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Dynamic analysis of a stochastic microorganism flocculation model with two complementary nutrients and nonlinear perturbation

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Abstract

This paper considers a stochastic model of microorganism flocculation incorporating two complementary nutrients. By introducing nonlinear perturbation, we analyze the influence of flocculations, microorganisms, and two nutrients on the model dynamic. The paper proves the existence and uniqueness of the stationary distribution in the stochastic model. Moreover, sufficient conditions for the extinction of microorganisms are established. Numerical simulations indicate that nonlinear perturbation makes the growth process of microorganisms more unpredictable, better reflecting the complicated variations in real-world environments. Noise interference is not always detrimental, but appropriate noise levels may promote the growth of microorganisms.

Keywords: Stochastic microorganism flocculation model; Nonlinear perturbation; Stationary distribution; Extinction

1 Introduction

The chemostat is a commonly used device in continuous microorganism cultivation and ecological research, playing a crucial role in mathematical ecology modeling. This device maintains a constant flow rate of the culture medium. In a chemostat, the concentration of supplied nutrients regulates the density of microorganisms, while the flow rate controls the growth rate. By adjusting the flow rate, the growth of microorganisms can be balanced with the inflow rate of nutrients, thus maintaining a stable population density of microorganisms. It is generally assumed that the reaction volume remains constant in the reactor, with equal inflow and outflow rates [1, 2]. Theoretically, the chemostat model can be represented using ODEs, PDEs, or SDEs, making it applicable to various fields, including chemical, pharmacology, ecology, and medical research. For example, in [3], a partial differential equation model describing two species competing for a limited nutrient source in a non-stirring chemostat was studied, in which one microorganism species was extinct. A stochastic chemostat model incorporating the Monod and Haldane consumption func-

Table 1 Terms in Model (1.1) and Their Descriptions

Term	Description
$r_1\varphi_1(C(t))\varphi_2(N(t))X(t)$	Uptake rate of carbon source
$r_2\varphi_1(C(t))\varphi_2(N(t))X(t)$	Uptake rate of nitrogen source
$r\varphi_1(C(t))\varphi_2(N(t))X(t)$	Growth rate of microorganisms
$m_1X(t)P(t)$	Flocculation rate of microorganisms
$m_2X(t)P(t)$	Loss rate of flocculants

tions for the species was analyzed in [4]. In [5], a stochastic chemostat model driven by white noise was investigated to study the dynamical properties of the system under environmental fluctuations. In reality, microorganisms are often too small to be easily collected directly, and flocculants are typically added to the reactor to collect large quantities of microorganisms. Flocculants [6] promote the aggregation of suspended particles into larger particles and are widely used in water treatment, textile, and other fields.

Given the broad application of microorganism flocculants and environmentally friendly characteristics, a dynamic analysis of the model is necessary. Tai et al. [7] developed a class of microorganism flocculation model with delay. Ni et al. [8] established a size-structured PDE model that considers the cell size of algae, depicting the evolutionary relationship among algal cell growth, nutrient uptake, and flocculation effect. By constructing a suitable positive invariant set and using the Lyapunov-LaSalle theorem, Guo and Ma [9] investigated the global stability of the equilibrium point under specific conditions. The persistence of the model was also studied, and an explicit expression for the eventual lower bound of microorganism concentration was provided. In addition, a microorganism flocculation delay model with a saturated functional response was presented in [10], demonstrating that microorganism collection is sustainable and gives an eventual lower bound on the microorganism concentration under certain conditions. Using the Lyapunov-LaSalle theorem, a sufficient condition for global stability was ultimately established. Wang et al. [11] proposed a dynamic model (1.1) with two different kinds of nutrients. Additionally, they investigated the global stability of the boundary equilibrium point and positive equilibrium point, as well as the persistence of the model (1.1)

$$\begin{cases} dC(t) = [1 - C(t) - r_1\varphi_1(C(t))\varphi_2(N(t))X(t)]dt, \\ dN(t) = [1 - N(t) - r_2\varphi_1(C(t))\varphi_2(N(t))X(t)]dt, \\ dX(t) = [r\varphi_1(C(t))\varphi_2(N(t))X(t) - X(t) - m_1X(t)P(t)]dt, \\ dP(t) = [1 - P(t) - m_2X(t)P(t)]dt, \end{cases} \tag{1.1}$$

where $C(t)$, $N(t)$, $X(t)$, and $P(t)$ denote the concentration of carbon source, nitrogen source, microorganisms and flocculants, respectively. The functions $\varphi_1(C(t))$ and $\varphi_2(N(t))$ are the Monod-type function

$$\varphi_1(C(t)) = \frac{C(t)}{K_1 + C(t)},\ \varphi_2(N(t)) = \frac{N(t)}{K_2 + N(t)},$$

where the constants $K_1 > 0$ and $K_2 > 0$. In the model (1.1), $r \geq 0$, $r_i \geq 0$ $(i = 1, 2)$ and $m_i \geq 0$ $(i = 1, 2)$ are the constants.

The influence of various external factors on deterministic systems is unavoidable in natural environments. Considering these environmental changes, it is both reasonable and practical to incorporate noise into the model. Therefore, analyzing the dynamic behavior of the system under stochastic perturbation is significant. To study the effects of stochastic perturbation, the researchers introduced perturbation into deterministic models using different methods. In [12], Yang et al. proposed a tri-trophic food chain model incorporating stochastic perturbation in the environment. Xu et al. [13] developed and examined a stochastic competition chemostat model. Recently, numerous researchers have concentrated on the ergodicity of the unique stationary distribution in the stochastic chemostat model and established sufficient conditions for microorganism survival and extinction within the model (Xu et al. [14], Lv et al. [15], Imhof and Walcher [16], Gao et al. [17], Zhang et al. [18], Sun and Zhang [19], Stephanopoulos et al. [20], Chi and Zhao [21]). Zhang et al. [22] introduced white noise as a linear perturbation to develop a microorganism flocculation model. Li et al. [23] studied a stochastic microorganism flocculation model with a Monod-type response function. Liu and Ma [24] proposed a stochastic model involving two complementary nutrients.

Noise characteristics are a key factor influencing the behavior of stochastic biological dynamical systems. Nonlinear noise (Zhou et al. [25], Yu and Yuan [26], Sun and Lu [27], and Li et al. [28]) means that there is a nonlinear relationship between the noise term and the system state. This type of noise is commonly used in models, particularly when describing complex dynamic behaviors.

The innovation of this paper is reflected in (i) Adding nonlinear perturbation to the microorganism flocculation model that involves two complementary nutrients. (ii) The interference of small nonlinear noise may positively impact the growth of microorganisms.

In Sect. 2, a stochastic model incorporating nonlinear perturbation is formulated. In Sect. 3, we show the existence and uniqueness of global positive solutions and the stationary distribution. Additionally, we demonstrate the extinction of microorganisms. Numerical simulations are employed in Sect. 4 to validate and support our theoretical results. The conclusions are presented in Sect. 5.

2 The model formulation and necessary lemmas

Stochastic dynamical systems have been widely employed to investigate various biological phenomena, including ecological stability, population dynamics, and disease transmission. Inspired by the success of stochastic epidemic models (He et al. [29], Tan et al. [30]), we aim to explore the stochastic dynamics of microorganism flocculation, a process influenced by complex interactions and environmental perturbations. In Sect. 2.1, we analyze a nonlinear perturbation model to describe the microorganism flocculation process.

2.1 The model formulation

In stochastic dynamical models, the "perturbation" of the deterministic model introduces randomness, simulating the natural or artificial effects on system dynamics. Below are three common perturbation methods: (i) Parameter Perturbation [31], (ii) Equilibrium Point Perturbation [32], and (iii) System-Wide Perturbation [33].

Regarding method (iii), Liu and Ma [24] proposed a stochastic model (2.1), incorporating the effects of linear perturbation on the system's behavior.

$$\begin{cases} dC(t) = [1 - C(t) - r_1\varphi_1(C(t))\varphi_2(N(t))X(t)]\,dt + \sigma_1 C(t)dB_1(t), \\ dN(t) = [1 - N(t) - r_2\varphi_1(C(t))\varphi_2(N(t))X(t)]\,dt + \sigma_2(t)N(t)dB_2(t), \\ dX(t) = [r\varphi_1(C(t))\varphi_2(N(t)) - 1 - m_1P(t)]\,X(t)dt + \sigma_3 X(t)dB_3(t), \\ dP(t) = [(1 - P(t)) - m_2X(t)P(t)]\,dt + \sigma_4 P(t)dB_4(t), \end{cases} \tag{2.1}$$

where $B_i(t)$ $(i = 1,2,3,4)$ represent independent standard Brownian motions satisfying $B_i(0) = 0$ $(i = 1,2,3,4)$, and σ_i^2 $(i = 1,2,3,4)$ denote the intensities of white noise. However, disturbances to the system are not always linear, so we introduce a nonlinear perturbation inspired by the methods outlined in [34, 35]. Given this, a stochastic model (2.2) with nonlinear perturbation is developed

$$\begin{cases} dC(t) = [1 - C(t) - r_1\varphi_1(C(t))\varphi_2(N(t))X(t)]\,dt \\ \qquad\qquad + \left(\sigma_{11}C^2(t) + \sigma_{12}C(t)\right) dB_1(t), \\ dN(t) = [1 - N(t) - r_2\varphi_1(C(t))\varphi_2(N(t))X(t)]\,dt \\ \qquad\qquad + \left(\sigma_{21}N^2(t) + \sigma_{22}(t)N(t)\right) dB_2(t), \\ dX(t) = [r\varphi_1(C(t))\varphi_2(N(t)) - 1 - m_1P(t)]\,X(t)dt \\ \qquad\qquad + \left(\sigma_{31}X^2(t) + \sigma_{32}X(t)\right) dB_3(t), \\ dP(t) = [1 - P(t) - m_2X(t)P(t)]\,dt \\ \qquad\qquad + \left(\sigma_{41}P^2(t) + \sigma_{42}P(t)\right) dB_4(t), \end{cases} \tag{2.2}$$

where $\sigma_{ij}^2 > 0$ $(i = 1,2,3,4, j = 1,2)$ are the intensities of noise. From the perspective of biology, the initials value of the model (2.2) are nonnegative

$$C(0) = C_0 \geq 0,\; N(0) = N_0 \geq 0,\; X(0) = X_0 \geq 0,\; P(0) = P_0 \geq 0,$$

where C_0, N_0, X_0, and P_0 are the concentrations of $C(t)$, $N(t)$, $X(t)$, and $P(t)$, respectively.

2.2 Some preliminaries and necessary lemmas

In the complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ with a natural filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions. On this space, we define the independent Brownian motions $B_i(t)$ for $i = 1,2,3,4$. Define the sets $\overline{\mathbb{R}}_+^4 = \{\beta = (\beta_1,\beta_2,\beta_3,\beta_4) \in \mathbb{R}^4 : \beta_i \geq 0,\ i = 1,2,3,4\}$ and $\mathbb{R}_+^4 = \{\beta = (\beta_1,\beta_2,\beta_3,\beta_4) \in \mathbb{R}^4 : \beta_i > 0,\ i = 1,2,3,4\}$. The solution of model (2.2) is represented by $W(t) = (C(t),N(t),X(t),P(t))^T$, with the initial value specified as $W_0 = (C_0,N_0,X_0,P_0)^T \in \mathbb{R}_+^4$.

