail, "\ker(w_1)"] \arrow[dl, tail, bend right, "i_0"] \arrow[ddll, dashed, "p_1"] & {[f_1]} \arrow[r, "w_1"'] \arrow[d, tail, "f_1^{\mathrm{i}}"'] & {[f_2]} \arrow[d, tail, "f_2^{\mathrm{i}}"] \\
 & b_0 \arrow[rr, tail, "v_0"] \arrow[d, "\mathrm{coker}(f_0)"'] \arrow[dr, bend right, "p_0"] & & b_1 \arrow[r, "v_1"] \arrow[d, "\mathrm{coker}(f_1)"'] & b_2 \arrow[d, "\mathrm{coker}(f_2)"] \\
z_1 \arrow[r, tail, "\ker(\overline{v_0}')"'] & c_0 \arrow[r, "\overline{v_0}'"'] & z_2 \arrow[r, tail, "\ker(\overline{v_1})"'] & c_1 \arrow[r, "\overline{v_1}"'] & c_2
\end{tikzcd}$$

deflation 同士の合成

$$
\begin{tikzcd}
 & d_0 \arrow[d, tail, red, "\ker(f_0)"'] & & d_1 \arrow[d, tail, red, "\ker(f_1)"] & d_2 \arrow[d, tail, red, "\ker(f_2)"] \\
 & a_0 \arrow[rr, tail, red, "u_0"] \arrow[d, cyan, "f_0^{\mathrm{d}}"'] & & a_1 \arrow[r, cyan, "u_1"] \arrow[d, cyan, "f_1^{\mathrm{d}}"'] \arrow[dr, dashed, cyan, "f_2^{\mathrm{d}} \circ u_1"] & a_2 \arrow[d, cyan, "f_2^{\mathrm{d}}"] \\
 & {[f_0]} \arrow[r, tail, red, bend left, "i_1"] \arrow[d, tail, red, "f_0^{\mathrm{i}}"'] & s \arrow[r, tail, red, "\ker(w_1)"] \arrow[dl, tail, red, bend right, "i_0"] \arrow[ddll, cyan, "p_1"] & {[f_1]} \arrow[r, cyan, "w_1"'] \arrow[d, tail, red, "f_1^{\mathrm{i}}"'] & {[f_2]} \arrow[d, tail, red, "f_2^{\mathrm{i}}"] \\
 & b_0 \arrow[rr, tail, red, "v_0"] \arrow[d, cyan, "\operatorname{coker}(f_0)"'] \arrow[dr, cyan, bend right, "p_0"] & & b_1 \arrow[r, cyan, "v_1"] \arrow[d, cyan, "\operatorname{coker}(f_1)"'] & b_2 \arrow[d, cyan, "\operatorname{coker}(f_2)"] \\
z_1 \arrow[r, tail, red, "\ker(\overline{v_0}')"'] & c_0 \arrow[r, cyan, "\overline{v_0}'"'] & z_2 \arrow[r, tail, red, "\ker(\overline{v_1})"'] & c_1 \arrow[r, cyan, "\overline{v_1}"'] & c_2
\end{tikzcd}
$$

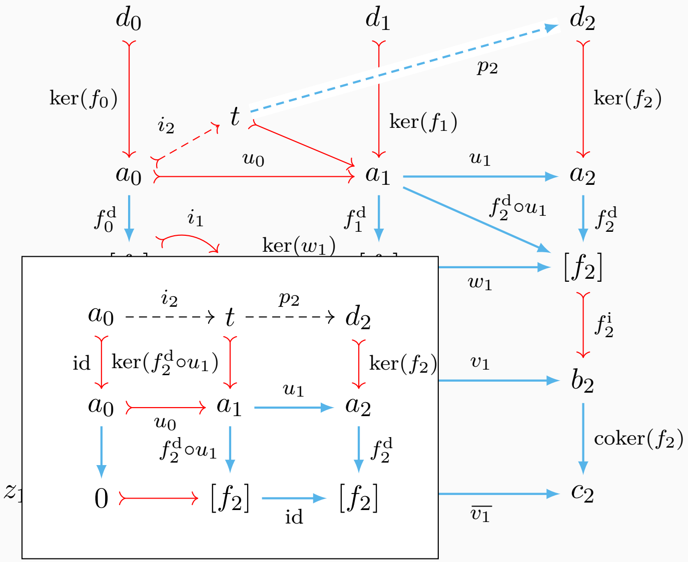
$f_2^{\mathrm{d}} \circ u_1$ の核

$$
\begin{tikzcd}
 & d_0 \arrow[dd, tail, "\ker(f_0)"'] & & d_1 \arrow[dd, tail, "\ker(f_1)"] & d_2 \arrow[dd, tail, "\ker(f_2)"] \\
 & & t \arrow[dr, tail, dashed, "\ker(f_2^{\mathrm{d}} \circ u_1)"] & & \\
 & a_0 \arrow[rr, tail, "u_0"] \arrow[d, "f_0^{\mathrm{d}}"'] & & a_1 \arrow[r, "u_1"] \arrow[d, "f_1^{\mathrm{d}}"'] \arrow[dr, "f_2^{\mathrm{d}} \circ u_1"] & a_2 \arrow[d, "f_2^{\mathrm{d}}"] \\
 & {[f_0]} \arrow[r, tail, bend left, "i_1"] \arrow[d, tail, "f_0^{\mathrm{i}}"'] & s \arrow[r, tail, "\ker(w_1)"] \arrow[dl, tail, bend right, "i_0"] \arrow[ddll, bend left, "p_1"] & {[f_1]} \arrow[r, "w_1"'] \arrow[d, tail, "f_1^{\mathrm{i}}"'] & {[f_2]} \arrow[d, tail, "f_2^{\mathrm{i}}"] \\
 & b_0 \arrow[rr, tail, "v_0"] \arrow[d, "\operatorname{coker}(f_0)"'] \arrow[dr, bend right, "p_0"] & & b_1 \arrow[r, "v_1"] \arrow[d, "\operatorname{coker}(f_1)"'] & b_2 \arrow[d, "\operatorname{coker}(f_2)"] \\
z_1 \arrow[r, tail, "\ker(\overline{v_0}')"'] & c_0 \arrow[r, "\overline{v_0}'"'] & z_2 \arrow[r, tail, "\ker(\overline{v_1})"'] & c_1 \arrow[r, "\overline{v_1}"'] & c_2
\end{tikzcd}
$$

普遍性から得られる射に 9 項補題 (上)

$$
\begin{tikzcd}
 & d_0 \arrow[dd, tail, "\ker(f_0)"'] & & d_1 \arrow[dd, tail, "\ker(f_1)"] & d_2 \arrow[dd, tail, "\ker(f_2)"] \\
 & & t \arrow[urr, dashed, "p_2"'] \arrow[dr, tail] & & \\
 & a_0 \arrow[ur, tail, dashed, "i_2"] \arrow[rr, tail, "u_0"] \arrow[d, "f_0^{\mathrm{d}}"'] & & a_1 \arrow[r, "u_1"] \arrow[dr, "f_2^{\mathrm{d}} \circ u_1"] \arrow[d, "f_1^{\mathrm{d}}"'] & a_2 \arrow[d, "f_2^{\mathrm{d}}"] \\
 & {[f_0]} \arrow[r, tail, bend left, "i_1"] \arrow[d, tail, "f_0^{\mathrm{i}}"'] & s \arrow[r, tail, "\ker(w_1)"] \arrow[dl, tail, bend right, "i_0"] \arrow[ddll, bend left, "p_1"] & {[f_1]} \arrow[r, "w_1"'] \arrow[d, tail, "f_1^{\mathrm{i}}"'] & {[f_2]} \arrow[d, tail, "f_2^{\mathrm{i}}"] \\
 & b_0 \arrow[rr, tail, "v_0"] \arrow[d, "\mathrm{coker}(f_0)"'] \arrow[dr, bend left, "p_0"] & & b_1 \arrow[r, "v_1"] \arrow[d, "\mathrm{coker}(f_1)"'] & b_2 \arrow[d, "\mathrm{coker}(f_2)"] \\
z_1 \arrow[r, tail, "\ker(\overline{v_0}')"'] & c_0 \arrow[r, "\overline{v_0}'"'] & z_2 \arrow[r, tail, "\ker(\overline{v_1})"'] & c_1 \arrow[r, "\overline{v_1}"'] & c_2
\end{tikzcd}
$$

