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Periodic solutions in a class of periodic switching delay differential equations

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Abstract

In this paper, we explore the dynamical properties of a class of nonlinear systems governed by delay differential equations with multitime periodic switching. The systems incorporate piecewise-smooth birth and death functions to capture complex population dynamics under seasonal variations. Assuming monotonicity for both birth and death functions, we obtain a novel equivalence result: when the delay is a positive integer multiple of the switching period, the existence and stability of periodic solutions for the systems are equivalent to those in the nondelay case. To illustrate and validate the theoretical findings, a logistic model with seasonal switching is presented. Numerical simulations further confirm that the system exhibits consistent dynamical behaviors across varying delay values.

Mathematics Subject Classification: 34K13; 34K20; 34K45

Keywords: Asymptotic stability; Delay differential equation; Periodic solutions; Time switching

1 Introduction

Nonlinear differential equations are widely used in mathematical biology as powerful tools for simulating complex biological processes. They effectively model the dynamical behaviors in population dynamics, disease spread, and ecosystem interactions. The application of such equations in biological mathematical modeling is well documented in the literature [1, 2]. Time delays, which are commonly observed in the life cycles of organisms, have motivated extensive studies on delay differential equations, driven by various biological phenomena. For example, in mosquito population growth models, delays account for the time required for newborn individuals to mature and reproduce [3, 4]. In epidemic models, delays are crucial for incorporating latent periods or delayed immune responses [5]. Moreover, in predator–prey dynamics, delays usually refer to the time for predators to search or digest prey [6]. Introducing time delay into biological mathematical models can result in rich and complex behaviors, such as oscillations, bifurcations, resonance, and chaotic dynamics [7–11]. These phenomena highlight the significant role of time delay in mathematical models, enabling them to capture real-world biological processes.

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In population dynamics, particularly when focusing on a single species, various specific models [10, 12–19] can be described by the following form of delay differential equation:

$$\dot{x}(t) = F(x(t-\tau)) + G(x(t)), \tag{1}$$

where $x(t)$ is the population density of a species at time t, $F(x)$ denotes the birth function of the species incorporating a maturation delay $\tau > 0$, and $G(x)$ is the death function of the species that only depends on the current population size. The dynamical behaviors of (1) with continuous versions of F and G have been extensively studied, including the monostable nonlinearity [12, 14, 15, 20] and bistability [15, 21–23]. Recently, there has been a growing interest in time-switching models, which explain periodic variations in population ecological environments, cyclic fishing practices, and certain human periodic behaviors. These models offer a framework to understand the periodic environment factors and periodic human interventions on species. A significant study on periodic succession in mosquito suppression model was conducted by Yu and Li in [24], where the authors assumed that the sterile mosquitoes are released impulsively and periodically at discrete time points $T_k = kT, k = 0, 1, 2, \ldots$, so that the suppression model finally consists of two subequations constantly switching between each other. They obtained sufficient and necessary conditions for the existence and uniqueness of a globally asymptotically stable T-periodic solution. For further studies and results on mosquito suppression models with time switching, see [25–28].

Motivated by the above works, we investigate the properties of equation (1) with multitime period switching. Specifically, we assume that the birth and the death functions satisfy the following conditions:

(A_1) Both F and G are piecewise-smooth T-periodic functions, expressed as

$$F(x(t-\tau)) = \begin{cases} f_1(x(t-\tau)), & t \in [iT, iT + t_1), \\ f_2(x(t-\tau)), & t \in [iT + t_1, iT + t_2), \\ \vdots & \\ f_n(x(t-\tau)), & t \in [iT + t_{n-1}, (i+1)T), \end{cases} \tag{2}$$

and

$$G(x(t)) = \begin{cases} g_1(x(t)), & t \in [iT, iT + t_1), \\ g_2(x(t)), & t \in [iT + t_1, iT + t_2), \\ \vdots & \\ g_n(x(t)), & t \in [iT + t_{n-1}, (i+1)T), \end{cases} \tag{3}$$

where $0 < t_1 < t_2 < \cdots < t_{n-1} < T$ and $i = 0, 1, 2, \ldots$

(A_2) $f_j(x)$ is strictly increasing and $g_j(x)$ is strictly decreasing for $j = 1, 2, \ldots, n$.

The assumption (A_2) may seem stringent, but it is commonly met in population models in various ecological scenarios. For instance, in [18], the author provided the birth and death functions of wild mosquitoes as

$$f(A(t-\tau)) = \frac{bA^2(t-\tau)}{2(A(t-\tau) + 2R(t-\tau))} \quad \text{and} \quad g(A(t)) = -m\left(1 + \frac{A(t)}{K}\right)A(t).$$

Another compelling example modeling a crocodilian population is found in [29], where the birth function

$$f(F_1(t-\tau)) = sbF_1(t-\tau)/(k_1 + F_1(t-\tau))$$

illustrates the increase in the number of adult female crocodiles and $g(F_1(t)) = -d_a F_1(t)$ denotes the death of adult female crocodiles. It is easy to verify that these functions satisfy our assumption (A_2), and the examples underscore the versatility of our model and validate its applicability to different ecological scenarios. Since both τ and T are positive, there exists a unique nonnegative integer p and a nonnegative number $q \in [0, T)$ such that $\tau = pT + q$, where $p = [\tau/T]$, $q \in [0, T)$. In this work, we focus on a special case, namely