Lemma 2.1 [35] *Let $(c(t),n(t))$ be the solutions of SDEs (2.3) with the initial value $(c(0) = C_0, n(0) = N_0)$,*

$$\begin{cases} dc(t) = [1 - c(t)]\,dt + (\sigma_{11}c(t) + \sigma_{12})\,c(t)dB_1(t), \\ dn(t) = [1 - n(t)]\,dt + (\sigma_{21}n(t) + \sigma_{22})\,n(t)dB_2(t). \end{cases} \tag{2.3}$$

We have

(i) $C(t) \le c(t)$, $N(t) \le n(t)$, *a.s.*

(ii) $\lim_{t\to\infty} \frac{1}{t}\int_0^t c^2(s)ds = \int_0^\infty x^2\pi_1(x)dx$, $\lim_{t\to\infty} \frac{1}{t}\int_0^t n^2(s)ds = \int_0^\infty y^2\pi_2(y)dy$, *a.s.*

The density functions $\pi_1(x)$ *and* $\pi_2(y)$ *are expressed*

$$\begin{cases} \pi_1(x) = Q_1 x^{-2(1+q_1)}\left(\sigma_{11}x+\sigma_{12}\right)^{-2(1-q_1)} \\ \qquad e^{-\frac{2}{\sigma_{12}(\sigma_{12}+\sigma_{11}x)}\left(\frac{1}{x}+\frac{2\sigma_{11}+\sigma_{12}}{\sigma_{12}}\right)}, \ (x\in(0,+\infty)), \\ \pi_2(y) = Q_2 y^{-2(1+q_2)}\left(\sigma_{21}y+\sigma_{22}\right)^{-2(1-q_2)} \\ \qquad e^{-\frac{2}{\sigma_{22}(\sigma_{22}+\sigma_{21}y)}\left(\frac{1}{y}+\frac{2\sigma_{21}+\sigma_{22}}{\sigma_{22}}\right)}, \ (y\in(0,+\infty)), \end{cases} \tag{2.4}$$

where $q_1 = \frac{2\sigma_{11}+\sigma_{12}}{\sigma_{12}^3}$, $q_2 = \frac{2\sigma_{21}+\sigma_{22}}{\sigma_{22}^3}$, Q_1 *and* Q_2 *are constants such that* $\int_0^\infty \pi_1(x)dx = 1$, $\int_0^\infty \pi_2(y)dy = 1$, *respectively.*

Lemma 2.2 [36] *Assume there has a bounded domain open* $D \subset \mathbb{E}^d$ *with a regular boundary* Γ *and satisfying the following conditions*

(i) *For a positive number* M, *the inequality* $\sum_{i,j=1}^n a_{ij}(x)\eta_i\eta_j \ge M|\eta|^2$ *holds for all* $x \in D$ *and any vector* $\eta \in \mathbb{E}^d$.

(ii) *There exists a nonnegative* C^2*-function* $V(x)$, *such that* $LV(x)$ *remains strictly negative for all* x *outside the domain* D, *i.e., for* $x \in \mathbb{E}^d \setminus D$.

Then, the Markov process $X(t)$ *possesses a unique ergodic stationary* $\pi(\cdot)$.

For all $x \in \mathbb{E}^d \setminus D$ *and every function* $f(\cdot)$ *that is integrable with respect to the measure* $\pi(\cdot)$, *we have*

$$\mathbf{P}\left\{\frac{1}{T}\int_0^T f(X(t))dt \underset{T\to\infty}{\longrightarrow} \int_{\mathbb{E}^d} f(x)\pi(dx)\right\} = 1,$$

where $X(t)$ *is the solution of SDE*

$$dX(t) = \mu(X)dt + \sum_{r=1}^k \sigma_r(X)dB_r(t),$$

the diffusion matrix

$$A(x) = (a_{ij}(x)),\ a_{ij}(x) = \sum_{r=1}^k \sigma_r^i(x)\sigma_r^i(x).$$

Lemma 2.3 [37] *If* $x \ge 0$, *we obtain two inequalities*

$$(i)\ \frac{x^3}{\left(x^2+1\right)} \ge \left(x-\frac{1}{2}\right),\ (ii)\ \frac{x^4}{\left(x^2+1\right)} \ge \left(\frac{3}{4}x^2-\frac{1}{4}\right).$$

3 Global dynamic analysis for model (2.2)

3.1 Existence and uniqueness

Theorem 3.1 *For any initial condition* $W_0 \in \mathbb{R}_+^4$, *there exists a unique solution* $W(t)$ *to model* (2.2) *that remains in* $\mathbb{R}_+^4$, *meaning that for all* $t \ge 0$, $W(t) \in \mathbb{R}_+^4$.

Proof The selection of a C^2-function for the proof process can be based on the methods outlined in [24]. Therefore, we omit it here. □

3.2 Stationary distribution

Define

$$R_0^s(\theta) = \frac{r}{a_1(\theta)a_2(\theta)a_3(\theta)(K_1+1)(K_2+1)},$$

where

$$a_1(\theta) = \left(1 + \frac{1}{2}\sigma_{12}^2 + 2\sqrt[3]{\frac{\sigma_{11}^2}{(1-\theta)^2}} + 2\sqrt{\frac{\sigma_{11}\sigma_{12}}{1-\theta}}\right),$$

$$a_2(\theta) = \left(1 + \frac{1}{2}\sigma_{22}^2 + 2\sqrt[3]{\frac{\sigma_{21}^2}{(1-\theta)^2}} + 2\sqrt{\frac{\sigma_{21}\sigma_{22}}{1-\theta}}\right),$$

$$a_3(\theta) = \Bigg(1 + m_1 + \frac{1}{2}\sigma_{32}^2 + \sqrt{\frac{r\sigma_{31}\sigma_{32}}{(1-\theta)r_1}} + \sqrt[3]{\left(\frac{r\sigma_{31}}{(1-\theta)r_1}\right)^2} + \sqrt{\frac{r\sigma_{31}\sigma_{32}}{(1-\theta)r_2}} + \sqrt[3]{\left(\frac{r\sigma_{31}}{(1-\theta)r_2}\right)^2}\Bigg).$$

When $\theta \to 0^+$, we obtain the stochastic threshold as follows

$$R_0^s = \frac{r}{a_1a_2a_3(K_1+1)(K_2+1)},$$

where

$$a_1 = \left(1 + \frac{1}{2}\sigma_{12}^2 + 2\sqrt[3]{\sigma_{11}^2} + 2\sqrt{\sigma_{11}\sigma_{12}}\right),$$

$$a_2 = \left(1 + \frac{1}{2}\sigma_{22}^2 + 2\sqrt[3]{\sigma_{21}^2} + 2\sqrt{\sigma_{21}\sigma_{22}}\right),$$

$$a_3 = \left(1 + m_1 + \frac{1}{2}\sigma_{32}^2 + \sqrt{\frac{r\sigma_{31}\sigma_{32}}{r_1}} + \sqrt[3]{\left(\frac{r\sigma_{31}}{r_1}\right)^2} + \sqrt{\frac{r\sigma_{31}\sigma_{32}}{r_2}} + \sqrt[3]{\left(\frac{r\sigma_{31}}{r_2}\right)^2}\right).$$

Theorem 3.2 *If $R_0^s > 1$, there exists a unique stationary distribution for model (2.2), and it has the ergodic property for any initial $W_0 \in \mathbb{R}_+^4$.*

Proof To check condition (i) in Lemma 2.2, it is enough to examine the diffusion matrix of model (2.2)

$$\sum_{i,j=1}^{4} a_{ij}(C,N,X,P)\xi_i\xi_j = \big((\sigma_{11}C^2 + \sigma_{12}C)\,\xi_1,\ (\sigma_{21}N^2 + \sigma_{22}N)\,\xi_2,$$

$$(\sigma_{31}X^2 + \sigma_{32}X)\,\xi_3,\ (\sigma_{41}P^2 + \sigma_{42}P)\,\xi_4\big)$$

$$
\begin{aligned}
&\begin{pmatrix} \left(\sigma_{11}C^2+\sigma_{12}C\right)\xi_1 \\ \left(\sigma_{21}N^2+\sigma_{22}N\right)\xi_2 \\ \left(\sigma_{31}X^2+\sigma_{32}X\right)\xi_3 \\ \left(\sigma_{41}P^2+\sigma_{42}P\right)\xi_4 \end{pmatrix} \\
&= \left(\sigma_{11}C^2+\sigma_{12}C\right)^2{\xi_1}^2 + \left(\sigma_{21}N^2+\sigma_{22}N\right)^2{\xi_2}^2 \\
&\quad + \left(\sigma_{31}X^2+\sigma_{32}X\right)^2{\xi_3}^2 + \left(\sigma_{41}P^2+\sigma_{42}P\right)^2{\xi_4}^2 \\
&\geq M^* \parallel \xi \parallel^2 \text{ for any } W \in \mathbb{R}_+^4, \\
&\xi = (\xi_1,\xi_2,\xi_3,\xi_4) \in D_\varepsilon,
\end{aligned}
$$

where $M^* = \min_{(C,N,X,P)\in D_\varepsilon}\{(\sigma_{11}C^2+\sigma_{12}C)^2, (\sigma_{21}N^2+\sigma_{22}N)^2, (\sigma_{31}X^2+\sigma_{32}X)^2, (\sigma_{41}P^2+\sigma_{42}P)^2\}$ and $D_\varepsilon = \left[\varepsilon, \frac{1}{\varepsilon}\right] \times \left[\varepsilon, \frac{1}{\varepsilon}\right] \times \left[\varepsilon, \frac{1}{\varepsilon}\right] \times \left[\varepsilon, \frac{1}{\varepsilon}\right]$. Therefore, condition (i) is satisfied.

We introduce a C^2-function

$$V^*(W) = MV_1^* + V_2^* + V_3^*,$$

where $V_1^* = -\ln X + m_1 P - c_1 \ln C - c_2 \ln N + c_3 C + c_4 N + c_1 U_1 + c_2 U_2 + U_3 + U_4$, $U_1 = \sum_{i=1}^2 \frac{\alpha_i (C+\beta_i)^\theta}{\theta}$, $U_2 = \sum_{i=1}^2 \frac{\eta_i (N+\gamma_i)^\theta}{\theta}$, $U_3 = \frac{1}{2}\left(k_1 C + \sum_{i=1}^2 \frac{\nu_i (X+\omega_i)^\theta}{\theta}\right)$, $U_4 = \frac{1}{2}\left(k_2 N + \sum_{i=1}^2 \frac{\vartheta_i (X+\varphi_i)^\theta}{\theta}\right)$, $V_2^* = \frac{(\sigma_{11}C+\sigma_{12})^\theta}{\theta} + \frac{(\sigma_{21}N+\sigma_{22})^\theta}{\theta} + \frac{(\sigma_{31}X+\sigma_{32})^\theta}{\theta} + \frac{(\sigma_{41}P+\sigma_{42})^\theta}{\theta}$, $V_3^* = -\ln C - \ln N - \ln P$, $0 < \theta < 1$, $0 < \alpha < 1$, $c_i > 0$ $(i = 1,2,3,4)$ and $M \geq 0$ denotes a sufficiently large constant satisfying

$$-M\lambda + G \leq -2,$$

where $\lambda = a_3(\theta)(R_0^s - 1)$ and $G = \sup_{W\in D_\varepsilon} \{A + F(C,N,P) - E(W)\}$,

$$
\begin{aligned}
&A = B + 3 + \sigma_{11}\sigma_{12}^{\theta-1} + \sigma_{21}\sigma_{22}^{\theta-1} + \sigma_{41}\sigma_{42}^{\theta-1}, \\
&E(W) = \frac{1-\theta}{4}\left(\sigma_{11}^{\theta+2}C^{\theta+2} + \sigma_{21}^{\theta+2}N^{\theta+2} + \sigma_{31}^{\theta+2}X^{\theta+2} + \sigma_{41}^{\theta+2}P^{\theta+2}\right), \\
&F(C,N,P) = \frac{1}{2}\left(\sigma_{11}C + \sigma_{12}\right)^2 + \frac{1}{2}\left(\sigma_{21}N + \sigma_{22}\right)^2 + \frac{1}{2}\left(\sigma_{41}P + \sigma_{42}\right)^2.
\end{aligned}
$$