普遍性から得られる射に 9 項補題 (上)



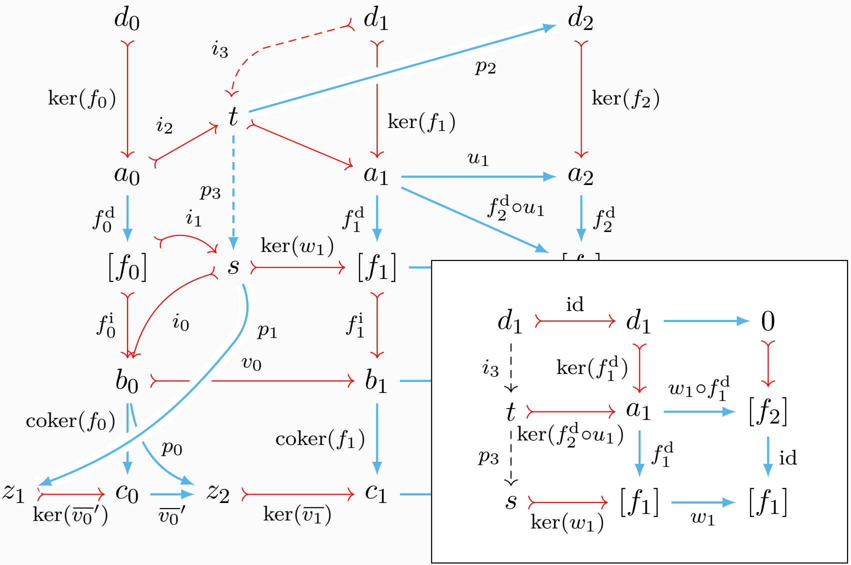
普遍性から得られる射に 9 項補題 (上)

$$
\begin{tikzcd}
 & d_0 \arrow[dd, tail, "\ker(f_0)"'] & & d_1 \arrow[dd, tail, "\ker(f_1)"] & d_2 \arrow[dd, tail, "\ker(f_2)"] \\
 & & t \arrow[urr, dashed, "p_2"'] \arrow[dr, tail] & & \\
 & a_0 \arrow[ur, tail, dashed, "i_2"] \arrow[rr, tail, "u_0"] \arrow[d, "f_0^{\mathrm{d}}"'] & & a_1 \arrow[r, "u_1"] \arrow[d, "f_1^{\mathrm{d}}"'] \arrow[dr, "f_2^{\mathrm{d}}\circ u_1"] & a_2 \arrow[d, "f_2^{\mathrm{d}}"] \\
 & {[f_0]} \arrow[r, tail, bend left, "i_1"] \arrow[d, tail, "f_0^{\mathrm{i}}"'] & s \arrow[r, tail, "\ker(w_1)"] \arrow[dl, tail, bend right, "i_0"] \arrow[ddll, "p_1"] & {[f_1]} \arrow[r, "w_1"'] \arrow[d, tail, "f_1^{\mathrm{i}}"'] & {[f_2]} \arrow[d, tail, "f_2^{\mathrm{i}}"] \\
 & b_0 \arrow[rr, tail, "v_0"] \arrow[d, "\mathrm{coker}(f_0)"'] \arrow[dr, bend right, "p_0"] & & b_1 \arrow[r, "v_1"] \arrow[d, "\mathrm{coker}(f_1)"'] & b_2 \arrow[d, "\mathrm{coker}(f_2)"] \\
z_1 \arrow[r, tail, "\ker(\overline{v_0}')"'] & c_0 \arrow[r, "\overline{v_0}'"'] & z_2 \arrow[r, tail, "\ker(\overline{v_1})"'] & c_1 \arrow[r, "\overline{v_1}"'] & c_2
\end{tikzcd}
$$

普遍性から得られる射に 9 項補題 (上)

$$
\begin{tikzcd}
 & d_0 \arrow[dd, tail, "\ker(f_0)"'] & & d_1 \arrow[dd, tail, "\ker(f_1)"] \arrow[dl, tail, dashed, bend right, "i_3"'] & d_2 \arrow[dd, tail, "\ker(f_2)"] \\
 & & t \arrow[urr, "p_2"] \arrow[dr, tail] \arrow[dd, dashed, "p_3"] & & \\
 & a_0 \arrow[ur, tail, "i_2"] \arrow[d, "f_0^{\mathrm{d}}"'] & & a_1 \arrow[r, "u_1"] \arrow[d, "f_1^{\mathrm{d}}"'] \arrow[dr, "f_2^{\mathrm{d}} \circ u_1"] & a_2 \arrow[d, "f_2^{\mathrm{d}}"] \\
 & {[f_0]} \arrow[r, tail, bend left, "i_1"] \arrow[d, tail, "f_0^{\mathrm{i}}"'] & s \arrow[r, tail, "\ker(w_1)"] \arrow[dl, tail, bend right, "i_0"] \arrow[ddll, bend left, "p_1"] & {[f_1]} \arrow[r, "w_1"'] \arrow[d, tail, "f_1^{\mathrm{i}}"'] & {[f_2]} \arrow[d, tail, "f_2^{\mathrm{i}}"] \\
 & b_0 \arrow[rr, tail, "v_0"] \arrow[d, "\mathrm{coker}(f_0)"'] \arrow[dr, bend left, "p_0"] & & b_1 \arrow[r, "v_1"] \arrow[d, "\mathrm{coker}(f_1)"'] & b_2 \arrow[d, "\mathrm{coker}(f_2)"] \\
z_1 \arrow[r, tail, "\ker(\overline{v_0}')"'] & c_0 \arrow[r, "\overline{v_0}'"'] & z_2 \arrow[r, tail, "\ker(\overline{v_1})"'] & c_1 \arrow[r, "\overline{v_1}"'] & c_2
\end{tikzcd}
$$

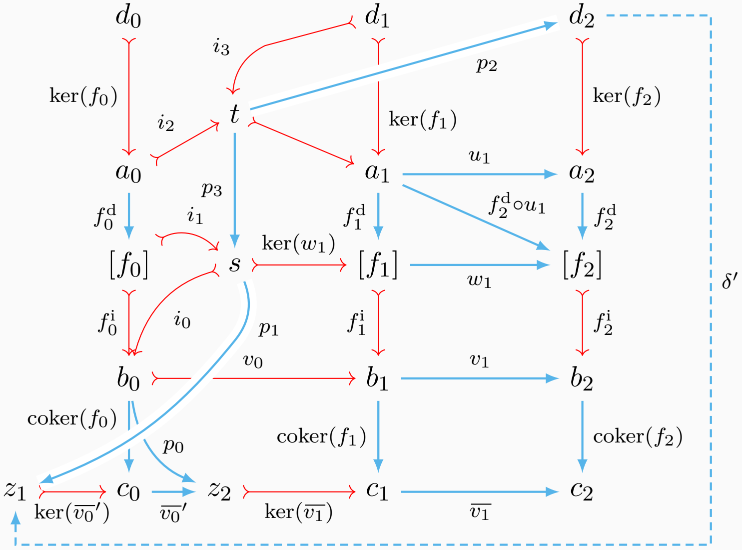
普遍性から得られる射に 9 項補題 (上)