(A_3) $\tau = pT$, $p = 1, 2, 3 \ldots$

Our objective is to demonstrate that under assumptions (A_1)–(A_3), equation (1) exhibits similar dynamical behaviors to those of an ordinary differential equation

$$\dot{x}(t) = F(x(t)) + G(x(t)). \tag{4}$$

Given an initial condition $\phi(t) \in C([\sigma - \tau, \sigma], \mathbb{R})$, $\sigma \in R$, $x(t) = x(t; \sigma, \phi)$ is said to be a solution of (1) if it satisfies (1) on $[\sigma, \infty)$ and $x(t) = \phi(t)$ on $t \in [\sigma - \tau, \sigma]$. Without loss of generality, we assume $\sigma \in [\tau, (p+1)T)$. Incorporating the notion of "good" solutions in [27, 28, 30], we introduce a special class of solutions of (1) whose initial functions are derived from solutions of (4). Specifically, a solution $x_u(t) = x(t; \sigma, \phi_u)$ of (1) is called a "good" solution if $\phi_u(t) = x(t; 0, u)$, $t \in [\sigma - \tau, \sigma]$, and $x(t; 0, u)$ is the solution to (4) which passes through the point $(0, u)$.

The remainder of this paper is organized as follows. In Sect. 2, we provide some basic lemmas which are used to prove our main results. In Sect. 3, we state and prove the main results in two steps, using the method of the upper and lower solutions and monotonic dynamical system theory. We then consider a logistic model with seasonal switching in Sect. 4 to validate our main results. Finally, we conclude with a brief discussion in Sect. 5.

2 Preliminaries

Definition 2.1

(1) A solution $\bar{x}(t)$ of (1) is said to be stable if for any $\sigma \in [\tau, (p+1)T)$ and $\varepsilon > 0$, there is $\delta(\varepsilon, \sigma) > 0$ such that $\phi \in C([\sigma - \tau, \sigma], \mathbb{R})$ and $|\phi(t) - \bar{x}(t)| < \delta$ for $t \in [\sigma - \tau, \sigma]$ imply

$$|x(t; \sigma, \phi) - \bar{x}(t)| < \varepsilon, \quad \text{for} \quad t \geq \sigma.$$

(2) A solution $\bar{x}(t)$ is said to be asymptotically stable if it is stable and there exists $\eta(\sigma) > 0$ such that whenever $\phi \in C([\sigma - \tau, \sigma], \mathbb{R})$ and $|\phi(t) - \bar{x}(t)| < \eta(\sigma)$ for $t \in [\sigma - \tau, \sigma]$, then

$$\lim_{t \to \infty} (x(t; \sigma, \phi) - \bar{x}(t)) = 0.$$

We next give several lemmas. The first shows that the solutions $\phi_u(t)$ have diverse periodic features.

Lemma 2.1 *For any given initial value $u > 0$ and $t \geq 0$, the following conclusions hold*:
(1) *If $\phi_u(T) > u$, then $\phi_u(t + T) > \phi_u(t)$, and the sequence $\{\phi_u(t + nT)\}$ is strictly increasing.*
(2) *If $\phi_u(T) = u$, then $\phi_u(nT) \equiv u$, $n = 0, 1, 2, \ldots$ Moreover, $\phi_u(t)$ is a T-periodic solution of equation (4).*
(3) *If $\phi_u(T) < u$, then $\phi_u(t + T) < \phi_u(t)$, and the sequence $\{\phi_u(t + nT)\}$ is strictly decreasing.*

Proof We only prove claim (1), as similar arguments can be applied to prove the other claims. Write

$$\tilde{x}(t) = \phi_u(t + T) = x(t + T; 0, u) = x(t; 0, \phi_u(T)).$$

Clearly, $\tilde{x}(t)$ is a solution of (4). It then follows from $\tilde{x}(0) = \phi_u(T) > u = \phi_u(0)$ that $\phi_u(t + T) > \phi_u(t)$. The proof is complete. □

In terms of Lemma 2.1, we know that if $\phi_u(T) > u$, then for any $\sigma \in [\tau, (p+1)T)$,

$$x_u(\sigma) = \phi_u(\sigma) > \phi_u(\sigma - T) > \phi_u(\sigma - 2T) > \cdots > \phi_u(\sigma - \tau) = x_u(\sigma - \tau).$$

Further, we have the following lemma about the monotonic and periodic dynamics of solution $x_u(t)$.

Lemma 2.2 *Suppose (A_1)–(A_3) hold. For any given initial value $u > 0$ and $t \geq \sigma$, the following conclusions hold*:
(1) *If $\phi_u(T) > u$, then $x_u(t) > x_u(t - \tau)$, and the sequence $\{x_u(t + n\tau)\}$ is strictly increasing.*
(2) *If $\phi_u(T) = u$, then $x_u(n\tau) \equiv u$.*
(3) *If $\phi_u(T) < u$, then $x_u(t) < x_u(t - \tau)$, and the sequence $\{x_u(t + n\tau)\}$ is strictly decreasing.*

Proof (1) Assume by contradiction that there is $\bar{t} > \sigma$ such that

$$\begin{cases} x_u(t) > x_u(t - \tau), \ \forall t \in [\sigma, \bar{t}), \\ x_u(\bar{t}) = x_u(\bar{t} - \tau), \\ x_u(t) < x_u(t - \tau), \ \forall \bar{t} < t < \bar{t} + \delta, \end{cases} \tag{5}$$

where $\delta > 0$ is sufficiently small. We can choose $\hat{t} \in (\bar{t}, \bar{t} + \delta)$ such that $\dot{x}_u(\hat{t}) \leq \dot{x}_u(\hat{t} - \tau)$. There are two situations, $\bar{t} \in (\sigma, \sigma + \tau)$ or $\bar{t} \geq \sigma + \tau$.

