pseudo double category

alg-d
http://alg-d.com/math/kan_extension/

2022 年 6 月 5 日

大雑把に言えば，**CAT** での internal category が double category である．ここでは少し条件を弱めた pseudo double category を扱う．

目次

1 pseudo double category 1
2 pseudo double category における Kan 拡張 30

1 psuedo double category

定義. pseudo double category $\mathbb{D}$ とは

(1) 圏 D^0, D^1 が与えられている．
(2) 関手 $L, R\colon D^1 \to D^0$，$U\colon D^0 \to D^1$ が与えられている．
(3) 関手 $\odot\colon D^1 \times_{D^0} D^1 \longrightarrow D^1$ が与えられている．但し $D^1 \times_{D^0} D^1$ は次の pullback である．

$$\begin{tikzcd} D^1 \times_{D^0} D^1 \arrow[r, "P_L"] \arrow[d, "P_R"'] & D^1 \arrow[d, "L"] \\ D^1 \arrow[r, "R"'] & D^0 \end{tikzcd}$$

(4) $L \circ U = \mathrm{id}$，$R \circ U = \mathrm{id}$，$L \circ \odot = L \circ P_R$，$R \circ \odot = R \circ P_L$ である．

(5) 次の自然同型 α, λ, ρ が与えられている．

$$\begin{tikzcd}
 & D^1 \times_{D^0} D^1 \times_{D^0} D^1 \arrow[dl, "\odot\times\mathrm{id}"'] \arrow[dr, "\mathrm{id}\times\odot"] & \\
D^1 \times_{D^0} D^1 \arrow[dr, "\odot"'] \arrow[rr, phantom, "\overset{\sim}{\underset{\alpha}{\Longrightarrow}}"] & & D^1 \times_{D^0} D^1 \arrow[dl, "\odot"] \\
 & D^1 &
\end{tikzcd}$$

$$\begin{tikzcd}
D^0 \times_{D^0} D^1 \arrow[rr, "\cong"] \arrow[dr, "U\times\mathrm{id}"'] & \arrow[d, phantom, "\lambda\Uparrow\wr"] & D^1 \\
 & D^1 \times_{D^0} D^1 \arrow[ur, "\odot"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
D^1 \times_{D^0} D^0 \arrow[rr, "\cong"] \arrow[dr, "\mathrm{id}\times U"'] & \arrow[d, phantom, "\rho\Uparrow\wr"] & D^1 \\
 & D^1 \times_{D^0} D^1 \arrow[ur, "\odot"'] &
\end{tikzcd}$$

(6) 次の自然変換の等式が成り立つ．($D^2 := D^1 \times_{D^0} D^1$，$D^3 := D^1 \times_{D^0} D^1 \times_{D^0} D^1$ などと書いた．)

$$\begin{tikzcd}
 & & D^4 \arrow[dll, "\odot\times\mathrm{id}\times\mathrm{id}"'] \arrow[drr, "\mathrm{id}\times\mathrm{id}\times\odot"] \arrow[d, "\mathrm{id}\times\odot\times\mathrm{id}"] & & \\
D^3 \arrow[dd, "\odot\times\mathrm{id}"'] & \arrow[r, phantom, "\overset{\alpha\times\mathrm{id}}{\Longrightarrow}"] & D^3 \arrow[ddll, "\odot\times\mathrm{id}"] \arrow[ddrr, "\mathrm{id}\times\odot"'] & \arrow[l, phantom, "\overset{\mathrm{id}\times\alpha}{\Longrightarrow}"] & D^3 \arrow[dd, "\mathrm{id}\times\odot"] \\
 & & & & \\
D^2 \arrow[drr, "\odot"'] & & \overset{\alpha}{\Longrightarrow} & & D^2 \arrow[dll, "\odot"] \\
 & & D^1 & &
\end{tikzcd}
=
\begin{tikzcd}
 & & D^4 \arrow[dll, "\odot\times\mathrm{id}\times\mathrm{id}"'] \arrow[drr, "\mathrm{id}\times\mathrm{id}\times\odot"] & & \\
D^3 \arrow[ddr, phantom, "\overset{\alpha}{\Longrightarrow}"] \arrow[dd, "\odot\times\mathrm{id}"'] \arrow[drr, "\mathrm{id}\times\odot"] & & = & & D^3 \arrow[dll, "\odot\times\mathrm{id}"'] \arrow[dd, "\mathrm{id}\times\odot"] \\
 & & D^2 \arrow[dd, "\odot"] & \overset{\alpha}{\Longrightarrow} & \\
D^2 \arrow[drr, "\odot"'] & & & & D^2 \arrow[dll, "\odot"] \\
 & & D^1 & &
\end{tikzcd}$$

(7) 次の自然変換の等式が成り立つ．

$$\begin{tikzcd}
 & D^2 \arrow[ddl, bend right, "\mathrm{id}"'] \arrow[ddr, bend left, "\mathrm{id}"] \arrow[d, "\mathrm{id}\times U\times\mathrm{id}"] & \\
\overset{\rho^{-1}\times\mathrm{id}}{\Longrightarrow} & D^3 \arrow[dl, "\odot\times\mathrm{id}"] \arrow[dr, "\mathrm{id}\times\odot"'] & \overset{\mathrm{id}\times\lambda}{\Longrightarrow} \\
D^2 \arrow[dr, "\odot"'] & \overset{\alpha}{\Longrightarrow} & D^2 \arrow[dl, "\odot"] \\
 & D^1 &
\end{tikzcd}
=
\begin{tikzcd}
 & D^2 \arrow[ddl, bend right, "\mathrm{id}"'] \arrow[ddr, bend left, "\mathrm{id}"] & \\
 & = & \\
D^2 \arrow[dr, "\odot"'] & & D^2 \arrow[dl, "\odot"] \\
 & D^1 &
\end{tikzcd}$$

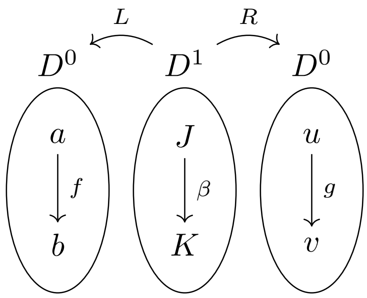
(8) $L \bullet \alpha, L \bullet \lambda, L \bullet \rho, R \bullet \alpha, R \bullet \lambda, R \bullet \rho$ は全て恒等変換である．

圏 D^0 の対象を $\mathbb{D}$ の対象 (object)，D^0 の射を $\mathbb{D}$ の垂直射 (vertical morphism) という．通常の射と同様，垂直射も記号 $f\colon a \to b$ で表す．また垂直射の合成とは D^0 での合成のことであり，$\circ$ で表す．

圏 D^1 の対象を $\mathbb{D}$ の水平射 (horizontal morphism) という．J を水平射とするとき，$L(J) \in \mathrm{Ob}(D^0)$ を J の domain，$R(J) \in \mathrm{Ob}(D^0)$ を J の codomain という．$L(J) = a$，$R(J) = b$ のとき記号で $J\colon a \nrightarrow b$ と書く．

D^1 の射を $\mathbb{D}$ の cell という．β を cell とするとき，D^1 の射としての domain, codomain を horizontal source, horizontal target といい，また $L(\beta) \in \mathrm{Mor}(D^0)$，$R(\beta) \in \mathrm{Mor}(D^0)$ を vertical source, vertical target という．

今 β の horizontal source, horizontal target が $J\colon a \nrightarrow u$, $K\colon b \nrightarrow v$ で vertical source, vertical target が f, g であるとする．即ち $L(\beta) = f$，$R(\beta) = g$ である．すると L, R が関手だから $f\colon a \to b$，$g\colon u \to v$ となる．



そこでこのような状況のときは図式で

$$
\begin{tikzcd}
a \arrow[r, "J", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\beta"] & u \arrow[d, "g"] \\
b \arrow[r, "K"', "\shortmid"{marking}] & v
\end{tikzcd}
$$

と表す．

β, γ を次の cell とする．

$$
\begin{tikzcd}
a \arrow[r, "J", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\beta"] & u \arrow[d, "g"] \\
b \arrow[r, "K"', "\shortmid"{marking}] & v
\end{tikzcd}
\qquad
\begin{tikzcd}
b \arrow[r, "K", "\shortmid"{marking}] \arrow[d, "h"'] \arrow[dr, phantom, "\Downarrow\gamma"] & v \arrow[d, "k"] \\
c \arrow[r, "M"', "\shortmid"{marking}] & w
\end{tikzcd}
$$

このとき D^1 の合成により定まる cell $\gamma\circ\beta$ を垂直合成という．(L が関手なので $L(\gamma\circ\beta) = L(\gamma)\circ L(\beta)$ である．R も同様．)

$$
\begin{tikzcd}
a \arrow[r, "J", "|"{marking}] \arrow[dd, "h\circ f"'] \arrow[rdd, phantom, "\Downarrow\gamma\circ\beta"] & u \arrow[dd, "k\circ g"] \\
 & \\
c \arrow[r, "M"', "|"{marking}] & w
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
a \arrow[r, "J", "|"{marking}] \arrow[d, "f"'] \arrow[rd, phantom, "\Downarrow\beta"] & u \arrow[d, "g"] \\
b \arrow[r, "K"', "|"{marking}] \arrow[d, "h"'] \arrow[rd, phantom, "\Downarrow\gamma"] & v \arrow[d, "k"] \\
c \arrow[r, "M"', "|"{marking}] & w
\end{tikzcd}
$$

更に

$$
\begin{tikzcd}
c \arrow[r, "M", "|"{marking}] \arrow[d, "p"'] \arrow[rd, phantom, "\Downarrow\sigma"] & w \arrow[d, "q"] \\
d \arrow[r, "N"', "|"{marking}] & x
\end{tikzcd}
$$

も cell とする．このとき D^0, D^1 は圏だから結合律が成り立ち，等式

$$
\begin{tikzcd}
a \arrow[r, "J", "|"{marking}] \arrow[dd, "h\circ f"'] \arrow[rdd, phantom, "\Downarrow\gamma\circ\beta"] & u \arrow[dd, "k\circ g"] \\
 & \\
c \arrow[r, "M"', "|"{marking}] \arrow[d, "p"'] \arrow[rd, phantom, "\Downarrow\sigma"] & w \arrow[d, "q"] \\
d \arrow[r, "N"', "|"{marking}] & x
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
a \arrow[r, "J", "|"{marking}] \arrow[d, "f"'] \arrow[rd, phantom, "\Downarrow\beta"] & u \arrow[d, "g"] \\
b \arrow[r, "K"', "|"{marking}] \arrow[dd, "p\circ h"'] \arrow[rdd, phantom, "\Downarrow\sigma\circ\gamma"] & v \arrow[dd, "q\circ k"] \\
 & \\
d \arrow[r, "N"', "|"{marking}] & x
\end{tikzcd}
$$

が成り立つ．よって，β, γ, σ の合成を

$$
\begin{tikzcd}
a \arrow[r, "J", "|"{marking}] \arrow[d, "f"'] \arrow[rd, phantom, "\Downarrow\beta"] & u \arrow[d, "g"] \\
b \arrow[r, "K"', "|"{marking}] \arrow[d, "h"'] \arrow[rd, phantom, "\Downarrow\gamma"] & v \arrow[d, "k"] \\
c \arrow[r, "M"', "|"{marking}] \arrow[d, "p"'] \arrow[rd, phantom, "\Downarrow\sigma"] & w \arrow[d, "q"] \\
d \arrow[r, "N"', "|"{marking}] & x
\end{tikzcd}
$$

で表しても特に問題はない．

水平射 $J\colon a \nrightarrow u$，即ち対象 $J \in D^1$ に対して (圏 D^1 における) 恒等射 id_J を考える．L, R は関手だから $L(\mathrm{id}_J) = \mathrm{id}_a$，$R(\mathrm{id}_J) = \mathrm{id}_u$ である．よって id_J は次のような cell である．

$$
\begin{tikzcd}
a \arrow[r, "J", "\shortmid"{marking}] \arrow[d, "\mathrm{id}_a"'] \arrow[dr, phantom, "\Downarrow\mathrm{id}_J"] & u \arrow[d, "\mathrm{id}_u"] \\
a \arrow[r, "J"', "\shortmid"{marking}] & u
\end{tikzcd}
$$

id_J は D^1 の恒等射だから

$$
\begin{tikzcd}
a \arrow[r, "J", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\beta"] & u \arrow[d, "g"] \\
b \arrow[r, "K"', "\shortmid"{marking}] \arrow[d, "\mathrm{id}_b"'] \arrow[dr, phantom, "\Downarrow\mathrm{id}_K"] & v \arrow[d, "\mathrm{id}_v"] \\
b \arrow[r, "K"', "\shortmid"{marking}] & v
\end{tikzcd}
=
\begin{tikzcd}
a \arrow[r, "J", "\shortmid"{marking}] \arrow[dd, "f"'] \arrow[dd, phantom, shift left=4ex, "\Downarrow\beta"] & u \arrow[dd, "g"] \\
& \\
b \arrow[r, "K"', "\shortmid"{marking}] & v
\end{tikzcd}
\qquad
\begin{tikzcd}
a \arrow[r, "J", "\shortmid"{marking}] \arrow[d, "\mathrm{id}_a"'] \arrow[dr, phantom, "\Downarrow\mathrm{id}_J"] & u \arrow[d, "\mathrm{id}_u"] \\
a \arrow[r, "J"', "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\beta"] & u \arrow[d, "g"] \\
b \arrow[r, "K"', "\shortmid"{marking}] & v
\end{tikzcd}
=
\begin{tikzcd}
a \arrow[r, "J", "\shortmid"{marking}] \arrow[dd, "f"'] \arrow[dd, phantom, shift left=4ex, "\Downarrow\beta"] & u \arrow[dd, "g"] \\
& \\
b \arrow[r, "K"', "\shortmid"{marking}] & v
\end{tikzcd}
$$

となる．

cell β が同型とは，D^1 の射として同型であることをいう．即ち，ある cell β^{-1} が存在して $\beta \circ \beta^{-1} = \mathrm{id}$，$\beta^{-1} \circ \beta = \mathrm{id}$ となることである．β, β^{-1} を

$$
\begin{tikzcd}
a \arrow[r, "J", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\beta"] & u \arrow[d, "g"] \\
b \arrow[r, "K"', "\shortmid"{marking}] & v
\end{tikzcd}
\qquad
\begin{tikzcd}
b \arrow[r, "K", "\shortmid"{marking}] \arrow[d, "h"'] \arrow[dr, phantom, "\Downarrow\beta^{-1}"] & v \arrow[d, "k"] \\
a \arrow[r, "J"', "\shortmid"{marking}] & u
\end{tikzcd}
$$

とすれば

$$
\begin{tikzcd}
a \arrow[r, "J", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\beta"] & u \arrow[d, "g"] \\
b \arrow[r, "K"', "\shortmid"{marking}] \arrow[d, "h"'] \arrow[dr, phantom, "\Downarrow\beta^{-1}"] & v \arrow[d, "k"] \\
a \arrow[r, "J"', "\shortmid"{marking}] & u
\end{tikzcd}
=
\begin{tikzcd}
a \arrow[r, "J", "\shortmid"{marking}] \arrow[dd, "\mathrm{id}_a"'] \arrow[dd, phantom, shift left=4ex, "\Downarrow\mathrm{id}_J"] & u \arrow[dd, "\mathrm{id}_u"] \\
& \\
a \arrow[r, "J"', "\shortmid"{marking}] & u
\end{tikzcd}
\qquad
\begin{tikzcd}
b \arrow[r, "K", "\shortmid"{marking}] \arrow[d, "h"'] \arrow[dr, phantom, "\Downarrow\beta^{-1}"] & v \arrow[d, "k"] \\
a \arrow[r, "J"', "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\beta"] & u \arrow[d, "g"] \\
b \arrow[r, "K"', "\shortmid"{marking}] & v
\end{tikzcd}
=
\begin{tikzcd}
b \arrow[r, "K", "\shortmid"{marking}] \arrow[dd, "\mathrm{id}_b"'] \arrow[dd, phantom, shift left=4ex, "\Downarrow\mathrm{id}_K"] & v \arrow[dd, "\mathrm{id}_v"] \\
& \\
b \arrow[r, "K"', "\shortmid"{marking}] & v
\end{tikzcd}
$$

となる．従って f, g, h, k も (D^0 の) 同型射であり $f^{-1} = h$，$g^{-1} = k$ となる．つまり，同型な cell の vertical source と vertical target は同型である．

次に β, γ を次の cell とする．

$$
\begin{tikzcd}
a \arrow[r, "J", "\shortmid" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\beta"] & u \arrow[d, "g"] \\
b \arrow[r, "K"', "\shortmid" marking] & v
\end{tikzcd}
\qquad
\begin{tikzcd}
u \arrow[r, "M", "\shortmid" marking] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow\gamma"] & s \arrow[d, "h"] \\
v \arrow[r, "N"', "\shortmid" marking] & t
\end{tikzcd}
$$

$R(\beta) = L(\gamma)$ だから $\langle \beta, \gamma \rangle \in D^1 \times_{D^0} D^1$ であり $P_R(\langle \beta, \gamma \rangle) = \beta$，$P_L(\langle \beta, \gamma \rangle) = \gamma$ となる．関手 $\odot\colon D^1 \times_{D^0} D^1 \to D^1$ により定まる cell $\beta \odot \gamma$ を水平合成という．定義により $L \circ \odot = L \circ P_R$，$R \circ \odot = R \circ P_L$ だから $L(\beta \odot \gamma) = L(\beta)$，$R(\beta \odot \gamma) = R(\gamma)$ となる．

$$
\begin{tikzcd}
a \arrow[r, "J \odot M", "\shortmid" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\beta\odot\gamma"] & s \arrow[d, "h"] \\
b \arrow[r, "K \odot N"', "\shortmid" marking] & t
\end{tikzcd}
=
\begin{tikzcd}
a \arrow[r, "J", "\shortmid" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\beta"] & u \arrow[r, "M", "\shortmid" marking] \arrow[d, "g"] \arrow[dr, phantom, "\Downarrow\gamma"] & s \arrow[d, "h"] \\
b \arrow[r, "K"', "\shortmid" marking] & v \arrow[r, "N"', "\shortmid" marking] & t
\end{tikzcd}
$$

また次の図式の状況のとき

$$
\begin{tikzcd}
a \arrow[r, "J", "\shortmid" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\beta"] & u \arrow[r, "N", "\shortmid" marking] \arrow[d, "g"] \arrow[dr, phantom, "\Downarrow\sigma"] & s \arrow[d, "p"] \\
b \arrow[r, "K"', "\shortmid" marking] \arrow[d, "h"'] \arrow[dr, phantom, "\Downarrow\gamma"] & v \arrow[r, "P"', "\shortmid" marking] \arrow[d, "k"] \arrow[dr, phantom, "\Downarrow\tau"] & t \arrow[d, "q"] \\
c \arrow[r, "M"', "\shortmid" marking] & w \arrow[r, "Q"', "\shortmid" marking] & x
\end{tikzcd}
$$

$\odot$ が関手だから $(\gamma \circ \beta) \odot (\tau \circ \sigma) = (\gamma \odot \tau) \circ (\beta \odot \sigma)$ が成り立つ．

$$
\begin{tikzcd}
a \arrow[r, "J", "\shortmid" marking] \arrow[dd, "h \circ f"'] \arrow[ddr, phantom, "\Downarrow\gamma\circ\beta"] & u \arrow[r, "N", "\shortmid" marking] \arrow[dd, "k \circ g"] \arrow[ddr, phantom, "\Downarrow\tau\circ\sigma"] & s \arrow[dd, "q \circ p"] \\
 & & \\
c \arrow[r, "M"', "\shortmid" marking] & w \arrow[r, "Q"', "\shortmid" marking] & x
\end{tikzcd}
=
\begin{tikzcd}
a \arrow[r, "J \odot N", "\shortmid" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\beta\odot\sigma"] & s \arrow[d, "p"] \\
b \arrow[r, "K \odot P"', "\shortmid" marking] \arrow[d, "h"'] \arrow[dr, phantom, "\Downarrow\gamma\odot\tau"] & t \arrow[d, "q"] \\
c \arrow[r, "M \odot Q"', "\shortmid" marking] & x
\end{tikzcd}
$$

次に

$$
\begin{tikzcd}
a \arrow[r, "J", "\shortmid" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\beta"] & u \arrow[d, "g"] \\
b \arrow[r, "K"', "\shortmid" marking] & v
\end{tikzcd}
\qquad
\begin{tikzcd}
u \arrow[r, "M", "\shortmid" marking] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow\gamma"] & s \arrow[d, "h"] \\
v \arrow[r, "N"', "\shortmid" marking] & t
\end{tikzcd}
\qquad
\begin{tikzcd}
s \arrow[r, "P", "\shortmid" marking] \arrow[d, "h"'] \arrow[dr, phantom, "\Downarrow\sigma"] & w \arrow[d, "k"] \\
t \arrow[r, "Q"', "\shortmid" marking] & x
\end{tikzcd}
$$