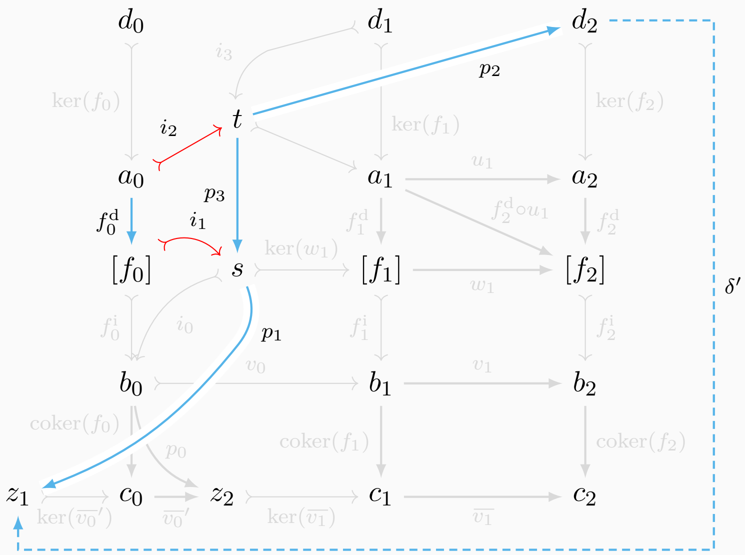
普遍性から得られる射に 9 項補題 (上)

$$
\begin{tikzcd}
 & d_0 \arrow[dd, tail, "\ker(f_0)"'] & & d_1 \arrow[dd, tail, "\ker(f_1)"] \arrow[dl, dashed, tail, "i_3"'] & d_2 \arrow[dd, tail, "\ker(f_2)"] \\
 & & t \arrow[urr, "p_2"] \arrow[dr, tail] \arrow[dd, dashed, "p_3"'] & & \\
 & a_0 \arrow[ur, tail, "i_2"] \arrow[d, "f_0^{\mathrm{d}}"'] & & a_1 \arrow[r, "u_1"] \arrow[d, "f_1^{\mathrm{d}}"'] \arrow[dr, "f_2^{\mathrm{d}} \circ u_1"] & a_2 \arrow[d, "f_2^{\mathrm{d}}"] \\
 & {[f_0]} \arrow[r, tail, bend left, "i_1"] \arrow[d, tail, "f_0^{\mathrm{i}}"'] & s \arrow[r, tail, "\ker(w_1)"] \arrow[dl, tail, bend right, "i_0"] \arrow[ddll, bend left, "p_1"] & {[f_1]} \arrow[r, "w_1"'] \arrow[d, tail, "f_1^{\mathrm{i}}"'] & {[f_2]} \arrow[d, tail, "f_2^{\mathrm{i}}"] \\
 & b_0 \arrow[rr, tail, "v_0"] \arrow[d, "\mathrm{coker}(f_0)"'] \arrow[dr, bend right, "p_0"] & & b_1 \arrow[r, "v_1"] \arrow[d, "\mathrm{coker}(f_1)"'] & b_2 \arrow[d, "\mathrm{coker}(f_2)"] \\
z_1 \arrow[r, tail, "\ker(\overline{v_0}')"'] & c_0 \arrow[r, "\overline{v_0}'"'] & z_2 \arrow[r, tail, "\ker(\overline{v_1})"'] & c_1 \arrow[r, "\overline{v_1}"'] & c_2
\end{tikzcd}
$$

普遍性から得られる射


d_0
d_1
d_2
$\ker(f_0)$
i_3
i_2
t
p_2
$\ker(f_1)$
$\ker(f_2)$
a_0
a_1
u_1
a_2
p_3
f_0^{d}
i_1
f_1^{d}
$f_2^{\mathrm{d}} \circ u_1$
f_2^{d}
$[f_0]$
s
$\ker(w_1)$
$[f_1]$
w_1
$[f_2]$
δ'
f_0^{i}
i_0
p_1
f_1^{i}
f_2^{i}
b_0
v_0
b_1
v_1
b_2
$\mathrm{coker}(f_0)$
p_0
$\mathrm{coker}(f_1)$
$\mathrm{coker}(f_2)$
z_1
c_0
z_2
c_1
c_2
$\ker(\overline{v_0}')$
$\overline{v_0}'$
$\ker(\overline{v_1})$
$\overline{v_1}$

普遍性から得られる射



d_0
d_1
d_2
i_3
$\ker(f_0)$
$\ker(f_1)$
$\ker(f_2)$
p_2
t
i_2
a_0
a_1
u_1
a_2
p_3
f_0^{d}
i_1
f_1^{d}
$f_2^{\mathrm{d}} \circ u_1$
f_2^{d}
$[f_0]$
s
$\ker(w_1)$
$[f_1]$
$[f_2]$
w_1
δ'
f_0^{i}
i_0
p_1
f_1^{i}
f_2^{i}
b_0
v_0
b_1
v_1
b_2
$\mathrm{coker}(f_0)$
p_0
$\mathrm{coker}(f_1)$
$\mathrm{coker}(f_2)$
z_1
c_0
z_2
c_1
c_2
$\ker(\overline{v_0}')$
$\overline{v_0}'$
$\ker(\overline{v_1})$
$\overline{v_1}$

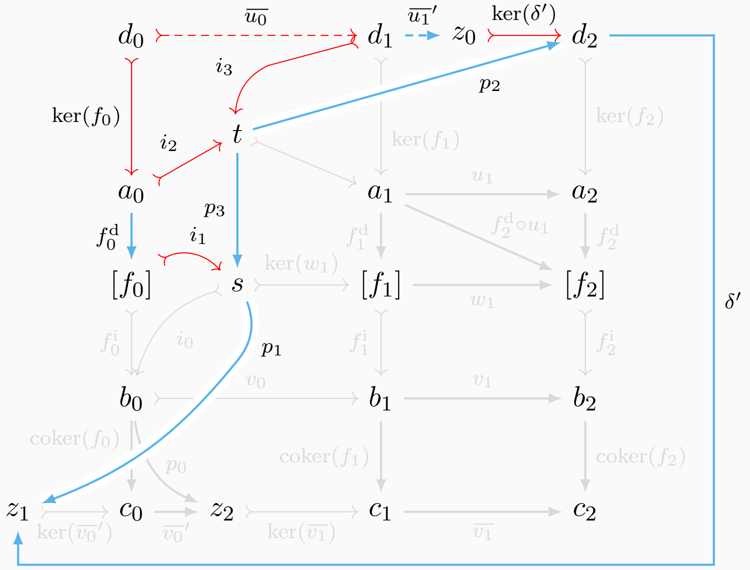
δ' の核

$$
\begin{tikzcd}
 & d_0 \arrow[dd, tail, "\ker(f_0)"'] & & d_1 \arrow[dl, tail, bend right, "i_3"'] \arrow[dd, tail, "\ker(f_1)"] & z_0 \arrow[r, tail, dashed, "\ker(\delta')"] & d_2 \arrow[dd, tail, "\ker(f_2)"] \arrow[dddddlllll, rounded corners, "\delta'"] \\
 & & t \arrow[urrr, "p_2"'] \arrow[dr, tail] \arrow[dd, "p_3"'] & & & \\
 & a_0 \arrow[ur, tail, "i_2"] \arrow[d, "f_0^{\mathrm{d}}"'] & & a_1 \arrow[rr, "u_1"] \arrow[drr, "f_2^{\mathrm{d}} \circ u_1"] \arrow[d, "f_1^{\mathrm{d}}"'] & & a_2 \arrow[d, "f_2^{\mathrm{d}}"] \\
 & {[f_0]} \arrow[r, tail, bend left, "i_1"] \arrow[d, tail, "f_0^{\mathrm{i}}"'] & s \arrow[r, tail, "\ker(w_1)"] \arrow[dl, tail, bend right, "i_0"] \arrow[ddll, bend left, "p_1"] & {[f_1]} \arrow[rr, "w_1"'] \arrow[d, tail, "f_1^{\mathrm{i}}"'] & & {[f_2]} \arrow[d, tail, "f_2^{\mathrm{i}}"] \\
 & b_0 \arrow[rr, tail, "v_0"] \arrow[d, "\operatorname{coker}(f_0)"'] \arrow[dr, bend right, "p_0"] & & b_1 \arrow[rr, "v_1"] \arrow[d, "\operatorname{coker}(f_1)"'] & & b_2 \arrow[d, "\operatorname{coker}(f_2)"] \\
z_1 \arrow[r, tail, "\ker(\overline{v_0}')"'] & c_0 \arrow[r, "\overline{v_0}'"'] & z_2 \arrow[r, tail, "\ker(\overline{v_1})"'] & c_1 \arrow[rr, "\overline{v_1}"'] & & c_2
\end{tikzcd}
$$