If $\bar{t} \in (\sigma, \sigma + \tau)$, then from (1) and (4) we have

$$\dot{x}_u(\hat{t}) = F(x_u(\hat{t} - \tau)) + G(x_u(\hat{t})) \text{ and } \dot{x}_u(\hat{t} - \tau) = F(x_u(\hat{t} - \tau)) + G(x_u(\hat{t} - \tau)).$$

There is a positive integer j such that $G(x_u(\hat{t})) = g_j(x_u(\hat{t}))$ and $G(x_u(\hat{t} - \tau)) = g_j(x_u(\hat{t} - \tau))$. Since g_j is strictly decreasing and $x_u(\hat{t}) < x_u(\hat{t} - \tau)$, we have $G(x_u(\hat{t})) > G(x_u(\hat{t} - \tau))$. Thus, we obtain $\dot{x}_u(\hat{t}) > \dot{x}_u(\hat{t} - \tau)$, a contradiction.

If $\bar{t} \geq \sigma + \tau$, then from (1) we obtain

$$\dot{x}_u(\hat{t}) = F(x_u(\hat{t} - \tau)) + G(x_u(\hat{t})) \text{ and } \dot{x}_u(\hat{t} - \tau) = F(x_u(\hat{t} - 2\tau)) + G(x_u(\hat{t} - \tau)).$$

There is a positive integer l such that

$$F(x_u(\hat{t}-\tau)) = f_l(x_u(\hat{t}-\tau)),\ G(x_u(\hat{t})) = g_l(x_u(\hat{t})),$$
$$F(x_u(\hat{t}-2\tau)) = f_l(x_u(\hat{t}-2\tau)),\ G(x_u(\hat{t}-\tau)) = g_l(x_u(\hat{t}-\tau)).$$

The monotonicity of f_l and g_l, together with $x_u(\hat{t}-\tau) > x_u(\hat{t})$ and $x_u(\hat{t}-\tau) > x_u(\hat{t}-2\tau)$, implies

$$F(x_u(\hat{t}-\tau)) > F(x_u(\hat{t}-2\tau)) \text{ and } G(x_u(\hat{t})) > G(x_u(\hat{t}-\tau)).$$

It indicates $\dot{x}_u(\hat{t}) > \dot{x}_u(\hat{t}-\tau)$, a contradiction again. Thus, we have established $x_u(t) > x_u(t-\tau)$. Following a similar procedure, it is easy to verify that $\{x_u\,(t+n\tau)\}$ is strictly increasing. Furthermore, analogous results hold if the inequalities are reversed or replaced with equalities. This completes the proof. □

We next give a comparison principle.

Lemma 2.3 *For any $\sigma \in \mathbb{R}$ and two given initial functions $\phi_i \in C([\sigma-\tau,\sigma],\mathbb{R})$, $i = 1,2$, if $\phi_1(t) \le \phi_2(t)$ for $t \in [\sigma-\tau,\sigma]$, then the solutions of (1) satisfy $x(t;\sigma,\phi_1) \le x(t;\sigma,\phi_2)$ for $t \ge \sigma$.*

Proof Let $x_1(t) = x(t;\sigma,\phi_1)$ and $x_2(t) = x(t;\sigma,\phi_2)$. Suppose the contrary. Then there is $\bar{s} \ge \sigma$ such that

$$x_1(t) \le x_2(t) \ \text{ for } \ t \in [\sigma-\tau,\bar{s}],\ \text{ and } \ x_1(t) > x_2(t) \ \text{ for } \ t \in (\bar{s},\bar{s}+\hat{\delta}), \tag{6}$$

where $\hat{\delta} > 0$ is sufficiently small. We can choose $s_1 \in (\bar{s},\bar{s}+\hat{\delta})$ such that $\dot{x}_1(s_1) > \dot{x}_2(s_1)$. There must be an integer k such that

$$\dot{x}_1(s_1) = f_k(x_1(s_1-\tau)) + g_k(x_1(s_1)) \text{ and } \dot{x}_2(s_1) = f_k(x_2(s_1-\tau)) + g_k(x_2(s_1)).$$

By the monotonicity of f_k and g_k, we arrive at $x_1(s_1-\tau) > x_2(s_1-\tau)$, which leads to a contradiction to (6). □

Based on Lemma 2.3, it is easy to derive the following lemma, which shows that there exist two solutions of (1), bounding $x(t;\sigma,\phi)$ below and above.

Lemma 2.4 *Assume that $\sigma \in \mathbb{R}$ and $\phi \in C([\sigma-\tau,\sigma],\mathbb{R})$. Let $m_1 < m_2$ be two positive constants such that*

$$m_1 \le \phi(t) \le m_2,\ \textit{for}\ \ t \in [\sigma-\tau,\sigma].$$

Then there must exist $u_1 \le m_1$ and $u_2 \ge m_2$ such that

$$x_{u_1}(t;\sigma) \le x(t;\sigma,\phi) \le x_{u_2}(t;\sigma),\ \textit{for}\ \ t \ge \sigma. \tag{7}$$

3 Main results

We concisely summarize the main results in the following theorem.

Theorem 3.1 *The solution $x_u(t) = x(t;\sigma,\phi_u)$ of (1) is a T-periodic solution if and only if $\phi_u(t)$ is a T-periodic solution of (4). Moreover, $x_u(t)$ is asymptotically stable if and only if $\phi_u(t)$ is asymptotically stable.*

Proof We divide the proof into the following two steps:

Step 1: $x_u(t) = x(t;\sigma,\phi_u)$ is a T-periodic solution if and only if $\phi_u(t)$ is a T-periodic solution.