Now, we define a nonnegative C^2-function $V(W)$ that holds for any $W \in \mathbb{R}_+^4$. Noting that $V^*(W)$ is a continuous function for any $(W) \in \mathbb{R}_+^4$ and

$$\lim_{\varepsilon\to 0, W\in\mathcal{D}_\varepsilon} V^*(W) = \infty.$$

Therefore, $V^*(W)$ must have a minimum point $W_m = (C_m, N_m, X_m, P_m) \in \mathbb{R}_+^4$. The C^2-function $V(W): \mathbb{R}_+^4 \to \mathbb{R}_+ \cup \{0\}$

$$V(W) = V^*(W) - V^*(W_m).$$

Let

$$c_1 = \frac{r}{(a_1(\theta))^2 (a_2(\theta)) (K_1+1)(K_2+1)},$$

$$c_2 = \frac{r}{(a_1(\theta))\,(a_2(\theta))^2\,(K_1+1)\,(K_2+1)},$$
$$c_3 = \frac{r}{(a_1(\theta))\,(a_2(\theta))\,(K_1+1)^2\,(K_2+1)},$$
$$c_4 = \frac{r}{(a_1(\theta))\,(a_2(\theta))\,(K_1+1)\,(K_2+1)^2}.$$

Using *Itô* formula for U_1, we have

$$\begin{aligned}
LU_1 &= \sum_{i=1}^{2} \alpha_i\,(C+\beta_i)^{\theta-1}\,(1-C-r_1\varphi_1(C)\varphi_2(N)X) \\
&\quad - \sum_{i=1}^{2} \frac{\alpha_i\,(1-\theta)}{2\,(C+\beta_i)^{2-\theta}}\left(\sigma_{11}C^2+\sigma_{12}C\right)^2 \\
&= \sum_{i=1}^{2} \frac{\alpha_i}{(C+\beta_i)^{1-\theta}}\,(1-C-r_1\varphi_1(C)\varphi_2(N)X) \\
&\quad - \sum_{i=1}^{2} \frac{\alpha_i\,(1-\theta)}{2\,(C+\beta_i)^{2-\theta}}\left(\sigma_{11}C^2+\sigma_{12}C\right)^2 \\
&\le \sum_{i=1}^{2} \frac{\alpha_i}{\beta_i^{1-\theta}} - \frac{\alpha_1\,(1-\theta)\,\beta_1^{\theta-2}\sigma_{11}^2 C^4}{2\left(\frac{C}{\beta_1}+1\right)^{2-\theta}} - \frac{\alpha_2(1-\theta)\beta_2^{\theta-2}\sigma_{11}\sigma_{12}C^3}{\left(\frac{C}{\beta_2}+1\right)^{2-\theta}} \\
&\le \sum_{i=1}^{2} \frac{\alpha_i}{\beta_i^{1-\theta}} - \frac{\alpha_1\,(1-\theta)\,\beta_1^{\theta+2}\sigma_{11}^2\left(\frac{C}{\beta_1}\right)^4}{2\left(\frac{C}{\beta_1}+1\right)^{2}} - \frac{\alpha_2(1-\theta)\beta_2^{\theta+1}\sigma_{11}\sigma_{12}\left(\frac{C}{\beta_2}\right)^3}{\left(\frac{C}{\beta_2}+1\right)^{2}} \\
&\le \sum_{i=1}^{2} \frac{\alpha_i}{\beta_i^{1-\theta}} - \frac{\alpha_1\,(1-\theta)\,\beta_1^{\theta+2}\sigma_{11}^2\left(\frac{C}{\beta_1}\right)^4}{4\left(\left(\frac{C}{\beta_1}\right)^2+1\right)} - \frac{\alpha_2(1-\theta)\beta_2^{\theta+1}\sigma_{11}\sigma_{12}\left(\frac{C}{\beta_2}\right)^3}{2\left(\left(\frac{C}{\beta_2}\right)^2+1\right)} \\
&\le \sum_{i=1}^{2} \frac{\alpha_i}{\beta_i^{1-\theta}} - \frac{\alpha_1\,(1-\theta)\,\beta_1^{\theta+2}\sigma_{11}^2}{4}\left[\frac{3}{4}\left(\frac{C}{\beta_1}\right)^2-\frac{1}{4}\right] \\
&\quad - \frac{\alpha_2(1-\theta)\beta_2^{\theta+1}\sigma_{11}\sigma_{12}}{2}\left(\frac{C}{\beta_2}-\frac{1}{2}\right) \\
&\le \left(\frac{\alpha_1}{\beta_1^{1-\theta}}+\frac{\alpha_1(1-\theta)\beta_1^{\theta+2}\sigma_{11}^2}{16}\right)+\left(\frac{\alpha_2}{\beta_2^{1-\theta}}+\frac{\alpha_2(1-\theta)\beta_2^{\theta+1}\sigma_{11}\sigma_{12}}{4}\right) \\
&\quad - \frac{3\alpha_1(1-\theta)\beta_1^{\theta}\sigma_{11}^2}{16}C^2 - \frac{\alpha_2(1-\theta)\beta_2^{\theta}\sigma_{11}\sigma_{12}}{2}C.
\end{aligned}$$

Let

$$\alpha_1 = \frac{8}{3(1-\theta)\beta_1^{\theta}},\ \alpha_2 = \frac{2}{(1-\theta)\beta_2^{\theta}},\ \beta_1 = 2\sqrt[3]{\frac{1}{(1-\theta)\sigma_{11}^2}},\ \beta_2 = 2\sqrt{\frac{1}{(1-\theta)\sigma_{11}\sigma_{12}}}.$$

Then

$$LU_1 \le 2\left[\sqrt[3]{\frac{\sigma_{11}^2}{(1-\theta)^2}}+\sqrt{\frac{\sigma_{11}\sigma_{12}}{1-\theta}}\right]-\frac{\sigma_{11}^2}{2}C^2-\sigma_{11}\sigma_{12}C.$$

Similarly, let

$$\eta_1=\frac{8}{3(1-\theta)\gamma_1^{\theta}},\ \eta_2=\frac{2}{(1-\theta)\gamma_2^{\theta}},\ \gamma_1=2\sqrt[3]{\frac{1}{(1-\theta)\sigma_{21}^2}},\ \gamma_2=2\sqrt{\frac{1}{(1-\theta)\sigma_{21}\sigma_{22}}}.$$

$$LU_2\le 2\left[\sqrt[3]{\frac{\sigma_{21}^2}{(1-\theta)^2}}+\sqrt{\frac{\sigma_{21}\sigma_{22}}{1-\theta}}\right]-\frac{\sigma_{21}^2}{2}N^2-\sigma_{11}\sigma_{12}N.$$

$$\begin{aligned}2LU_3=&k_1\left(1-C-r_1\varphi_1(C)\varphi_2(N)X\right)+\sum_{i=1}^{2}\left[\nu_i(X+\omega_i)^{\theta-1}(r\varphi_1(C)\varphi_2(N)X\right.\\&\left.-X-m_1XP)-\frac{\nu_i\,(1-\theta)}{2\,(X+\omega_i)^{2-\theta}}\left(\sigma_{31}X^2+\sigma_{32}X\right)^2\right]\\\le&k_1+\left(\sum_{i=1}^{2}\nu_i\omega_i^{\theta-1}r-k_1r_1\right)\varphi_1(C)\varphi_2(N)X+\frac{\nu_1(1-\theta)\omega_1^{\theta+2}\sigma_{31}^2}{16}\\&+\frac{\nu_2(1-\theta)\omega_2^{\theta+1}\sigma_{31}\sigma_{32}}{4}-\frac{3\nu_1(1-\theta)\omega_1^{\theta}\sigma_{31}^2}{16}X^2-\frac{\nu_2(1-\theta)\omega_2^{\theta}\sigma_{31}\sigma_{32}}{2}X.\end{aligned}$$

Let

$$\begin{aligned}&k_1=\frac{\sum_{i=1}^2\nu_i\omega_i^{\theta-1}r}{r_1},\ \nu_1=\frac{8}{3(1-\theta)\omega_1^{\theta}},\ \nu_2=\frac{2}{(1-\theta)\omega_2^{\theta}},\\&\omega_1=2\sqrt[3]{\frac{r}{r_1(1-\theta)\sigma_{31}^2}},\ \omega_2=2\sqrt{\frac{r}{r_1(1-\theta)\sigma_{31}\sigma_{32}}}.\end{aligned}$$

Then, we have

$$2LU_3\le 2\left[\sqrt{\frac{r\sigma_{31}\sigma_{32}}{(1-\theta)r_1}}+\sqrt[3]{\left(\frac{r\sigma_{31}}{(1-\theta)r_1}\right)^2}\right]-\frac{\sigma_{31}^2}{2}X^2-\sigma_{31}\sigma_{32}X.$$

Similarly, let

$$\begin{aligned}&k_2=\frac{\sum_{i=1}^2\vartheta_i\varpi_i^{\theta-1}r}{r_2},\ \vartheta_1=\frac{8}{3(1-\theta)\varpi_1^{\theta}},\ \vartheta_2=\frac{2}{(1-\theta)\varpi_2^{\theta}},\\&\varpi_1=2\sqrt[3]{\frac{r}{r_2(1-\theta)\sigma_{31}^2}},\ \varpi_2=2\sqrt{\frac{r}{r_2(1-\theta)\sigma_{31}\sigma_{32}}},\end{aligned}$$

we have

$$2LU_4\le 2\left[\sqrt{\frac{r\sigma_{31}\sigma_{32}}{(1-\theta)r_2}}+\sqrt[3]{\left(\frac{r\sigma_{31}}{(1-\theta)r_2}\right)^2}\right]-\frac{\sigma_{31}^2}{2}X^2-\sigma_{31}\sigma_{32}X.$$