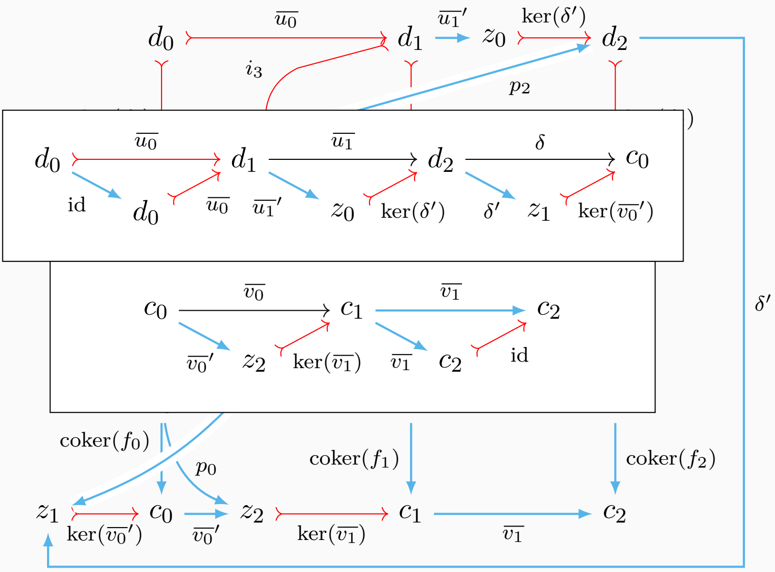
普遍性から得られる射に 9 項補題 (上)

$$\begin{tikzcd}
 & d_0 \arrow[rr, tail, dashed, "\overline{u_0}"] \arrow[dd, tail, "\ker(f_0)"'] & & d_1 \arrow[r, dashed, "\overline{u_1}'"] \arrow[dd, tail, "\ker(f_1)"] \arrow[dl, tail, bend right, "i_3"'] & z_0 \arrow[r, tail, "\ker(\delta')"] & d_2 \arrow[dd, tail, "\ker(f_2)"] \arrow[dddddlllll, bend left=60, "\delta'"] \\
 & & t \arrow[urrr, "p_2"'] \arrow[dr, tail] \arrow[dd, "p_3"] & & & \\
 & a_0 \arrow[ur, tail, "i_2"] \arrow[d, "f_0^{\mathrm{d}}"'] & & a_1 \arrow[rr, "u_1"] \arrow[drr, "f_2^{\mathrm{d}} \circ u_1"] \arrow[d, "f_1^{\mathrm{d}}"'] & & a_2 \arrow[d, "f_2^{\mathrm{d}}"] \\
 & {[f_0]} \arrow[r, tail, bend left, "i_1"] \arrow[d, tail, "f_0^{\mathrm{i}}"'] & s \arrow[r, tail, "\ker(w_1)"] \arrow[dl, tail, bend right, "i_0"'] \arrow[ddll, bend left, "p_1"] & {[f_1]} \arrow[rr, "w_1"'] \arrow[d, tail, "f_1^{\mathrm{i}}"'] & & {[f_2]} \arrow[d, tail, "f_2^{\mathrm{i}}"] \\
 & b_0 \arrow[rr, tail, "v_0"] \arrow[d, "\operatorname{coker}(f_0)"'] \arrow[dr, bend right, "p_0"] & & b_1 \arrow[rr, "v_1"] \arrow[d, "\operatorname{coker}(f_1)"'] & & b_2 \arrow[d, "\operatorname{coker}(f_2)"] \\
z_1 \arrow[r, tail, "\ker(\overline{v_0}')"'] & c_0 \arrow[r, "\overline{v_0}'"'] & z_2 \arrow[r, tail, "\ker(\overline{v_1})"'] & c_1 \arrow[rr, "\overline{v_1}"'] & & c_2
\end{tikzcd}$$

普遍性から得られる射に 9 項補題 (上)



こうして完全列が得られた



□

【定理】 (蛇の補題)

次の可換図式で横列は両方とも完全列とする．

$$
\begin{tikzcd}
 & a_0 \arrow[r, "u_0"] \arrow[d, "f_0"'] & a_1 \arrow[r, "u_1"] \arrow[d, "f_1"] & a_2 \arrow[r] \arrow[d, "f_2"] & 0 \\
0 \arrow[r] & b_0 \arrow[r, "v_0"'] & b_1 \arrow[r, "v_1"'] & b_2 &
\end{tikzcd}
$$

また f_i は admissible で f_i の核と余核をそれぞれ
$\ker(f_i)\colon d_i \to a_i$，$\operatorname{coker}(f_i)\colon b_i \to c_i$ とする．このとき完全列

$$
d_0 \xrightarrow{\overline{u_0}} d_1 \xrightarrow{\overline{u_1}} d_2 \xrightarrow{\delta} c_0 \xrightarrow{\overline{v_0}} c_1 \xrightarrow{\overline{v_1}} c_2
$$

が存在する．

弱完全圏

証明.

まず蛇の補題の図式はこうなっている.

$$\begin{tikzcd}
 & a_0 \arrow[r, "u_0"] \arrow[d, "f_0^{\mathrm{d}}"'] & a_1 \arrow[r, "u_1"] \arrow[d, "f_1^{\mathrm{d}}"] & a_2 \arrow[r] \arrow[d, "f_2^{\mathrm{d}}"] & 0 \\
 & {[f_0]} \arrow[d, tail, "f_0^{\mathrm{i}}"'] & {[f_1]} \arrow[d, tail, "f_1^{\mathrm{i}}"] & {[f_2]} \arrow[d, tail, "f_2^{\mathrm{i}}"] & \\
0 \arrow[r] & b_0 \arrow[r, "v_0"'] & b_1 \arrow[r, "v_1"'] & b_2 &
\end{tikzcd}$$

弱完全圏

証明.

定理の条件からこのように書き換えられる.

$$\begin{tikzcd}
a_0 \arrow[r, "u_0^{\mathrm{d}}"] \arrow[d, "f_0^{\mathrm{d}}"'] & {[u_0]} \arrow[r, tail, "u_0^{\mathrm{i}}"] & a_1 \arrow[rr, "u_1"] \arrow[d, "f_1^{\mathrm{d}}"] & & a_2 \arrow[d, "f_2^{\mathrm{d}}"] \\
{[f_0]} \arrow[d, tail, "f_0^{\mathrm{i}}"'] & & {[f_1]} \arrow[d, tail, "f_1^{\mathrm{i}}"] & & {[f_2]} \arrow[d, tail, "f_2^{\mathrm{i}}"] \\
b_0 \arrow[rr, tail, "v_0"'] & & b_1 \arrow[r, "v_1^{\mathrm{d}}"'] & {[v_1]} \arrow[r, tail, "v_1^{\mathrm{i}}"'] & b_2
\end{tikzcd}$$

証明.

$\ker(u_0) = \ker(u_0^{\mathrm{d}})$ を取る.

$$
\begin{tikzcd}
z \arrow[r, tail, dashed, "\ker(u_0)"] & a_0 \arrow[r, "u_0^{\mathrm{d}}"] \arrow[d, "f_0^{\mathrm{d}}"'] & {[u_0]} \arrow[r, tail, "u_0^{\mathrm{i}}"] & a_1 \arrow[rr, "u_1"] \arrow[d, "f_1^{\mathrm{d}}"] & & a_2 \arrow[d, "f_2^{\mathrm{d}}"] \\
& {[f_0]} \arrow[d, tail, "f_0^{\mathrm{i}}"'] & & {[f_1]} \arrow[d, tail, "f_1^{\mathrm{i}}"] & & {[f_2]} \arrow[d, tail, "f_2^{\mathrm{i}}"] \\
& b_0 \arrow[rr, tail, "v_0"'] & & b_1 \arrow[r, "v_1^{\mathrm{d}}"'] & {[v_1]} \arrow[r, tail, "v_1^{\mathrm{i}}"'] & b_2
\end{tikzcd}
$$

証明.

$v_0 \circ f_0^{\mathrm{i}} \circ f_0^{\mathrm{d}} \circ \ker(u_0) = 0$ である.

$$
\begin{tikzcd}
z \arrow[r, tail, "\ker(u_0)"] & a_0 \arrow[r, "u_0^{\mathrm{d}}"] \arrow[d, "f_0^{\mathrm{d}}"'] & {[u_0]} \arrow[r, tail, "u_0^{\mathrm{i}}"] & a_1 \arrow[rr, "u_1"] \arrow[d, "f_1^{\mathrm{d}}"] & & a_2 \arrow[d, "f_2^{\mathrm{d}}"] \\
& {[f_0]} \arrow[d, tail, "f_0^{\mathrm{i}}"'] & & {[f_1]} \arrow[d, tail, "f_1^{\mathrm{i}}"] & & {[f_2]} \arrow[d, tail, "f_2^{\mathrm{i}}"] \\
& b_0 \arrow[rr, tail, "v_0"'] & & b_1 \arrow[r, "v_1^{\mathrm{d}}"'] & {[v_1]} \arrow[r, tail, "v_1^{\mathrm{i}}"'] & b_2
\end{tikzcd}
$$

証明.