We start by proving the necessity. It suffices to show that $x_u(t)$ is T-periodic if $\phi_u(T) = u$. Since equation (1) is autonomous, for any $t > \sigma$, we have $x_u(t) = x(t;\sigma,\phi_u) = x(t + T;\sigma + T,\phi_u)$. By the T-periodicity of $\phi_u(t)$, we have $x(t+T;\sigma+T,\phi_u) = x(t+T;\sigma,\phi_u) = x_u(t+T)$, and hence $x_u(t) = x_u(t+T)$, which implies that $x_u(t)$ is a T-periodic solution of (4). On the other hand, if $x_u(t)$ is a T-periodic solution, then

$$\dot{x}_u(t) = f_j(x(t-\tau)) + g_j(x(t)) = f_j(x(t-pT)) + g_j(x(t)) = f_j(x(t)) + g_j(x(t)).$$

Hence, $\phi_u(t)$ is a T-periodic solution of equation (4).

Step 2: The T-periodic solution $x_u(t) = x(t;\sigma,\phi_u)$ is asymptotically stable for (1) if and only if $\phi_u(t)$ is asymptotically stable for (4).

Assume that $\phi_u(t)$ is asymptotically stable for (4). We prove that $x_u(t)$ is asymptotically stable for (1). We first show the stability of $x_u(t)$. For any $\sigma \in [\tau,(p+1)T)$ and $\varepsilon > 0$, it suffices to show that there is $\delta = \delta(\varepsilon,\sigma) > 0$ such that $\phi \in C([\sigma-\tau,\sigma],\mathbb{R})$ and $|\phi(t)-x_u(t)| < \delta$ for $t \in [\sigma-\tau,\sigma]$, imply

$$|x(t;\sigma,\phi) - x_u(t)| < \varepsilon, \quad \text{for} \quad t \geq \sigma. \tag{8}$$

Since $\phi_u(t)$ is stable for (4), there must exist $\tilde{\delta} > 0$ such that $|u - v| < \tilde{\delta}$ implies

$$|\phi_u(t) - \phi_v(t)| < \frac{1}{2}\varepsilon \quad \text{for} \quad t \geq \sigma. \tag{9}$$

Let $\delta_1 = \min\limits_{t\in[\sigma-\tau,\sigma]} \left(\phi_{u+\frac{1}{2}\tilde{\delta}}(t) - \phi_u(t)\right)$ and $\delta_2 = \min\limits_{t\in[\sigma-\tau,\sigma]} \left(\phi_u(t) - \phi_{u-\frac{1}{2}\tilde{\delta}}(t)\right)$. We then choose $\delta = \min\{\delta_1,\delta_2\}$, and prove that (8) is true. It follows from $|\phi(t)-x_u(t)| < \delta$ for $t \in [\sigma-\tau,\sigma]$ that

$$\phi(t) > x_u(t) - \delta \geq x_u(t) - \delta_2 \geq \phi_{u-\frac{1}{2}\tilde{\delta}}(t), \text{ and } \phi(t) < x_u(t) + \delta \leq x_u(t) + \delta_1 \leq \phi_{u+\frac{1}{2}\tilde{\delta}}(t),$$

which, by Lemma 2.3, leads to $x_{u-\frac{1}{2}\tilde{\delta}}(t) < x(t;\sigma,\phi) < x_{u+\frac{1}{2}\tilde{\delta}}(t)$ for $t \geq \sigma$. On the other hand, we have $x_{u-\frac{1}{2}\tilde{\delta}}(t) < x_u(t) < x_{u+\frac{1}{2}\tilde{\delta}}(t)$, for $t \geq \sigma$. It follows that

$$|x(t;\sigma,\phi) - x_u(t)| < x_{u+\frac{1}{2}\tilde{\delta}}(t) - x_{u-\frac{1}{2}\tilde{\delta}}(t), \quad \text{for} \quad t \geq \sigma.$$

To verify (8), we only need to show

$$x_{u+\frac{1}{2}\tilde{\delta}}(t) - x_{u-\frac{1}{2}\tilde{\delta}}(t) < \varepsilon, \quad \text{for} \quad t \geq \sigma.$$

Since $\phi_u(t)$ is asymptotically stable for (4),

$$\phi_{u+\frac{1}{2}\tilde{\delta}}(T) < u + \frac{1}{2}\tilde{\delta} \text{ and } \phi_{u-\frac{1}{2}\tilde{\delta}}(T) > u - \frac{1}{2}\tilde{\delta}.$$

Recruiting Lemma 2.2 for $t \in [\sigma, \sigma + \tau]$ gives

$$x_{u+\frac{1}{2}\tilde{\delta}}(t) \le x_{u+\frac{1}{2}\tilde{\delta}}(t-\tau) = \phi_{u+\frac{1}{2}\tilde{\delta}}(t-\tau) \text{ and } x_{u-\frac{1}{2}\tilde{\delta}}(t) \ge x_{u-\frac{1}{2}\tilde{\delta}}(t-\tau) = \phi_{u-\frac{1}{2}\tilde{\delta}}(t-\tau).$$

It then follows from (9) that

$$x_{u+\frac{1}{2}\tilde{\delta}}(t) - x_u(t) \le \phi_{u+\frac{1}{2}\tilde{\delta}}(t-\tau) - x_u(t-\tau) < \frac{1}{2}\varepsilon,$$
$$x_u(t) - x_{u-\frac{1}{2}\tilde{\delta}}(t;\sigma) \le x_u(t-\tau) - \phi_{u+\frac{1}{2}\tilde{\delta}}(t-\tau) < \frac{1}{2}\varepsilon.$$

Thus, $x_{u+\frac{1}{2}\tilde{\delta}}(t) - x_{u-\frac{1}{2}\tilde{\delta}}(t) < \varepsilon$, and hence (8) is true for $t \in [\sigma, \sigma + \tau]$. By induction, we can achieve (8) for $t \in [\sigma + k\tau, \sigma + (k+1)\tau]$, $k = 1, 2, \dots$ Therefore, (8) is true and the T-periodic solution $x_u(t)$ of (1) is stable.