を cell とすると，二通りの水平合成 $(\beta\odot\gamma)\odot\sigma$ と $\beta\odot(\gamma\odot\sigma)$ を考えることができる．この合成の「結合律」を与えるのが α である．今 $\langle J, M, P\rangle \in D^1 \times_{D^0} D^1 \times_{D^0} D^1$ であるから，α_{JMP} は D^1 の射，即ち cell である．また $L\bullet\alpha = \mathrm{id}$，$R\bullet\alpha = \mathrm{id}$ だから $L(\alpha_{JMP}) = \mathrm{id}_a$，$R(\alpha_{JMP}) = \mathrm{id}_d$ となる．図式で書くと次のようになる．(但し，id は等号で表した．以降も同様とする．)

$$
\begin{tikzcd}
a \arrow[r, "(J\odot M)\odot P", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\alpha_{JMP}"] & w \arrow[d, equal] \\
a \arrow[r, "J\odot(M\odot P)"', "\shortmid"{marking}] & w
\end{tikzcd}
$$

また α が自然変換であることから

$$
\begin{tikzcd}
(J\odot M)\odot P \arrow[r, "\alpha_{JMP}"] \arrow[d, "(\beta\odot\gamma)\odot\sigma"'] & J\odot(M\odot P) \arrow[d, "\beta\odot(\gamma\odot\sigma)"] \\
(K\odot N)\odot Q \arrow[r, "\alpha_{KNQ}"'] & K\odot(N\odot Q)
\end{tikzcd}
$$

が可換である．これは $\mathbb{D}$ の図式で書けば

$$
\begin{tikzcd}
a \arrow[r, "J\odot M", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\beta\odot\gamma"] & s \arrow[r, "P", "\shortmid"{marking}] \arrow[d, "h"] \arrow[dr, phantom, "\Downarrow\sigma"] & w \arrow[d, "k"] \\
b \arrow[r, "K\odot N"', "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\alpha_{KNQ}"] & t \arrow[r, "Q"', "\shortmid"{marking}] & x \arrow[d, equal] \\
b \arrow[r, "K"', "\shortmid"{marking}] & v \arrow[r, "N\odot Q"', "\shortmid"{marking}] & x
\end{tikzcd}
=
\begin{tikzcd}
a \arrow[r, "J\odot M", "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\alpha_{JMP}"] & s \arrow[r, "P", "\shortmid"{marking}] & w \arrow[d, equal] \\
a \arrow[r, "J", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\beta"] & u \arrow[r, "M\odot P", "\shortmid"{marking}] \arrow[d, "g"] \arrow[dr, phantom, "\Downarrow\gamma\odot\sigma"] & w \arrow[d, "k"] \\
b \arrow[r, "K"', "\shortmid"{marking}] & v \arrow[r, "N\odot Q"', "\shortmid"{marking}] & x
\end{tikzcd}
\tag{1}
$$

である．更に条件 6 により $a \xrightarrow{J} b \xrightarrow{K} c \xrightarrow{M} d \xrightarrow{N} e$ に対して

$$
\begin{tikzcd}
a \arrow[r, "(J\odot K)\odot M", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\alpha_{JKM}"] & d \arrow[r, "N", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mathrm{id}"] & e \arrow[d, equal] \\
a \arrow[r, "J\odot(K\odot M)"', "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\alpha_{J,K\odot M,N}"] & d \arrow[r, "N"', "\shortmid"{marking}] & e \arrow[d, equal] \\
a \arrow[r, "J"', "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mathrm{id}"] & b \arrow[r, "(K\odot M)\odot N"', "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\alpha_{KMN}"] & e \arrow[d, equal] \\
a \arrow[r, "J"', "\shortmid"{marking}] & b \arrow[r, "K\odot(M\odot N)"', "\shortmid"{marking}] & e
\end{tikzcd}
=
\begin{tikzcd}
a \arrow[r, "(J\odot K)\odot M", "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\alpha_{J\odot K,M,N}"] & d \arrow[r, "N", "\shortmid"{marking}] & e \arrow[d, equal] \\
a \arrow[r, "J\odot K"', "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\alpha_{J,K,M\odot N}"] & c \arrow[r, "M\odot N"', "\shortmid"{marking}] & e \arrow[d, equal] \\
a \arrow[r, "J"', "\shortmid"{marking}] & b \arrow[r, "K\odot(M\odot N)"', "\shortmid"{marking}] & e
\end{tikzcd}
\tag{2}
$$

が成り立つ．

$a \in \mathbb{D}$ を対象とする．$U\colon D^0 \to D^1$ が関手だから Ua は水平射である．$LU = \mathrm{id}$, $RU = \mathrm{id}$ だから $Ua\colon a \nrightarrow a$ となる．更に $f\colon a \to b$ を垂直射とすると Uf は

$$\begin{tikzcd} a \arrow[r, "Ua", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow Uf"] & a \arrow[d, "f"] \\ b \arrow[r, "Ub"', "\shortmid"{marking}] & b \end{tikzcd}$$

の形の cell である．また $\mathrm{id}_a\colon a \to a$ に対して $U(\mathrm{id}_a) = \mathrm{id}_{Ua}$ である．

$J\colon a \nrightarrow b$ とすると，α の場合と同様にして，λ_J, ρ_J は次のような同型な cell である．

$$\begin{tikzcd} a \arrow[r, "Ua", "\shortmid"{marking}] \arrow[d, equal] & a \arrow[r, "J", "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow\lambda_J"] & b \arrow[d, equal] \\ a \arrow[rr, "J"', "\shortmid"{marking}] & & b \end{tikzcd} \qquad \begin{tikzcd} a \arrow[r, "J", "\shortmid"{marking}] \arrow[d, equal] & b \arrow[r, "Ub", "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow\rho_J"] & b \arrow[d, equal] \\ a \arrow[rr, "J"', "\shortmid"{marking}] & & b \end{tikzcd}$$

故に Ua は id の役割を持つ水平射である．そこで以降では，Ua も単に等号で表す．λ, ρ が自然変換だから，cell

$$\begin{tikzcd} a \arrow[r, "J", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\beta"] & u \arrow[d, "g"] \\ b \arrow[r, "K"', "\shortmid"{marking}] & v \end{tikzcd}$$

に対して

$$\begin{tikzcd} a \arrow[r, equal] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow Uf"] & a \arrow[r, "J", "\shortmid"{marking}] \arrow[d, "f"] \arrow[dr, phantom, "\Downarrow\beta"] & u \arrow[d, "g"] \\ b \arrow[r, equal] \arrow[d, equal] & b \arrow[r, "K"', "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow\lambda_K"] & v \arrow[d, equal] \\ b \arrow[rr, "K"', "\shortmid"{marking}] & & v \end{tikzcd} \quad = \quad \begin{tikzcd} a \arrow[r, equal] \arrow[d, equal] & a \arrow[r, "J", "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow\lambda_J"] & u \arrow[d, equal] \\ a \arrow[rr, "J", "\shortmid"{marking}] \arrow[d, "f"'] & \arrow[d, phantom, "\Downarrow\beta"] & u \arrow[d, "g"] \\ b \arrow[rr, "K"', "\shortmid"{marking}] & & v \end{tikzcd} \tag{3}$$

$$\begin{tikzcd} a \arrow[r, "J", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\beta"] & u \arrow[r, equal] \arrow[d, "g"] \arrow[dr, phantom, "\Downarrow Ug"] & u \arrow[d, "g"] \\ b \arrow[r, "K"', "\shortmid"{marking}] \arrow[d, equal] & v \arrow[r, equal] \arrow[d, phantom, "\Downarrow\rho_K"] & v \arrow[d, equal] \\ b \arrow[rr, "K"', "\shortmid"{marking}] & & v \end{tikzcd} \quad = \quad \begin{tikzcd} a \arrow[r, "J", "\shortmid"{marking}] \arrow[d, equal] & u \arrow[r, equal] \arrow[d, phantom, "\Downarrow\rho_J"] & u \arrow[d, equal] \\ a \arrow[rr, "J", "\shortmid"{marking}] \arrow[d, "f"'] & \arrow[d, phantom, "\Downarrow\beta"] & u \arrow[d, "g"] \\ b \arrow[rr, "K"', "\shortmid"{marking}] & & v \end{tikzcd} \tag{4}$$

が成り立つ．また条件 7 により $a \xrightarrow{J} u \xrightarrow{K} s$ に対して

$$
\begin{tikzcd}
a \arrow[rr, "J", "\shortmid"{marking}] \arrow[d, equal] \arrow[rr, phantom, shift right=6, "\Downarrow\rho_J^{-1}"] & & u \arrow[r, "K", "\shortmid"{marking}] \arrow[d, equal] \arrow[r, phantom, shift right=6, "\Downarrow\mathrm{id}"] & s \arrow[d, equal] \\
a \arrow[rr, "J\odot Uu"', "\shortmid"{marking}] \arrow[d, equal] \arrow[rrr, phantom, shift right=6, "\Downarrow\alpha_{J,Uu,K}"] & & u \arrow[r, "K"', "\shortmid"{marking}] & s \arrow[d, equal] \\
a \arrow[r, "J"', "\shortmid"{marking}] \arrow[d, equal] \arrow[r, phantom, shift right=6, "\Downarrow\mathrm{id}"] & u \arrow[rr, "Uu\odot K"', "\shortmid"{marking}] \arrow[d, equal] \arrow[rr, phantom, shift right=6, "\Downarrow\lambda_K"] & & s \arrow[d, equal] \\
a \arrow[r, "J"', "\shortmid"{marking}] & u \arrow[rr, "K"', "\shortmid"{marking}] & & s
\end{tikzcd}
=
\begin{tikzcd}
a \arrow[r, "J", "\shortmid"{marking}] \arrow[ddd, equal] & u \arrow[r, "K", "\shortmid"{marking}] & s \arrow[ddd, equal] \\
& \arrow[phantom, "\Downarrow\mathrm{id}_{J\odot K}"] & \\
& & \\
a \arrow[r, "J"', "\shortmid"{marking}] & u \arrow[r, "K"', "\shortmid"{marking}] & s
\end{tikzcd}
\qquad (5)
$$

が成り立つ．

水平射 $J\colon a \nrightarrow u$ に対して cell ι_J, ι'_J を

$$
\begin{tikzcd}
a \arrow[r, equal] \arrow[dd, equal] & a \arrow[r, "J", "\shortmid"{marking}] & u \arrow[dd, equal] \\
& \arrow[phantom, "\Downarrow\iota_J"] & \\
a \arrow[r, "J"', "\shortmid"{marking}] & u \arrow[r, equal] & u
\end{tikzcd}
=
\begin{tikzcd}
a \arrow[r, equal] \arrow[d, equal] & a \arrow[r, "J", "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow\lambda_J"] & u \arrow[d, equal] \\
a \arrow[rr, "J"', "\shortmid"{marking}] \arrow[d, equal] & \arrow[d, phantom, "\Downarrow\rho_J^{-1}"] & u \arrow[d, equal] \\
a \arrow[r, "J"', "\shortmid"{marking}] & u \arrow[r, equal] & u
\end{tikzcd}
$$

$$
\begin{tikzcd}
a \arrow[r, "J", "\shortmid"{marking}] \arrow[dd, equal] & u \arrow[r, equal] & u \arrow[dd, equal] \\
& \arrow[phantom, "\Downarrow\iota'_J"] & \\
a \arrow[r, equal] & a \arrow[r, "J"', "\shortmid"{marking}] & u
\end{tikzcd}
=
\begin{tikzcd}
a \arrow[r, "J", "\shortmid"{marking}] \arrow[d, equal] & u \arrow[r, equal] \arrow[d, phantom, "\Downarrow\rho_J"] & u \arrow[d, equal] \\
a \arrow[rr, "J"', "\shortmid"{marking}] \arrow[d, equal] & \arrow[d, phantom, "\Downarrow\lambda_J^{-1}"] & u \arrow[d, equal] \\
a \arrow[r, equal] & a \arrow[r, "J"', "\shortmid"{marking}] & u
\end{tikzcd}
$$

で定義しておく．

bicategory の場合と同様に，次の補題が成り立つ．

補題 6. 水平射 $J\colon a \nrightarrow u$, $K\colon u \nrightarrow s$ に対して等式

$$
\begin{tikzcd}
a \arrow[rr, "J\odot K", "\shortmid"{marking}] \arrow[d, equal] & & s \arrow[r, equal] \arrow[d, phantom, "\Downarrow\alpha_{JKUa}"] & s \arrow[d, equal] \\
a \arrow[r, "J", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mathrm{id}"] & u \arrow[rr, "K\odot Ua", "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\rho_K"] & & s \arrow[d, equal] \\
a \arrow[r, "J"', "\shortmid"{marking}] & u \arrow[rr, "K"', "\shortmid"{marking}] & & s
\end{tikzcd}
=
\begin{tikzcd}
a \arrow[rr, "J\odot K", "\shortmid"{marking}] \arrow[d, equal] \arrow[drrr, phantom, "\Downarrow\rho_{J\odot K}"] & & s \arrow[r, equal] & s \arrow[d, equal] \\
a \arrow[r, "J"', "\shortmid"{marking}] & u \arrow[rr, "K"', "\shortmid"{marking}] & & s
\end{tikzcd}
$$

が成り立つ.

証明. 省略 □

補題 7. $\lambda_{Ua} = \rho_{Ua}$ である.

$$
\begin{tikzcd}
a \arrow[r, equal] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\lambda_{Ua}"] & a \arrow[r, equal] & a \arrow[d, equal] \\
a \arrow[rr, equal] & & a
\end{tikzcd}
=
\begin{tikzcd}
a \arrow[r, equal] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\rho_{Ua}"] & a \arrow[r, equal] & a \arrow[d, equal] \\
a \arrow[rr, equal] & & a
\end{tikzcd}
$$

証明. ρ の自然性から等式

$$
\begin{tikzcd}
a \arrow[r, "Ua\odot Ua", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\rho_{Ua}"] & a \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow U\mathrm{id}"] & a \arrow[d, equal] \\
a \arrow[r, equal] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\rho_{Ua}"] & a \arrow[r, equal] & a \arrow[d, equal] \\
a \arrow[rr, equal] & & a
\end{tikzcd}
=
\begin{tikzcd}
a \arrow[r, "Ua\odot Ua", "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\rho_{Ua\odot Ua}"] & a \arrow[r, equal] & a \arrow[d, equal] \\
a \arrow[r, equal] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\rho_{Ua}"] & a \arrow[r, equal] & a \arrow[d, equal] \\
a \arrow[rr, equal] & & a
\end{tikzcd}
$$

が成り立つ．ここで ρ_{Ua} は同型だから

$$
\begin{tikzcd}
a \arrow[r, "Ua\odot Ua", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\rho_{Ua}"] & a \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow U\mathrm{id}"] & a \arrow[d, equal] \\
a \arrow[r, equal] & a \arrow[r, equal] & a
\end{tikzcd}
=
\begin{tikzcd}
a \arrow[r, "Ua\odot Ua", "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\rho_{Ua\odot Ua}"] & a \arrow[r, equal] & a \arrow[d, equal] \\
a \arrow[r, equal] & a \arrow[r, equal] & a
\end{tikzcd}
$$

となる.

$$
\begin{tikzcd}
a \arrow[rr, "Ua\odot Ua", "\shortmid"{marking}] \arrow[d, equal] \arrow[drrr, phantom, "\Downarrow\alpha"] & & a \arrow[r, equal] & a \arrow[d, equal] \\
a \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mathrm{id}"] & a \arrow[rr, "Ua\odot Ua", "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\lambda_{Ua}"] & & a \arrow[d, equal] \\
a \arrow[r, equal] & a \arrow[rr, equal] & & a
\end{tikzcd}
=
\begin{tikzcd}
a \arrow[r, "Ua\odot Ua", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\rho_{Ua}"] & a \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow U\mathrm{id}"] & a \arrow[d, equal] \\
a \arrow[r, equal] & a \arrow[r, equal] & a
\end{tikzcd}
$$

$$
=\quad
\begin{tikzcd}
a \arrow[r, "Ua\odot Ua", "|" marking] \arrow[d, equal] & a \arrow[r, equal] & a \arrow[d, equal] \\
a \arrow[r, equal] \arrow[rr, phantom, "\Downarrow\rho_{Ua\odot Ua}"] & a \arrow[r, equal] & a
\end{tikzcd}
\quad=\quad
\begin{tikzcd}
a \arrow[r, "Ua\odot Ua", "|" marking] \arrow[d, equal] & a \arrow[r, equal] \arrow[d, phantom, "\Downarrow\alpha"] & a \arrow[d, equal] \\
a \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mathrm{id}"] & a \arrow[r, "Ua\odot Ua", "|" marking] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\rho_{Ua}"] & a \arrow[d, equal] \\
a \arrow[r, equal] & a \arrow[r, equal] & a
\end{tikzcd}
$$

□

命題 8. pseudo double category $\mathbb{D}$ に対して，次のように定義すると $\mathcal{V}(\mathbb{D})$ は strict 2-category になる．($\mathcal{V}(\mathbb{D})$ を vertical 2-category という．)

- $\mathcal{V}(\mathbb{D})$ の対象は $\mathbb{D}$ の対象とする．
- $\mathcal{V}(\mathbb{D})$ の 1-morphism は $\mathbb{D}$ の垂直射とする．
- $\mathcal{V}(\mathbb{D})$ の 2-morphism は次の形の cell とする．

$$
\begin{tikzcd}
a \arrow[r, equal] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\beta"] & a \arrow[d, "g"] \\
b \arrow[r, equal] & b
\end{tikzcd}
$$

  但しこのとき $\mathrm{dom}(\beta)=f$，$\mathrm{cod}(\beta)=g$ とする．
- 1-morphism の合成は垂直射の合成とする．
- 2-morphism $\beta\colon f\Rightarrow g$，$\gamma\colon g\Rightarrow h$ の垂直合成 $\gamma*\beta$ は $\rho_{Ua}\circ(\beta\odot\gamma)\circ\lambda_{Ua}^{-1}$ とする．

$$
\begin{tikzcd}
a \arrow[r, equal] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\gamma*\beta"] & a \arrow[d, "h"] \\
b \arrow[r, equal] & b
\end{tikzcd}
\quad:=\quad
\begin{tikzcd}
a \arrow[rr, equal] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\lambda^{-1}"] & & a \arrow[d, equal] \\
a \arrow[r, equal] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\beta"] & a \arrow[r, equal] \arrow[d, "g"] \arrow[dr, phantom, "\Downarrow\gamma"] & a \arrow[d, "h"] \\
b \arrow[r, equal] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\rho"] & b \arrow[r, equal] & b \arrow[d, equal] \\
b \arrow[rr, equal] & & b
\end{tikzcd}
$$

- 2-morphism の水平合成は D^1 の合成とする．

証明. まず $a,b\in\mathbb{D}$ に対して $\mathcal{V}(\mathbb{D})(a,b)$ は圏である．

$\because$) まず結合律を示す．そのために $f,g,h,k\colon a\to b$ として $\beta\colon f\Rightarrow g$，$\gamma\colon g\Rightarrow h$，

$\sigma\colon h \Rightarrow k$ とする．このとき

$$
\begin{tikzcd}
a \arrow[r, equal] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\sigma*(\gamma*\beta)"] & a \arrow[d, "k"] \\
b \arrow[r, equal] & b
\end{tikzcd}
=
\begin{tikzcd}
a \arrow[rr, equal] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\lambda^{-1}"] & & a \arrow[d, equal] \\
a \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\lambda^{-1}"] & a \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mathrm{id}"] & a \arrow[d, equal] \\
a \arrow[r, "+"marking, "Ua\odot Ua"'] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\beta\odot\gamma"] & a \arrow[r, equal] \arrow[d, "h"] \arrow[dr, phantom, "\Downarrow\sigma"] & a \arrow[d, "k"] \\
b \arrow[r, "+"marking, "Ua\odot Ua"'] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\rho"] & b \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mathrm{id}"] & b \arrow[d, equal] \\
b \arrow[r, equal] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\rho"] & b \arrow[r, equal] & b \arrow[d, equal] \\
b \arrow[rr, equal] & & b
\end{tikzcd}
$$

$$
=
\begin{tikzcd}
a \arrow[rrr, equal] \arrow[d, equal] \arrow[drrr, phantom, "\Downarrow\lambda^{-1}"] & & & a \arrow[d, equal] \\
a \arrow[rr, equal] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\rho^{-1}"] & & a \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mathrm{id}"] & a \arrow[d, equal] \\
a \arrow[rr, "+"marking, "Ua\odot Ua"'] \arrow[d, equal] \arrow[drrr, phantom, "\Downarrow\alpha"] & & a \arrow[r, equal] & a \arrow[d, equal] \\
a \arrow[r, equal] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\beta"] & a \arrow[rr, "+"marking, "Ua\odot Ua"] \arrow[d, "g"] \arrow[drr, phantom, "\Downarrow\gamma\odot\sigma"] & & a \arrow[d, "k"] \\
b \arrow[r, equal] \arrow[d, equal] \arrow[drrr, phantom, "\Downarrow\alpha^{-1}"] & b \arrow[rr, "+"marking, "Ua\odot Ua"'] & & b \arrow[d, equal] \\
b \arrow[rr, "+"marking, "Ua\odot Ua"] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\lambda"] & & b \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mathrm{id}"] & b \arrow[d, equal] \\
b \arrow[rr, equal] \arrow[d, equal] \arrow[drrr, phantom, "\Downarrow\rho"] & & b \arrow[r, equal] & b \arrow[d, equal] \\
b \arrow[rrr, equal] & & & b
\end{tikzcd}
=
\begin{tikzcd}
a \arrow[rr, equal] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\lambda^{-1}"] & & a \arrow[d, equal] \\
a \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mathrm{id}"] & a \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\lambda^{-1}"] & a \arrow[d, equal] \\
a \arrow[r, equal] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\beta"] & a \arrow[r, "+"marking, "Ua\odot Ua"] \arrow[d, "g"] \arrow[dr, phantom, "\Downarrow\gamma\odot\sigma"] & a \arrow[d, "k"] \\
b \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mathrm{id}"] & b \arrow[r, "+"marking, "Ua\odot Ua"'] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\rho"] & b \arrow[d, equal] \\
b \arrow[r, equal] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\rho"] & b \arrow[r, equal] & b \arrow[d, equal] \\
b \arrow[rr, equal] & & b
\end{tikzcd}
$$

$$
= \quad (\sigma * \gamma) * \beta
$$

だから結合律が成り立つ．

恒等射については，$\mathrm{id}_f := Uf$ と定義すれば

$$
\beta * \mathrm{id}_f \quad = \quad
\begin{tikzcd}
a \arrow[rr, equal] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\lambda^{-1}"] & & a \arrow[d, equal] \\
a \arrow[r, equal] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\mathrm{id}_f"] & a \arrow[r, equal] \arrow[d, "f"] \arrow[dr, phantom, "\Downarrow\beta"] & a \arrow[d, "g"] \\
b \arrow[r, equal] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\rho"] & b \arrow[r, equal] & b \arrow[d, equal] \\
b \arrow[rr, equal] & & b
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
a \arrow[rr, equal] \arrow[d, "f"'] \arrow[drr, phantom, "\Downarrow\beta"] & & a \arrow[d, "g"] \\
b \arrow[rr, equal] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\lambda^{-1}"] & & b \arrow[d, equal] \\
b \arrow[r, equal] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\rho"] & b \arrow[r, equal] & b \arrow[d, equal] \\
b \arrow[rr, equal] & & b
\end{tikzcd}
\quad = \quad \beta
$$

$$
\mathrm{id}_g * \beta \quad = \quad
\begin{tikzcd}
a \arrow[rr, equal] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\lambda^{-1}"] & & a \arrow[d, equal] \\
a \arrow[r, equal] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\beta"] & a \arrow[r, equal] \arrow[d, "g"] \arrow[dr, phantom, "\Downarrow\mathrm{id}_g"] & a \arrow[d, "g"] \\
b \arrow[r, equal] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\rho"] & b \arrow[r, equal] & b \arrow[d, equal] \\
b \arrow[rr, equal] & & b
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
a \arrow[rr, equal] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\lambda^{-1}"] & & a \arrow[d, equal] \\
a \arrow[r, equal] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\rho"] & a \arrow[r, equal] & a \arrow[d, equal] \\
a \arrow[rr, equal] \arrow[d, "f"'] \arrow[drr, phantom, "\Downarrow\beta"] & & a \arrow[d, "g"] \\
b \arrow[rr, equal] & & b
\end{tikzcd}
\quad = \quad \beta
$$

となるから成り立つ.