$v_0 \circ f_0^{\mathrm{i}} \circ f_0^{\mathrm{d}} \circ \ker(u_0) = 0$ である.

$$
\begin{tikzcd}
z \arrow[r, tail, "\ker(u_0)"] & a_0 \arrow[r, "u_0^{\mathrm{d}}"] \arrow[d, "f_0^{\mathrm{d}}"'] & {[u_0]} \arrow[r, tail, "u_0^{\mathrm{i}}"] & a_1 \arrow[rr, "u_1"] \arrow[d, "f_1^{\mathrm{d}}"] & & a_2 \arrow[d, "f_2^{\mathrm{d}}"] \\
 & {[f_0]} \arrow[d, tail, "f_0^{\mathrm{i}}"'] & & {[f_1]} \arrow[d, tail, "f_1^{\mathrm{i}}"] & & {[f_2]} \arrow[d, tail, "f_2^{\mathrm{i}}"] \\
 & b_0 \arrow[rr, tail, "v_0"'] & & b_1 \arrow[r, "v_1^{\mathrm{d}}"'] & {[v_1]} \arrow[r, tail, "v_1^{\mathrm{i}}"'] & b_2
\end{tikzcd}
$$

証明.

inflation はモノ射だから $f_0^{\mathrm{d}} \circ \ker(u_0) = 0$ である．

$$
\begin{tikzcd}
z \arrow[r, tail, "\ker(u_0)"] & a_0 \arrow[r, "u_0^{\mathrm{d}}"] \arrow[d, "f_0^{\mathrm{d}}"'] & {[u_0]} \arrow[r, tail, "u_0^{\mathrm{i}}"] & a_1 \arrow[rr, "u_1"] \arrow[d, "f_1^{\mathrm{d}}"] & & a_2 \arrow[d, "f_2^{\mathrm{d}}"] \\
& {[f_0]} \arrow[d, tail, "f_0^{\mathrm{i}}"'] & & {[f_1]} \arrow[d, tail, "f_1^{\mathrm{i}}"] & & {[f_2]} \arrow[d, tail, "f_2^{\mathrm{i}}"] \\
& b_0 \arrow[rr, tail, "v_0"'] & & b_1 \arrow[r, "v_1^{\mathrm{d}}"'] & {[v_1]} \arrow[r, tail, "v_1^{\mathrm{i}}"'] & b_2
\end{tikzcd}
$$

弱完全圏

証明.

inflation はモノ射だから $f_0^{\mathrm{d}} \circ \ker(u_0) = 0$ である.

$$
\begin{tikzcd}
z \arrow[r, tail, "\ker(u_0)"] \arrow[dr, bend right] & a_0 \arrow[r, "u_0^{\mathrm{d}}"] \arrow[d, "f_0^{\mathrm{d}}"'] & {[u_0]} \arrow[r, tail, "u_0^{\mathrm{i}}"] & a_1 \arrow[rr, "u_1"] \arrow[d, "f_1^{\mathrm{d}}"] & & a_2 \arrow[d, "f_2^{\mathrm{d}}"] \\
& {[f_0]} \arrow[d, tail, "f_0^{\mathrm{i}}"'] & & {[f_1]} \arrow[d, tail, "f_1^{\mathrm{i}}"] & & {[f_2]} \arrow[d, tail, "f_2^{\mathrm{i}}"] \\
& b_0 \arrow[rr, tail, "v_0"'] & & b_1 \arrow[r, "v_1^{\mathrm{d}}"'] & {[v_1]} \arrow[r, tail, "v_1^{\mathrm{i}}"'] & b_2
\end{tikzcd}
$$

弱完全圏

証明.

$u_0^{\mathrm{d}} \cong \mathrm{coker}(\ker(u_0))$ の普遍性から $h\colon s \to x_0$ が得られる.

$$
\begin{tikzcd}
z \arrow[r, tail, "\ker(u_0)"] & a_0 \arrow[r, "u_0^{\mathrm{d}}"] \arrow[d, "f_0^{\mathrm{d}}"'] & {[u_0]} \arrow[r, tail, "u_0^{\mathrm{i}}"] \arrow[dl, dashed, "h"] & a_1 \arrow[rr, "u_1"] \arrow[d, "f_1^{\mathrm{d}}"] & & a_2 \arrow[d, "f_2^{\mathrm{d}}"] \\
 & {[f_0]} \arrow[d, tail, "f_0^{\mathrm{i}}"'] & & {[f_1]} \arrow[d, tail, "f_1^{\mathrm{i}}"] & & {[f_2]} \arrow[d, tail, "f_2^{\mathrm{i}}"] \\
 & b_0 \arrow[rr, tail, "v_0"'] & & b_1 \arrow[r, "v_1^{\mathrm{d}}"'] & {[v_1]} \arrow[r, tail, "v_1^{\mathrm{i}}"'] & b_2
\end{tikzcd}
$$

弱完全圏

証明.

弱完全圏の定義より h は deflation で，これは可換である．

$$
\begin{tikzcd}
z \arrow[r, tail, "\ker(u_0)"] & a_0 \arrow[r, "u_0^{\mathrm{d}}"] \arrow[d, "f_0^{\mathrm{d}}"'] & {[u_0]} \arrow[r, tail, "u_0^{\mathrm{i}}"] \arrow[dl, "h"] & a_1 \arrow[rr, "u_1"] \arrow[d, "f_1^{\mathrm{d}}"] & & a_2 \arrow[d, "f_2^{\mathrm{d}}"] \\
 & {[f_0]} \arrow[d, tail, "f_0^{\mathrm{i}}"'] & & {[f_1]} \arrow[d, tail, "f_1^{\mathrm{i}}"] & & {[f_2]} \arrow[d, tail, "f_2^{\mathrm{i}}"] \\
 & b_0 \arrow[rr, tail, "v_0"'] & & b_1 \arrow[r, "v_1^{\mathrm{d}}"'] & {[v_1]} \arrow[r, tail, "v_1^{\mathrm{i}}"'] & b_2
\end{tikzcd}
$$

証明.

同様にして inflation k が得られて可換である．

$$
\begin{tikzcd}
z \arrow[r, tail, "\ker(u_0)"] & a_0 \arrow[r, "u_0^{\mathrm{d}}"] \arrow[d, "f_0^{\mathrm{d}}"'] & {[u_0]} \arrow[r, tail, "u_0^{\mathrm{i}}"] \arrow[dl, "h"] & a_1 \arrow[rr, "u_1"] \arrow[d, "f_1^{\mathrm{d}}"] & & a_2 \arrow[d, "f_2^{\mathrm{d}}"] \\
& {[f_0]} \arrow[d, tail, "f_0^{\mathrm{i}}"'] & & {[f_1]} \arrow[d, tail, "f_1^{\mathrm{i}}"] & & {[f_2]} \arrow[d, tail, "f_2^{\mathrm{i}}"] \arrow[dl, tail, "k"'] \\
& b_0 \arrow[rr, tail, "v_0"'] & & b_1 \arrow[r, "v_1^{\mathrm{d}}"'] & {[v_1]} \arrow[r, tail, "v_1^{\mathrm{i}}"'] & b_2
\end{tikzcd}
$$

証明.

この部分に蛇の補題 (弱) を適用することで

$$\begin{tikzcd}
z \arrow[r, tail, "\ker(u_0)"] & a_0 \arrow[r, "u_0^{\mathrm{d}}"] \arrow[d, "f_0^{\mathrm{d}}"'] & {[u_0]} \arrow[r, tail, "u_0^{\mathrm{i}}"] \arrow[dl, "h"] & a_1 \arrow[rr, "u_1"] \arrow[d, "f_1^{\mathrm{d}}"] & & a_2 \arrow[d, "f_2^{\mathrm{d}}"] \\
& {[f_0]} \arrow[d, tail, "f_0^{\mathrm{i}}"'] & & {[f_1]} \arrow[d, tail, "f_1^{\mathrm{i}}"] & & {[f_2]} \arrow[d, tail, "f_2^{\mathrm{i}}"] \arrow[dl, tail, "k"'] \\
& b_0 \arrow[rr, tail, "v_0"'] & & b_1 \arrow[r, "v_1^{\mathrm{d}}"'] & {[v_1]} \arrow[r, tail, "v_1^{\mathrm{i}}"'] & b_2
\end{tikzcd}$$

証明.