We next prove the attractivity of $x_u(t)$ for (1). It suffices to show that for any $\phi \in C([\sigma - \tau, \sigma], \mathbb{R})$ and $|\phi(t) - x_u(t)| < \delta$ for $t \in [\sigma - \tau, \sigma]$,

$$\lim_{t\to\infty} |x(t;\sigma,\phi) - x_u(t)| = 0. \tag{10}$$

Employing Lemma 2.4, there are u_1, u_2 and $u \in (u_1, u_2)$ such that $x_{u_1}(t) \le x(t;\sigma,\phi) \le x_{u_2}(t)$. Thus, to show (10), we only need to prove

$$\lim_{t\to\infty} |x_{u_1}(t) - x_u(t)| = 0 \text{ and } \lim_{t\to\infty} |x_{u_2}(t) - x_u(t)| = 0. \tag{11}$$

We can guarantee that both u_1 and u_2 are within the attraction domain of $\phi_u(t)$. It follows that $\phi_{u_1}(T) > u_1$ and $\phi_{u_2}(T) < u_2$, which implies that $\{x_{u_1}(t + n\tau)\}$ is strictly increasing and $\{x_{u_2}(t + n\tau)\}$ is strictly decreasing. Note that $x_u(t)$ is bounded. Set

$$w_{u_1}(t) = \lim_{n\to\infty} x_{u_1}(t + n\tau) \text{ and } w_{u_2}(t) = \lim_{n\to\infty} x_{u_2}(t + n\tau). \tag{12}$$

Clearly, both $w_{u_1}(t)$ and $w_{u_2}(t)$ are τ-periodic functions.

We now prove that $w_{u_1}(t)$ and $w_{u_2}(t)$ are continuous and piecewise differentiable on $[\sigma, \infty)$ and we only verify this for $w_{u_1}(t)$ due to similarity. In fact, for any t', $t'' > \sigma$, (12) shows that for any $\varepsilon > 0$, there exist $N_1 = N_1(t')$ and $N_2 = N_2(t'')$ such that

$$|w_{u_1}(t') - x_{u_1}(t' + n\tau)| < \frac{\varepsilon}{3} \text{ and } |w_{u_2}(t'') - x_{u_2}(t'' + n\tau)| < \frac{\varepsilon}{3} \text{ if } n > N_1 + N_2.$$

It is obvious that $x_{u_1}(t + n\tau)$ is uniformly continuous on $t \in [\sigma, \infty)$. Then there is $\delta = \delta(\varepsilon) > 0$ such that if $|t' - t''| < \delta$, then $|x_{u_1}(t' + n\tau) - x_{u_1}(t'' + n\tau)| < \varepsilon/3$. Hence, when $|t' - t''| < \delta$, we have

$$\begin{aligned} |w_{u_1}(t') - w_{u_2}(t'')| &< |x_{u_1}(t' + n\tau) - x_{u_1}(t'' + n\tau)| \\ &< |w_{u_1}(t') - x_{u_1}(t' + n\tau)| + |w_{u_2}(t'') - x_{u_2}(t'' + n\tau)| \end{aligned}$$

$$+ |x_{u_1}(t' + n\tau) - x_{u_1}(t'' + n\tau)| < \varepsilon,$$

which implies that $w_{u_1}(t)$ is continuous on $[\sigma, \infty)$. Moreover, for any nonnegative integer k and $t' \in \bigcup_{j=0}^{n-2}(kT + t_j, kT + t_{j+1}) \cup (kT + t_{n-1}, (k+1)T)$, we have

$$\begin{aligned}
\lim_{t\to t'} \frac{w_{u_1}(t) - w_{u_1}(t')}{t - t'} &= \lim_{t\to t'} \lim_{n\to\infty} \frac{x_{u_1}(t + n\tau) - x_{u_1}(t' + n\tau)}{t - t'} \\
&= \lim_{n\to\infty} \lim_{t\to t'} \frac{x_{u_1}(t + n\tau) - x_{u_1}(t' + n\tau)}{t - t'} \\
&= \lim_{n\to\infty} \dot{x}_{u_1}(t' + n\tau) = F(w_{u_1}(t')) + G(w_{u_1}(t')).
\end{aligned}$$

It indicates that $w_{u_1}(t)$ is differentiable at the point t'. Similarly, we can show that $w_{u_1}(t)$ is left differentiable at the points $t = kT$ and $t = kT + t_{j+1}$. Thus, $w_{u_1}(t)$ is piecewise differentiable on $t \in [\sigma, \infty)$. We can conclude that $w_{u_1}(t)$ is a τ-periodic solution to (4). Therefore, $w_{u_1}(t) = w_{u_2}(t) = x_u(t)$, which proves (11), and thus the attractivity of $x_u(t)$ for (1).