Therefore, we obtain

$$
\begin{aligned}
LV_1^* \leq& \left(-\frac{c_1}{C} + c_3\right)(1 - C - r_1\varphi_1(C)\varphi_2(N)X) + \frac{c_1}{2C^2}\left(\sigma_{11}C^2 + \sigma_{12}C\right)^2 \\
&+ \left(-\frac{c_2}{N} + c_4\right)(1 - N - r_2\varphi_1(C)\varphi_2(N)X) + \frac{c_2}{2N^2}\left(\sigma_{21}N^2 + \sigma_{22}N\right)^2 \\
&- \frac{1}{X}(r\varphi_1(C)\varphi_2(N)X - X - m_1XP) + \frac{1}{2X^2}\left(\sigma_{31}X^2 + \sigma_{32}X\right)^2 \\
&+ m_1(1 - P - m_2XP) + c_1\left(2\sqrt[3]{\frac{\sigma_{11}^2}{(1-\theta)^2}} + 2\sqrt{\frac{\sigma_{11}\sigma_{12}}{1-\theta}} - \frac{\sigma_{11}^2}{2}C^2\right. \\
&\left.- \sigma_{11}\sigma_{12}C\right) + c_2\left(2\sqrt[3]{\frac{\sigma_{21}^2}{(1-\theta)^2}} + 2\sqrt{\frac{\sigma_{21}\sigma_{22}}{1-\theta}} - \frac{\sigma_{21}^2}{2}N^2 - \sigma_{21}\sigma_{22}N\right) \\
&+ \sqrt{\frac{r\sigma_{31}\sigma_{32}}{(1-\theta)r_1}} + \sqrt[3]{\left(\frac{r\sigma_{31}}{(1-\theta)r_1}\right)^2} + \sqrt{\frac{r\sigma_{31}\sigma_{32}}{(1-\theta)r_2}} + \sqrt[3]{\left(\frac{r\sigma_{31}}{(1-\theta)r_2}\right)^2} \\
&- \frac{\sigma_{31}^2}{2}X^2 - \sigma_{31}\sigma_{32}X \\
\leq& -\left[\frac{rCN}{(K_1+C)(K_2+N)} + \frac{c_1}{C} + \frac{c_2}{N} + c_3(K_1+C) + c_4(K_2+N)\right] \\
&+ \frac{c_1r_1NX + c_2r_2CX}{(K_1+C)(K_2+N)} + c_3(K_1+1) + c_4(K_2+1) \\
&+ 1 + m_1 + \frac{1}{2}\sigma_{32}^2 + \sqrt{\frac{r\sigma_{31}\sigma_{32}}{(1-\theta)r_1}} + \sqrt[3]{\left(\frac{r\sigma_{31}}{(1-\theta)r_1}\right)^2} + \sqrt{\frac{r\sigma_{31}\sigma_{32}}{(1-\theta)r_2}} \\
&+ \sqrt[3]{\left(\frac{r\sigma_{31}}{(1-\theta)r_2}\right)^2} + c_1\left(1 + \frac{1}{2}\sigma_{12}^2 + 2\sqrt[3]{\frac{\sigma_{11}^2}{(1-\theta)^2}} + 2\sqrt{\frac{\sigma_{11}\sigma_{12}}{1-\theta}}\right) \\
&+ c_2\left(1 + \frac{1}{2}\sigma_{22}^2 + 2\sqrt[3]{\frac{\sigma_{21}^2}{(1-\theta)^2}} + 2\sqrt{\frac{\sigma_{21}\sigma_{22}}{1-\theta}}\right) \\
\leq& -5\sqrt[5]{rc_1c_2c_3c_4} + \frac{c_1r_1NX + c_2r_2CX}{(K_1+C)(K_2+N)} + c_3(K_1+1) \\
&+ c_4(K_2+1) + c_1\left(1 + \frac{1}{2}\sigma_{12}^2 + 2\sqrt[3]{\frac{\sigma_{11}^2}{(1-\theta)^2}} + 2\sqrt{\frac{\sigma_{11}\sigma_{12}}{1-\theta}}\right) \\
&+ c_2\left(1 + \frac{1}{2}\sigma_{22}^2 + 2\sqrt[3]{\frac{\sigma_{21}^2}{(1-\theta)^2}} + 2\sqrt{\frac{\sigma_{21}\sigma_{22}}{1-\theta}}\right) + 1 + m_1 + \frac{1}{2}\sigma_{32}^2 \\
&+ \sqrt{\frac{r\sigma_{31}\sigma_{32}}{(1-\theta)r_1}} + \sqrt[3]{\left(\frac{r\sigma_{31}}{(1-\theta)r_1}\right)^2} + \sqrt{\frac{r\sigma_{31}\sigma_{32}}{(1-\theta)r_2}} + \sqrt[3]{\left(\frac{r\sigma_{31}}{(1-\theta)r_2}\right)^2} \\
=& -\frac{r}{(a_1(\theta))(a_2(\theta))(K_1+1)(K_2+1)} + a_3(\theta) + \frac{c_1r_1NX + c_2r_2CX}{(K_1+C)(K_2+N)} \\
=& -a_3(\theta)(R_0^s - 1) + \frac{c_1r_1NX + c_2r_2CX}{(K_1+C)(K_2+N)} \\
=& -\lambda + \frac{c_1r_1NX + c_2r_2CX}{(K_1+C)(K_2+N)}.
\end{aligned}
$$

Similarly using *Itô* formula to V_2^*, we have

$$
\begin{aligned}
LV_2^* =& \sigma_{11}\left(\sigma_{11}C+\sigma_{12}\right)^{\theta-1}\left(1-C-r_1\varphi_1\left(C\right)\varphi_2\left(N\right)X\right)\\
&+\frac{1}{2}(\theta-1)\sigma_{11}^2\left(\sigma_{11}C+\sigma_{12}\right)^{\theta-2}\left(\sigma_{11}C^2+\sigma_{12}C\right)^2\\
&+\sigma_{21}\left(\sigma_{21}N+\sigma_{22}\right)^{\theta-1}\left(1-N-r_2\varphi_1\left(C\right)\varphi_2\left(N\right)X\right)\\
&+\frac{1}{2}(\theta-1)\sigma_{21}^2\left(\sigma_{21}N+\sigma_{22}\right)^{\theta-2}\left(\sigma_{21}N^2+\sigma_{22}N\right)^2\\
&+\sigma_{31}\left(\sigma_{31}X+\sigma_{32}\right)^{\theta-1}\left(r\varphi_1\left(C\right)\varphi_2\left(N\right)X-X-m_1XP\right)\\
&+\frac{1}{2}(\theta-1)\sigma_{31}^2\left(\sigma_{31}X+\sigma_{32}\right)^{\theta-2}\left(\sigma_{31}X^2+\sigma_{32}X\right)^2\\
&+\sigma_{41}\left(\sigma_{41}P+\sigma_{42}\right)^{\theta-1}\left(1-P-m_2XP\right)\\
&+\frac{1}{2}(\theta-1)\sigma_{41}^2\left(\sigma_{41}P+\sigma_{42}\right)^{\theta-2}\left(\sigma_{41}P^2+\sigma_{42}P\right)^2\\
\le& \sigma_{11}\sigma_{12}^{\theta-1}-\frac{1-\theta}{2}\sigma_{11}^{2+\theta}C^{\theta+2}+\sigma_{21}\sigma_{22}^{\theta-1}-\frac{1-\theta}{2}\sigma_{21}^{2+\theta}N^{\theta+2}\\
&+\sigma_{31}\sigma_{32}^{\theta-1}r\varphi_1(C)\varphi_2(N)X-\frac{1-\theta}{2}\sigma_{31}^{2+\theta}X^{\theta+2}+\sigma_{41}\sigma_{42}^{\theta-1}\\
&-\frac{1-\theta}{2}\sigma_{41}^{2+\theta}P^{\theta+2}\\
\le& B+\sigma_{11}\sigma_{12}^{\theta-1}+\sigma_{21}\sigma_{22}^{\theta-1}+\sigma_{41}\sigma_{42}^{\theta-1}\\
&-\frac{1-\theta}{4}\left(\sigma_{11}^{\theta+2}C^{\theta+2}+\sigma_{21}^{\theta+2}N^{\theta+2}+\sigma_{31}^{\theta+2}X^{\theta+2}+\sigma_{41}^{\theta+2}P^{\theta+2}\right),
\end{aligned}
$$

where

$$
\begin{aligned}
B=\sup_{U\in\mathbb{R}_+^4}\Bigg\{&-\frac{1-\theta}{4}\left(\sigma_{11}^{\theta+2}C^{\theta+2}+\sigma_{21}^{\theta+2}N^{\theta+2}+\sigma_{31}^{\theta+2}X^{\theta+2}+\sigma_{41}^{\theta+2}P^{\theta+2}\right)\\
&+\sigma_{31}\sigma_{32}^{\theta-1}r\varphi_1(C)\varphi_2(N)X\Bigg\}.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
LV_3^* =& -\frac{1}{C}-\frac{1}{N}-\frac{1}{P}+3+\frac{r_1NX+r_2CX}{(K_1+C)(K_2+N)}\\
&+m_2X+\frac{1}{2}\left(\sigma_{11}C+\sigma_{12}\right)^2+\frac{1}{2}\left(\sigma_{21}N+\sigma_{22}\right)^2+\frac{1}{2}\left(\sigma_{41}P+\sigma_{42}\right)^2.
\end{aligned}
$$

Therefore, we obtain

$$
\begin{aligned}
LV \le& M\left[-\lambda+\frac{c_1r_1NX+c_2r_2CX}{(K_1+C)(K_2+N)}\right]-\frac{1}{C}-\frac{1}{N}\\
&-\frac{1}{P}+3+\frac{r_1NX+r_2CX}{(K_1+C)(K_2+N)}+m_2X\\
&+\frac{1}{2}\left(\sigma_{11}C+\sigma_{12}\right)^2+\frac{1}{2}\left(\sigma_{21}N+\sigma_{22}\right)^2+\frac{1}{2}\left(\sigma_{41}P+\sigma_{42}\right)^2
\end{aligned}
$$

$$+ B + \sigma_{11}\sigma_{12}^{\theta-1} + \sigma_{21}\sigma_{22}^{\theta-1} + \sigma_{41}\sigma_{42}^{\theta-1}$$
$$- \frac{1-\theta}{4}\left(\sigma_{11}^{\theta+2}C^{\theta+2} + \sigma_{21}^{\theta+2}N^{\theta+2} + \sigma_{31}^{\theta+2}X^{\theta+2} + \sigma_{41}^{\theta+2}P^{\theta+2}\right).$$

Let us show that, for any $W \in \mathbb{R}_+^4 \backslash D_\varepsilon$, the inequality $LV \leq -1$ holds. The set of $\mathbb{R}_+^4 \backslash D_\varepsilon$ can be divided into eight domains

$$D_1 = \{W \in \mathbb{R}_+^4 : C < \varepsilon\}, D_2 = \{W \in \mathbb{R}_+^4 : N < \varepsilon\},$$
$$D_3 = \{W \in \mathbb{R}_+^4 : X < \varepsilon\}, D_4 = \{W \in \mathbb{R}_+^4 : P < \varepsilon\},$$
$$D_5 = \left\{W \in \mathbb{R}_+^4 : C > \frac{1}{\varepsilon}\right\}, D_6 = \left\{W \in \mathbb{R}_+^4 : N > \frac{1}{\varepsilon}\right\},$$
$$D_7 = \left\{W \in \mathbb{R}_+^4 : X > \frac{1}{\varepsilon}\right\}, D_8 = \left\{W \in \mathbb{R}_+^4 : P > \frac{1}{\varepsilon}\right\}.$$

We select ε to be sufficiently small so that
(i) $-\frac{1}{\varepsilon} + G \leq -1$,
(ii) $\frac{(Mc_1+1)r_1\varepsilon}{K_1} + \frac{(Mc_2+1)r_2\varepsilon}{K_2} + m_2\varepsilon \leq 1$,
(iii) $-\frac{1-\theta}{8}\frac{1}{\varepsilon^{\theta+2}}\sigma_{11}^{\theta+2} + H_1 \leq -1$,
(iv) $-\frac{1-\theta}{8}\frac{1}{\varepsilon^{\theta+2}}\sigma_{21}^{\theta+2} + H_2 \leq -1$,
(v) $-\frac{1-\theta}{8}\frac{1}{\varepsilon^{\theta+2}}\sigma_{31}^{\theta+2} + H_3 \leq -1$,
(vi) $-\frac{1-\theta}{8}\frac{1}{\varepsilon^{\theta+2}}\sigma_{41}^{\theta+2} + H_4 \leq -1$.