水平合成は関手 $M^{abc}\colon \mathcal{V}(\mathbb{D})(b,c) \times \mathcal{V}(\mathbb{D})(a,b) \to \mathcal{V}(\mathbb{D})(a,c)$ を定める.

$\because$) $\mathcal{V}(\mathbb{D})$ において次の状況のとき $(\tau * \sigma) \bullet (\gamma * \beta) = (\tau \bullet \gamma) * (\sigma \bullet \beta)$ を示す.

$$
\begin{tikzcd}[column sep=huge]
a \arrow[r, bend left=50, "f"{name=F}] \arrow[r, "g"{name=G, near end}] \arrow[r, bend right=50, "h"'{name=H}] \arrow[r, phantom, bend left=25, "\beta\Downarrow"] \arrow[r, phantom, bend right=25, "\gamma\Downarrow"] & b \arrow[r, bend left=50, "k"] \arrow[r, "l"{near start}] \arrow[r, bend right=50, "m"'] \arrow[r, phantom, bend left=25, "\sigma\Downarrow"] \arrow[r, phantom, bend right=25, "\tau\Downarrow"] & c
\end{tikzcd}
$$

即ち

$$\begin{tikzcd}
a \arrow[r, equal] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\gamma*\beta"] & a \arrow[d, "h"] \\
b \arrow[r, equal] \arrow[d, "k"'] \arrow[dr, phantom, "\Downarrow\tau*\sigma"] & b \arrow[d, "m"] \\
c \arrow[r, equal] & c
\end{tikzcd}
=
\begin{tikzcd}
a \arrow[rr, equal] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\lambda^{-1}"] & & a \arrow[d, equal] \\
a \arrow[r, equal] \arrow[d, "k\circ f"'] \arrow[dr, phantom, "\Downarrow\sigma\bullet\beta"] & a \arrow[r, equal] \arrow[d, "l\circ g"] \arrow[dr, phantom, "\Downarrow\tau\bullet\gamma"] & a \arrow[d, "m\circ h"] \\
c \arrow[r, equal] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\rho"] & c \arrow[r, equal] & c \arrow[d, equal] \\
c \arrow[rr, equal] & & c
\end{tikzcd}$$

を示せばよいがそれは明らか.

この M は明らかに次の可換図式を満たす.

$$\begin{tikzcd}
& \mathcal{V}(\mathbb{D})(c,d) \times \mathcal{V}(\mathbb{D})(b,c) \times \mathcal{V}(\mathbb{D})(a,b) \arrow[dl, "M^{bcd}\times\mathrm{id}"'] \arrow[dr, "\mathrm{id}\times M^{abc}"] & \\
\mathcal{V}(\mathbb{D})(b,d) \times \mathcal{V}(\mathbb{D})(a,b) \arrow[dr, "M^{abd}"'] \arrow[rr, phantom, "="] & & \mathcal{V}(\mathbb{D})(c,d) \times \mathcal{V}(\mathbb{D})(a,c) \arrow[dl, "M^{acd}"] \\
& \mathcal{V}(\mathbb{D})(a,d) &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{1} \times \mathcal{V}(\mathbb{D})(a,b) \arrow[rr, "\cong"] \arrow[dr, "I^b\times\mathrm{id}"'] & \| & \mathcal{V}(\mathbb{D})(a,b) \\
& \mathcal{V}(\mathbb{D})(b,b) \times \mathcal{V}(\mathbb{D})(a,b) \arrow[ur, "M^{abb}"'] &
\end{tikzcd}
\quad
\begin{tikzcd}
\mathcal{V}(\mathbb{D}) \times \mathbb{1} \arrow[rr, "\cong"] \arrow[dr, "\mathrm{id}\times I^a"'] & \| & \mathcal{V}(\mathbb{D})(a,b) \\
& \mathcal{V}(\mathbb{D})(a,b) \times \mathcal{V}(\mathbb{D})(a,a) \arrow[ur, "M^{aab}"'] &
\end{tikzcd}$$

従って $\mathcal{V}(\mathbb{D})$ は strict 2-category である. □

命題 9. pseudo double category $\mathbb{D}$ に対して, 次のように定義すると $\mathcal{H}(\mathbb{D})$ は bicategory になる. ($\mathcal{H}(\mathbb{D})$ を horizontal bicategory という.)

- $\mathcal{H}(\mathbb{D})$ の対象は $\mathbb{D}$ の対象とする.
- $\mathcal{H}(\mathbb{D})$ の 1-morphism は $\mathbb{D}$ の水平射とする.
- $\mathcal{H}(\mathbb{D})$ の 2-morphism は次の形の cell とする.

$$\begin{tikzcd}
a \arrow[r, "J", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\beta"] & b \arrow[d, equal] \\
a \arrow[r, "K"', "\shortmid"{marking}] & b
\end{tikzcd}$$

但しこのとき $\mathrm{dom}(\beta) = J,\ \mathrm{cod}(\beta) = K$ とする.

- 1-morphism $a \xrightarrow{J} b \xrightarrow{K} c$ に対して合成を $K \circ J := J \odot K$ で定める.
- 2-morphism の垂直合成は D^1 の合成とする.
- 2-morphism $\begin{tikzcd} a \arrow[r, bend left, "J"] \arrow[r, bend right, "K"'] \arrow[r, phantom, "\Downarrow\beta"] & b \arrow[r, bend left, "M"] \arrow[r, bend right, "N"'] \arrow[r, phantom, "\Downarrow\gamma"] & c \end{tikzcd}$ に対して水平合成を $\gamma \bullet \beta := \beta \odot \gamma$ で定める.

証明. まず $a, b \in \mathbb{D}$ に対して $\mathcal{H}(\mathbb{D})(a,b)$ は圏である.

$\because$) $\mathcal{H}(\mathbb{D})(a,b)$ の合成は圏 D^1 の合成であるから明らか.

次に水平合成は関手 $M^{abc}\colon \mathcal{H}(\mathbb{D})(b,c) \times \mathcal{H}(\mathbb{D})(a,b) \to \mathcal{H}(\mathbb{D})(a,c)$ を定める.

$\because$) $\mathcal{H}(\mathbb{D})$ において次の状況のとき $(\tau * \sigma) \bullet (\gamma * \beta) = (\tau \bullet \gamma) * (\sigma \bullet \beta)$ を示す.

$$\begin{tikzcd}[column sep=large] a \arrow[r, bend left=50, "J"] \arrow[r, "K" description] \arrow[r, bend right=50, "M"'] \arrow[r, phantom, shift left=3ex, "\beta\Downarrow"] \arrow[r, phantom, shift right=3ex, "\gamma\Downarrow"] & b \arrow[r, bend left=50, "N"] \arrow[r, "P" description] \arrow[r, bend right=50, "Q"'] \arrow[r, phantom, shift left=3ex, "\sigma\Downarrow"] \arrow[r, phantom, shift right=3ex, "\tau\Downarrow"] & c \end{tikzcd}$$

即ち

$$\begin{tikzcd} a \arrow[r, "J"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\gamma\circ\beta"] & s \arrow[r, "N"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\tau\circ\sigma"] & v \arrow[d, equal] \\ a \arrow[r, "M"'] & s \arrow[r, "Q"'] & v \end{tikzcd} = \begin{tikzcd} a \arrow[r, "J\odot N"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\beta\odot\sigma"] & v \arrow[d, equal] \\ a \arrow[r, "K\odot P"'] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\gamma\odot\tau"] & v \arrow[d, equal] \\ a \arrow[r, "M\odot Q"'] & v \end{tikzcd}$$

を示せばよいがそれは $\mathbb{D}$ が pseudo double category であるから成り立つ.

1-morphism $a \xrightarrow{J} b \xrightarrow{K} c \xrightarrow{M} d$ に対して $\alpha'_{MKJ}\colon (M \circ K) \circ J \Rightarrow M \circ (K \circ J)$ を

$\alpha'_{MKJ} := \alpha^{-1}_{JKM}$ で定める．これは自然同型

$$
\begin{tikzcd}
 & \mathcal{H}(\mathbb{D})(c,d)\times\mathcal{H}(\mathbb{D})(b,c)\times\mathcal{H}(\mathbb{D})(a,b) \arrow[dl, "M^{bcd}\times\mathrm{id}"'] \arrow[dr, "\mathrm{id}\times M^{abc}"] & \\
\mathcal{H}(\mathbb{D})(b,d)\times\mathcal{H}(\mathbb{D})(a,b) \arrow[dr, "M^{abd}"'] \arrow[rr, phantom, "\overset{\sim}{\underset{\alpha'}{\Longrightarrow}}"] & & \mathcal{H}(\mathbb{D})(c,d)\times\mathcal{H}(\mathbb{D})(a,c) \arrow[dl, "M^{acd}"] \\
 & \mathcal{H}(\mathbb{D})(a,d) &
\end{tikzcd}
$$

を与える．

∵) α'_{MKJ} は $\mathcal{H}(\mathbb{D})(a,d)$ の同型射である．よって $\mathcal{H}(\mathbb{D})$ において

$$
\begin{tikzcd}
a \arrow[r, bend left, "J"] \arrow[r, bend right, "K"'] \arrow[r, phantom, "\Downarrow\beta"] & b \arrow[r, bend left, "M"] \arrow[r, bend right, "N"'] \arrow[r, phantom, "\Downarrow\gamma"] & c \arrow[r, bend left, "P"] \arrow[r, bend right, "Q"'] \arrow[r, phantom, "\Downarrow\sigma"] & d
\end{tikzcd}
$$

のとき

$$
\begin{tikzcd}
(P\circ M)\circ J \arrow[r, "\alpha'_{PMJ}"] \arrow[d, "(\sigma\bullet\gamma)\bullet\beta"'] & P\circ(M\circ J) \arrow[d, "\sigma\bullet(\gamma\bullet\beta)"] \\
(Q\circ N)\circ K \arrow[r, "\alpha'_{QNK}"'] & Q\circ(N\circ K)
\end{tikzcd}
$$

が可換であることを示せばよい．これは定義より $\mathbb{D}$ の図式で描けば

$$
\begin{tikzcd}
a \arrow[r, "J", "\mid"marking] \arrow[d, equal] & b \arrow[rr, "M\odot P", "\mid"marking] \arrow[d, phantom, "\Downarrow\alpha^{-1}_{JMP}"] & & d \arrow[d, equal] \\
a \arrow[rr, "J\odot M"', "\mid"marking] \arrow[d, equal] \arrow[rr, phantom, shift right=4ex, "\Downarrow\beta\odot\gamma"] & & c \arrow[r, "P"', "\mid"marking] \arrow[d, equal] \arrow[r, phantom, shift right=4ex, "\Downarrow\sigma"] & d \arrow[d, equal] \\
a \arrow[rr, "K\odot N"', "\mid"marking] & & c \arrow[r, "N\odot Q"', "\mid"marking] & d
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
a \arrow[r, "J", "\mid"marking] \arrow[d, equal] \arrow[d, phantom, shift left=4ex, "\Downarrow\beta"] & b \arrow[rr, "M\odot P", "\mid"marking] \arrow[d, equal] & \arrow[d, phantom, "\Downarrow\gamma\odot\sigma"] & d \arrow[d, equal] \\
a \arrow[r, "K"', "\mid"marking] \arrow[d, equal] & b \arrow[rr, "N\odot Q"', "\mid"marking] \arrow[d, phantom, "\Downarrow\alpha^{-1}_{KNQ}"] & & d \arrow[d, equal] \\
a \arrow[rr, "K\odot N"', "\mid"marking] & & c \arrow[r, "Q"', "\mid"marking] & d
\end{tikzcd}
$$

となるが，これは (1) より成り立つ．

1-morphism $a \overset{J}{\longrightarrow} b$ に対して $\lambda'_J := \rho_{Jb}$，$\rho'_J := \lambda_{aJ}$ と定める．これは自然同型

$$
\begin{tikzcd}
\mathbb{1}\times\mathcal{H}(\mathbb{D})(a,b) \arrow[rr, "\cong"] \arrow[dr, "I^b\times\mathrm{id}"'] & {} \arrow[d, phantom, "\lambda'\Uparrow\wr"] & \mathcal{H}(\mathbb{D})(a,b) \\
& \mathcal{H}(\mathbb{D})\times\mathcal{H}(\mathbb{D})(a,b) \arrow[ur, "M^{abb}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{H}(\mathbb{D})(a,b)\times\mathbb{1} \arrow[rr, "\cong"] \arrow[dr, "\mathrm{id}\times I^a"'] & {} \arrow[d, phantom, "\rho'\Uparrow\wr"] & \mathcal{H}(\mathbb{D})(a,b) \\
& \mathcal{H}(\mathbb{D})(a,b)\times\mathcal{H}(\mathbb{D})(a,a) \arrow[ur, "M^{aab}"'] &
\end{tikzcd}
$$

を与える．

∵) λ'_J, ρ'_J は $\mathcal{H}(\mathbb{D})(a,b)$ の同型射である．よって $\mathcal{H}(\mathbb{D})$ における 2-morphism

$$
\begin{tikzcd}
a \arrow[r, bend left, "J"] \arrow[r, bend right, "K"'] \arrow[r, phantom, "\Downarrow\beta"] & b
\end{tikzcd}
$$

に対して

$$
\begin{tikzcd}
\mathrm{id}_b\circ J \arrow[r, "\lambda'_J"] \arrow[d, "\mathrm{id}\bullet\beta"'] & J \arrow[d, "\beta"] \\
\mathrm{id}_b\circ K \arrow[r, "\lambda'_K"'] & K
\end{tikzcd}
\qquad
\begin{tikzcd}
J\circ\mathrm{id}_a \arrow[r, "\rho'_J"] \arrow[d, "\beta\bullet\mathrm{id}"'] & J \arrow[d, "\beta"] \\
K\circ\mathrm{id}_a \arrow[r, "\rho'_K"'] & K
\end{tikzcd}
$$

が可換であることを示せばよい．これは定義より $\mathbb{D}$ の図式で描けば

$$
\begin{tikzcd}
a \arrow[r, "J"] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\beta"] & u \arrow[r, equal] \arrow[d, "g"] \arrow[dr, phantom, "\Downarrow Ug"] & u \arrow[d, "g"] \\
b \arrow[r, "K"'] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\rho_K"] & v \arrow[r, equal] & v \arrow[d, equal] \\
b \arrow[rr, "K"'] & & v
\end{tikzcd}
=
\begin{tikzcd}
a \arrow[r, "J"] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\rho_J"] & u \arrow[r, equal] & u \arrow[d, equal] \\
a \arrow[rr, "J"] \arrow[d, "f"'] \arrow[drr, phantom, "\Downarrow\beta"] & & u \arrow[d, "g"] \\
b \arrow[rr, "K"'] & & v
\end{tikzcd}
$$

$$
\begin{tikzcd}
a \arrow[r, equal] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow Uf"] & a \arrow[r, "J"] \arrow[d, "f"] \arrow[dr, phantom, "\Downarrow\beta"] & u \arrow[d, "g"] \\
b \arrow[r, equal] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\lambda_K"] & b \arrow[r, "K"'] & v \arrow[d, equal] \\
b \arrow[rr, "K"'] & & v
\end{tikzcd}
=
\begin{tikzcd}
a \arrow[r, equal] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\lambda_J"] & a \arrow[r, "J"] & u \arrow[d, equal] \\
a \arrow[rr, "J"] \arrow[d, "f"'] \arrow[drr, phantom, "\Downarrow\beta"] & & u \arrow[d, "g"] \\
b \arrow[rr, "K"'] & & v
\end{tikzcd}
$$

となるが，これは (3) と (4) により成り立つ．

このとき次の自然変換の等式が成り立つ.

$$
\begin{tikzcd}[column sep=small]
& \mathcal{H}edcba \arrow[dl, "M^{cde}\times\mathrm{id}\times\mathrm{id}"'] \arrow[dr, "\mathrm{id}\times\mathrm{id}\times M^{abc}"] \arrow[dd, "\mathrm{id}\times M^{bcd}\times\mathrm{id}"] & \\
\mathcal{H}ecba \arrow[dd, "M^{bce}\times\mathrm{id}"'] & & \mathcal{H}edca \arrow[dd, "\mathrm{id}\times M^{acd}"] \\
{} \arrow[r, phantom, "\overset{\alpha'\times\mathrm{id}}{\Longrightarrow}"] & \mathcal{H}edba \arrow[dl, "M^{bde}\times\mathrm{id}"] \arrow[dr, "\mathrm{id}\times M^{abd}"'] \arrow[r, phantom, "\overset{\mathrm{id}\times\alpha'}{\Longrightarrow}"] & {} \\
\mathcal{H}eba \arrow[dr, "M^{abe}"'] & \overset{\alpha'}{\Longrightarrow} & \mathcal{H}eda \arrow[dl, "M^{ade}"] \\
& \mathcal{H}ea &
\end{tikzcd}
=
\begin{tikzcd}[column sep=small]
& \mathcal{H}edcba \arrow[dl, "M^{cde}\times\mathrm{id}\times\mathrm{id}"'] \arrow[dr, "\mathrm{id}\times\mathrm{id}\times M^{abc}"] \arrow[d, phantom, "="] & \\
\mathcal{H}ecba \arrow[dd, "M^{bce}\times\mathrm{id}"'] \arrow[dr, "\mathrm{id}\times M^{abc}"] & {} & \mathcal{H}edca \arrow[dd, "\mathrm{id}\times M^{acd}"] \arrow[dl, "M^{cde}\times\mathrm{id}"'] \\
{} & \mathcal{H}ace \arrow[dd, "M^{ace}"] & {} \\
\mathcal{H}eba \arrow[dr, "M^{abe}"'] \arrow[r, phantom, "\overset{\alpha'}{\Longrightarrow}"] & {} & \mathcal{H}eda \arrow[dl, "M^{ade}"] \arrow[l, phantom, "\overset{\alpha'}{\Longleftarrow}"'] \\
& \mathcal{H}ea &
\end{tikzcd}
$$

∵) $\mathcal{H}(\mathbb{D})$ の 1-morphism $a \xrightarrow{J} b \xrightarrow{K} c \xrightarrow{M} d \xrightarrow{N} e$ に対して

$$
(N \bullet \alpha'_{MKJ}) * \alpha'_{N,M\circ K,J} * (\alpha'_{NMK} \bullet J) = \alpha'_{N,M,K\circ J} * \alpha'_{N\circ M,K,J}
$$

を示せばよい. 即ち定義より

$$
\begin{tikzcd}
a \arrow[r, "J", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mathrm{id}"] & b \arrow[r, "K\odot(M\odot N)", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\alpha^{-1}_{KMN}"] & e \arrow[d, equal] \\
a \arrow[r, "J"', "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\alpha^{-1}_{J,K\odot M,N}"] & b \arrow[r, "(K\odot M)\odot N"', "\shortmid"{marking}] & e \arrow[d, equal] \\
a \arrow[r, "J\odot(K\odot M)"', "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\alpha^{-1}_{JKM}"] & d \arrow[r, "N"', "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mathrm{id}"] & e \arrow[d, equal] \\
a \arrow[r, "(J\odot K)\odot M"', "\shortmid"{marking}] & d \arrow[r, "N"', "\shortmid"{marking}] & e
\end{tikzcd}
=
\begin{tikzcd}
a \arrow[r, "J", "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\alpha^{-1}_{J,K,M\odot N}"] & b \arrow[r, "K\odot(M\odot N)", "\shortmid"{marking}] & e \arrow[d, equal] \\
a \arrow[r, "J\odot K"', "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\alpha^{-1}_{J\odot K,M,N}"] & c \arrow[r, "M\odot N"', "\shortmid"{marking}] & e \arrow[d, equal] \\
a \arrow[r, "(J\odot K)\odot M"', "\shortmid"{marking}] & d \arrow[r, "N"', "\shortmid"{marking}] & e
\end{tikzcd}
$$

を示せばよいがこれは (2) より成り立つ.