Finally, we prove the sufficiency by contradiction. We claim that there exists $\eta \in (0, \delta)$ such that

$$\phi_\nu(T) < \nu, \text{ for } \nu \in (u, u + \eta), \text{ and } \phi_\nu(T) > \nu, \text{ for } \nu \in (u - \eta, u). \tag{13}$$

In fact, if not, then for any $\eta > 0$, we can find $\nu' \in (u, u + \eta)$ such that $\phi_\nu(T) \geq \nu$, or $\nu' \in (u - \eta, u)$ such that $\phi_\nu(T) \leq \nu$. For $\nu' \in (u, u + \eta)$, if $\phi_\nu(T) = \nu$, then $x_\nu(t)$ is another T-periodic solution, a contradiction to the asymptotic stability of $x_u(t)$. If $\phi_\nu(T) > \nu$, then Lemma 2.2 tells that $\{x_\nu(n\tau)\}$ is strictly increasing, which contradicts the convergence of $\{x_\nu(n\tau)\}$. Similarly, for $u \in (u - \eta, u)$, we can arrive at the same contradiction. Hence, (13) is true. This completes the proof. □

We provide Fig. 1 to illustrate the stability of $x_u(t)$ in Theorem 3.1.

Corollary 3.1 *An equilibrium solution of* (1) *is asymptotically stable if and only if the corresponding equilibrium solution of* (4) *is asymptotically stable.*

4 Logistic model with seasonal switching

In this section, we provide an example of logistic model with seasonal switching to verify our theoretical results. The continuous version of the logistic model is given by the following equation

$$\frac{dx}{dt} = rx\left(1 - \frac{x}{K}\right),$$

where x represents the number of a population, r is the rate of maximum population growth, and K is carrying capacity of the population. To better capture the influence of environmental variations on population dynamics, we divide each environmental cycle into four seasons: $\mathcal{S}_1$, $\mathcal{S}_2$, $\mathcal{S}_3$, and $\mathcal{S}_4$. Assume that, in each environmental cycle, the season $\mathcal{S}_1$ starts at $t = 0$ and ends at $t = t_1$; the season $\mathcal{S}_2$ starts at $t = t_1$ and ends at $t = t_2$; the season $\mathcal{S}_3$ starts at $t = t_2$ and ends at $t = t_3$, and the season $\mathcal{S}_4$ starts at $t = t_3$ and lasts until $t = T$.

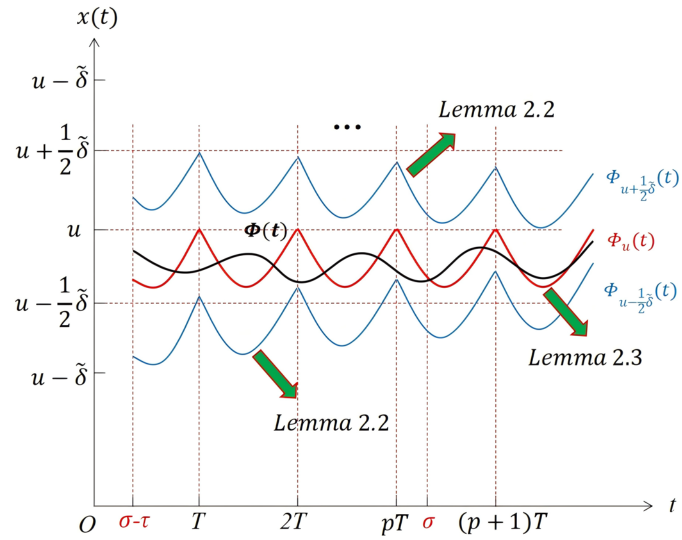

Figure 1 The main idea for proving the stability of $x_u(t)$ in Theorem 3.1. The two blue curves represent the solutions $x_{u+\frac{1}{2}\tilde{\delta}}(t)$ and $x_{u-\frac{1}{2}\tilde{\delta}}(t)$. The red and black curves represent the solutions $x_u(t)$ and $x(t;\sigma,\phi)$, respectively. The monotonic dynamics exhibited by Lemma 2.2 guarantees that the distance between $x_{u+\frac{1}{2}\tilde{\delta}}(t)$ and $x_{u-\frac{1}{2}\tilde{\delta}}(t)$ tends to 0. It forces the distance between $x_u(t)$ and $x(t;\sigma,\phi)$ tend to 0

Let r_i and K_i be the maximum growth rates and the carrying capacity of the population during $\mathcal{S}_i$, $i=1,2,3,4$, respectively. We assume that resources are abundant and gradually increase with seasonal changes throughout the year. According to Malthusian population theory [31], we know that resource-rich environments typically support higher reproduction rates and larger population sizes. Thus, we further assume that $r_i > r_j$ and $K_i > K_j$ for $i > j$. Then we get the following time switching differential equations:

$$\frac{dx}{dt} = r_1 x\left(1-\frac{x}{K_1}\right) = f_1(x),\quad t\in[mT, mT+t_1), \tag{14}$$

$$\frac{dx}{dt} = r_2 x\left(1-\frac{x}{K_2}\right) = f_2(x),\quad t\in[mT+t_1, mT+t_2), \tag{15}$$

$$\frac{dx}{dt} = r_3 x\left(1-\frac{x}{K_3}\right) = f_3(x),\quad t\in[mT+t_2, mT+t_3), \tag{16}$$

$$\frac{dx}{dt} = r_4 x\left(1-\frac{x}{K_4}\right) = f_4(x),\quad t\in[mT+t_3, (m+1)T), \tag{17}$$

where $m=0,1,2,\ldots$

The solution $x(t)$ for different time intervals is defined as follows:

- For $t\in[0,t_1)$, $x(t)$ satisfies (14), with the initial value $x(0)=u$.
- For $t\in[t_1,t_2)$, $x(t)$ satisfies (15), with the initial value $x(t_1)=\lim_{t\to t_1^-} x(t)$.
- For $t\in[t_2,t_3)$, $x(t)$ satisfies (16), with the initial value $x(t_2)=\lim_{t\to t_2^-} x(t)$.
- For $t\in[t_3,T)$, $x(t)$ satisfies (17), with the initial value $x(t_3)=\lim_{t\to t_3^-} x(t)$.