この部分に蛇の補題 (弱) を適用することで

$$
\begin{tikzcd}
 & & d_0' \arrow[d, tail, "\ker(h)"'] & d_1 \arrow[d, tail, "\ker(f_1)"] & & d_2 \arrow[d, tail, "\ker(f_2)"] \\
z \arrow[r, tail, "\ker(u_0)"] & a_0 \arrow[r, "u_0^{\mathrm{d}}"] \arrow[d, "f_0^{\mathrm{d}}"'] & {[u_0]} \arrow[r, tail, "u_0^{\mathrm{i}}"] \arrow[dl, "h"] & a_1 \arrow[rr, "u_1"] \arrow[d, "f_1^{\mathrm{d}}"] & & a_2 \arrow[d, "f_2^{\mathrm{d}}"] \\
 & {[f_0]} \arrow[d, tail, "f_0^{\mathrm{i}}"'] & & {[f_1]} \arrow[d, tail, "f_1^{\mathrm{i}}"] & & {[f_2]} \arrow[dl, tail, "k"'] \arrow[d, tail, "f_2^{\mathrm{i}}"] \\
 & b_0 \arrow[rr, tail, "v_0"'] \arrow[d, "\operatorname{coker}(f_0)"'] & & b_1 \arrow[r, "v_1^{\mathrm{d}}"'] \arrow[d, "\operatorname{coker}(f_1)"'] & {[v_1]} \arrow[r, tail, "v_1^{\mathrm{i}}"'] \arrow[d, "\operatorname{coker}(k)"] & b_2 \\
 & c_0 & & c_1 & c_2' &
\end{tikzcd}
$$

証明.

この部分に蛇の補題 (弱) を適用することで

$$
\begin{tikzcd}
 & & d_0' \arrow[r, tail, "\overline{u_0^{\mathrm{i}}}"] \arrow[d, tail, "\ker(h)"'] & d_1 \arrow[rr, "\overline{u_1}"] \arrow[d, tail, "\ker(f_1)"] & & d_2 \arrow[r, dash, "\delta"] \arrow[d, tail, "\ker(f_2)"] & \cdots \\
z \arrow[r, tail, "\ker(u_0)"] & a_0 \arrow[r, "u_0^{\mathrm{d}}"] \arrow[d, "f_0^{\mathrm{d}}"'] & {[u_0]} \arrow[r, tail, "u_0^{\mathrm{i}}"] \arrow[dl, "h"] & a_1 \arrow[rr, "u_1"] \arrow[d, "f_1^{\mathrm{d}}"] & & a_2 \arrow[d, "f_2^{\mathrm{d}}"] & \\
 & {[f_0]} \arrow[d, tail, "f_0^{\mathrm{i}}"'] & & {[f_1]} \arrow[d, tail, "f_1^{\mathrm{i}}"] & & {[f_2]} \arrow[dl, tail, "k"'] \arrow[d, tail, "f_2^{\mathrm{i}}"] & \\
 & b_0 \arrow[rr, tail, "v_0"'] \arrow[d, "\operatorname{coker}(f_0)"'] & & b_1 \arrow[r, "v_1^{\mathrm{d}}"'] \arrow[d, "\operatorname{coker}(f_1)"'] & {[v_1]} \arrow[r, tail, "v_1^{\mathrm{i}}"'] \arrow[d, "\operatorname{coker}(k)"] & b_2 & \\
\cdots \arrow[r] & c_0 \arrow[rr, "\overline{v_0}"'] & & c_1 \arrow[r, "\overline{v_1^{\mathrm{d}}}"'] & c_2' & &
\end{tikzcd}
$$

証明.

$\ker(f_0)$ を考える.

$$\begin{tikzcd}
 & d_0 \arrow[r, dashed, "\overline{u_0^{\mathrm{d}}}"] \arrow[d, tail, "\ker(f_0)"'] & d_0' \arrow[r, tail, "\overline{u_0^{\mathrm{i}}}"] \arrow[d, tail, "\ker(h)"'] & d_1 \arrow[rr, "\overline{u_1}"] \arrow[d, tail, "\ker(f_1)"] & & d_2 \arrow[r, dash, "\delta"] \arrow[d, tail, "\ker(f_2)"] & \cdots \\
z \arrow[r, tail, "\ker(u_0)"] & a_0 \arrow[r, "u_0^{\mathrm{d}}"] \arrow[d, "f_0^{\mathrm{d}}"'] & {[u_0]} \arrow[r, tail, "u_0^{\mathrm{i}}"] \arrow[dl, "h"] & a_1 \arrow[rr, "u_1"] \arrow[d, "f_1^{\mathrm{d}}"] & & a_2 \arrow[d, "f_2^{\mathrm{d}}"] & \\
 & {[f_0]} \arrow[d, tail, "f_0^{\mathrm{i}}"'] & & {[f_1]} \arrow[d, tail, "f_1^{\mathrm{i}}"] & & {[f_2]} \arrow[d, tail, "f_2^{\mathrm{i}}"] \arrow[dl, tail, "k"'] & \\
 & b_0 \arrow[rr, tail, "v_0"'] \arrow[d, "\mathrm{coker}(f_0)"'] & & b_1 \arrow[r, "v_1^{\mathrm{d}}"'] \arrow[d, "\mathrm{coker}(f_1)"'] & {[v_1]} \arrow[r, tail, "v_1^{\mathrm{i}}"'] \arrow[d, "\mathrm{coker}(k)"] & b_2 & \\
\cdots \arrow[r] & c_0 \arrow[rr, "\overline{v_0}"'] & & c_1 \arrow[r, "\overline{v_1^{\mathrm{d}}}"'] & c_2' & &
\end{tikzcd}$$

証明.

$\ker(f_0)$ を考える.

$$
\begin{tikzcd}
 & d_0 \arrow[r, dashed, "\overline{u_0^{\mathrm{d}}}"] \arrow[d, tail, "\ker(f_0)"'] & d_0' \arrow[r, tail, "\overline{u_0^{\mathrm{i}}}"] \arrow[d, tail, "\ker(h)"'] & d_1 \arrow[rr, "\overline{u_1}"] \arrow[d, tail, "\ker(f_1)"] & & d_2 \arrow[r, "\delta"] \arrow[d, tail, "\ker(f_2)"] & \cdots \\
z \arrow[r, tail, "\ker(u_0)"] & a_0 \arrow[r, "u_0^{\mathrm{d}}"] \arrow[d, "f_0^{\mathrm{d}}"'] & {[u_0]} \arrow[r, tail, "u_0^{\mathrm{i}}"] \arrow[dl, "h"] & a_1 \arrow[rr, "u_1"] \arrow[d, "f_1^{\mathrm{d}}"] & & a_2 \arrow[d, "f_2^{\mathrm{d}}"] & \\
 & {[f_0]} \arrow[d, tail, "f_0^{\mathrm{i}}"'] & & {[f_1]} \arrow[d, tail, "f_1^{\mathrm{i}}"] & & {[f_2]} \arrow[d, tail, "f_2^{\mathrm{i}}"] \arrow[dl, tail, "k"'] & \\
 & b_0 \arrow[rr, tail, "v_0"'] \arrow[d, "\mathrm{coker}(f_0)"'] & & b_1 \arrow[r, "v_1^{\mathrm{d}}"'] \arrow[d, "\mathrm{coker}(f_1)"'] & {[v_1]} \arrow[r, tail, "v_1^{\mathrm{i}}"'] \arrow[d, "\mathrm{coker}(k)"] & b_2 & \\
\cdots \arrow[r] & c_0 \arrow[rr, "\overline{v_0}"'] & & c_1 \arrow[r, "\overline{v_1^{\mathrm{d}}}"'] & c_2' & &
\end{tikzcd}
$$