また次の自然変換の等式が成り立つ.

$$
\begin{tikzcd}
 & \mathcal{H}cba \arrow[dl, bend right, "\mathrm{id}"'] \arrow[dr, bend left, "\mathrm{id}"] \arrow[d, "\mathrm{id}\times I^b\times\mathrm{id}"] & \\
 \arrow[r, phantom, "\xRightarrow{(\rho')^{-1}\times\mathrm{id}}"] & \mathcal{H}cbba \arrow[dl, "M^{bbc}\times\mathrm{id}"] \arrow[dr, "\mathrm{id}\times M^{abb}"'] \arrow[r, phantom, "\xRightarrow{\mathrm{id}\times\lambda'}"] & \\
\mathcal{H}cba \arrow[dr, "M^{abc}"'] \arrow[rr, phantom, "\xRightarrow{\alpha'}"] & & \mathcal{H}cba \arrow[dl, "M^{abc}"] \\
 & \mathcal{H}ca &
\end{tikzcd}
=
\begin{tikzcd}
 & \mathcal{H}cba \arrow[dl, bend right, "\mathrm{id}"'] \arrow[dr, bend left, "\mathrm{id}"] & \\
\mathcal{H}cba \arrow[dr, "M^{abc}"'] \arrow[rr, phantom, "="] & & \mathcal{H}cba \arrow[dl, "M^{abc}"] \\
 & \mathcal{H}ca &
\end{tikzcd}
$$

$\because$) $\mathcal{H}(\mathbb{D})$ の 1-morphism $a \xrightarrow{J} b \xrightarrow{K} c$ に対して

$$(K \bullet \lambda'_J) * \alpha'_{K\mathrm{id}_b J} * ((\rho'_K)^{-1} \bullet J) = \mathrm{id}$$

を示せばよい．即ち定義より

$$
\begin{tikzcd}
a \arrow[r, "J", "|"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\rho_J^{-1}"] & u \arrow[r, "K", "|"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mathrm{id}"] & s \arrow[d, equal] \\
a \arrow[r, "J\odot Uu"', "|"{marking}] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\alpha_{J,Uu,K}"] & u \arrow[r, "K"', "|"{marking}] & s \arrow[d, equal] \\
a \arrow[r, "J"', "|"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mathrm{id}"] & u \arrow[r, "Uu\odot K"', "|"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\lambda_K"] & s \arrow[d, equal] \\
a \arrow[r, "J"', "|"{marking}] & u \arrow[r, "K"', "|"{marking}] & s
\end{tikzcd}
=
\begin{tikzcd}
a \arrow[r, "J", "|"{marking}] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\mathrm{id}"] & u \arrow[r, "K", "|"{marking}] & s \arrow[d, equal] \\
a \arrow[r, "J"', "|"{marking}] & u \arrow[r, "K"', "|"{marking}] & s
\end{tikzcd}
$$

を示せばよいがこれは (5) より成り立つ.

以上により $\mathcal{H}(\mathbb{D})$ は bicategory である. □

定義. pseudo double category $\mathbb{D}$ に対して，関手 $L, R\colon D^1 \to D^0$ から直積の普遍性により得られる関手を $P\colon D^1 \to D^0 \times D^0$ とする.

(1) $\mathbb{D}$ が framed bicategory $\iff$ P が Grothendieck fibration である.

(2) cell β が cartesian $\iff$ D^1 の射として β が P-cartesian である[*1].

[*1] P-cartesian の定義は「fibration」の PDF を参照.

(3) cell β が opcartesian $\iff$ D^1 の射として β が P-opcartesian である．

即ち，cell

$$\begin{tikzcd}
a \arrow[r, "J", "\shortmid" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\beta"] & u \arrow[d, "g"] \\
b \arrow[r, "K"', "\shortmid" marking] & v
\end{tikzcd}$$

が cartesian とは，任意の cell

$$\begin{tikzcd}
s \arrow[r, "M", "\shortmid" marking] \arrow[d, "h"'] & t \arrow[d, "k"] \\
a \arrow[r, phantom, "\Downarrow\theta"] \arrow[d, "f"'] & u \arrow[d, "g"] \\
b \arrow[r, "K"', "\shortmid" marking] & v
\end{tikzcd}$$

に対して，cell θ' が一意に存在して等式

$$\begin{tikzcd}
s \arrow[r, "M", "\shortmid" marking] \arrow[d, "h"'] \arrow[dr, phantom, "\Downarrow\theta'"] & t \arrow[d, "k"] \\
a \arrow[r, "J"', "\shortmid" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\beta"] & u \arrow[d, "g"] \\
b \arrow[r, "K"', "\shortmid" marking] & v
\end{tikzcd}
=
\begin{tikzcd}
s \arrow[r, "M", "\shortmid" marking] \arrow[d, "h"'] & t \arrow[d, "k"] \\
a \arrow[r, phantom, "\Downarrow\theta"] \arrow[d, "f"'] & u \arrow[d, "g"] \\
b \arrow[r, "K"', "\shortmid" marking] & v
\end{tikzcd}$$

が成り立つことである．opcartesian についても同様である．また，pseudo double category $\mathbb{D}$ が framed bicategory とは，任意の図式

$$\begin{tikzcd}
a \arrow[d, "f"'] & u \arrow[d, "g"] \\
b \arrow[r, "K"', "\shortmid" marking] & v
\end{tikzcd}$$

に対して，cartesian cell

$$\begin{tikzcd}
a \arrow[r, "J", "\shortmid" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\beta"] & u \arrow[d, "g"] \\
b \arrow[r, "K"', "\shortmid" marking] & v
\end{tikzcd}$$

が存在することである．cartesian cell の普遍性から，このような J は同型を除いて一意である．そこでこのとき J を f, g に沿った K の制限 (restriction) といい，J を記号で $K(f,g)$ と書く．$K = U(a)$ のときは $a(f,g)$ と書く．

定義． 垂直射 $f\colon a \to b$ の companion とは次のような 3 つ組 $\langle f_\star, {}_f\varepsilon, {}_f\eta \rangle$ であって

$$
\begin{tikzcd}
a \arrow[r, "f_\star", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow {}_f\varepsilon"] & b \arrow[d, equal] \\
b \arrow[r, equal] & b
\end{tikzcd}
\qquad
\begin{tikzcd}
a \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow {}_f\eta"] & a \arrow[d, "f"] \\
a \arrow[r, "f_\star"', "\shortmid"{marking}] & b
\end{tikzcd}
$$

次の 2 等式を満たすものをいう．

$$
\begin{tikzcd}
a \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow {}_f\eta"] & a \arrow[r, "f_\star", "\shortmid"{marking}] \arrow[d, "f"] \arrow[dr, phantom, "\Downarrow {}_f\varepsilon"] & b \arrow[d, equal] \\
a \arrow[r, "f_\star"', "\shortmid"{marking}] & b \arrow[r, equal] & b
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
a \arrow[r, equal] \arrow[d, equal] & a \arrow[r, "f_\star", "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow \iota_{f_\star}"] & b \arrow[d, equal] \\
a \arrow[r, "f_\star"', "\shortmid"{marking}] & b \arrow[r, equal] & b
\end{tikzcd}
$$

$$
\begin{tikzcd}
a \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow {}_f\eta"] & a \arrow[d, "f"] \\
a \arrow[r, "f_\star"', "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow {}_f\varepsilon"] & b \arrow[d, equal] \\
b \arrow[r, equal] & b
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
a \arrow[r, equal] \arrow[d, equal] & a \arrow[d, "f"] \\
a \arrow[r, phantom, "\Downarrow Uf"] \arrow[d, "f"'] & b \arrow[d, equal] \\
b \arrow[r, equal] & b
\end{tikzcd}
$$

また f の conjoint とは次のような 3 つ組 $\langle f^\star, \varepsilon_f, \eta_f \rangle$ であって

$$
\begin{tikzcd}
b \arrow[r, "f^\star", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow \varepsilon_f"] & a \arrow[d, "f"] \\
b \arrow[r, equal] & b
\end{tikzcd}
\qquad
\begin{tikzcd}
a \arrow[r, equal] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow \eta_f"] & a \arrow[d, equal] \\
b \arrow[r, "f^\star"', "\shortmid"{marking}] & a
\end{tikzcd}
$$

次の 2 等式を満たすものをいう．

$$
\begin{tikzcd}
b \arrow[r, "f^\star", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow \varepsilon_f"] & a \arrow[r, equal] \arrow[d, "f"] \arrow[dr, phantom, "\Downarrow \eta_f"] & a \arrow[d, equal] \\
b \arrow[r, equal] & b \arrow[r, "f^\star"', "\shortmid"{marking}] & a
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
b \arrow[r, "f^\star", "\shortmid"{marking}] \arrow[d, equal] & a \arrow[r, equal] \arrow[d, phantom, "\Downarrow \iota'_{f^\star}"] & a \arrow[d, equal] \\
b \arrow[r, equal] & b \arrow[r, "f^\star"', "\shortmid"{marking}] & a
\end{tikzcd}
$$

$$
\begin{tikzcd}
a \arrow[r, equal] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\eta_f"] & a \arrow[d, equal] \\
b \arrow[r, "f^\star"', "+" marking] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\varepsilon_f"] & a \arrow[d, "f"] \\
b \arrow[r, equal] & b
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
a \arrow[r, equal] \arrow[d, "f"'] & a \arrow[d, equal] \\
b \arrow[r, phantom, "\Downarrow Uf"] \arrow[d, equal] & a \arrow[d, "f"] \\
b \arrow[r, equal] & b
\end{tikzcd}
$$

命題 10. ${}_f\varepsilon$ は cartesian である.

証明. 任意の cell

$$
\begin{tikzcd}
s \arrow[r, "H", "+" marking] \arrow[d, "h"'] & t \arrow[d, "k"] \\
a \arrow[r, phantom, "\Downarrow\theta"] \arrow[d, "f"'] & b \arrow[d, equal] \\
b \arrow[r, equal] & b
\end{tikzcd}
$$

を取る.

$$
\theta' \quad := \quad
\begin{tikzcd}
s \arrow[rr, "H", "+" marking] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\lambda^{-1}"] & & t \arrow[d, equal] \\
s \arrow[r, equal] \arrow[d, "h"'] \arrow[dr, phantom, "\Downarrow Uh"] & s \arrow[r, "H", "+" marking] \arrow[d, "h"] & t \arrow[dd, "k"] \\
a \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow{}_f\eta"] & a \arrow[d, "f"] \arrow[r, phantom, "\Downarrow\theta"] & \\
a \arrow[r, "f_\star"', "+" marking] \arrow[d, equal] & b \arrow[r, equal] \arrow[d, phantom, "\Downarrow\rho"] & b \arrow[d, equal] \\
a \arrow[rr, "f_\star"', "+" marking] & & b
\end{tikzcd}
$$

と定義すると

$$
\begin{tikzcd}
s \arrow[r, "H", "\shortmid"{marking}] \arrow[d, "h"'] \arrow[dr, phantom, "\Downarrow\theta'"] & t \arrow[d, "k"] \\
a \arrow[r, "f_\star", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow {}_f\varepsilon"] & b \arrow[d, equal] \\
b \arrow[r, equal] & b
\end{tikzcd}
=
\begin{tikzcd}
s \arrow[rr, "H", "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\lambda^{-1}"] & & t \arrow[d, equal] \\
s \arrow[r, equal] \arrow[d, "h"'] \arrow[dr, phantom, "\Downarrow Uh"] & s \arrow[r, "H", "\shortmid"{marking}] \arrow[d, "h"] \arrow[ddr, phantom, "\Downarrow\theta"] & t \arrow[dd, "k"] \\
a \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow {}_f\eta"] & a \arrow[d, "f"] & \\
a \arrow[r, "f_\star"', "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\rho"] & b \arrow[r, equal] & b \arrow[d, equal] \\
a \arrow[rr, "f_\star"', "\shortmid"{marking}] \arrow[d, "f"'] \arrow[drr, phantom, "\Downarrow {}_f\varepsilon"] & & b \arrow[d, equal] \\
b \arrow[rr, equal] & & b
\end{tikzcd}
=
\begin{tikzcd}
s \arrow[rr, "H", "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\lambda^{-1}"] & & t \arrow[d, equal] \\
s \arrow[r, equal] \arrow[d, "h"'] \arrow[dr, phantom, "\Downarrow Uh"] & s \arrow[r, "H", "\shortmid"{marking}] \arrow[d, "h"] \arrow[ddr, phantom, "\Downarrow\theta"] & t \arrow[dd, "k"] \\
a \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow {}_f\eta"] & a \arrow[d, "f"] & \\
a \arrow[r, "f_\star"', "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow {}_f\varepsilon"] & b \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mathrm{id}"] & b \arrow[d, equal] \\
b \arrow[r, equal] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\rho"] & b \arrow[r, equal] & b \arrow[d, equal] \\
b \arrow[rr, equal] & & b
\end{tikzcd}
$$

$$
=
\begin{tikzcd}
s \arrow[rr, "H", "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\lambda^{-1}"] & & t \arrow[d, equal] \\
s \arrow[r, equal] \arrow[d, "h"'] \arrow[dr, phantom, "\Downarrow Uh"] & s \arrow[r, "H", "\shortmid"{marking}] \arrow[d, "h"] \arrow[ddr, phantom, "\Downarrow\theta"] & t \arrow[dd, "k"] \\
a \arrow[r, equal] \arrow[d, equal] & a \arrow[d, "f"] & \\
a \arrow[r, phantom, "\Downarrow Uf"] \arrow[d, "f"'] & b \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mathrm{id}"] & b \arrow[d, equal] \\
b \arrow[r, equal] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\rho"] & b \arrow[r, equal] & b \arrow[d, equal] \\
b \arrow[rr, equal] & & b
\end{tikzcd}
= \quad \theta
$$

である．一意性を示すため，ψ が

$$
\begin{tikzcd}
s \arrow[r, "H", "\shortmid"{marking}] \arrow[d, "h"'] \arrow[dr, phantom, "\Downarrow\psi"] & t \arrow[d, "k"] \\
a \arrow[r, "f_\star", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow {}_f\varepsilon"] & b \arrow[d, equal] \\
b \arrow[r, equal] & b
\end{tikzcd}
=
\begin{tikzcd}
s \arrow[r, "H", "\shortmid"{marking}] \arrow[d, "h"'] & t \arrow[d, "k"] \\
a \arrow[r, phantom, "\Downarrow\theta"] \arrow[d, "f"'] & b \arrow[d, equal] \\
b \arrow[r, equal] & b
\end{tikzcd}
$$

を満たしたとすると

$$
\theta' \quad = \quad
\begin{tikzcd}
s \arrow[rr, "H", "|"{marking}] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\lambda^{-1}"] & & t \arrow[d, equal] \\
s \arrow[r, equal] \arrow[d, "h"'] \arrow[dr, phantom, "\Downarrow Uh"] & s \arrow[r, "H", "|"{marking}] \arrow[d, "h"] & t \arrow[dd, "k"] \\
a \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow{}_f\eta"] & a \arrow[d, "f"] \arrow[r, phantom, "\Downarrow\theta"] & \\
a \arrow[r, "f_\star"', "|"{marking}] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\rho"] & b \arrow[r, equal] & b \arrow[d, equal] \\
a \arrow[rr, "f_\star"', "|"{marking}] & & b
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
s \arrow[rr, "H", "|"{marking}] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\lambda^{-1}"] & & t \arrow[d, equal] \\
s \arrow[r, equal] \arrow[d, "h"'] \arrow[dr, phantom, "\Downarrow Uh"] & s \arrow[r, "H", "|"{marking}] \arrow[d, "h"] \arrow[dr, phantom, "\Downarrow\psi"] & t \arrow[d, "k"] \\
a \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow{}_f\eta"] & a \arrow[r, "f_\star", "|"{marking}] \arrow[d, "f"] \arrow[dr, phantom, "\Downarrow{}_f\varepsilon"] & b \arrow[d, equal] \\
a \arrow[r, "f_\star"', "|"{marking}] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\rho"] & b \arrow[r, equal] & b \arrow[d, equal] \\
a \arrow[rr, "f_\star"', "|"{marking}] & & b
\end{tikzcd}
$$

$$
= \quad
\begin{tikzcd}
s \arrow[rr, "H", "|"{marking}] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\lambda^{-1}"] & & t \arrow[d, equal] \\
s \arrow[r, equal] \arrow[d, "h"'] \arrow[dr, phantom, "\Downarrow Uh"] & s \arrow[r, "H", "|"{marking}] \arrow[d, "h"] \arrow[dr, phantom, "\Downarrow\psi"] & t \arrow[d, "k"] \\
a \arrow[r, equal] \arrow[dd, equal] \arrow[ddrr, phantom, "\Downarrow\lambda"] & a \arrow[r, "f_\star"', "|"{marking}] & b \arrow[dd, equal] \\
& & \\
a \arrow[rr, "f_\star"', "|"{marking}] & & b
\end{tikzcd}
\quad = \quad \psi
$$

である. □

同様にして次の命題も成り立つ.

命題 11. ε_f は cartesian で, ${}_f\eta$ と η_f は opcartesian である. □

逆に, 次の命題も成り立つ.