On other intervals, the solution can be defined in the same way. To describe the periodic dynamics of solution $x(t)$, we define

$$h(u) := x(T; 0, u) \text{ and } h_i(u) := x(t_i; 0, u) \text{ for } i = 1, 2, 3,$$

where h is the Poincaré map of the system (14)–(17). It is easy to see that the solution $x(t; 0, u)$ of the system (14)–(17) is T-periodic if and only if u is a fixed point of h. We use similar arguments to those in [24–26, 32, 33] to discuss the fixed points of h.

The next lemma shows the nonexistence of the fixed points.

Lemma 4.1 *If $u \in (0, K_1]$, then $h(u) > u$. If $u \in [K_4, \infty)$, then $h(u) < u$.*

Proof When $u \in (0, K_1]$, a solution $x(t; 0, u)$ of the system (14)–(17) is determined by (14) for $t \in [0, t_1)$ and satisfies $dx/dt \geq 0$, which leads to $u \leq h_1(u)$. Then initiated from $h_1(u)$, a solution $x(t, t_1, h_1(u))$ is driven by (15) for $t \in [t_1, t_2)$ and satisfies $dx/dt > 0$, which yields $h_1(u) < h_2(u)$. Similarly, we can achieve $h_2(u) < h_3(u) < h(u)$. Hence, we have $h(u) > u$ for $u \in (0, K_1]$. When $u \in [K_4, \infty)$, we have $dx/dt < 0$ for $t \in [0, t_1)$, which leads to $u > h_1(u)$. If $h_1(u) > K_2$, then $u > h_1(u) > h_2(u)$. If $h_1(u) \leq K_2$, then $h_2(u) \leq K_2 < u$. Both cases illustrate that $u > h_2(u)$. Following a similar procedure, we can finally arrive at $u > h(u)$. The proof is complete. □

Lemma 4.1 implies that the system (14)–(17) has no T-periodic solutions in $(0, K_1] \cup [K_4, \infty)$ and has at least one T-periodic solution in (K_1, K_4). Furthermore, we have

Theorem 4.1 *The system (14)–(17) has a unique T-periodic solution which is globally asymptotically stable.*

Proof Let u^* be a fixed point of h in (K_1, K_2). By the approach in [32], it suffices to show the uniqueness of u^* and $(u - u^*)(h(u) - u) < 0$ for all $u > 0$. In fact, we can calculate that

$$h'(u) = \frac{B_1(h_1(u))}{B_1(h_2(u))} \frac{B_2(h_2(u))}{B_2(h_3(u))} \frac{B_3(h_3(u))}{B_3(h(u))} \quad \text{at } u = u^*,$$

where

$$B_1(u) = \frac{f_1(u)}{f_2(u)}, \quad B_2(u) = \frac{f_1(u)}{f_3(u)}, \quad \text{and} \quad B_3(u) = \frac{f_1(u)}{f_4(u)}. \tag{18}$$

Direct computation yields

$$B_1'(u) = \frac{r_1 K_2 (K_1 - K_2)}{r_2 K_1 (K_2 - u)^2}, \; B_2'(u) = \frac{r_1 K_3 (K_1 - K_3)}{r_3 K_1 (K_3 - u)^2}, \; \text{and } B_3'(u) = \frac{r_1 K_4 (K_1 - K_4)}{r_4 K_1 (K_4 - u)^2},$$

which indicate that $B_1(u)$ is a strictly decreasing function in the intervals (K_1, K_2), (K_2, K_4), $B_2(u)$ is a strictly decreasing function in the intervals (K_1, K_3), (K_3, K_4), and $B_3(u)$ is a strictly decreasing function in the interval (K_1, K_4).

We now prove that h is a contraction mapping. It suffices to show that $h'(u) < 1$ for $u \in (K_1, K_4)$. Due to similarity, we only focus on the interval $u \in (K_1, K_2)$, in which we have

$$h_2(u) > h_1(u), \quad h_3(u) > h_2(u), \quad \text{and} \quad h(u) > h_3(u).$$

By the monotonicity of $B_1(u)$, $B_2(u)$, and $B_3(u)$, we then obtain

$$B_1(h_2(u)) < B_1(h_1(u)), \quad B_2(h_3(u)) < B_2(h_2(u)), \quad \text{and} \quad B_3(h(u)) < B_3(h_3(u)).$$

Since $K_1 < h_1(u) < h_2(u) < K_2$, we have $B_1(h_2(u)) < B_1(h_1(u)) < 0$ and $B_2(h_3(u)) < B_2(h_2(u)) < 0$. Besides, from $K_1 < h_3(u) < h(u) < K_4$, we obtain $B_3(h(u)) < B_3(h_3(u)) < 0$. Hence, $h'(u) < 1$ at each $u \in (K_1, K_2)$ satisfying $h(u) = u$, which suffices to show the uniqueness of the fixed points in (K_1, K_2). The proof is complete. □