- *Case 1:* If $W \in D_1$, we have

$$\begin{aligned} LV \leq & -\frac{1}{C} + \frac{(Mc_1+1)r_1NX + (Mc_2+1)r_2CX}{(K_1+C)(K_2+N)} \\ & + m_2X + F + A - E \\ := & -\frac{1}{C} + G \leq -1. \end{aligned}$$

- *Case 2:* If $W \in D_2$, we have

$$\begin{aligned} LV \leq & -\frac{1}{N} + \frac{(Mc_1+1)r_1NX + (Mc_2+1)r_2CX}{(K_1+C)(K_2+N)} \\ & + m_2X + F + A - E \\ := & -\frac{1}{N} + G \leq -1. \end{aligned}$$

- *Case 3:* If $W \in D_3$, we have

$$\begin{aligned} LV \leq & -M\lambda + \frac{(Mc_1+1)r_1NX + (Mc_2+1)r_2CX}{(K_1+C)(K_2+N)} \\ & + m_2X + F + A - E \\ \leq & -M\lambda + \frac{(Mc_1+1)r_1\varepsilon}{K_1} + \frac{(Mc_2+1)r_2\varepsilon}{K_2} \\ & + m_2\varepsilon + F + A - E \end{aligned}$$

$$
\begin{aligned}
&:= -M\lambda + \frac{(Mc_1+1)r_1\varepsilon}{K_1} + \frac{(Mc_2+1)r_2\varepsilon}{K_2} + m_2\varepsilon + G \\
&\le -1.
\end{aligned}
$$

- *Case 4:* If $W \in D_4$, we have

$$
\begin{aligned}
LV \le& -\frac{1}{P} + \frac{(Mc_1+1)r_1NX + (Mc_2+1)r_2CX}{(K_1+C)(K_2+N)} \\
&+ m_2X + F + A - E \\
:=& -\frac{1}{P} + G \le -1.
\end{aligned}
$$

- *Case 5:* If $W \in D_5$, we have

$$
\begin{aligned}
LV \le& -\frac{1-\theta}{8}\sigma_{11}^{\theta+2}C^{\theta+2} - \frac{1-\theta}{8}\sigma_{11}^{\theta+2}C^{\theta+2} \\
&+ \frac{(Mc_1+1)r_1NX + (Mc_2+1)r_2CX}{(K_1+C)(K_2+N)} + +m_2X + F + A \\
&- \frac{1-\theta}{4}\left(\sigma_{21}^{\theta+2}N^{\theta+2} + \sigma_{31}^{\theta+2}X^{\theta+2} + \sigma_{41}^{\theta+2}P^{\theta+2}\right) \\
\le& -\frac{1-\theta}{8}\frac{1}{\varepsilon^{\theta+2}}\sigma_{11}^{\theta+2} - \frac{1-\theta}{8}\sigma_{11}^{\theta+2}C^{\theta+2} \\
&+ \frac{(Mc_1+1)r_1NX + (Mc_2+1)r_2CX}{(K_1+C)(K_2+N)} + m_2X + F + A \\
&- \frac{1-\theta}{4}\left(\sigma_{21}^{\theta+2}N^{\theta+2} + \sigma_{31}^{\theta+2}X^{\theta+2} + \sigma_{41}^{\theta+2}P^{\theta+2}\right) \\
:=& \frac{1-\theta}{8}\frac{1}{\varepsilon^{\theta+2}}\sigma_{11}^{\theta+2} + \phi_1 \\
\le& -\frac{1-\theta}{8}\frac{1}{\varepsilon^{\theta+2}}\sigma_{11}^{\theta+2} + H_1 \le -1,
\end{aligned}
$$

where $H_1 = \sup\left\{\phi_1 \mid W \in \mathbb{R}_+^4\right\}$.

- *Case 6:* If $W \in D_6$, we have

$$
\begin{aligned}
LV \le& -\frac{1-\theta}{8}\sigma_{21}^{\theta+2}N^{\theta+2} - \frac{1-\theta}{8}\sigma_{21}^{\theta+2}N^{\theta+2} \\
&+ \frac{(Mc_1+1)r_1NX + (Mc_2+1)r_2CX}{(K_1+C)(K_2+N)} + m_2X + F + A \\
&- \frac{1-\theta}{4}\left(\sigma_{11}^{\theta+2}C^{\theta+2} + \sigma_{31}^{\theta+2}X^{\theta+2} + \sigma_{41}^{\theta+2}P^{\theta+2}\right) \\
\le& -\frac{1-\theta}{8}\frac{1}{\varepsilon^{\theta+2}}\sigma_{21}^{\theta+2} - \frac{1-\theta}{8}\sigma_{21}^{\theta+2}N^{\theta+2} \\
&+ \frac{(Mc_1+1)r_1NX + (Mc_2+1)r_2CX}{(K_1+C)(K_2+N)} + +m_2X + F + A \\
&- \frac{1-\theta}{4}\left(\sigma_{11}^{\theta+2}C^{\theta+2} + \sigma_{31}^{\theta+2}X^{\theta+2} + \sigma_{41}^{\theta+2}P^{\theta+2}\right) \\
:=& -\frac{1-\theta}{8}\frac{1}{\varepsilon^{\theta+2}}\sigma_{21}^{\theta+2} + \phi_2
\end{aligned}
$$

$$\leq -\frac{1-\theta}{8}\frac{1}{\varepsilon^{\theta+2}}\sigma_{21}^{\theta+2} + H_2 \leq -1,$$

where $H_2 = \sup\{\phi_2 \mid W \in \mathbb{R}_+^4\}$.

- *Case 7:* If $W \in D_7$, we have

$$\begin{aligned} LV \leq & -\frac{1-\theta}{8}\sigma_{31}^{\theta+2}X^{\theta+2} - \frac{1-\theta}{8}\sigma_{31}^{\theta+2}X^{\theta+2} \\ & + \frac{(Mc_1+1)r_1NX + (Mc_2+1)r_2CX}{(K_1+C)(K_2+N)} + +m_2X + F + A \\ & - \frac{1-\theta}{4}\left(\sigma_{11}^{\theta+2}C^{\theta+2} + \sigma_{21}^{\theta+2}N^{\theta+2} + \sigma_{41}^{\theta+2}P^{\theta+2}\right) \\ \leq & -\frac{1-\theta}{8}\frac{1}{\varepsilon^{\theta+2}}\sigma_{31}^{\theta+2} - \frac{1-\theta}{8}\sigma_{31}^{\theta+2}X^{\theta+2} \\ & + \frac{(Mc_1+1)r_1NX + (Mc_2+1)r_2CX}{(K_1+C)(K_2+N)} + m_2X + F + A \\ & - \frac{1-\theta}{4}\left(\sigma_{11}^{\theta+2}C^{\theta+2} + \sigma_{21}^{\theta+2}N^{\theta+2} + \sigma_{41}^{\theta+2}P^{\theta+2}\right) \\ := & -\frac{1-\theta}{8}\frac{1}{\varepsilon^{\theta+2}}\sigma_{31}^{\theta+2} + \phi_3 \\ \leq & -\frac{1-\theta}{8}\frac{1}{\varepsilon^{\theta+2}}\sigma_{31}^{\theta+2} + H_3 \leq -1, \end{aligned}$$

where $H_3 = \sup\{\phi_3 \mid W \in \mathbb{R}_+^4\}$.

- *Case 8:* If $W \in D_8$, we have

$$\begin{aligned} LV \leq & -\frac{1-\theta}{8}\sigma_{41}^{\theta+2}P^{\theta+2} - \frac{1-\theta}{8}\sigma_{41}^{\theta+2}P^{\theta+2} \\ & + \frac{(Mc_1+1)r_1NX + (Mc_2+1)r_2CX}{(K_1+C)(K_2+N)} + m_2X + F + A \\ & - \frac{1-\theta}{4}\left(\sigma_{11}^{\theta+2}C^{\theta+2} + \sigma_{21}^{\theta+2}N^{\theta+2} + \sigma_{31}^{\theta+2}X^{\theta+2}\right) \\ \leq & -\frac{1-\theta}{8}\frac{1}{\varepsilon^{\theta+2}}\sigma_{41}^{\theta+2} - \frac{1-\theta}{8}\sigma_{41}^{\theta+2}P^{\theta+2} \\ & + \frac{(Mc_1+1)r_1NX + (Mc_2+1)r_2CX}{(K_1+C)(K_2+N)} + +m_2X + F + A \\ & - \frac{1-\theta}{4}\left(\sigma_{11}^{\theta+2}C^{\theta+2} + \sigma_{21}^{\theta+2}N^{\theta+2} + \sigma_{31}^{\theta+2}X^{\theta+2}\right) \\ := & -\frac{1-\theta}{8}\frac{1}{\varepsilon^{\theta+2}}\sigma_{41}^{\theta+2} + \phi_4 \\ \leq & -\frac{1-\theta}{8}\frac{1}{\varepsilon^{\theta+2}}\sigma_{41}^{\theta+2} + H_4 \leq -1, \end{aligned}$$

where $H_4 = \sup\{\phi_4 \mid W \in \mathbb{R}_+^4\}$.

Thus, a sufficiently small ε exists such that

$$LV(W) \leq -1, \quad \forall\, W \in \mathbb{R}_+^4 \setminus D_\varepsilon,$$

i.e., the condition (ii) is also verified. □

For model (2.2), the existence of a unique stationary distribution indicates that microorganisms can persist when the stochastic threshold $R_0^s > 1$.

3.3 Extinction

First, define

$$R_1 = \frac{r\left(\int_0^\infty x^2\pi_1(x)dx\right)^{\frac{1}{2}}\left(\int_0^\infty y^2\pi_2(y)dy\right)^{\frac{1}{2}}}{K_1K_2 + (K_1+K_2+1)\left(\int_0^\infty x^2\pi_1(x)dx\right)^{\frac{1}{2}}\left(\int_0^\infty y^2\pi_2(y)dy\right)^{\frac{1}{2}}\left(1+\frac{\sigma_{32}^2}{2}\right)},$$

$$\tilde{R}_1 = \frac{r}{(K_1+K_2+1)\left(1+\frac{\sigma_{32}^2}{2}\right)}.$$

Theorem 3.3 *Given any initial value $W_0 \in \mathbb{R}_+^4$, let $W(t)$ denote the solution of model (2.2). If*

$$R_1 < 1 \text{ or } \tilde{R}_1 < 1,$$

then

$$\lim_{t\to\infty} X(t) = 0 \ a.s.$$

Namely, under the nonlinear perturbation, the microorganisms in model (2.2) will go extinct exponentially with probability one.