命題 12. cell

$$
\begin{tikzcd}
a \arrow[r, "J", "|"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\beta"] & b \arrow[d, equal] \\
b \arrow[r, equal] & b
\end{tikzcd}
$$

が cartesian ならば，ある cell

$$\begin{tikzcd}
a \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\gamma"] & a \arrow[d, "f"] \\
a \arrow[r, "J"', "\shortmid"{marking}] & b
\end{tikzcd}$$

が存在して $\langle J, \beta, \gamma \rangle$ が f の companion となる．

証明. β が cartesian だから，ある γ が存在して

$$\begin{tikzcd}
a \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\gamma"] & a \arrow[d, "f"] \\
a \arrow[r, "J"', "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\beta"] & b \arrow[d, equal] \\
b \arrow[r, equal] & b
\end{tikzcd}
=
\begin{tikzcd}
a \arrow[r, equal] \arrow[d, equal] & a \arrow[d, "f"] \\
a \arrow[r, phantom, "\Downarrow Uf"] \arrow[d, "f"'] & b \arrow[d, equal] \\
b \arrow[r, equal] & b
\end{tikzcd}$$

とできる．このとき

$$\begin{tikzcd}
a \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\gamma"] & a \arrow[r, "J", "\shortmid"{marking}] \arrow[d, "f"] \arrow[dr, phantom, "\Downarrow\beta"] & b \arrow[d, equal] \\
a \arrow[r, "J"', "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\rho"] & b \arrow[r, equal] & b \arrow[d, equal] \\
a \arrow[rr, "J"', "\shortmid"{marking}] \arrow[d, "f"'] \arrow[drr, phantom, "\Downarrow\beta"] & & b \arrow[d, equal] \\
b \arrow[rr, equal] & & b
\end{tikzcd}
=
\begin{tikzcd}
a \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\gamma"] & a \arrow[r, "J", "\shortmid"{marking}] \arrow[d, "f"] \arrow[dr, phantom, "\Downarrow\beta"] & b \arrow[d, equal] \\
a \arrow[r, "J"', "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\beta"] & b \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mathrm{id}"] & b \arrow[d, equal] \\
a \arrow[r, equal] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\rho"] & b \arrow[r, equal] & b \arrow[d, equal] \\
a \arrow[rr, equal] & & b
\end{tikzcd}
=
\begin{tikzcd}
a \arrow[r, equal] \arrow[d, equal] & a \arrow[r, "J", "\shortmid"{marking}] \arrow[d, "f"] \arrow[dr, phantom, "\Downarrow\beta"] & b \arrow[d, equal] \\
a \arrow[r, phantom, "\Downarrow Uf"] \arrow[d, "f"'] & b \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mathrm{id}"] & b \arrow[d, equal] \\
a \arrow[r, equal] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\lambda"] & b \arrow[r, equal] & b \arrow[d, equal] \\
a \arrow[rr, equal] & & b
\end{tikzcd}$$

$$=
\begin{tikzcd}
a \arrow[r, equal] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow Uf"] & a \arrow[r, "J", "\shortmid"{marking}] \arrow[d, "f"] \arrow[dr, phantom, "\Downarrow\beta"] & b \arrow[d, equal] \\
b \arrow[r, equal] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\lambda"] & b \arrow[r, equal] & b \arrow[d, equal] \\
b \arrow[rr, equal] & & b
\end{tikzcd}
=
\begin{tikzcd}
a \arrow[r, equal] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\lambda"] & a \arrow[r, "J", "\shortmid"{marking}] & b \arrow[d, equal] \\
a \arrow[rr, "J", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[drr, phantom, "\Downarrow\beta"] & & b \arrow[d, equal] \\
b \arrow[rr, equal] & & b
\end{tikzcd}$$

となるから，cartesian cell β の普遍性により

$$\begin{tikzcd}
a \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\gamma"] & a \arrow[r, "J", "\shortmid"{marking}] \arrow[d, "f"] \arrow[dr, phantom, "\Downarrow\beta"] & b \arrow[d, equal] \\
a \arrow[r, "J"', "\shortmid"{marking}] \arrow[d, equal] & b \arrow[r, equal] \arrow[d, phantom, "\Downarrow\rho"] & b \arrow[d, equal] \\
a \arrow[rr, "J"', "\shortmid"{marking}] & & b
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
a \arrow[r, equal] \arrow[dd, equal] & a \arrow[r, "J", "\shortmid"{marking}] \arrow[dd, phantom, "\Downarrow\lambda"] & b \arrow[dd, equal] \\
 & & \\
a \arrow[rr, "J"', "\shortmid"{marking}] & & b
\end{tikzcd}$$

となり，$\langle J, \beta, \gamma \rangle$ は f の companion である． $\square$

同様のことが ${}_f\varepsilon, {}_f\eta, \eta_f$ についても成り立つ (省略)．更に次の補題が成り立つ．

補題 13. $f\colon a \to b$ の companion と $g\colon u \to v$ の conjoint が存在するとする．このとき $J\colon b \nrightarrow v$ に対して

$$\beta \quad := \quad \begin{tikzcd}
a \arrow[r, "f_\star", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow{}_f\varepsilon"] & b \arrow[r, "J", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mathrm{id}_J"] & v \arrow[r, "g^\star", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\varepsilon_g"] & u \arrow[d, "g"] \\
b \arrow[r, equal] \arrow[d, equal] & b \arrow[r, "J"', "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow\lambda"] & v \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mathrm{id}"] & v \arrow[d, equal] \\
b \arrow[rr, "J"', "\shortmid"{marking}] \arrow[d, equal] & & v \arrow[r, equal] \arrow[d, phantom, "\Downarrow\rho"] & v \arrow[d, equal] \\
b \arrow[rrr, "J"', "\shortmid"{marking}] & & & v
\end{tikzcd}$$

は cartesian である．

証明. β が cartesian であることを示すため，任意の cell

$$\begin{tikzcd}
s \arrow[r, "M", "\shortmid"{marking}] \arrow[d, "h"'] & t \arrow[d, "k"] \\
a \arrow[r, phantom, "\Downarrow\theta"] \arrow[d, "f"'] & u \arrow[d, "g"] \\
b \arrow[r, "J"', "\shortmid"{marking}] & v
\end{tikzcd}$$

を取る．このとき

$$
\theta' \quad := \quad
\begin{tikzcd}
s \arrow[rrr, "M", "|" marking] \arrow[d, equal] \arrow[drrr, phantom, "\Downarrow\rho^{-1}"] & & & t \arrow[d, equal] \\
s \arrow[rr, "M", "|" marking] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\lambda^{-1}"] & & t \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mathrm{id}"] & t \arrow[d, equal] \\
s \arrow[r, equal] \arrow[d, "h"'] \arrow[dr, phantom, "\Downarrow Uh"] & s \arrow[r, "M", "|" marking] \arrow[d, "h"] \arrow[dr, phantom, "\Downarrow\theta"] & t \arrow[r, equal] \arrow[d, "k"'] \arrow[dr, phantom, "\Downarrow Uk"] & t \arrow[d, "k"] \\
a \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow {}_f\eta"] & a \arrow[d, "f"] & u \arrow[r, equal] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow\eta_g"] & u \arrow[d, equal] \\
a \arrow[r, "f_\star"', "|" marking] & b \arrow[r, "J"', "|" marking] & v \arrow[r, "g^\star"', "|" marking] & u
\end{tikzcd}
$$

と定義すると明らかに $\beta \circ \theta' = \theta$ である．θ' の一意性も明らかだから β は cartesian である． □

同様にして

補題 14. $f\colon a \to b$ の companion と $g\colon u \to v$ の conjoint が存在するとする．このとき $J\colon a \nrightarrow u$ に対して

$$
\begin{tikzcd}
a \arrow[rrr, "J", "|" marking] \arrow[d, equal] \arrow[drrr, phantom, "\Downarrow\rho^{-1}"] & & & u \arrow[d, equal] \\
a \arrow[rr, "J", "|" marking] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\lambda^{-1}"] & & u \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mathrm{id}"] & u \arrow[d, equal] \\
a \arrow[r, equal] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\eta_f"] & a \arrow[r, "J", "|" marking] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mathrm{id}_J"] & u \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow {}_g\eta"] & u \arrow[d, "g"] \\
b \arrow[r, "f^\star"', "|" marking] & a \arrow[r, "J"', "|" marking] & u \arrow[r, "g_\star"', "|" marking] & v
\end{tikzcd}
$$

は opcartesian である． □

命題 15. pseudo double category $\mathbb{D}$ が framed bicategory である
$\iff$ 任意の垂直射が companion と conjoint を持つ．

証明. ($\Longrightarrow$) $\mathbb{D}$ が framed bicategory であるとする．次の左の図式に対して，右の図式の

cartesian cell を取る．

$$\begin{tikzcd} a \arrow[d, "f"'] & b \arrow[d, equal] \\ b \arrow[r, equal] & b \end{tikzcd} \qquad \begin{tikzcd} a \arrow[r, "f_\star", "|"] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow {}_f\varepsilon"] & b \arrow[d, equal] \\ b \arrow[r, equal] & b \end{tikzcd}$$

このとき命題 12 により，${}_f\eta$ が存在して $\langle f_\star, {}_f\varepsilon, {}_f\eta\rangle$ が companion となる．

同様に，次の左の図式に対して，右の図式の cartesian cell を取る．

$$\begin{tikzcd} b \arrow[d, equal] & a \arrow[d, "f"] \\ b \arrow[r, equal] & b \end{tikzcd} \qquad \begin{tikzcd} b \arrow[r, "f^\star", "|"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow {}_f\varepsilon"] & a \arrow[d, "f"] \\ b \arrow[r, equal] & b \end{tikzcd}$$

このとき命題 12 と同様にして，η_f が存在して $\langle f^\star, \varepsilon_f, \eta_f\rangle$ が conjoint となる．

($\Longleftarrow$) 補題 13 より，任意の

$$\begin{tikzcd} a \arrow[d, "f"'] & u \arrow[d, "g"] \\ b \arrow[r, "|", "K"'] & v \end{tikzcd}$$

に対して cartesian cell

$$\begin{tikzcd} a \arrow[r, "J", "|"'] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\beta"] & u \arrow[d, "g"] \\ b \arrow[r, "|", "K"'] & v \end{tikzcd}$$

が存在する． □

補題 16. framed bicategory $\mathbb{D}$ における図式

$$\begin{tikzcd} a \arrow[r, "J", "|"'] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\beta"] & u \arrow[r, "M", "|"'] \arrow[d, "g"] \arrow[dr, phantom, "\Downarrow\gamma"] & x \arrow[d, equal] \\ b \arrow[r, "|", "K"'] & v \arrow[r, "|", "N"'] & x \end{tikzcd}$$

について，β が opcartesian で γ が cartesian ならば $\beta \odot \gamma$ は opcartesian である．

証明．β が opcartesian だから，ある σ が存在して

$$
\begin{tikzcd}
a \arrow[rr, "J", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[drr, phantom, "\Downarrow\beta"] & & u \arrow[d, "g"] \\
b \arrow[rr, "K", "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\sigma"] & & v \arrow[d, equal] \\
b \arrow[r, "f^\star \odot J"', "\shortmid"{marking}] & u \arrow[r, "g_\star"', "\shortmid"{marking}] & v
\end{tikzcd}
=
\begin{tikzcd}
a \arrow[rrr, "J", "\shortmid"{marking}] \arrow[d, equal] \arrow[drrr, phantom, "\Downarrow\rho^{-1}"] & & & u \arrow[d, equal] \\
a \arrow[rr, "J", "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\lambda^{-1}"] & & u \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mathrm{id}"] & u \arrow[d, equal] \\
a \arrow[r, equal] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\eta_f"] & a \arrow[r, "J", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mathrm{id}_J"] & u \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow{}_g\eta"] & u \arrow[d, "g"] \\
b \arrow[r, "f^\star"', "\shortmid"{marking}] & a \arrow[r, "J"', "\shortmid"{marking}] & u \arrow[r, "g_\star"', "\shortmid"{marking}] & v
\end{tikzcd}
$$

が成り立つ．補題 14 より右辺全体は opcartesian だから，opcartesian の普遍性により σ は同型である．

次に γ が cartesian だから，ある τ が存在して

$$
\begin{tikzcd}
u \arrow[r, "g_\star", "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\tau"] & v \arrow[r, "N", "\shortmid"{marking}] & x \arrow[d, equal] \\
u \arrow[rr, "M", "\shortmid"{marking}] \arrow[d, "g"'] \arrow[drr, phantom, "\Downarrow\gamma"] & & x \arrow[d, equal] \\
v \arrow[rr, "N"', "\shortmid"{marking}] & & x
\end{tikzcd}
=
\begin{tikzcd}
u \arrow[r, "g_\star", "\shortmid"{marking}] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow{}_g\varepsilon"] & v \arrow[r, "N", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mathrm{id}_N"] & x \arrow[d, equal] \\
v \arrow[r, equal] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\lambda"] & v \arrow[r, "N"', "\shortmid"{marking}] & x \arrow[d, equal] \\
v \arrow[rr, "N"', "\shortmid"{marking}] & & x
\end{tikzcd}
$$

が成り立つ．補題 13 より右辺全体は cartesian だから，τ は同型である．

このとき

$$
\begin{tikzcd}
a \arrow[r, "J", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mathrm{id}_J"] & u \arrow[r, "g_\star \odot N", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\tau"] & x \arrow[d, equal] \\
a \arrow[r, "J", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\beta"] & u \arrow[r, "M", "\shortmid"{marking}] \arrow[d, "g"] \arrow[dr, phantom, "\Downarrow\gamma"] & x \arrow[d, equal] \\
b \arrow[r, "K", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\sigma"] & v \arrow[r, "N", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mathrm{id}_N"] & x \arrow[d, equal] \\
b \arrow[r, "(f^\star \odot J) \odot g_\star"', "\shortmid"{marking}] & v \arrow[r, "N"', "\shortmid"{marking}] & x
\end{tikzcd}
=
\begin{tikzcd}
a \arrow[rrr, "J", "\shortmid"{marking}] \arrow[d, equal] \arrow[drrr, phantom, "\Downarrow\rho^{-1}"] & & & u \arrow[r, "g_\star", "\shortmid"{marking}] \arrow[d, equal] & v \arrow[r, "N", "\shortmid"{marking}] & x \arrow[d, equal] \\
a \arrow[rr, "J", "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\lambda^{-1}"] & & u \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mathrm{id}"] & u \arrow[d, equal] \arrow[rr, phantom, "\Downarrow\mathrm{id}"] & & x \arrow[d, equal] \\
a \arrow[r, equal] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\eta_f"] & a \arrow[r, "J", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mathrm{id}_J"] & u \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow{}_g\eta"] & u \arrow[r, "g_\star", "\shortmid"{marking}] \arrow[d, "g"] \arrow[dr, phantom, "\Downarrow{}_g\varepsilon"] & v \arrow[r, "N", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mathrm{id}_N"] & x \arrow[d, equal] \\
b \arrow[r, "f^\star"', "\shortmid"{marking}] \arrow[d, equal] \arrow[drrr, phantom, "\Downarrow\mathrm{id}"] & a \arrow[r, "J"', "\shortmid"{marking}] & u \arrow[r, "g_\star"', "\shortmid"{marking}] & v \arrow[r, equal] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\lambda"] & v \arrow[r, "N"', "\shortmid"{marking}] & x \arrow[d, equal] \\
b \arrow[r, "f^\star"', "\shortmid"{marking}] & a \arrow[r, "J"', "\shortmid"{marking}] & u \arrow[r, "g_\star"', "\shortmid"{marking}] & v \arrow[rr, "N"', "\shortmid"{marking}] & & x
\end{tikzcd}
$$

$$= \eta_f \odot \mathrm{id}$$

となる．補題 14 よりこれは opcartesian である．σ, τ が同型だから $\beta \odot \gamma$ も opcartesian である． $\square$

同様にして次も成り立つ．

補題 17. framed bicategory $\mathbb{D}$ における図式

$$\begin{tikzcd} a \arrow[r, "J", "|"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\beta"] & u \arrow[r, "M", "|"{marking}] \arrow[d, "g"] \arrow[dr, phantom, "\Downarrow\gamma"] & x \arrow[d, equal] \\ b \arrow[r, "K"', "|"{marking}] & v \arrow[r, "N"', "|"{marking}] & x \end{tikzcd}$$

について，β が cartesian で γ が opcartesian ならば $\beta \odot \gamma$ は cartesian である． $\square$

補題 18. framed bicategory $\mathbb{D}$ における図式

$$\begin{tikzcd} x \arrow[r, "J", "|"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\beta"] & a \arrow[r, "M", "|"{marking}] \arrow[d, "f"] \arrow[dr, phantom, "\Downarrow\gamma"] & u \arrow[d, "g"] \\ x \arrow[r, "K"', "|"{marking}] & b \arrow[r, "N"', "|"{marking}] & v \end{tikzcd}$$

について，β が opcartesian で γ が cartesian ならば $\beta \odot \gamma$ は cartesian である． $\square$

補題 19. framed bicategory $\mathbb{D}$ における図式

$$\begin{tikzcd} x \arrow[r, "J", "|"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\beta"] & a \arrow[r, "M", "|"{marking}] \arrow[d, "f"] \arrow[dr, phantom, "\Downarrow\gamma"] & u \arrow[d, "g"] \\ x \arrow[r, "K"', "|"{marking}] & b \arrow[r, "N"', "|"{marking}] & v \end{tikzcd}$$

について，β が cartesian で γ が opcartesian ならば $\beta \odot \gamma$ は opcartesian である． $\square$

2 pseudo double category における Kan 拡張

定義. $J\colon a \nrightarrow b$, $g\colon a \to c$ とする．J に沿った g の左 Kan 拡張とは組 $\langle l, \eta \rangle$ であって以下の条件を満たすものである．

(1) $l\colon u \to b$ で η は次の cell である．

$$\begin{tikzcd} a \arrow[r, "J", "|" marking] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow\eta"] & b \arrow[d, "l"] \\ c \arrow[r, equal] & c \end{tikzcd}$$

(2) 任意の cell

$$\begin{tikzcd} a \arrow[r, "J", "|" marking] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow\theta"] & b \arrow[d, "k"] \\ c \arrow[r, equal] & c \end{tikzcd}$$

に対して，cell τ が一意に存在して，次の等式が成り立つ．

$$\begin{tikzcd} a \arrow[rr, "J", "|" marking] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\rho^{-1}"] & & b \arrow[d, equal] \\ a \arrow[r, "J", "|" marking] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow\eta"] & b \arrow[r, equal] \arrow[d, "l"] \arrow[dr, phantom, "\Downarrow\tau"] & b \arrow[d, "k"] \\ c \arrow[r, equal] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\rho"] & c \arrow[r, equal] & c \arrow[d, equal] \\ c \arrow[rr, equal] & & c \end{tikzcd} \quad = \quad \begin{tikzcd} a \arrow[r, "J", "|" marking] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow\theta"] & b \arrow[d, "k"] \\ c \arrow[r, equal] & c \end{tikzcd}$$

これは次の意味で strict 2-category の場合の一般化になっている．

定理 20. $\mathcal{V}(\mathbb{D})$ の図式

$$\begin{tikzcd} b \arrow[dr, "l"] & \\ a \arrow[u, "f"] \arrow[r, "g"'] \arrow[u, phantom, shift left=3ex, "\eta\Uparrow"] & c \end{tikzcd}$$

を考える．即ち $\mathbb{D}$ の図式

$$\begin{tikzcd} a \arrow[r, equal] \arrow[dd, "g"'] \arrow[ddr, phantom, "\Downarrow\eta"] & a \arrow[d, "f"] \\ & b \arrow[d, "l"] \\ c \arrow[r, equal] & c \end{tikzcd}$$

である．垂直射 $f\colon a \to b$ の companion が存在するとして，η を

$$\begin{tikzcd}
a \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow_f\eta"] & a \arrow[d, "f"] \\
a \arrow[r, "f_\star"] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow\eta'"] & b \arrow[d, "l"] \\
c \arrow[r, equal] & c
\end{tikzcd}
=
\begin{tikzcd}
a \arrow[r, equal] \arrow[dd, "g"'] & a \arrow[d, "f"] \\
\arrow[r, phantom, "\Downarrow\eta"] & b \arrow[d, "l"] \\
c \arrow[r, equal] & c
\end{tikzcd}$$

と分解する．このとき

$$\langle l, \eta' \rangle \text{ が } f_\star \text{ に沿った } g \text{ の左 Kan 拡張である}$$
$$\iff \mathcal{V}(\mathbb{D}) \text{ において } \langle l, \eta \rangle \text{ が } f \text{ に沿った } g \text{ の左 Kan 拡張である．}$$

証明．$(\Longrightarrow)$ $\mathcal{V}(\mathbb{D})$ の図式

$$\begin{tikzcd}
b \arrow[dr, "k"] & \\
a \arrow[u, "f"] \arrow[r, "g"'] \arrow[ur, phantom, "\theta\Uparrow"] & c
\end{tikzcd}$$

を考える．θ を $\mathbb{D}$ において

$$\begin{tikzcd}
a \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow_f\eta"] & a \arrow[d, "f"] \\
a \arrow[r, "f_\star"] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow\theta'"] & b \arrow[d, "k"] \\
c \arrow[r, equal] & c
\end{tikzcd}
=
\begin{tikzcd}
a \arrow[r, equal] \arrow[dd, "g"'] & a \arrow[d, "f"] \\
\arrow[r, phantom, "\Downarrow\theta"] & b \arrow[d, "k"] \\
c \arrow[r, equal] & c
\end{tikzcd}$$

と分解する．$\langle l, \eta' \rangle$ が $f_\star$ に沿った g の左 Kan 拡張だから，cell τ が一意に存在して

$$\begin{tikzcd}
a \arrow[rr, "f_\star", "|"{marking}] \arrow[d, equal] & \arrow[d, phantom, "\Downarrow\rho^{-1}"] & b \arrow[d, equal] \\
a \arrow[r, "f_\star", "|"{marking}] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow\eta'"] & b \arrow[r, equal] \arrow[d, "l"] \arrow[dr, phantom, "\Downarrow\tau"] & b \arrow[d, "k"] \\
c \arrow[r, equal] \arrow[d, equal] & c \arrow[r, equal] \arrow[d, phantom, "\Downarrow\rho"] & c \arrow[d, equal] \\
c \arrow[rr, equal] & & c
\end{tikzcd}
=
\begin{tikzcd}
a \arrow[rr, "f_\star", "|"{marking}] \arrow[d, "g"'] & \arrow[d, phantom, "\Downarrow\theta'"] & b \arrow[d, "k"] \\
c \arrow[rr, equal] & & c
\end{tikzcd}$$