証明．

9 項補題 (上) により $\overline{u_0^{\mathrm{d}}}$ は deflation である．

$$
\begin{tikzcd}
z \arrow[r, dashed] \arrow[d, tail, "\mathrm{id}"] & d_0 \arrow[r, dashed, "\overline{u_0^{\mathrm{d}}}"] \arrow[d, tail, "\ker(f_0)"'] & d_0' \arrow[r, tail, "\overline{u_0^{\mathrm{i}}}"] \arrow[d, tail, "\ker(h)"'] & d_1 \arrow[rr, "\overline{u_1}"] \arrow[d, tail, "\ker(f_1)"] & & d_2 \arrow[r, dash, "\delta"] \arrow[d, tail, "\ker(f_2)"] & \cdots \\
z \arrow[r, tail, "\ker(u_0)"] \arrow[d] & a_0 \arrow[r, "u_0^{\mathrm{d}}"] \arrow[d, "f_0^{\mathrm{d}}"'] & {[u_0]} \arrow[r, tail, "u_0^{\mathrm{i}}"] \arrow[d, "h"] & a_1 \arrow[rr, "u_1"] \arrow[d, "f_1^{\mathrm{d}}"] & & a_2 \arrow[d, "f_2^{\mathrm{d}}"] & \\
0 \arrow[r, tail] & {[f_0]} \arrow[r, "\mathrm{id}"'] \arrow[d, tail, "f_0^{\mathrm{i}}"'] & x_0 & {[f_1]} \arrow[d, tail, "f_1^{\mathrm{i}}"] & & {[f_2]} \arrow[dl, tail, "k"'] \arrow[d, tail, "f_2^{\mathrm{i}}"] & \\
& b_0 \arrow[rr, tail, "v_0"'] \arrow[d, "\mathrm{coker}(f_0)"'] & & b_1 \arrow[r, "v_1^{\mathrm{d}}"'] \arrow[d, "\mathrm{coker}(f_1)"'] & {[v_1]} \arrow[r, tail, "v_1^{\mathrm{i}}"'] \arrow[d, "\mathrm{coker}(k)"] & b_2 & \\
\cdots \arrow[r] & c_0 \arrow[rr, "\overline{v_0}"'] & & c_1 \arrow[r, "\overline{v_1^{\mathrm{d}}}"'] & c_2' & &
\end{tikzcd}
$$

証明.

同様にして $\overline{v_1^{\mathrm{i}}}$ を取れば完全列が得られる.

$$
\begin{tikzcd}
 & d_0 \arrow[r, "\overline{u_0^{\mathrm{d}}}"] \arrow[rr, bend left, "\overline{u_0}"] \arrow[d, tail, "\ker(f_0)"'] & d_0' \arrow[r, tail, "\overline{u_0^{\mathrm{i}}}"] \arrow[d, tail, "\ker(h)"'] & d_1 \arrow[rr, "\overline{u_1}"] \arrow[d, tail, "\ker(f_1)"] & & d_2 \arrow[r, dash, "\delta"] \arrow[d, tail, "\ker(f_2)"] & \cdots \\
z \arrow[r, tail, "\ker(u_0)"] & a_0 \arrow[r, "u_0^{\mathrm{d}}"] \arrow[d, "f_0"'] & s \arrow[r, tail, "u_0^{\mathrm{i}}"] & a_1 \arrow[rr, "u_1"] \arrow[d, "f_1"] & & a_2 \arrow[d, "f_2"] & \\
 & b_0 \arrow[rr, tail, "v_0"'] \arrow[d, "\mathrm{coker}(f_0)"'] & & b_1 \arrow[r, "v_1^{\mathrm{d}}"'] \arrow[d, "\mathrm{coker}(f_1)"'] & t \arrow[r, tail, "v_1^{\mathrm{i}}"'] \arrow[d, "\mathrm{coker}(k)"] & b_2 \arrow[d, "\mathrm{coker}(f_2)"] & \\
\cdots \arrow[r] & c_0 \arrow[rr, "\overline{v_0}"'] & & c_1 \arrow[r, "\overline{v_1^{\mathrm{d}}}"'] \arrow[rr, bend right, "\overline{v_0}"'] & c_2' \arrow[r, tail, "\overline{v_1^{\mathrm{i}}}"'] & c_2 &
\end{tikzcd}
$$

□

【定理】 (5 項補題)

inflation の合成が inflation になると仮定する．次の可換図式で，横の列は完全列であるとする．

$$
\begin{array}{ccccccccc}
a_0 & \xrightarrow{u_0} & a_1 & \xrightarrow{u_1} & a_2 & \xrightarrow{u_2} & a_3 & \xrightarrow{u_3} & a_4 \\
{\scriptstyle f_0}\downarrow & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \downarrow{\scriptstyle f_4} \\
b_0 & \xrightarrow[v_0]{} & b_1 & \xrightarrow[v_1]{} & b_2 & \xrightarrow[v_2]{} & b_3 & \xrightarrow[v_3]{} & b_4
\end{array}
$$

また f_2 が admissible とする．このとき

(1) f_0 が deflation で，f_1, f_3 が inflation ならば f_2 も inflation．

(2) f_4 が inflation で，f_1, f_3 が deflation ならば f_2 も deflation．

【定義】

$f\colon a \to b$ が正則エピ射 (regular epimorphism)
$\iff$ ある射 $g, h\colon x \to a$ が存在して $x \overset{g}{\underset{h}{\rightrightarrows}} a \xrightarrow{f} b$ が coequalizer となる．

弱完全圏において deflation は正則エピ射である．そこで次の定義をする．

弱完全圏

【定義】

$f\colon a \to b$ が正則エピ射 (regular epimorphism)

$\iff$ ある射 $g, h\colon x \to a$ が存在して $x \underset{h}{\overset{g}{\rightrightarrows}} a \xrightarrow{f} b$ が coequalizer となる．

弱完全圏において deflation は正則エピ射である．そこで次の定義をする．

【定義】

零対象を持つ圏 C が weakly homological

$\iff D := \{f \in \mathrm{Mor}(C) \mid f$ は正則エピ射 $\}$ により C が弱完全圏になる．

【定義】

正則圏 (regular category) とは次の条件を満たす圏 C をいう．

(1) C は有限完備である．

(2) 正則エピ射の pullback は正則エピ射である．

(3) 任意の射 $f\colon a \to b$ の kernel pair は coequalizer を持つ．

弱完全圏

【定義】

正則圏 (regular category) とは次の条件を満たす圏 C をいう．

(1) C は有限完備である．

(2) 正則エピ射の pullback は正則エピ射である．

(3) 任意の射 $f\colon a \to b$ の kernel pair は coequalizer を持つ．

【定理】

零対象を持つ正則圏 C に対して

$$\text{weakly homological}$$

$$\iff \text{正則エピ射 } f \text{ に対して } \operatorname{coker}(\ker(f)) \cong f. \qquad \square$$

【例】

$\mathbf{Grp}$ は正則圏である．また任意のエピ射が正則エピ射であり，エピ射 f に対して $\mathrm{coker}(\ker(f)) \cong f$ である．従って $\mathbf{Grp}$ は weakly homological であるから

$$D := \{ f \in \mathrm{Mor}(\mathbf{Grp}) \mid f \text{ はエピ射} \}$$

とすることで $\mathbf{Grp}$ は弱完全圏になることが分かる． $\square$

【命題】

C は weak homological な弱完全圏で，余核を持ち正規かつ余正規であるとする．図式 $a_0 \xrightarrow{f_0} a_1 \xrightarrow{f_1} \cdots \xrightarrow{f_n} a_{n+1}$ について

これが完全列 $\Longleftrightarrow$ 各 $0 \leq i < n$ について $\mathrm{Im}(f_i) \cong \ker(f_{i+1})$.

弱完全圏

証明.

$(\Longrightarrow)$

$$\begin{tikzcd}
\cdots & a_i \arrow[rr, "f_i"] \arrow[dr, "f_i^{\mathrm{d}}"'] & & a_{i+1} \arrow[rr, "f_{i+1}"] \arrow[dr, "f_{i+1}^{\mathrm{d}}"'] & & a_{i+2} & \cdots \\
& & {[f_i]} \arrow[ur, tail, "f_i^{\mathrm{i}}"'] & & {[f_{i+1}]} \arrow[ur, tail, "f_{i+1}^{\mathrm{i}}"'] & &
\end{tikzcd}$$

このとき $\mathrm{Im}(f_i) \cong \ker(\mathrm{coker}(f_i)) \cong \ker(\mathrm{coker}(f_i^{\mathrm{i}})) \cong \ker(f_{i+1}^{\mathrm{d}}) \cong \ker(f_{i+1})$ である．

$(\Longleftarrow)$ $f_i = (a_i \xrightarrow{f_i'} x_i \xrightarrow{\mathrm{Im}(f_i)} a_{i+1})$ と分解する．$\mathrm{Im}(f_i)$ がモノ射だから $\ker(f_{i+1}') \cong \ker(f_{i+1}) \cong \mathrm{Im}(f_i)$ である．故に f_i' が正則エピ射であることを示せばよい．f_i' はエピ射であり，C が余正規だから正則エピ射である． $\square$