Proof *Itô* formula [38] to *lnX*

$$d\ln X = \left(\frac{rCN}{(K_1+C)(K_2+N)} - 1 - m_1P - \frac{\sigma_{32}^2}{2} - \sigma_{31}\sigma_{32}X - \frac{\sigma_{31}^2}{2}X^2\right)dt + (\sigma_{31}X + \sigma_{32})\,dB_3(t), \tag{3.1}$$

then integrating both sides of (3.1) from 0 to t and dividing each term by t

$$\begin{aligned}\frac{\ln X(t) - \ln X(0)}{t} &= \frac{r}{t}\int_0^t \frac{C(s)N(s)}{(K_1+C(s))(K_2+N(s))}ds - \left(1+\frac{\sigma_{32}^2}{2}\right)\\ &\quad - \frac{m_1}{t}\int_0^t P(s)ds - \frac{\sigma_{31}\sigma_{32}}{t}\int_0^t X(s)ds - \frac{\sigma_{31}^2}{2t}\int_0^t X^2(s)ds\\ &\quad + \frac{\sigma_{31}}{t}\int_0^t X(s)dB_3(s) + \sigma_{32}\frac{B_3(t)}{t}\\ &\le \frac{r}{t}\int_0^t \frac{C(s)N(s)}{(K_1+C(s))(K_2+N(s))}ds - \left(1+\frac{\sigma_{32}^2}{2}\right)\\ &\quad - \frac{\sigma_{31}^2}{2t}\int_0^t X^2(s)ds + \sigma_{32}\frac{B_3(t)}{t} + \frac{M(t)}{t},\end{aligned} \tag{3.2}$$

where $M(t) = \sigma_{31}\int_0^t X(s)dB_3(s)$ is a martingale and its quadratic variation is

$$\langle M,M\rangle(t) = \sigma_{31}^2\int_0^t X^2(s)ds.$$

Considering the exponential martingales inequality [36]

$$\mathbb{P}\left\{\sup_{0\leq t\leq k}\left[M(t)-\frac{1}{2}\langle M,M\rangle(t)\right]>2\ln k\right\}\leq\frac{1}{k^2}.$$

Using the Borel-Cantelli lemma [36], it can be found that for almost all $\omega\in\Omega$, there exits $k_0(\omega)$ such that for $k\geq k_0(\omega)$, we get

$$\sup_{0\leq t\leq k}\left[M(t)-\frac{1}{2}\langle M,M\rangle(t)\right]\leq 2\ln k,$$

then

$$M(t)\leq 2\ln k+\frac{1}{2}\langle M,M\rangle(t)=2\ln k+\frac{1}{2}\sigma_{31}^2\int_0^t X^2(s)ds, a.s. \tag{3.3}$$

For $0\leq k-1\leq t\leq k$, substituting (3.3) into (3.2) yields

$$\begin{aligned}\frac{\ln X(t)}{t}\leq{}&\frac{\ln X(0)}{t}+\frac{r}{t}\int_0^t\frac{C(s)N(s)}{(K_1+C(s))(K_2+N(s))}ds-\left(1+\frac{\sigma_{32}^2}{2}\right)\\&+\sigma_{32}\frac{B_3(t)}{t}+\frac{2\ln k}{k-1}.\end{aligned} \tag{3.4}$$

Using Hölder's inequality and Lemma 2.1, we have

$$\begin{aligned}\frac{1}{t}\int_0^t C(s)N(s)ds&\leq\left(\frac{1}{t}\int_0^t C^2(s)ds\right)^{\frac{1}{2}}\left(\frac{1}{t}\int_0^t N^2(s)ds\right)^{\frac{1}{2}}\\&\leq\left(\frac{1}{t}\int_0^t c^2(s)ds\right)^{\frac{1}{2}}\left(\frac{1}{t}\int_0^t n^2(s)ds\right)^{\frac{1}{2}}.\end{aligned} \tag{3.5}$$

From Lemma 4.1, Case (i) of [24] and $\lim_{t\to\infty}\frac{\sigma_{32}}{t}B_3(t)=0$ *a.s.*, we note that $limsup_{t\to\infty}\frac{1}{t}\int_0^t C(s)ds\leq\lim_{t\to\infty}\frac{1}{t}\int_0^t c(s)ds=1$ and $limsup_{t\to\infty}\frac{1}{t}\int_0^t N(s)ds\leq\lim_{t\to\infty}\frac{1}{t}\int_0^t n(s)ds=1$ *a.s.* Based on (3.4), we obtain

$$\begin{aligned}\lim_{t\to\infty}sup\frac{\ln X(t)}{t}\leq&\frac{r\left(\int_0^\infty x^2\pi_1(x)dx\right)^{\frac{1}{2}}\left(\int_0^\infty y^2\pi_2(y)dy\right)^{\frac{1}{2}}}{K_1K_2+(K_1+K_2+1)\left(\int_0^\infty x^2\pi_1(x)dx\right)^{\frac{1}{2}}\left(\int_0^\infty y^2\pi_2(y)dy\right)^{\frac{1}{2}}}\\&-\left(1+\frac{\sigma_{32}^2}{2}\right)=\left(1+\frac{\sigma_{32}^2}{2}\right)(R_1-1)\leq 0.\end{aligned}$$

Similarly, from the Lemma 4.1, Case (ii) of [24], we obtain

$$\lim_{t\to\infty}\sup\frac{\ln X(t)}{t}\leq\frac{r}{K_1+K_2+1}-1-\frac{\sigma_{32}^2}{2}=\left(1+\frac{\sigma_{32}^2}{2}\right)(\tilde{R}_1-1)\leq 0.$$

So, $\lim_{t\to\infty}X(t)=0$ *a.s.* □

The threshold R_0^s of the nonlinear noise model (2.2) can be transformed into the threshold $\bar{R}_0^s = \frac{r}{\left(1+\frac{1}{2}\sigma_1^2\right)\left(1+\frac{1}{2}\sigma_2^2\right)(K_1+1)(K_2+1)\left(1+m_1+\frac{1}{2}\sigma_3^2\right)}$ of the linear noise model (2.1) or the basic reproduction number $R_0 = \frac{r}{(1+m_1)(K_1+1)(K_2+1)}$ of model (1.1) when $\sigma_{i1} = 0$ $(i = 1,2,3,4)$ or $\sigma_{ij} = 0$ $(i = 1,2,3,4, j = 1,2)$.

4 Numerical simulations

The Euler-Milstein method has significant advantages compared to the Euler-Maruyama method, as described in [36, 39, 40], and [41]. Its strong convergence order of 1 provides higher accuracy and demonstrates superior performance in handling systems with nonlinear diffusion terms. The numerical discretization of the Euler-Milstein method is given by $X_{n+1} = X_n + \mu(X_n,t_n)\Delta t + \sigma(X_n,t_n)\Delta W_n + \frac{1}{2}\sigma(X_n,t_n)\frac{\partial\sigma(X_n,t_n)}{\partial X}\left((\Delta W_n)^2 - \Delta t\right)$, we can get the numerical discretization scheme for system (2.2)

$$\begin{cases} C_{n+1} = C_n + \left(1 - C_n - \dfrac{r_1 C_n N_n X_n}{(K_1 + C_n)(K_2 + N_n)}\right)\Delta t + (\sigma_{11}C_n + \sigma_{12})C_n\sqrt{\Delta t}\xi_n \\ \qquad + \dfrac{C_n}{2}(2\sigma_{11}^2 C_n^2 + 3\sigma_{11}\sigma_{12}C_n + \sigma_{12}^2)(\xi_n^2 - 1)\Delta t, \\ N_{n+1} = N_n + \left(1 - N_n - \dfrac{r_2 C_n N_n X_n}{(K_1 + C_n)(K_2 + N_n)}\right)\Delta t + (\sigma_{21}N_n + \sigma_{22})N_n\sqrt{\Delta t}\varsigma_n \\ \qquad + \dfrac{N_n}{2}(2\sigma_{21}^2 N_n^2 + 3\sigma_{21}\sigma_{22}N_n + \sigma_{22}^2)(\varsigma_n^2 - 1)\Delta t, \\ X_{n+1} = X_n + \left(\dfrac{rC_n N_n X_n}{(K_1 + C_n)(K_2 + N_n)} - X_n - m_1 X_n P_n\right)\Delta t + (\sigma_{31}X_n + \sigma_{32})X_n \\ \qquad \sqrt{\Delta t}\zeta_n + \dfrac{X_n}{2}(2\sigma_{31}^2 X_n^2 + 3\sigma_{31}\sigma_{32}X_n + \sigma_{32}^2)(\zeta_n^2 - 1)\Delta t, \\ P_{N+1} = P_n + (1 - P_n - m_2 X_n P_n)\Delta t + (\sigma_{41}P_n + \sigma_{42})P_n\sqrt{\Delta t}\ell_n \\ \qquad + \dfrac{P_n}{2}(2\sigma_{41}^2 P_n^2 + 3\sigma_{41}\sigma_{42}P_n + \sigma_{42}^2)(\ell_n^2 - 1)\Delta t, \end{cases}$$

where Δt indicates the time increments. The variables ξ_n, ς_n, ζ_n, and ℓ_n are independent Gaussian random variables, each following the distribution $N(0,1)$. We conduct numerical simulations to illustrate and validate our main results, offering deeper insights of the system behavior under various conditions.

The parameters listed in Table 2 are used.

Table 2 Parameter Symbols and Their Descriptions

Symbol	Description	Unit
r	Microorganism growth rate	hour^{-1}
r_1	Consumption rate of carbon source	hour^{-1}
r_2	Consumption rate of nitrogen source	hour^{-1}
m_1	Interaction coefficient of microorganisms and flocculants	mL/(μg · hour)
m_2	Degradation rate of flocculants	mL/(μg · hour)
K_1	Half-saturation constant of carbon source response	μg/mL
K_2	Half-saturation constant of nitrogen source response	μg/mL

4.1 Stationary distribution

Example 1 According to [24], we choose appropriate parameters in model (2.2) as follows

$$r = 8.0, r_1 = 6.0, r_2 = 8.0, m_1 = 0.01, m_2 = 1.0, K_1 = 0.1, K_2 = 0.6,$$

$$\sigma_{11} = \sigma_{21} = \sigma_{31} = \sigma_{41} = 0.01, \sigma_{12} = \sigma_{22} = \sigma_{32} = \sigma_{42} = 0.2.$$

By calculation, we obtain

$$R_0^s = \frac{r}{a_1 a_2 a_3 (K_1 + 1)(K_2 + 1)} \approx 1.3773 > 1,$$

in accordance with the conditions specified in Theorem 3.2, the model (2.2) exists as a unique stationary distribution. Figures 1–3 shows the corresponding numerical simulation results. The deterministic model has a positive equilibrium point

$$E^* = (C^*, N^*, X^*, P^*) \approx (0.3376, 0.1168, 0.8786, 0.5323).$$

For comparison with system (2.1), we choose $\sigma_1 = \sigma_2 = \sigma_3 = \sigma_4 = 0.2$. This is equivalent to $\sigma_{11} = \sigma_{21} = \sigma_{31} = \sigma_{41} = 0$ in system (2.2). Figure 4 shows the corresponding numerical simulation results. Then, the nonlinear perturbation makes the microorganism growth process more complicated.

Example 2 To demonstrate the effect of nonlinear perturbation on model (2.2), we select $\sigma_{11} = \sigma_{21} = \sigma_{31} = \sigma_{41} = 0.2, \sigma_{12} = \sigma_{22} = \sigma_{32} = \sigma_{42} = 0$. The selection of all other parameters is identical to that specified in Example 1, maintaining consistency for comparison purposes. Figure 5 shows the corresponding numerical simulation results.

4.2 Extinction of microorganisms

Example 3 If the flocculation effect (m_1) is increased $m_1 = 0.01$ to $m_1 = 1.62$, the other parameters are the same as in Example 1. By calculating, $R_0^s \approx 0.4931$ does not satisfy Theorem 3.2. The microorganisms go extinct. Figure 6 shows the corresponding numerical simulation results.

Example 4 The selection of parameters is as follows

$$r = 1.7,\ r_1 = 0.96,\ r_2 = 1.001,\ m_1 = 0.005,\ m_2 = 4.0,\ K_1 = 0.36,\ K_2 = 0.035,$$

$$\sigma_{11} = \sigma_{21} = \sigma_{31} = \sigma_{41} = 0.01,\ \sigma_{12} = \sigma_{22} = \sigma_{32} = \sigma_{42} = 0.7.$$

We have $R_1 \approx 0.97066$ and $\tilde{R}_1 \approx 0.97883$. So, the microorganisms extinct with probability one. Figure 7 shows the corresponding numerical simulation results.