となる．このとき

$$
\theta \quad = \quad
\begin{tikzcd}
a \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow_f\eta"] & a \arrow[d, "f"] \\
a \arrow[r, "f_\star"', "\shortmid" marking] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow\theta'"] & b \arrow[d, "k"] \\
c \arrow[r, equal] & c
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
a \arrow[rr, equal] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow_f\eta"] & & a \arrow[d, "f"] \\
a \arrow[rr, "f_\star"', "\shortmid" marking] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\rho^{-1}"] & & b \arrow[d, equal] \\
a \arrow[r, "f_\star", "\shortmid" marking] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow\eta'"] & b \arrow[r, equal] \arrow[d, "l"] \arrow[dr, phantom, "\Downarrow\tau"] & b \arrow[d, "k"] \\
c \arrow[r, equal] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\rho"] & c \arrow[r, equal] & c \arrow[d, equal] \\
c \arrow[rr, equal] & & c
\end{tikzcd}
$$

$$
= \quad
\begin{tikzcd}
a \arrow[rr, equal] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\rho^{-1}"] & & a \arrow[d, equal] \\
a \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow_f\eta"] & a \arrow[r, equal] \arrow[d, "f"] \arrow[dr, phantom, "\Downarrow Uf"] & a \arrow[d, "f"] \\
a \arrow[r, "f_\star", "\shortmid" marking] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow\eta'"] & b \arrow[r, equal] \arrow[d, "l"] \arrow[dr, phantom, "\Downarrow\tau"] & b \arrow[d, "k"] \\
c \arrow[r, equal] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\rho"] & c \arrow[r, equal] & c \arrow[d, equal] \\
c \arrow[rr, equal] & & c
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
a \arrow[rr, equal] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\rho^{-1}"] & & a \arrow[d, equal] \\
a \arrow[r, equal] \arrow[d, equal] & a \arrow[r, equal] \arrow[d, "f"] & a \arrow[d, "f"] \\
a \arrow[d, "g"'] \arrow[r, phantom, "\Downarrow\eta"] & b \arrow[d, "l"] \arrow[r, phantom, "\Downarrow\tau\circ Uf"] & b \arrow[d, "k"] \\
c \arrow[r, equal] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\rho"] & c \arrow[r, equal] & c \arrow[d, equal] \\
c \arrow[rr, equal] & & c
\end{tikzcd}
$$

となるから，$\mathcal{V}(\mathbb{D})$ において $(\tau \bullet f) * \eta = \theta$，即ち

$$
\begin{tikzcd}
b \arrow[dr, bend left=50, "k"] \arrow[dr, "l"] \arrow[dr, phantom, bend left=25, "\Nearrow\tau"] & \\
a \arrow[u, "f"] \arrow[r, "g"'] \arrow[ur, phantom, "\eta\Uparrow"] & c
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
b \arrow[dr, bend left=50, "k"] & \\
a \arrow[u, "f"] \arrow[r, "g"'] \arrow[ur, phantom, "\theta\Uparrow"] & c
\end{tikzcd}
$$

である．τ の一意性も容易に分かるから $\mathcal{V}(\mathbb{D})$ において $\langle l, \eta \rangle$ が f に沿った g の左 Kan 拡張である．

($\Longleftarrow$) 任意の cell

$$
\begin{tikzcd}
a \arrow[r, "f_\star", "\shortmid" marking] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow\theta"] & b \arrow[d, "k"] \\
c \arrow[r, equal] & c
\end{tikzcd}
$$

を取る．合成

$$
\begin{tikzcd}
a \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow {}_f\eta"] & a \arrow[d, "f"] \\
a \arrow[r, "f_\star"] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow\theta"] & b \arrow[d, "k"] \\
c \arrow[r, equal] & c
\end{tikzcd}
$$

を $\mathcal{V}(\mathbb{D})$ の 2-morphism とみなして，左 Kan 拡張の普遍性から

$$
\begin{tikzcd}
b \arrow[rr, bend left, "k"] \arrow[drr, "l"] & & \\
a \arrow[u, "f"] \arrow[rr, "g"'] \arrow[ur, phantom, "\eta\Uparrow"] & & c
\end{tikzcd}
\quad \Rightarrow\!\!\tau \quad
=
\begin{tikzcd}
b \arrow[rr, bend left, "k"] & & \\
a \arrow[u, "f"] \arrow[rr, "g"'] \arrow[ur, phantom, "\theta\circ{}_f\eta\Uparrow"] & & c
\end{tikzcd}
$$

となる τ を取る．このとき

$$
\begin{tikzcd}
a \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow {}_f\eta"] & a \arrow[d, "f"] \\
a \arrow[r, "f_\star"', "|" marking] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow\theta"] & b \arrow[d, "k"] \\
c \arrow[r, equal] & c
\end{tikzcd}
=
\begin{tikzcd}
a \arrow[rr, equal] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\rho^{-1}"] & & a \arrow[d, equal] \\
a \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow {}_f\eta"] & a \arrow[r, equal] \arrow[d, "f"] \arrow[dr, phantom, "\Downarrow Uf"] & a \arrow[d, "f"] \\
a \arrow[r, "f_\star", "|" marking] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow\eta'"] & b \arrow[r, equal] \arrow[d, "l"] \arrow[dr, phantom, "\Downarrow\tau"] & b \arrow[d, "k"] \\
c \arrow[r, equal] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\rho"] & c \arrow[r, equal] & c \arrow[d, equal] \\
c \arrow[rr, equal] & & c
\end{tikzcd}
=
\begin{tikzcd}
a \arrow[rr, equal] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow {}_f\eta"] & & a \arrow[d, "f"] \\
a \arrow[rr, "f_\star", "|" marking] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\rho^{-1}"] & & b \arrow[d, equal] \\
a \arrow[r, "f_\star", "|" marking] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow\eta'"] & b \arrow[r, equal] \arrow[d, "l"] \arrow[dr, phantom, "\Downarrow\tau"] & b \arrow[d, "k"] \\
c \arrow[r, equal] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\rho"] & c \arrow[r, equal] & c \arrow[d, equal] \\
c \arrow[rr, equal] & & c
\end{tikzcd}
$$

となるから ${}_f\eta$ の普遍性により

$$
\begin{tikzcd}
a \arrow[r, "f_\star", "|" marking] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow\theta"] & b \arrow[d, "k"] \\
c \arrow[r, equal] & c
\end{tikzcd}
=
\begin{tikzcd}
a \arrow[rr, "f_\star", "|" marking] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\rho^{-1}"] & & b \arrow[d, equal] \\
a \arrow[r, "f_\star", "|" marking] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow\eta'"] & b \arrow[r, equal] \arrow[d, "l"] \arrow[dr, phantom, "\Downarrow\tau"] & b \arrow[d, "k"] \\
c \arrow[r, equal] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\rho"] & c \arrow[r, equal] & c \arrow[d, equal] \\
c \arrow[rr, equal] & & c
\end{tikzcd}
$$

である． □

定義． $J\colon a \nrightarrow b$, $g\colon a \to c$ とする．J に沿った g の各点左 Kan 拡張とは組 $\langle l, \eta\rangle$ であって以下の条件を満たすものである．

(1) $l\colon b \to c$ で η は次の cell である．

$$\begin{tikzcd} a \arrow[r, "J", "\shortmid"{marking}] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow\eta"] & b \arrow[d, "l"] \\ c \arrow[r, equal] & c \end{tikzcd}$$

(2) 任意の $H\colon b \nrightarrow u$, $k\colon u \to c$ と cell

$$\begin{tikzcd} a \arrow[r, "J", "\shortmid"{marking}] \arrow[d, "g"'] & b \arrow[r, "H", "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow\theta"] & u \arrow[d, "k"] \\ c \arrow[rr, equal] & & c \end{tikzcd}$$

に対して，cell τ が一意に存在して，次の等式が成り立つ．

$$\begin{tikzcd} a \arrow[r, "J", "\shortmid"{marking}] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow\eta"] & b \arrow[r, "H", "\shortmid"{marking}] \arrow[d, "l"] \arrow[dr, phantom, "\Downarrow\tau"] & u \arrow[d, "k"] \\ c \arrow[r, equal] \arrow[d, equal] & c \arrow[r, equal] \arrow[d, phantom, "\Downarrow\rho"] & c \arrow[d, equal] \\ c \arrow[rr, equal] & & c \end{tikzcd} = \begin{tikzcd} a \arrow[r, "J", "\shortmid"{marking}] \arrow[dd, "g"'] & b \arrow[r, "H", "\shortmid"{marking}] \arrow[dd, phantom, "\Downarrow\theta"] & u \arrow[dd, "k"] \\ & & \\ c \arrow[rr, equal] & & c \end{tikzcd}$$

命題 21． 各点左 Kan 拡張は左 Kan 拡張である．

証明． $\langle l, \eta\rangle$ が J に沿った g の各点左 Kan 拡張であるとする．任意の cell

$$\begin{tikzcd} a \arrow[r, "J", "\shortmid"{marking}] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow\theta"] & b \arrow[d, "k"] \\ c \arrow[r, equal] & c \end{tikzcd}$$

を取る．$\langle l, \eta\rangle$ が各点左 Kan 拡張だから，cell τ が一意に存在して，次の等式が成り立つ．

$$\begin{tikzcd} a \arrow[r, "J", "\shortmid"{marking}] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow\eta"] & b \arrow[r, equal] \arrow[d, "l"] \arrow[dr, phantom, "\Downarrow\tau"] & b \arrow[d, "k"] \\ c \arrow[r, equal] \arrow[d, equal] & c \arrow[r, equal] \arrow[d, phantom, "\Downarrow\rho"] & c \arrow[d, equal] \\ c \arrow[rr, equal] & & c \end{tikzcd} = \begin{tikzcd} a \arrow[r, "J", "\shortmid"{marking}] \arrow[d, equal] & b \arrow[r, equal] \arrow[d, phantom, "\Downarrow\rho"] & b \arrow[d, equal] \\ a \arrow[rr, "J", "\shortmid"{marking}] \arrow[d, "g"'] & \arrow[d, phantom, "\Downarrow\theta"] & b \arrow[d, "k"] \\ c \arrow[rr, equal] & & c \end{tikzcd}$$

従って

$$
\begin{tikzcd}
a \arrow[rr, "J" description, "|" marking] \arrow[d, equal] & & b \arrow[d, equal] \\
a \arrow[r, "J", "|" marking] \arrow[d, "g"'] \arrow[rr, phantom, "\Downarrow\rho^{-1}", shift left=20] & b \arrow[r, equal] \arrow[d, "l"] & b \arrow[d, "k"] \\
c \arrow[r, equal] \arrow[d, equal] \arrow[ur, phantom, "\Downarrow\eta"] & c \arrow[r, equal] \arrow[ur, phantom, "\Downarrow\tau"] & c \arrow[d, equal] \\
c \arrow[rr, equal] \arrow[urr, phantom, "\Downarrow\rho"] & & c
\end{tikzcd}
=
\begin{tikzcd}
a \arrow[r, "J", "|" marking] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow\theta"] & b \arrow[d, "k"] \\
c \arrow[r, equal] & c
\end{tikzcd}
$$

である．τ の一意性も明らかであるから $\langle l, \eta \rangle$ は左 Kan 拡張である． $\square$

定義. 水平射 $J\colon a \nrightarrow b$ の tabulation とは組 $\langle \langle J \rangle, \pi \rangle$ であって

(1) $\langle J \rangle \in \mathbb{D}$ は対象で π は次の cell である．

$$
\begin{tikzcd}
\langle J \rangle \arrow[r, equal] \arrow[d, "p_a"'] \arrow[dr, phantom, "\Downarrow\pi"] & \langle J \rangle \arrow[d, "p_b"] \\
a \arrow[r, "|" marking, "J"'] & b
\end{tikzcd}
$$

(2) (1 次元的普遍性) 任意の cell

$$
\begin{tikzcd}
u \arrow[r, equal] \arrow[d, "q_a"'] \arrow[dr, phantom, "\Downarrow\varphi"] & u \arrow[d, "q_b"] \\
a \arrow[r, "|" marking, "J"'] & b
\end{tikzcd}
$$

に対して，垂直射 $h\colon u \to \langle J \rangle$ が一意に存在して

$$
\begin{tikzcd}
u \arrow[r, equal] \arrow[d, "h"'] \arrow[dr, phantom, "\Downarrow Uh"] & u \arrow[d, "h"] \\
\langle J \rangle \arrow[r, equal] \arrow[d, "p_a"'] \arrow[dr, phantom, "\Downarrow\pi"] & \langle J \rangle \arrow[d, "p_b"] \\
a \arrow[r, "|" marking, "J"'] & b
\end{tikzcd}
=
\begin{tikzcd}
u \arrow[r, equal] \arrow[d, "q_a"'] \arrow[dr, phantom, "\Downarrow\varphi"] & u \arrow[d, "q_b"] \\
a \arrow[r, "|" marking, "J"'] & b
\end{tikzcd}
$$

となる．

(3) (2 次元的普遍性) 2 つの cell

$$
\begin{tikzcd}
u \arrow[r, equal] \arrow[d, "q_a"'] \arrow[dr, phantom, "\Downarrow\varphi"] & u \arrow[d, "q_b"] \\
a \arrow[r, "J"', "\shortmid" marking] & b
\end{tikzcd}
\qquad
\begin{tikzcd}
v \arrow[r, equal] \arrow[d, "r_a"'] \arrow[dr, phantom, "\Downarrow\psi"] & v \arrow[d, "r_b"] \\
a \arrow[r, "J"', "\shortmid" marking] & b
\end{tikzcd}
$$

に対して，1 次元的普遍性により

$$
\begin{tikzcd}
u \arrow[r, equal] \arrow[d, "h"'] \arrow[dr, phantom, "\Downarrow Uh"] & u \arrow[d, "h"] \\
\langle J\rangle \arrow[r, equal] \arrow[d, "p_a"'] \arrow[dr, phantom, "\Downarrow\pi"] & \langle J\rangle \arrow[d, "p_b"] \\
a \arrow[r, "J"', "\shortmid" marking] & b
\end{tikzcd}
=
\begin{tikzcd}
u \arrow[r, equal] \arrow[dd, "q_a"'] \arrow[ddr, phantom, "\Downarrow\varphi"] & u \arrow[dd, "q_b"] \\
 & \\
a \arrow[r, "J"', "\shortmid" marking] & b
\end{tikzcd}
\qquad
\begin{tikzcd}
v \arrow[r, equal] \arrow[d, "k"'] \arrow[dr, phantom, "\Downarrow Uk"] & v \arrow[d, "k"] \\
\langle J\rangle \arrow[r, equal] \arrow[d, "p_a"'] \arrow[dr, phantom, "\Downarrow\pi"] & \langle J\rangle \arrow[d, "p_b"] \\
a \arrow[r, "J"', "\shortmid" marking] & b
\end{tikzcd}
=
\begin{tikzcd}
v \arrow[r, equal] \arrow[dd, "r_a"'] \arrow[ddr, phantom, "\Downarrow\psi"] & v \arrow[dd, "r_b"] \\
 & \\
a \arrow[r, "J"', "\shortmid" marking] & b
\end{tikzcd}
$$

となるように h, k を取る．更に水平射 $H\colon u \nrightarrow v$ と cell ξ_a, ξ_b が等式

$$
\begin{tikzcd}
u \arrow[rr, "H", "\shortmid" marking] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\lambda^{-1}"] & & v \arrow[d, equal] \\
u \arrow[r, equal] \arrow[d, "q_a"'] \arrow[dr, phantom, "\Downarrow\varphi"] & u \arrow[r, "H", "\shortmid" marking] \arrow[d, "q_b"] \arrow[dr, phantom, "\Downarrow\xi_b"] & v \arrow[d, "r_b"] \\
a \arrow[r, "J"', "\shortmid" marking] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\rho"] & b \arrow[r, equal] & b \arrow[d, equal] \\
a \arrow[rr, "J"', "\shortmid" marking] & & b
\end{tikzcd}
=
\begin{tikzcd}
u \arrow[rr, "H", "\shortmid" marking] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\rho^{-1}"] & & v \arrow[d, equal] \\
u \arrow[r, "H", "\shortmid" marking] \arrow[d, "q_a"'] \arrow[dr, phantom, "\Downarrow\xi_a"] & v \arrow[r, equal] \arrow[d, "r_a"] \arrow[dr, phantom, "\Downarrow\psi"] & v \arrow[d, "r_b"] \\
a \arrow[r, equal] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\lambda"] & a \arrow[r, "J"', "\shortmid" marking] & b \arrow[d, equal] \\
a \arrow[rr, "J"', "\shortmid" marking] & & b
\end{tikzcd}
$$

を満たすとする．このとき cell

$$
\begin{tikzcd}
u \arrow[r, "H", "\shortmid" marking] \arrow[d, "h"'] \arrow[dr, phantom, "\Downarrow\xi"] & v \arrow[d, "k"] \\
\langle J\rangle \arrow[r, equal] & \langle J\rangle
\end{tikzcd}
$$

が一意に存在して等式

$$
\begin{tikzcd}
u \arrow[r, "H", "\shortmid" marking] \arrow[d, "h"'] \arrow[dr, phantom, "\Downarrow\xi"] & v \arrow[d, "k"] \\
\langle J\rangle \arrow[r, equal] \arrow[d, "p_a"'] \arrow[dr, phantom, "\Downarrow Up_a"] & \langle J\rangle \arrow[d, "p_a"] \\
a \arrow[r, equal] & a
\end{tikzcd}
=
\begin{tikzcd}
u \arrow[r, "H", "\shortmid" marking] \arrow[dd, "q_a"'] \arrow[ddr, phantom, "\Downarrow\xi_a"] & v \arrow[dd, "r_a"] \\
 & \\
a \arrow[r, equal] & a
\end{tikzcd}
\qquad
\begin{tikzcd}
u \arrow[r, "H", "\shortmid" marking] \arrow[d, "h"'] \arrow[dr, phantom, "\Downarrow\xi"] & v \arrow[d, "k"] \\
\langle J\rangle \arrow[r, equal] \arrow[d, "p_b"'] \arrow[dr, phantom, "\Downarrow Up_b"] & \langle J\rangle \arrow[d, "p_b"] \\
b \arrow[r, equal] & b
\end{tikzcd}
=
\begin{tikzcd}
u \arrow[r, "H", "\shortmid" marking] \arrow[dd, "q_b"'] \arrow[ddr, phantom, "\Downarrow\xi_b"] & v \arrow[dd, "r_b"] \\
 & \\
b \arrow[r, equal] & b
\end{tikzcd}
$$

が成り立つ.

更に π が opcartesian のとき，$\langle\langle J\rangle, \pi\rangle$ を opcartesian tabulation という.

tabulation は次の意味で「コンマ対象の double category バージョン」である.

命題 22. $f\colon a \to x$，$g\colon b \to x$ を $\mathbb{D}$ の垂直射として，制限 $x(f,g)$ が存在して更にその tabulation $\langle\langle x(f,g)\rangle, \pi\rangle$ が存在するとする.

$$
\begin{tikzcd}
\langle x(f,g)\rangle \arrow[r, equal] \arrow[d, "p_a"'] \arrow[dr, phantom, "\Downarrow\pi"] & \langle x(f,g)\rangle \arrow[d, "p_b"] \\
a \arrow[r, "x(f,g)", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\sigma"] & b \arrow[d, "g"] \\
x \arrow[r, equal] & x
\end{tikzcd}
$$

このとき $\mathcal{V}(\mathbb{D})$ において $\langle x(f,g)\rangle$ は f と g のコンマ対象となる.