【例】

アーベル圏は正則圏である．また任意のエピ射が正則エピ射であり，エピ射 f に対して $\mathrm{coker}(\ker(f)) \cong f$ である．従ってアーベル圏は weakly homological である．アーベル圏は前命題の仮定を満たすから，アーベル圏における (弱完全圏の意味での) 完全列は (通常の意味での) 完全列と一致する． $\square$

おまけ

おまけ

このスライドは beamer で作成しています．

beamer は Ti*k*Z と相性がよく，今回のような図式の表現ができます．

beamer における `\pause`，`\only`，`\uncover` などは tikzpicture 環境においてもそのまま使うことができます．

(Ti*k*Z の使い方は『Ti*k*Z の使い方 (圏論編)』を参照．)

おまけ

```
\begin{tikzpicture}
\node (a) at (0,2) {$a$};
\node (b) at (0,0) {$b$};
\node (x) at (2,2) {$x$};
\draw[->] (a) --(x);
\draw[->] (a) --(b);

\node (y) at (2,0) {$y$};
\draw[->] (x) --(y);

\draw[->] (b) --(y);
\end{tikzpicture}
```

$$
\begin{tikzcd}
a \arrow[r] \arrow[d] & x \arrow[d] \\
b \arrow[r] & y
\end{tikzcd}
$$

おまけ

```
\begin{tikzpicture}
\node (a) at (0,2) {$a$};
\node (b) at (0,0) {$b$};
\node (x) at (2,2) {$x$};
\draw[->] (a) --(x);
\draw[->] (a) --(b);
\pause
\node (y) at (2,0) {$y$};
\draw[->] (x) --(y);
\pause
\draw[->] (b) --(y);
\end{tikzpicture}
```

$$\begin{tikzcd} a \arrow[r] \arrow[d] & x \\ b & \end{tikzcd}$$

1 ページ目

おまけ

```
\begin{tikzpicture}
\node (a) at (0,2) {$a$};
\node (b) at (0,0) {$b$};
\node (x) at (2,2) {$x$};
\draw[->] (a) --(x);
\draw[->] (a) --(b);
\pause
\node (y) at (2,0) {$y$};
\draw[->] (x) --(y);
\pause
\draw[->] (b) --(y);
\end{tikzpicture}
```

$$\begin{tikzcd} a \arrow[r] \arrow[d] & x \arrow[d] \\ b & y \end{tikzcd}$$

2 ページ目

おまけ

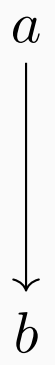
```
\begin{tikzpicture}
\node (a) at (0,2) {$a$};
\node (b) at (0,0) {$b$};
\node (x) at (2,2) {$x$};
\draw[->] (a) --(x);
\draw[->] (a) --(b);
\pause
\node (y) at (2,0) {$y$};
\draw[->] (x) --(y);
\pause
\draw[->] (b) --(y);
\end{tikzpicture}
```

$$\begin{tikzcd} a \arrow[r] \arrow[d] & x \arrow[d] \\ b \arrow[r] & y \end{tikzcd}$$

3 ページ目

おまけ

```
\begin{tikzpicture}
\node (a) at (0,2) {$a$};
\uncover<1>{
\node (b) at (0,0) {$b$};
\draw[->] (a) --(b);
}
\uncover<2>{
\node (x) at (2,2) {$x$};
\draw[->] (a) --(x);
}
\uncover<3>{
\node (y) at (2,0) {$y$};
\draw[->] (a) --(y);
}
\end{tikzpicture}
```



1 ページ目

おまけ

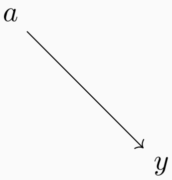
```
\begin{tikzpicture}
\node (a) at (0,2) {$a$};
\uncover<1>{
\node (b) at (0,0) {$b$};
\draw[->] (a) --(b);
}
\uncover<2>{
\node (x) at (2,2) {$x$};
\draw[->] (a) --(x);
}
\uncover<3>{
\node (y) at (2,0) {$y$};
\draw[->] (a) --(y);
}
\end{tikzpicture}
```

$$a \longrightarrow x$$

2 ページ目

おまけ

```
\begin{tikzpicture}
\node (a) at (0,2) {$a$};
\uncover<1>{
\node (b) at (0,0) {$b$};
\draw[->] (a) --(b);
}
\uncover<2>{
\node (x) at (2,2) {$x$};
\draw[->] (a) --(x);
}
\uncover<3>{
\node (y) at (2,0) {$y$};
\draw[->] (a) --(y);
}
\end{tikzpicture}
```



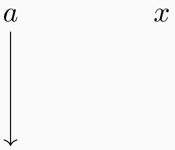
3 ページ目

おまけ

```
\usetikzlibrary{overlay-beamer-styles}
```

おまけ

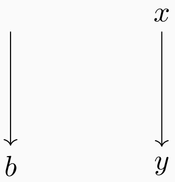
```
\begin{tikzpicture}
\node[visible on=<1>] (a) at (0,2) {$a$};
\node[visible on=<2>] (b) at (0,0) {$b$};
\node (x) at (2,2) {$x$};
\draw[->,visible on=<3>] (a) --(x);
\draw[->] (a) --(b);
\pause
\node (y) at (2,0) {$y$};
\draw[->] (x) --(y);
\pause
\draw[->] (b) --(y);
\end{tikzpicture}
```



1 ページ目

おまけ

```
\begin{tikzpicture}
\node[visible on=<1>] (a) at (0,2) {$a$};
\node[visible on=<2>] (b) at (0,0) {$b$};
\node (x) at (2,2) {$x$};
\draw[->,visible on=<3>] (a) --(x);
\draw[->] (a) --(b);
\pause
\node (y) at (2,0) {$y$};
\draw[->] (x) --(y);
\pause
\draw[->] (b) --(y);
\end{tikzpicture}
```



2 ページ目

おまけ

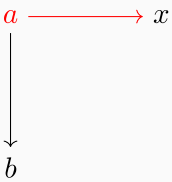
```
\begin{tikzpicture}
\node[visible on=<1>] (a) at (0,2) {$a$};
\node[visible on=<2>] (b) at (0,0) {$b$};
\node (x) at (2,2) {$x$};
\draw[->,visible on=<3>] (a) --(x);
\draw[->] (a) --(b);
\pause
\node (y) at (2,0) {$y$};
\draw[->] (x) --(y);
\pause
\draw[->] (b) --(y);
\end{tikzpicture}
```

$$\begin{tikzcd} {} \arrow[r] \arrow[d] & x \arrow[d] \\ {} \arrow[r] & y \end{tikzcd}$$

3 ページ目

おまけ

```
\begin{tikzpicture}
\node[alt=<1>{red}{blue}]
      (a) at (0,2) {$a$};
\node (b) at (0,0) {$b$};
\node (x) at (2,2) {$x$};
\draw[alt=<1-2>{->,red}{<-,blue}]
          (a) --(x);
\draw[->] (a) --(b);
\pause
\node (y) at (2,0) {$y$};
\draw[->] (x) --(y);
\pause
\draw[->] (b) --(y);
\end{tikzpicture}
```



1 ページ目

おまけ

```
\begin{tikzpicture}
\node[alt=<1>{red}{blue}]
      (a) at (0,2) {$a$};
\node (b) at (0,0) {$b$};
\node (x) at (2,2) {$x$};
\draw[alt=<1-2>{->,red}{<-,blue}]
          (a) --(x);
\draw[->] (a) --(b);
\pause
\node (y) at (2,0) {$y$};
\draw[->] (x) --(y);
\pause
\draw[->] (b) --(y);
\end{tikzpicture}
```

$$
\begin{tikzcd}
a \arrow[r] \arrow[d] & x \arrow[d] \\
b & y
\end{tikzcd}
$$

2 ページ目

おまけ

```
\begin{tikzpicture}
\node[alt=<1>{red}{blue}]
      (a) at (0,2) {$a$};
\node (b) at (0,0) {$b$};
\node (x) at (2,2) {$x$};
\draw[alt=<1-2>{->,red}{<-,blue}]
          (a) --(x);
\draw[->] (a) --(b);
\pause
\node (y) at (2,0) {$y$};
\draw[->] (x) --(y);
\pause
\draw[->] (b) --(y);
\end{tikzpicture}
```

$$\begin{tikzcd} a \arrow[d] & x \arrow[l] \arrow[d] \\ b \arrow[r] & y \end{tikzcd}$$

3ページ目