Example 5 The selection of parameters is as follows

$$r = 3.5,\ r_1 = 0.96,\ r_2 = 1.001,\ m_1 = 1.5,\ m_2 = 4.0,\ K_1 = 0.36,\ K_2 = 0.3,$$

$$\sigma_{11} = \sigma_{21} = \sigma_{31} = \sigma_{41} = 0.01,\ \sigma_{12} = \sigma_{22} = \sigma_{32} = \sigma_{42} = 0.2.$$

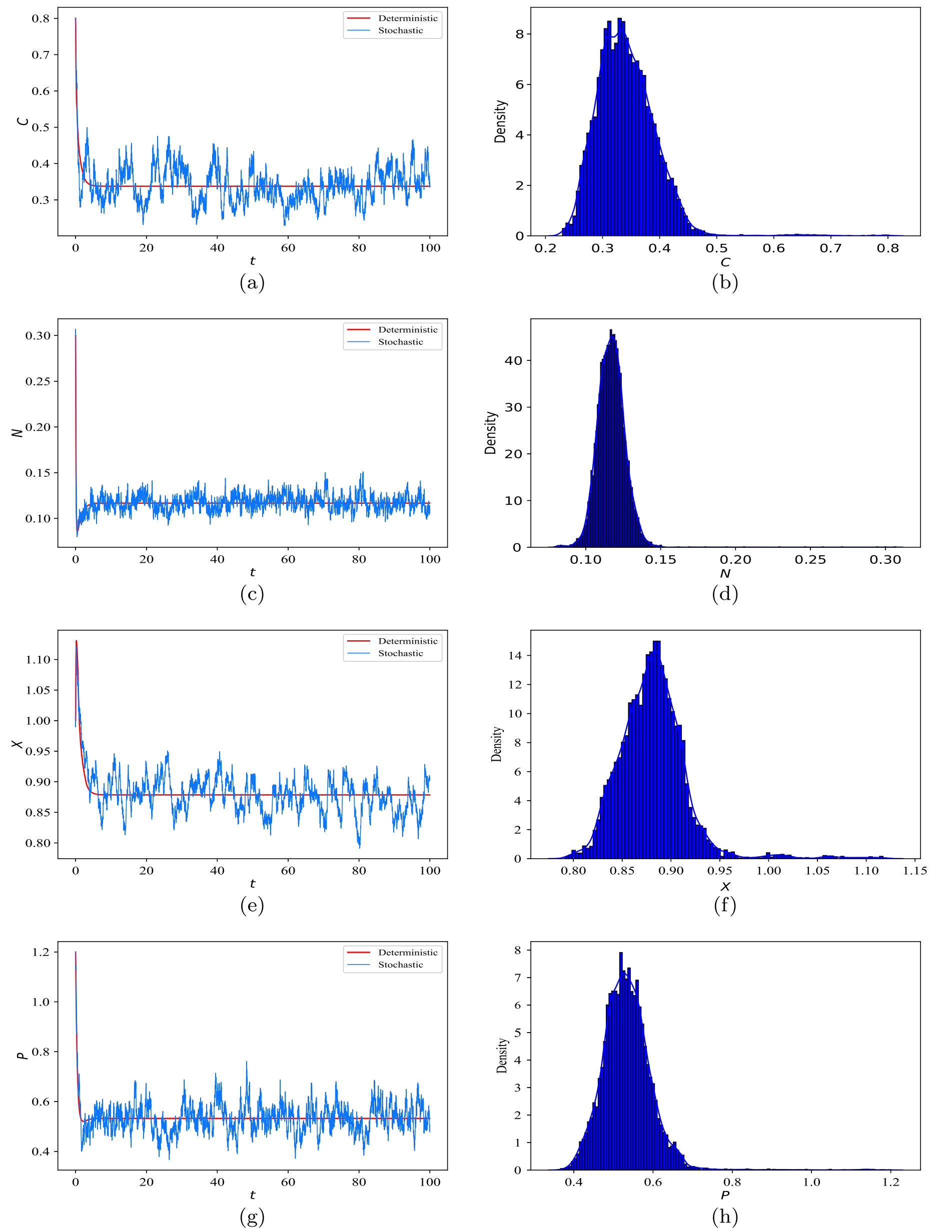

Figure 1 The blue fluctuating lines in the left figures stand for the solutions of system (2.2), while the red smooth curves correspond to the solutions of system (1.1). The right figures show the empirical density distributions. The initial value is set as $(C_0, N_0, X_0, P_0) = (0.8, 0.3, 1.0, 1.2)$

We have $R_1 \approx 1.77197$ and $\tilde{R}_1 \approx 2.09794$. The microorganisms are not extinct, but deterministic model (1.1) has a stable boundary equilibrium point $E = (1, 1, 0, 1)$. Figure 8 shows the corresponding numerical simulation results.

Example 6 Based on the parameters in Example 4, we increase $\sigma_{11} = \sigma_{21} = \sigma_{31} = \sigma_{41} = 0.01$ to $\sigma_{11} = \sigma_{21} = \sigma_{31} = \sigma_{41} = 0.4$. The probability of microorganisms extinction is one. Figure 9 shows the corresponding numerical simulation results.

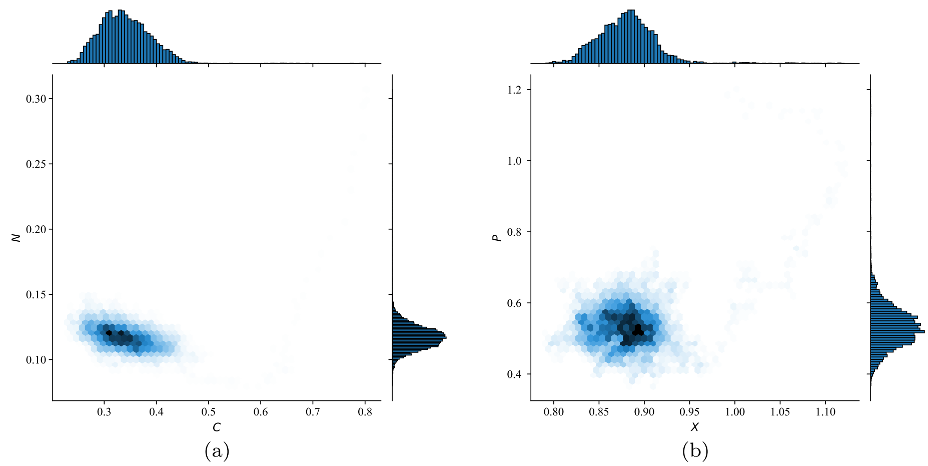

Figure 2 Figures (*a*) and (*b*) are the hexbin plots illustrating the 2*D* distributions of $(C(t), N(t))$ and $(X(t), P(t))$, respectively

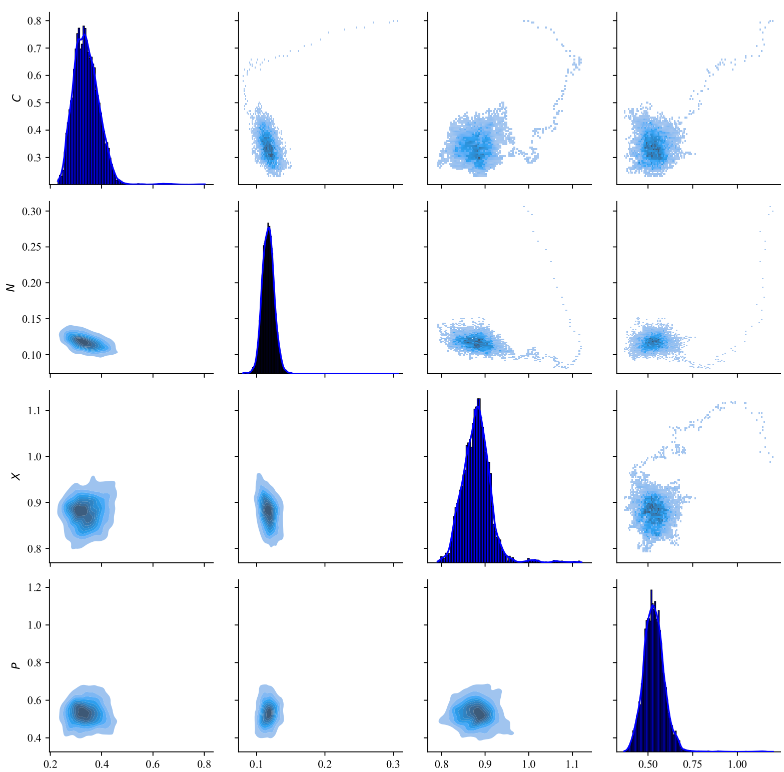

Figure 3 The discrete and continuous empirical density distributions for $C(t)$, $N(t)$, $X(t)$, and $P(t)$ are presented as their respective representations

Figure 1 shows that the paths of the stochastic model ultimately oscillate around the positive equilibrium point of the ODEs model and possess a stationary distribution. In Fig. 4, although the amplitude of oscillations around the positive equilibrium point for carbon source, nitrogen source, and flocculant in the two stochastic models does not vary significantly, it is evident that nonlinear perturbation has a greater impact on microorganism

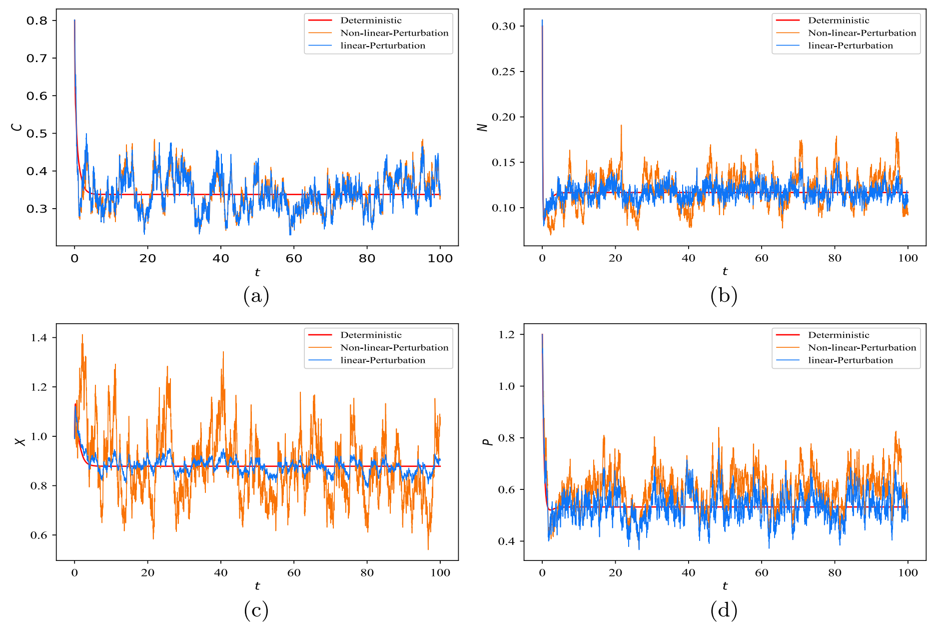

Figure 4 In the comparison of stochastic nonlinear and linear perturbation, the initial value is set as $(C_0, N_0, X_0, P_0) = (0.8, 0.3, 1.0, 1.2)$. In the figures, the orange fluctuating lines stand for the solutions of system (2.2). The red smooth curves correspond to the solutions of system (1.1). The blue fluctuating lines stand for the solutions of system (2.1)