証明. 上記の cell の合成 $\eta := \sigma \circ \pi$ は $\mathcal{V}(\mathbb{D})$ における次の図式を与える.

$$
\begin{tikzcd}
b \arrow[r, "g"] & x \\
\langle x(f,g)\rangle \arrow[u, "p_b"] \arrow[r, "p_a"'] \arrow[ur, phantom, "\Nwarrow\eta"] & a \arrow[u, "f"']
\end{tikzcd}
$$

これがコンマ対象を与えることを示そう.
(1 次元的普遍性) $\mathcal{V}(\mathbb{D})$ の任意の図式

$$
\begin{tikzcd}
b \arrow[r, "g"] & x \\
u \arrow[u, "q_b"] \arrow[r, "q_a"'] \arrow[ur, phantom, "\Nwarrow\varphi"] & a \arrow[u, "f"']
\end{tikzcd}
$$

を取る. これを $\mathbb{D}$ における cell とみなして

$$
\begin{tikzcd}
u \arrow[r, equal] \arrow[d, "q_a"'] \arrow[dr, phantom, "\Downarrow\varphi'"] & u \arrow[d, "q_b"] \\
a \arrow[r, "x(f,g)", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\sigma"] & b \arrow[d, "g"] \\
x \arrow[r, equal] & x
\end{tikzcd}
=
\begin{tikzcd}
u \arrow[r, equal] \arrow[d, "q_a"'] & u \arrow[d, "q_b"] \\
a \arrow[d, "f"'] \arrow[r, phantom, "\Downarrow\varphi"] & b \arrow[d, "g"] \\
x \arrow[r, equal] & x
\end{tikzcd}
$$

と分解する．この φ' に対して tabulation の 1 次元的普遍性により $h\colon u \to \langle x(f,g)\rangle$ が一意に存在して

$$
\begin{tikzcd}
u \arrow[r, equal] \arrow[d, "h"'] \arrow[dr, phantom, "\Downarrow Uh"] & u \arrow[d, "h"] \\
\langle x(f,g)\rangle \arrow[r, equal] \arrow[d, "p_a"'] \arrow[dr, phantom, "\Downarrow\pi"] & \langle x(f,g)\rangle \arrow[d, "p_b"] \\
a \arrow[r, "x(f,g)"', "|"] & b
\end{tikzcd}
=
\begin{tikzcd}
u \arrow[r, equal] \arrow[d, "q_a"'] \arrow[dr, phantom, "\Downarrow\varphi'"] & u \arrow[d, "q_b"] \\
a \arrow[r, "x(f,g)"', "|"] & b
\end{tikzcd}
$$

となる．このとき

$$
\begin{tikzcd}
& b \arrow[r, "g"] & x \\
& \langle x(f,g)\rangle \arrow[u, "p_b"] \arrow[r, "p_a"'] \arrow[ur, phantom, "\Nwarrow\eta"] & a \arrow[u, "f"'] \\
u \arrow[ur, "h"'] \arrow[uur, bend left, "q_b"] \arrow[urr, bend right, "q_a"'] & &
\end{tikzcd}
=
\begin{tikzcd}
& b \arrow[r, "g"] & x \\
& \arrow[ur, phantom, "\Nwarrow\varphi"] & a \arrow[u, "f"'] \\
u \arrow[uur, bend left, "q_b"] \arrow[urr, bend right, "q_a"'] & &
\end{tikzcd}
$$

である．h の一意性も分かる．

(2 次元的普遍性) $\mathcal{V}(\mathbb{D})$ の図式

$$
\begin{tikzcd}
b \arrow[r, "g"] & x \\
u \arrow[u, "q_b"] \arrow[r, "q_a"'] \arrow[ur, phantom, "\Nwarrow\varphi"] & a \arrow[u, "f"']
\end{tikzcd}
\qquad
\begin{tikzcd}
b \arrow[r, "g"] & x \\
u \arrow[u, "r_b"] \arrow[r, "r_a"'] \arrow[ur, phantom, "\Nwarrow\psi"] & a \arrow[u, "f"']
\end{tikzcd}
$$

を取り，これらに 1 次元的普遍性で対応する 1-morphism を $h, k\colon u \to \langle x(f,g)\rangle$ とする．更に 2-morphism $\xi_a\colon q_a \Rightarrow r_a$，$\xi_b\colon q_b \Rightarrow r_b$ が

$$
\begin{tikzcd}
b \arrow[r, "g"] & x \\
u \arrow[u, bend left, "r_b"] \arrow[u, bend right, "q_b"'] \arrow[u, phantom, "\underset{\xi_b}{\Leftarrow}"] \arrow[r, bend right, "q_a"'] \arrow[ur, phantom, "\Nwarrow\varphi"] & a \arrow[u, "f"']
\end{tikzcd}
=
\begin{tikzcd}
a \arrow[r, "g"] & x \\
u \arrow[u, bend left, "r_b"] \arrow[r, bend left, "r_a"] \arrow[r, bend right, "q_a"'] \arrow[r, phantom, "\Uparrow\xi_a"] \arrow[ur, phantom, "\Nwarrow\psi"] & a \arrow[u, "f"']
\end{tikzcd}
$$

を満たすとする．φ, ψ を $\mathbb{D}$ における cell とみなして

$$
\begin{tikzcd}
u \arrow[r, equal] \arrow[d, "q_a"'] \arrow[dr, phantom, "\Downarrow\varphi'"] & u \arrow[d, "q_b"] \\
a \arrow[r, "x(f,g)", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\sigma"] & b \arrow[d, "g"] \\
x \arrow[r, equal] & x
\end{tikzcd}
=
\begin{tikzcd}
u \arrow[r, equal] \arrow[d, "q_a"'] & u \arrow[d, "q_b"] \\
a \arrow[r, phantom, "\Downarrow\varphi"] \arrow[d, "f"'] & b \arrow[d, "g"] \\
x \arrow[r, equal] & x
\end{tikzcd}
\qquad
\begin{tikzcd}
u \arrow[r, equal] \arrow[d, "r_a"'] \arrow[dr, phantom, "\Downarrow\psi'"] & u \arrow[d, "r_b"] \\
a \arrow[r, "x(f,g)", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\sigma"] & b \arrow[d, "g"] \\
x \arrow[r, equal] & x
\end{tikzcd}
=
\begin{tikzcd}
u \arrow[r, equal] \arrow[d, "r_a"'] & u \arrow[d, "r_b"] \\
a \arrow[r, phantom, "\Downarrow\psi"] \arrow[d, "f"'] & b \arrow[d, "g"] \\
x \arrow[r, equal] & x
\end{tikzcd}
$$

と分解すれば

$$
\begin{tikzcd}
u \arrow[r, equal] \arrow[d, "h"'] \arrow[dr, phantom, "\Downarrow Uh"] & u \arrow[d, "h"] \\
\langle x(f,g) \rangle \arrow[r, equal] \arrow[d, "p_a"'] \arrow[dr, phantom, "\Downarrow\pi"] & \langle x(f,g) \rangle \arrow[d, "p_b"] \\
a \arrow[r, "c(f,g)"', "\shortmid"{marking}] & b
\end{tikzcd}
=
\begin{tikzcd}
u \arrow[r, equal] \arrow[d, "q_a"'] \arrow[dr, phantom, "\Downarrow\varphi'"] & u \arrow[d, "q_b"] \\
a \arrow[r, "x(f,g)"', "\shortmid"{marking}] & b
\end{tikzcd}
$$

$$
\begin{tikzcd}
u \arrow[r, equal] \arrow[d, "k"'] \arrow[dr, phantom, "\Downarrow Uk"] & u \arrow[d, "k"] \\
\langle x(f,g) \rangle \arrow[r, equal] \arrow[d, "p_a"'] \arrow[dr, phantom, "\Downarrow\pi"] & \langle x(f,g) \rangle \arrow[d, "p_b"] \\
a \arrow[r, "x(f,g)"', "\shortmid"{marking}] & b
\end{tikzcd}
=
\begin{tikzcd}
u \arrow[r, equal] \arrow[d, "r_a"'] \arrow[dr, phantom, "\Downarrow\psi'"] & u \arrow[d, "r_b"] \\
a \arrow[r, "x(f,g)"', "\shortmid"{marking}] & b
\end{tikzcd}
$$

である．また等式

$$
\begin{tikzcd}
u \arrow[rr, equal] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\lambda^{-1}"] & & u \arrow[d, equal] \\
u \arrow[r, equal] \arrow[d, "q_a"'] \arrow[dr, phantom, "\Downarrow\varphi"] & u \arrow[r, equal] \arrow[d, "q_b"] \arrow[dr, phantom, "\Downarrow\xi_b"] & u \arrow[d, "r_b"] \\
a \arrow[r, "x(f,g)"', "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\rho"] & b \arrow[r, equal] & b \arrow[d, equal] \\
a \arrow[rr, "x(f,g)"', "\shortmid"{marking}] & & b
\end{tikzcd}
=
\begin{tikzcd}
u \arrow[rr, equal] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\rho^{-1}"] & & u \arrow[d, equal] \\
u \arrow[r, equal] \arrow[d, "q_a"'] \arrow[dr, phantom, "\Downarrow\xi_a"] & u \arrow[r, equal] \arrow[d, "r_a"] \arrow[dr, phantom, "\Downarrow\psi"] & u \arrow[d, "r_b"] \\
a \arrow[r, equal] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\lambda"] & a \arrow[r, "x(f,g)"', "\shortmid"{marking}] & b \arrow[d, equal] \\
a \arrow[rr, "x(f,g)"', "\shortmid"{marking}] & & b
\end{tikzcd}
$$

が成り立つ．故に tabulation の 2 次元的普遍性により，cell

$$
\begin{tikzcd}
u \arrow[r, equal] \arrow[d, "h"'] \arrow[dr, phantom, "\Downarrow\xi"] & u \arrow[d, "k"] \\
\langle x(f,g) \rangle \arrow[r, equal] & \langle x(f,g) \rangle
\end{tikzcd}
$$

が一意に存在して等式

$$
\begin{tikzcd}
u \arrow[r, equal] \arrow[d, "h"'] \arrow[dr, phantom, "\Downarrow\xi"] & u \arrow[d, "k"] \\
\langle x(f,g)\rangle \arrow[r, equal] \arrow[d, "p_a"'] \arrow[dr, phantom, "\Downarrow Up_a"] & \langle x(f,g)\rangle \arrow[d, "p_a"] \\
a \arrow[r, equal] & a
\end{tikzcd}
=
\begin{tikzcd}
u \arrow[r, equal] \arrow[dd, "q_a"'] \arrow[ddr, phantom, "\Downarrow\xi_a"] & u \arrow[dd, "r_a"] \\
& \\
a \arrow[r, equal] & a
\end{tikzcd}
$$

$$
\begin{tikzcd}
u \arrow[r, equal] \arrow[d, "h"'] \arrow[dr, phantom, "\Downarrow\xi"] & u \arrow[d, "k"] \\
\langle x(f,g)\rangle \arrow[r, equal] \arrow[d, "p_b"'] \arrow[dr, phantom, "\Downarrow Up_b"] & \langle x(f,g)\rangle \arrow[d, "p_b"] \\
b \arrow[r, equal] & b
\end{tikzcd}
=
\begin{tikzcd}
u \arrow[r, equal] \arrow[dd, "q_b"'] \arrow[ddr, phantom, "\Downarrow\xi_b"] & u \arrow[dd, "r_b"] \\
& \\
b \arrow[r, equal] & b
\end{tikzcd}
$$

が成り立つ．これは $\mathcal{V}(\mathbb{D})$ において

$$
\begin{tikzcd}
b \\
\langle x(f,g)\rangle \arrow[u, "p_b"] \\
u \arrow[u, bend left, "k"] \arrow[u, bend right, "h"'] \arrow[u, phantom, "\overset{\xi}{\Leftarrow}"]
\end{tikzcd}
=
\begin{tikzcd}
b \\
\\
u \arrow[uu, bend left, "r_b"] \arrow[uu, bend right, "q_b"'] \arrow[uu, phantom, "\overset{\xi_a}{\Leftarrow}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
u \arrow[r, bend left, "k"] \arrow[r, bend right, "h"'] \arrow[r, phantom, "\Uparrow\xi"] & \langle x(f,g)\rangle \arrow[r, "p_a"] & a
\end{tikzcd}
=
\begin{tikzcd}
u \arrow[r, bend left, "r_a"] \arrow[r, bend right, "q_a"'] \arrow[r, phantom, "\Uparrow\xi_a"] & a
\end{tikzcd}
$$

である． □

補題 23. $\mathbb{D}$ を opcartesian tabulation を持つ framed bicategory とする．η を cell

$$
\begin{tikzcd}
a \arrow[r, "J", "|" marking] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow\eta"] & b \arrow[d, "l"] \\
c \arrow[r, equal] & c
\end{tikzcd}
$$

としたとき，$\langle l, \eta\rangle$ が J に沿った g の各点左 Kan 拡張
$\Longleftrightarrow$ $j\colon x \to a$ を任意の垂直射としたとき，制限 $J(\mathrm{id}_a, j)$ を与える cartesian cell β に対

して $\langle l \circ j, \eta \circ \beta \rangle$ が $J(\mathrm{id}_a, j)$ に沿った g の左 Kan 拡張となる．

$$
\begin{tikzcd}
a \arrow[r, "{J(\mathrm{id}_a,j)}", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\beta"] & x \arrow[d, "j"] \\
a \arrow[r, "J", "\shortmid"{marking}] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow\eta"] & b \arrow[d, "l"] \\
c \arrow[r, equal] & c
\end{tikzcd}
$$

証明. ($\Longrightarrow$) 任意の cell

$$
\begin{tikzcd}
a \arrow[r, "{J(\mathrm{id}_a,j)}", "\shortmid"{marking}] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow\theta"] & x \arrow[d, "k"] \\
c \arrow[r, equal] & c
\end{tikzcd}
$$

を取る．β が cartesian だから，cell σ が存在して

$$
\varphi \quad := \quad
\begin{tikzcd}
a \arrow[r, "J", "\shortmid"{marking}] \arrow[d, equal] & b \arrow[r, "j^\star", "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow\sigma"] & x \arrow[d, equal] \\
a \arrow[rr, "{J(\mathrm{id}_a,j)}", "\shortmid"{marking}] \arrow[d, equal] & \arrow[d, phantom, "\Downarrow\beta"] & x \arrow[d, "j"] \\
a \arrow[rr, "J", "\shortmid"{marking}] & & b
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
a \arrow[r, "J", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mathrm{id}_J"] & b \arrow[r, "j^\star", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\varepsilon_j"] & x \arrow[d, "j"] \\
a \arrow[r, "J"', "\shortmid"{marking}] \arrow[d, equal] & b \arrow[r, equal] \arrow[d, phantom, "\Downarrow\rho"] & b \arrow[d, equal] \\
a \arrow[rr, "J"', "\shortmid"{marking}] & & b
\end{tikzcd}
$$

となる．補題 13 より右辺全体は cartesian だから σ は同型である．今 $\langle l, \eta \rangle$ が各点左 Kan 拡張だから，τ が一意に存在して

$$
\begin{tikzcd}
a \arrow[r, "J", "\shortmid"{marking}] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow\eta"] & b \arrow[r, "j^\star", "\shortmid"{marking}] \arrow[d, "l"] \arrow[dr, phantom, "\Downarrow\tau"] & x \arrow[d, "k"] \\
c \arrow[r, equal] \arrow[d, equal] & c \arrow[r, equal] \arrow[d, phantom, "\Downarrow\rho"] & c \arrow[d, equal] \\
c \arrow[rr, equal] & & c
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
a \arrow[r, "J", "\shortmid"{marking}] \arrow[d, equal] & b \arrow[r, "j^\star", "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow\sigma"] & x \arrow[d, equal] \\
a \arrow[rr, "{J(\mathrm{id}_a,j)}", "\shortmid"{marking}] \arrow[d, "g"'] & \arrow[d, phantom, "\Downarrow\theta"] & x \arrow[d, "k"] \\
c \arrow[rr, equal] & & c
\end{tikzcd}
$$

となる．ここで

$$
\varphi \odot \eta_j \quad = \quad
\begin{tikzcd}
a \arrow[r, "J", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow \mathrm{id}_J"] & b \arrow[r, "j^\star", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow \varepsilon_j"] & x \arrow[r, equal] \arrow[d, "j"] \arrow[dr, phantom, "\Downarrow \eta_j"] & x \arrow[d, equal] \\
a \arrow[r, "J"', "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow \rho"] & b \arrow[r, equal] & b \arrow[r, "j^\star", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow \mathrm{id}_{j^\star}"] & x \arrow[d, equal] \\
a \arrow[rr, "J"', "\shortmid"{marking}] & & b \arrow[r, "j^\star"', "\shortmid"{marking}] & x
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
a \arrow[r, "J \odot j^\star", "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow \rho"] & x \arrow[r, equal] & x \arrow[d, equal] \\
a \arrow[r, "J"', "\shortmid"{marking}] & b \arrow[r, "j^\star"', "\shortmid"{marking}] & x
\end{tikzcd}
$$

である．故に

$$
\theta \circ \sigma \quad = \quad \theta \circ \sigma \circ \mathrm{id}_{J \odot j^\star} \quad = \quad
\begin{tikzcd}
a \arrow[r, "J", "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow \rho^{-1}"] & b \arrow[r, "j^\star", "\shortmid"{marking}] & x \arrow[d, equal] \\
a \arrow[r, "J \odot j^\star", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow \varphi"] & x \arrow[r, equal] \arrow[d, "j"] \arrow[dr, phantom, "\Downarrow \eta_j"] & x \arrow[d, equal] \\
a \arrow[r, "J", "\shortmid"{marking}] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow \eta"] & b \arrow[r, "j^\star", "\shortmid"{marking}] \arrow[d, "l"] \arrow[dr, phantom, "\Downarrow \tau"] & x \arrow[d, "k"] \\
c \arrow[r, equal] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow \rho"] & c \arrow[r, equal] & c \arrow[d, equal] \\
c \arrow[rr, equal] & & c
\end{tikzcd}
$$

$$
= \quad
\begin{tikzcd}
a \arrow[r, "J", "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow \rho^{-1}"] & b \arrow[r, "j^\star", "\shortmid"{marking}] & x \arrow[d, equal] \\
a \arrow[r, "J \odot j^\star", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow \sigma"] & x \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow \mathrm{id}"] & x \arrow[d, equal] \\
a \arrow[r, "J(\mathrm{id}_a, j)", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow \beta"] & x \arrow[r, equal] \arrow[d, "j"] \arrow[dr, phantom, "\Downarrow \eta_j"] & x \arrow[d, equal] \\
a \arrow[r, "J", "\shortmid"{marking}] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow \eta"] & b \arrow[r, "j^\star", "\shortmid"{marking}] \arrow[d, "l"] \arrow[dr, phantom, "\Downarrow \tau"] & x \arrow[d, "k"] \\
c \arrow[r, equal] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow \rho"] & c \arrow[r, equal] & c \arrow[d, equal] \\
c \arrow[rr, equal] & & c
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
a \arrow[r, "J", "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow \sigma"] & b \arrow[r, "j^\star", "\shortmid"{marking}] & x \arrow[d, equal] \\
a \arrow[rr, "J(\mathrm{id}_a, j)", "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow \rho^{-1}"] & & x \arrow[d, equal] \\
a \arrow[r, "J(\mathrm{id}_a, j)", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow \beta"] & x \arrow[r, equal] \arrow[d, "j"] \arrow[dr, phantom, "\Downarrow \eta_j"] & x \arrow[d, equal] \\
a \arrow[r, "J", "\shortmid"{marking}] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow \eta"] & b \arrow[r, "j^\star", "\shortmid"{marking}] \arrow[d, "l"] \arrow[dr, phantom, "\Downarrow \tau"] & x \arrow[d, "k"] \\
c \arrow[r, equal] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow \rho"] & c \arrow[r, equal] & c \arrow[d, equal] \\
c \arrow[rr, equal] & & c
\end{tikzcd}
$$

となるが，σ が同型だったから

$$
\begin{tikzcd}
a \arrow[rr, "J(\mathrm{id}_a,j)", "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\rho^{-1}"] & & x \arrow[d, equal] \\
a \arrow[r, "J(\mathrm{id}_a,j)", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\beta"] & x \arrow[r, equal] \arrow[d, "j"] \arrow[dr, phantom, "\Downarrow\eta_j"] & x \arrow[d, equal] \\
a \arrow[r, "J", "\shortmid"{marking}] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow\eta"] & b \arrow[r, "j^\star", "\shortmid"{marking}] \arrow[d, "l"] \arrow[dr, phantom, "\Downarrow\tau"] & x \arrow[d, "k"] \\
c \arrow[r, equal] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\rho"] & c \arrow[r, equal] & c \arrow[d, equal] \\
c \arrow[rr, equal] & & c
\end{tikzcd}
=
\begin{tikzcd}
a \arrow[r, "J(\mathrm{id}_a,j)", "\shortmid"{marking}] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow\theta"] & x \arrow[d, "k"] \\
c \arrow[r, equal] & c
\end{tikzcd}
$$

が分かる．一意性も容易に分かるから $\langle l\circ j, \eta\circ\beta\rangle$ が左 Kan 拡張と分かった．

($\Longleftarrow$) 任意の cell

$$
\begin{tikzcd}
a \arrow[r, "J", "\shortmid"{marking}] \arrow[d, "g"'] \arrow[drr, phantom, "\Downarrow\theta"] & b \arrow[r, "H", "\shortmid"{marking}] & u \arrow[d, "k"] \\
c \arrow[rr, equal] & & c
\end{tikzcd}
$$

を取る．H の opcartesian tabulation π と cartesian cell β を取り右の図式のように合成すると，仮定より cell τ が存在して次の等式が成り立つ．

$$
\begin{tikzcd}
a \arrow[rr, "J", "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\rho^{-1}"] & & \langle H\rangle \arrow[d, equal] \\
a \arrow[r, "J(\mathrm{id},p_b)", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\beta"] & \langle H\rangle \arrow[r, equal] \arrow[d, "p_b"] \arrow[dr, phantom, "\Downarrow\tau"] & \langle H\rangle \arrow[d, "p_u"] \\
a \arrow[r, "J", "\shortmid"{marking}] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow\eta"] & b \arrow[d, "l"] & u \arrow[d, "k"] \\
c \arrow[r, equal] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\rho"] & c \arrow[r, equal] & c \arrow[d, equal] \\
c \arrow[rr, equal] & & c
\end{tikzcd}
=
\begin{tikzcd}
a \arrow[rr, "J", "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\rho^{-1}"] & & \langle H\rangle \arrow[d, equal] \\
a \arrow[r, "J(\mathrm{id},p_b)", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\beta"] & \langle H\rangle \arrow[r, equal] \arrow[d, "p_b"] \arrow[dr, phantom, "\Downarrow\pi"] & \langle H\rangle \arrow[d, "p_u"] \\
a \arrow[r, "J"', "\shortmid"{marking}] \arrow[d, "g"'] \arrow[drr, phantom, "\Downarrow\theta"] & b \arrow[r, "H"', "\shortmid"{marking}] & u \arrow[d, "k"] \\
c \arrow[rr, equal] & & c
\end{tikzcd}
$$

今 π が opcartesian だから次のように分解できる．

$$
\begin{tikzcd}[row sep=small]
\langle H\rangle \arrow[r, equal] \arrow[d, "p_b"'] \arrow[dr, phantom, "\Downarrow\pi"] & \langle H\rangle \arrow[d, "p_u"] \\
b \arrow[r, "H"', "\shortmid"{marking}] \arrow[d, "l"'] \arrow[dr, phantom, "\Downarrow\tau'"] & u \arrow[d, "k"] \\
c \arrow[r, equal] & c
\end{tikzcd}
=
\begin{tikzcd}[row sep=small]
\langle H\rangle \arrow[r, equal] \arrow[d, "q_a"'] & \langle H\rangle \arrow[d, "p_u"] \\
b \arrow[d, "l"'] \arrow[r, phantom, "\Downarrow\tau"] & u \arrow[d, "k"] \\
c \arrow[r, equal] & c
\end{tikzcd}
$$