We consider the following time delay system:

$$\frac{dx}{dt} = r_1 x(t - pT) - \frac{r_1 x^2(t)}{K_1}, \quad t \in [mT, mT + t_1), \tag{19}$$

$$\frac{dx}{dt} = r_2 x(t - pT) - \frac{r_2 x^2(t)}{K_2}, \quad t \in [mT + t_1, mT + t_2), \tag{20}$$

$$\frac{dx}{dt} = r_3 x(t - pT) - \frac{r_3 x^2(t)}{K_3}, \quad t \in [mT + t_2, mT + t_3), \tag{21}$$

$$\frac{dx}{dt} = r_4 x(t - pT) - \frac{r_4 x^2(t)}{K_4}, \quad t \in [mT + t_3, (m+1)T), \tag{22}$$

where $p, m = 0, 1, 2, \ldots$

Consider the parameters

$$(r_1, r_2, r_3, r_4, K_1, K_2, K_3, K_4, t_1, t_2, t_3, T) = (2, 4, 6, 8, 100, 150, 200, 250, 7, 12, 17, 20). \tag{23}$$

Let $p = 1$. Figures 2(A) and 3(A) demonstrate that the system (19)–(22) has a unique globally asymptotically stable τ-periodic solution. Similarly, when $p = 4$, Figs. 2(B) and 3(B) exhibit the same dynamic behavior for the system (19)–(22). In fact, the system exhibits similar dynamic behavior regardless of the value of p, where p is a positive integer. This shows that the qualitative characteristics of the system, such as the existence and stability of periodic solutions, remain the same for different integer multiples of the environmental period.

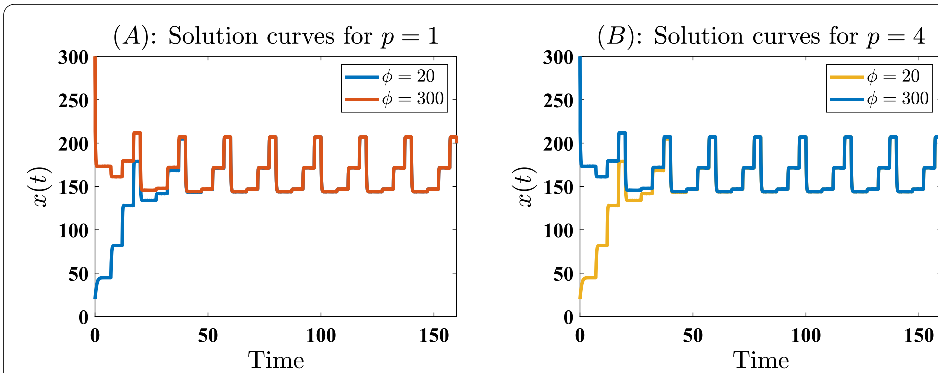

Figure 2 Solution curves of system (19)–(22) with parameters given in (23)

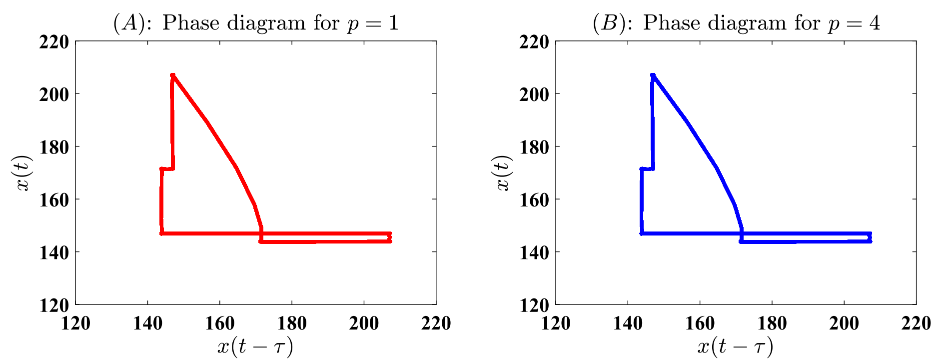

Figure 3 Phase-space plots of system (19)–(22) with parameters given in (23)

5 Discussion

In the population ecological environment, the periodic environment is widely present in the natural and social systems, such as seasonal climate change, species reproduction laws, or periodic fluctuations in the economy. The behavior of the system often fluctuates regularly with the periodic changes in external conditions. Studying the impact of periodic fluctuations on the system can help us understand the long-term behavior of the system, predict its future changes, and even discover potential critical points. This makes the periodic environment an important background for analyzing the behavior of complex systems, and has a wide range of application value in many fields such as natural sciences, engineering, and economics.

In this work, we developed a class of nonlinear delay differential equations to model systems in a periodic environment. We focused on the critical case where the delay is a positive integer multiple of the environmental period. By constructing upper and lower solutions and applying the theory of monotonic dynamical systems, we demonstrated that, under the conditions (A_1)–(A_3), the existence and stability of periodic solutions and equilibria in time delay systems are equivalent to those in the corresponding nondelay case. Applying our general results to the logistic model with seasonal switching, the numerical simulations validated our theoretical findings. Our study extended and improved the results in [27, 28, 30], the model we presented has a general form with wide applicability.

There are some directions in which our model can be extended. The first extension is to remove the restriction of the time delay τ, that is, delay could be a nonpositive integer multiple of the environmental period, which is more in line with various practical issues in practical applications. Another direction in our future research plan is to extend our model by relaxing the monotonicity constraints on the birth and death functions. In the current framework, the presence of monotonicity simplifies the stability analysis of the periodic solutions. However, many real-world systems exhibit nonmonotonic behavior due to factors such as density dependence, environmental fluctuations, or complex interactions between species. By removing these limitations, we can explore a wider range of biological and ecological models that more realistically represent population dynamics.

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Author contributions
The authors have equal contributions to the manuscript. All authors read and approved the final manuscript.

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Data availability
Data sharing is not applicable to this paper as no datasets were generated or analyzed during the current study.

Declarations

Competing interests
The authors declare no conflict of interest.

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