補題 19 より $\beta\odot\pi$ は opcartesian だから

$$
\begin{tikzcd}
a \arrow[r, "J", "\shortmid"{marking}] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow\eta"] & b \arrow[r, "H", "\shortmid"{marking}] \arrow[d, "l"] \arrow[dr, phantom, "\Downarrow\tau'"] & u \arrow[d, "k"] \\
c \arrow[r, equal] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\rho"] & c \arrow[r, equal] & c \arrow[d, equal] \\
c \arrow[rr, equal] & & c
\end{tikzcd}
=
\begin{tikzcd}
a \arrow[r, "J", "\shortmid"{marking}] \arrow[dd, "g"'] \arrow[ddrr, phantom, "\Downarrow\theta"] & b \arrow[r, "H", "\shortmid"{marking}] & u \arrow[dd, "k"] \\
& & \\
c \arrow[rr, equal] & & c
\end{tikzcd}
$$

が成り立つ．τ' の一意性を示すため，φ が等式

$$
\begin{tikzcd}
a \arrow[r, "J", "\shortmid"{marking}] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow\eta"] & b \arrow[r, "H", "\shortmid"{marking}] \arrow[d, "l"] \arrow[dr, phantom, "\Downarrow\varphi"] & u \arrow[d, "k"] \\
c \arrow[r, equal] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\rho"] & c \arrow[r, equal] & c \arrow[d, equal] \\
c \arrow[rr, equal] & & c
\end{tikzcd}
=
\begin{tikzcd}
a \arrow[r, "J", "\shortmid"{marking}] \arrow[dd, "g"'] \arrow[ddrr, phantom, "\Downarrow\theta"] & b \arrow[r, "H", "\shortmid"{marking}] & u \arrow[dd, "k"] \\
& & \\
c \arrow[rr, equal] & & c
\end{tikzcd}
$$

を満たしたとすると，左 Kan 拡張 $\langle l\circ j, \eta\circ\beta\rangle$ の普遍性により

$$
\begin{tikzcd}[row sep=small]
\langle H\rangle \arrow[r, equal] \arrow[d, "p_b"'] \arrow[dr, phantom, "\Downarrow\pi"] & \langle H\rangle \arrow[d, "p_u"] \\
b \arrow[r, "H"', "\shortmid"{marking}] \arrow[d, "l"'] \arrow[dr, phantom, "\Downarrow\varphi"] & u \arrow[d, "k"] \\
c \arrow[r, equal] & c
\end{tikzcd}
=
\begin{tikzcd}[row sep=small]
\langle H\rangle \arrow[r, equal] \arrow[d, "q_a"'] & \langle H\rangle \arrow[d, "p_u"] \\
b \arrow[d, "l"'] \arrow[r, phantom, "\Downarrow\tau"] & u \arrow[d, "k"] \\
c \arrow[r, equal] & c
\end{tikzcd}
$$

が成り立つ．よって π の普遍性により $\varphi=\tau'$ である． □

補題 24. 図式

$$
\begin{tikzcd}
a \arrow[r, "J", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\beta"] & u \arrow[d, equal] \\
b \arrow[r, "K", "\shortmid"{marking}] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow\eta"] & u \arrow[d, "l"] \\
c \arrow[r, equal] & c
\end{tikzcd}
$$

において β は opcartesian とする．このとき

$$
\begin{aligned}
&\langle l, \eta \rangle \text{ が } K \text{ に沿った } g \text{ の左 Kan 拡張} \\
\iff &\langle l, \eta \circ \beta \rangle \text{ が } J \text{ に沿った } g \circ f \text{ の左 Kan 拡張}
\end{aligned}
$$

証明. ($\Longrightarrow$) 任意の cell

$$
\begin{tikzcd}
a \arrow[r, "J", "\shortmid"{marking}] \arrow[d, "f"'] & u \arrow[dd, "k"] \\
b \arrow[d, "g"'] \arrow[r, phantom, "\Downarrow\theta"] & {} \\
c \arrow[r, equal] & c
\end{tikzcd}
$$

を取る．これを

$$
\begin{tikzcd}
a \arrow[r, "J", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\beta"] & u \arrow[d, equal] \\
b \arrow[r, "K", "\shortmid"{marking}] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow\theta'"] & u \arrow[d, "k"] \\
c \arrow[r, equal] & c
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
a \arrow[r, "J", "\shortmid"{marking}] \arrow[d, "f"'] & u \arrow[dd, "k"] \\
b \arrow[d, "g"'] \arrow[r, phantom, "\Downarrow\theta"] & {} \\
c \arrow[r, equal] & c
\end{tikzcd}
$$

と分解する．$\langle l, \eta \rangle$ が左 Kan 拡張だから，cell τ が一意に存在して，次の等式が成り立つ．

$$
\begin{tikzcd}
a \arrow[rr, "K", "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\rho^{-1}"] & & u \arrow[d, equal] \\
a \arrow[r, "K", "\shortmid"{marking}] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow\eta"] & u \arrow[r, equal] \arrow[d, "l"] \arrow[dr, phantom, "\Downarrow\tau"] & u \arrow[d, "k"] \\
c \arrow[r, equal] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\rho"] & c \arrow[r, equal] & c \arrow[d, equal] \\
c \arrow[rr, equal] & & c
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
b \arrow[r, "K", "\shortmid"{marking}] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow\theta'"] & u \arrow[d, "k"] \\
c \arrow[r, equal] & c
\end{tikzcd}
$$

このとき

$$
\begin{tikzcd}
a \arrow[rr, "J", "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\rho^{-1}"] & & u \arrow[d, equal] \\
a \arrow[r, "J", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\beta"] & u \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mathrm{id}"] & u \arrow[d, equal] \\
b \arrow[r, "K", "\shortmid"{marking}] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow\eta"] & u \arrow[r, equal] \arrow[d, "l"] \arrow[dr, phantom, "\Downarrow\tau"] & u \arrow[d, "k"] \\
c \arrow[r, equal] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\rho"] & c \arrow[r, equal] & c \arrow[d, equal] \\
c \arrow[rr, equal] & & c
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
a \arrow[rr, "J", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[drr, phantom, "\Downarrow\beta"] & & u \arrow[d, equal] \\
a \arrow[rr, "K", "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\rho^{-1}"] & & u \arrow[d, equal] \\
b \arrow[r, "K", "\shortmid"{marking}] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow\eta"] & u \arrow[r, equal] \arrow[d, "l"] \arrow[dr, phantom, "\Downarrow\tau"] & u \arrow[d, "k"] \\
c \arrow[r, equal] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\rho"] & c \arrow[r, equal] & c \arrow[d, equal] \\
c \arrow[rr, equal] & & c
\end{tikzcd}
$$

$$
= \quad
\begin{tikzcd}
a \arrow[r, "J", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\beta"] & u \arrow[d, equal] \\
a \arrow[r, "K", "\shortmid"{marking}] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow\theta'"] & u \arrow[d, "k"] \\
c \arrow[r, equal] & c
\end{tikzcd}
\quad = \quad \theta
$$

となるから $\langle l, \eta \circ \beta \rangle$ は左 Kan 拡張である．

($\Longleftarrow$) 任意の cell

$$
\begin{tikzcd}
a \arrow[r, "K", "\shortmid"{marking}] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow\theta"] & u \arrow[d, "k"] \\
c \arrow[r, equal] & c
\end{tikzcd}
$$

を取る．$\langle l, \eta \circ \beta \rangle$ が左 Kan 拡張だから cell τ が一意に存在して

$$
\begin{tikzcd}
a \arrow[rr, "J", "\shortmid"{marking}] \arrow[d, equal] & \arrow[d, phantom, "\Downarrow\rho^{-1}"] & u \arrow[d, equal] \\
a \arrow[r, "J", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\beta"] & u \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mathrm{id}"] & u \arrow[d, equal] \\
b \arrow[r, "K", "\shortmid"{marking}] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow\eta"] & u \arrow[r, equal] \arrow[d, "l"] \arrow[dr, phantom, "\Downarrow\tau"] & u \arrow[d, "k"] \\
c \arrow[r, equal] \arrow[d, equal] & c \arrow[r, equal] \arrow[d, phantom, "\Downarrow\rho"] & c \arrow[d, equal] \\
c \arrow[rr, equal] & & c
\end{tikzcd}
=
\begin{tikzcd}
a \arrow[r, "J", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\beta"] & u \arrow[d, "k"] \\
b \arrow[r, "K", "\shortmid"{marking}] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow\theta"] & u \arrow[d, "k"] \\
c \arrow[r, equal] & c
\end{tikzcd}
$$

となる．このとき β が opcartesian だから

$$
\begin{tikzcd}
a \arrow[rr, "J", "\shortmid"{marking}] \arrow[d, equal] & \arrow[d, phantom, "\Downarrow\rho^{-1}"] & b \arrow[d, equal] \\
a \arrow[r, "J", "\shortmid"{marking}] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow\eta"] & b \arrow[r, equal] \arrow[d, "l"] \arrow[dr, phantom, "\Downarrow\tau"] & b \arrow[d, "k"] \\
c \arrow[r, equal] \arrow[d, equal] & c \arrow[r, equal] \arrow[d, phantom, "\Downarrow\rho"] & c \arrow[d, equal] \\
c \arrow[rr, equal] & & c
\end{tikzcd}
=
\begin{tikzcd}
a \arrow[r, "J", "\shortmid"{marking}] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow\theta"] & b \arrow[d, "k"] \\
c \arrow[r, equal] & c
\end{tikzcd}
$$

が分かる． □

補題 25. $J\colon a \nrightarrow b$ の opcartesian tabulation

$$
\begin{tikzcd}
\langle J \rangle \arrow[r, equal] \arrow[d, "p_a"'] \arrow[dr, phantom, "\Downarrow\pi"] & \langle J \rangle \arrow[d, "p_b"] \\
a \arrow[r, "J"', "\shortmid"{marking}] & b
\end{tikzcd}
$$

が存在するとして，更に p_a の conjoint と p_b の companion が存在するとする．π を

$$
\begin{tikzcd}
\langle J\rangle \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow_{p_b}\eta"] & \langle J\rangle \arrow[d, "p_b"] \\
\langle J\rangle \arrow[r, "(p_b)_\star", "\shortmid"{marking}] \arrow[d, "p_a"'] \arrow[dr, phantom, "\Downarrow\pi'"] & b \arrow[d, equal] \\
a \arrow[r, "J"', "\shortmid"{marking}] & b
\end{tikzcd}
=
\begin{tikzcd}
\langle J\rangle \arrow[r, equal] \arrow[d, "p_a"'] \arrow[dr, phantom, "\Downarrow\pi"] & \langle J\rangle \arrow[d, "p_b"] \\
a \arrow[r, "J"', "\shortmid"{marking}] & b
\end{tikzcd}
$$

と分解する．このとき

$$
\begin{gathered}
\langle l,\eta\rangle \text{ が } K \text{ に沿った } g \text{ の左 Kan 拡張} \\
\Longleftrightarrow \langle l,\eta\circ\pi'\rangle \text{ が } J \text{ に沿った } g\circ f \text{ の左 Kan 拡張}
\end{gathered}
$$

$$
\begin{tikzcd}
\langle J\rangle \arrow[r, "(p_b)_\star", "\shortmid"{marking}] \arrow[d, "p_a"'] \arrow[dr, phantom, "\Downarrow\pi'"] & b \arrow[d, equal] \\
a \arrow[r, "J", "\shortmid"{marking}] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow\eta"] & b \arrow[d, "l"] \\
c \arrow[r, equal] & c
\end{tikzcd}
$$

証明. π と ${}_{p_b}\eta$ が opcartesian だから π' も opcartesian である．よって補題 24 より成り立つ． □

定理 26. $\mathbb{D}$ を opcartesian tabulation を持つ framed bicategory とする．cell

$$
\begin{tikzcd}
a \arrow[r, equal] \arrow[dd, "g"'] \arrow[ddr, phantom, "\Downarrow\eta"] & a \arrow[d, "f"] \\
 & b \arrow[d, "l"] \\
c \arrow[r, equal] & c
\end{tikzcd}
$$

を

$$
\begin{tikzcd}
a \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow_f\eta"] & a \arrow[d, "f"] \\
a \arrow[r, "f_\star", "\shortmid"{marking}] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow\eta'"] & b \arrow[d, "l"] \\
c \arrow[r, equal] & c
\end{tikzcd}
=
\begin{tikzcd}
a \arrow[r, equal] \arrow[dd, "g"'] \arrow[ddr, phantom, "\Downarrow\eta"] & a \arrow[d, "f"] \\
 & b \arrow[d, "l"] \\
c \arrow[r, equal] & c
\end{tikzcd}
$$

と分解する．このとき

$$\langle l, \eta' \rangle \text{ が } f_* \text{ に沿った } g \text{ の各点左 Kan 拡張である}$$
$$\iff \mathcal{V}(\mathbb{D}) \text{ において } \langle l, \eta \rangle \text{ が } f \text{ に沿った } g \text{ の c-各点左 Kan 拡張である.}$$

証明. 任意の垂直射 $j\colon x \to b$ に対して cartesian cell

$$\begin{tikzcd}
a \arrow[r, "J", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\beta"] & x \arrow[d, "j"] \\
a \arrow[r, "f_\star"', "\shortmid"{marking}] & b
\end{tikzcd}$$

を考える.

$$\gamma \quad := \quad \begin{tikzcd}
a \arrow[r, "J", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\beta"] & x \arrow[d, "j"] \\
a \arrow[r, "f_\star", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow{}_f\varepsilon"] & b \arrow[d, equal] \\
b \arrow[r, equal] & b
\end{tikzcd}$$

と定めると，β も ${}_f\varepsilon$ も cartesian だから γ も cartesian である．故に $J = b(f, j)$ である．また

$$\begin{tikzcd}
a \arrow[rr, "{b(f,j)}", "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\lambda^{-1}"] & & x \arrow[d, equal] \\
a \arrow[r, equal] \arrow[ddd, "g"'] & a \arrow[r, "{b(f,j)}", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\gamma"] \arrow[ldd, phantom, "\Downarrow\eta"] & x \arrow[d, "j"] \\
& b \arrow[r, equal] \arrow[d, "l"] \arrow[dr, phantom, "\Downarrow Ul"] & b \arrow[d, "l"] \\
c \arrow[r, equal] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\lambda"] & c \arrow[r, equal] & c \arrow[d, equal] \\
c \arrow[rr, equal] & & c
\end{tikzcd}
=
\begin{tikzcd}
a \arrow[rr, "{b(f,j)}", "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\lambda^{-1}"] & & x \arrow[d, equal] \\
a \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mathrm{id}"] & a \arrow[r, "{b(f,j)}", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\beta"] & x \arrow[d, "j"] \\
a \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow{}_f\eta"] & a \arrow[r, "f_\star", "\shortmid"{marking}] \arrow[d, "f"] \arrow[dr, phantom, "\Downarrow{}_f\varepsilon"] & b \arrow[d, equal] \\
a \arrow[r, "f_\star", "\shortmid"{marking}] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow\eta'"] & b \arrow[r, equal] \arrow[d, "l"] \arrow[dr, phantom, "\Downarrow Ul"] & b \arrow[d, "l"] \\
c \arrow[r, equal] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\rho"] & c \arrow[r, equal] & c \arrow[d, equal] \\
c \arrow[rr, equal] & & c
\end{tikzcd}
=
\begin{tikzcd}
a \arrow[rr, "{b(f,j)}", "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\beta"] & & x \arrow[d, "j"] \\
a \arrow[rr, "f_\star", "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\lambda^{-1}"] & & b \arrow[d, equal] \\
a \arrow[r, equal] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\iota_{f_\star}"] & a \arrow[r, "f_\star", "\shortmid"{marking}] & b \arrow[d, equal] \\
a \arrow[r, "f_\star", "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\rho"] & b \arrow[r, equal] & b \arrow[d, equal] \\
a \arrow[rr, "f_\star", "\shortmid"{marking}] \arrow[d, "g"'] \arrow[drr, phantom, "\Downarrow\eta'"] & & b \arrow[d, "l"] \\
c \arrow[rr, equal] & & c
\end{tikzcd}$$

$$
= \quad \begin{tikzcd}
a \arrow[r, "b(f,j)", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\beta"] & x \arrow[d, "j"] \\
a \arrow[r, "f_\star", "\shortmid"{marking}] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow\eta'"] & b \arrow[d, "l"] \\
c \arrow[r, equal] & c
\end{tikzcd}
\tag{27}
$$

となる．次に $b(f,j)$ の opcartesian tabulation

$$
\begin{tikzcd}
\langle b(f,j) \rangle \arrow[r, equal] \arrow[d, "p_a"'] \arrow[dr, phantom, "\Downarrow\pi"] & \langle b(f,j) \rangle \arrow[d, "p_x"] \\
a \arrow[r, "b(f,j)"', "\shortmid"{marking}] & x
\end{tikzcd}
$$

を取り

$$
\begin{tikzcd}
\langle b(f,j) \rangle \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow p_x\eta"] & \langle b(f,j) \rangle \arrow[d, "p_x"] \\
\langle b(f,j) \rangle \arrow[r, "(p_x)_\star", "\shortmid"{marking}] \arrow[d, "p_a"'] \arrow[dr, phantom, "\Downarrow\pi'"] & x \arrow[d, equal] \\
a \arrow[r, "b(f,j)"', "\shortmid"{marking}] & x
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\langle b(f,j) \rangle \arrow[r, equal] \arrow[d, "p_a"'] \arrow[dr, phantom, "\Downarrow\pi"] & \langle b(f,j) \rangle \arrow[d, "p_x"] \\
a \arrow[r, "b(f,j)"', "\shortmid"{marking}] & x
\end{tikzcd}
$$

と分解する．これらを使うと

$$
\begin{tikzcd}
a \arrow[r, "f_\star", "\shortmid"{marking}] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow\eta"] & b \arrow[d, "l"] \\
c \arrow[r, equal] & c
\end{tikzcd}
\ \text{が各点左 Kan 拡張である}
$$

$$
\Longleftrightarrow \text{任意の } j \text{ に対して }
\begin{tikzcd}
a \arrow[r, "b(f,j)", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\beta"] & x \arrow[d, "j"] \\
a \arrow[r, "f_\star", "\shortmid"{marking}] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow\eta'"] & b \arrow[d, "l"] \\
c \arrow[r, equal] & c
\end{tikzcd}
\ \text{が左 Kan 拡張である (補題 23)}
$$

$$
\Longleftrightarrow \text{任意の } j \text{ に対して }
\begin{tikzcd}
a \arrow[rr, "{b(f,j)}", "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\lambda^{-1}"] & & x \arrow[d, equal] \\
a \arrow[r, equal] \arrow[dd, "g"'] \arrow[dr, phantom, "\Downarrow\eta"] & a \arrow[r, "{b(f,j)}", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\gamma"] & x \arrow[d, "j"] \\
 & b \arrow[r, equal] \arrow[d, "l"'] \arrow[dr, phantom, "\Downarrow Ul"] & b \arrow[d, "l"] \\
c \arrow[r, equal] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\lambda"] & c \arrow[r, equal] & c \arrow[d, equal] \\
c \arrow[rr, equal] & & c
\end{tikzcd}
\text{ が左 Kan 拡張である (27)}
$$

$$
\Longleftrightarrow \text{任意の } j \text{ に対して }
\begin{tikzcd}
\langle b(f,j)\rangle \arrow[rr, "(p_x)_\star", "\shortmid"{marking}] \arrow[d, "p_a"] \arrow[drr, phantom, "\Downarrow\pi'"] & & x \arrow[d, equal] \\
a \arrow[rr, "{b(f,j)}", "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\lambda^{-1}"] & & x \arrow[d, equal] \\
a \arrow[r, equal] \arrow[dd, "g"'] \arrow[dr, phantom, "\Downarrow\eta"] & a \arrow[r, "{b(f,j)}", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\gamma"] & x \arrow[d, "j"] \\
 & b \arrow[r, equal] \arrow[d, "l"'] \arrow[dr, phantom, "\Downarrow Ul"] & b \arrow[d, "l"] \\
c \arrow[r, equal] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\lambda"] & c \arrow[r, equal] & c \arrow[d, equal] \\
c \arrow[rr, equal] & & c
\end{tikzcd}
\text{ が左 Kan 拡張である (補題 25)}
$$

$\Longleftrightarrow$ $\mathcal{V}(\mathbb{D})$ において $l \circ j$ が p_x に沿った $g \circ p_a$ の左 Kan 拡張である (定理 20)

$\square$

参考文献

[1] M. A. Shulman, Framed Bicategories and Monoidal Fibrations, `https://arxiv.org/abs/0706.1286`

[2] S. R. Koudenburg, On pointwise Kan extensions in double categories, `https://arxiv.org/abs/1402.0250`

[3] P. Schultz, Regular and exact (virtual) double categories, https://arxiv.org/abs/1505.00712