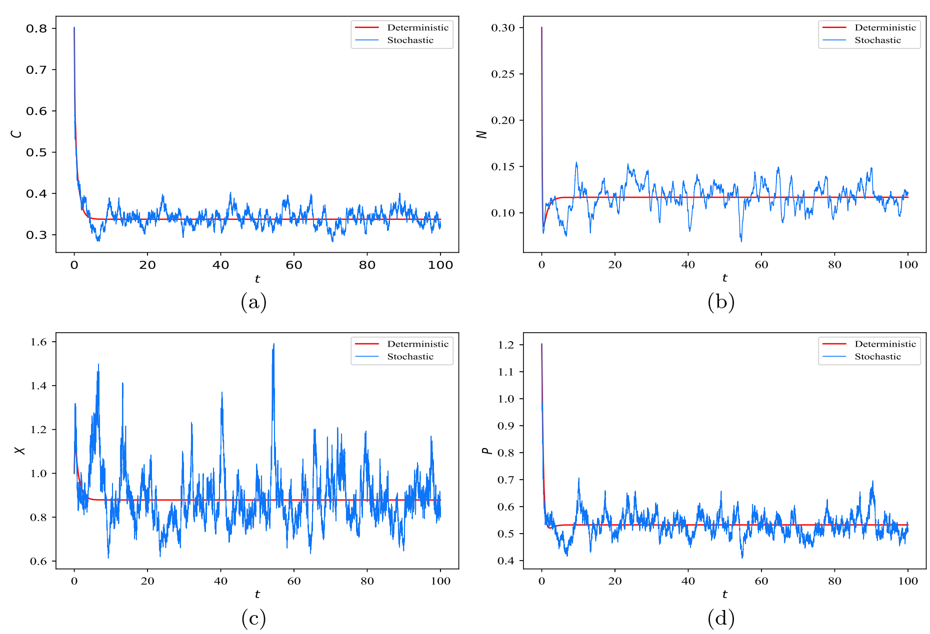

Figure 5 The initial value is set as $(C_0, N_0, X_0, P_0) = (0.8, 0.3, 1.0, 1.2)$. In the figures, the blue fluctuating lines stand for the solutions of system (2.2), and the red smooth curves correspond to the solutions of system (1.1)

growth compared to linear perturbation. Figure 5 further demonstrates that the nonlinear perturbation model significantly influences microbial growth. Figure 6 illustrates that as the parameter m_1 increases, microorganisms in the stochastic model go extinct, whereas

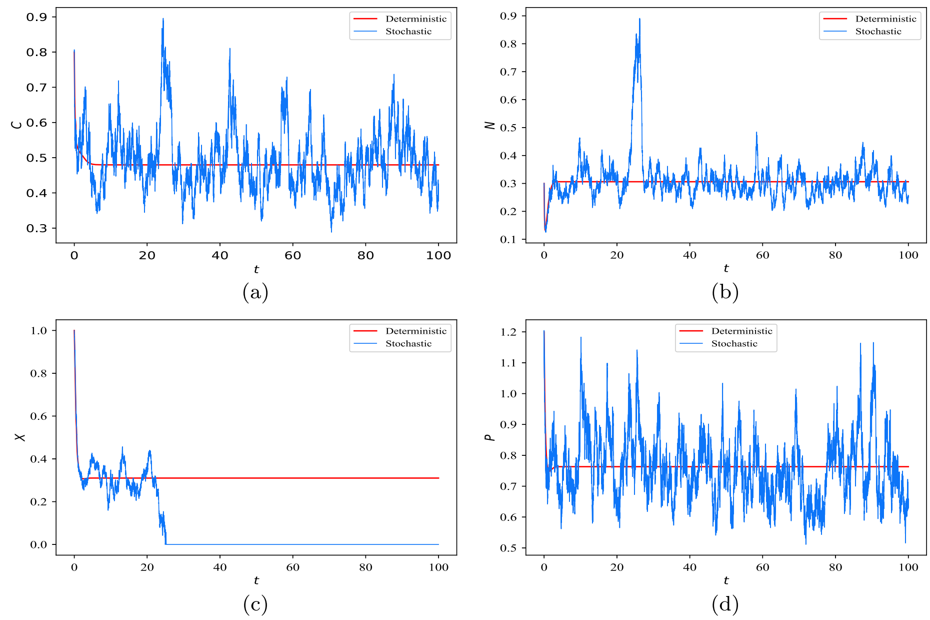

Figure 6 The initial value is set as $(C_0, N_0, X_0, P_0) = (0.8, 0.3, 1.0, 1.2)$. In the figures, the blue fluctuating lines stand for the solutions of system (2.2) with $m_1 = 1.62$, and the red smooth curves correspond to the solutions of system (1.1)

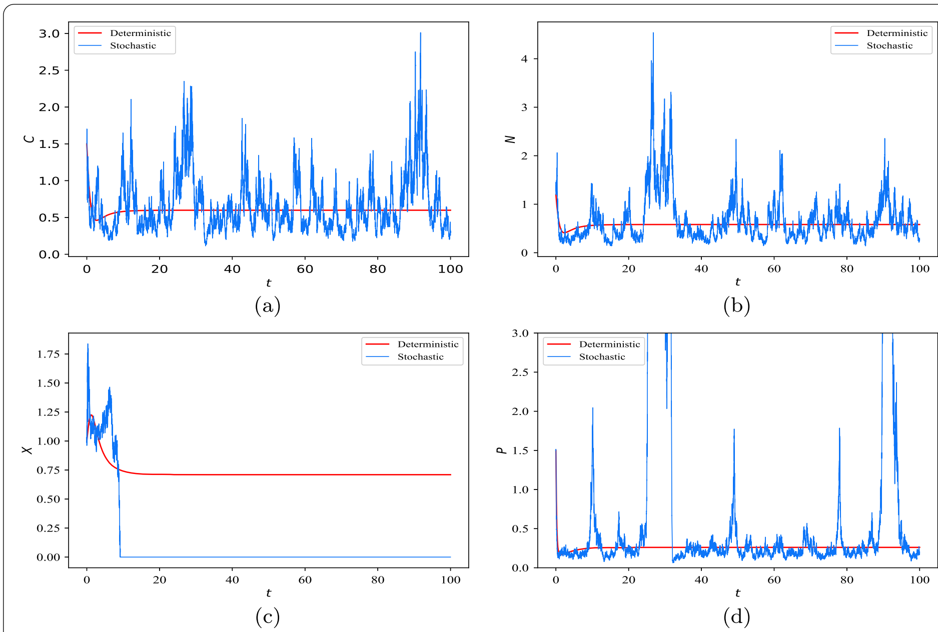

Figure 7 The initial value is set as $(C_0, N_0, X_0, P_0) = (1.5, 1.2, 1.0, 1.5)$. In the figures, the blue fluctuating lines stand for the solutions of system (2.2), and the red smooth curves correspond the solutions of system (1.1)

the ODEs model retains a positive equilibrium point. Similarly, Fig. 7 shows that increasing the noise level leads to a similar extinction effect in the stochastic model, consistent with the trend observed in Fig. 6. In Fig. 8, while microorganisms in the ODEs model go ex-

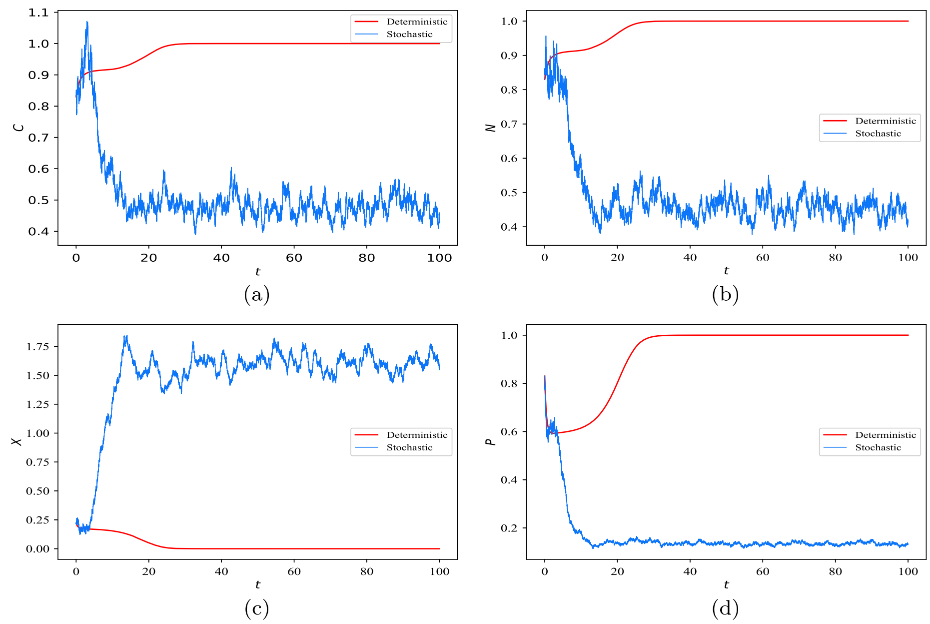

Figure 8 The initial value is set as $(C_0, N_0, X_0, P_0) = (0.84, 0.84, 0.22, 0.84)$. In the figures, the blue fluctuating lines stand for the solutions of system (2.2) with $\sigma_{11} = \sigma_{21} = \sigma_{31} = \sigma_{41} = 0.01$, and the red smooth curves correspond the solutions of system (1.1)

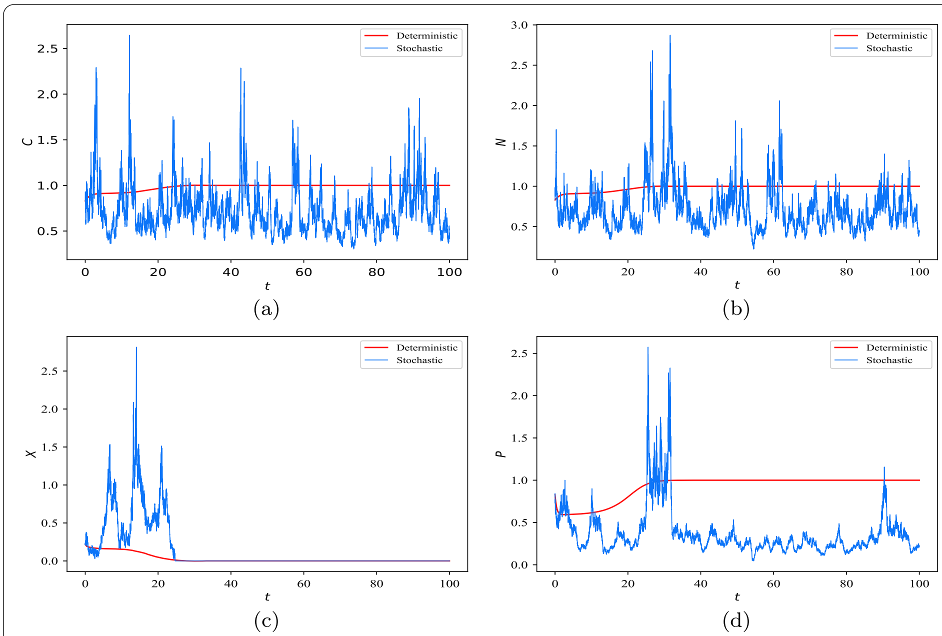

Figure 9 The initial value is set as $(C_0, N_0, X_0, P_0) = (0.84, 0.84, 0.22, 0.84)$. In the figures, the blue fluctuating lines stand for the solutions of system (2.2) with $\sigma_{11} = \sigma_{21} = \sigma_{31} = \sigma_{41} = 0.4$, and the red smooth curves correspond the solutions of system (1.1)

tinct, those in the stochastic model can survive under an appropriate noise level. However, Fig. 9 shows that further increasing the noise level based on Fig. 8 results in the extinction of microorganisms in both models.

5 Conclusion

Environmental noises, such as temperature, humidity, nutrient concentration, oxygen concentration, and other factors, can affect the growth rate of microorganisms. Therefore, incorporating environmental noises into the model of microorganism flocculation is both crucial and meaningful, as it more accurately reflects real-world dynamics and uncertainties. First, we obtain the existence and uniqueness of the solution to the stochastic model (2.2). Next, we prove that the model (2.2) has a unique ergodic stationary distribution if $R_0^s > 1$, while the microorganism will go extinct if $R_1 < 1$ or $\tilde{R}_1 < 1$. We emphasize that the selection of the stochastic threshold is not unique. Finally, we verify that the conclusions depend on numerical simulations. The finding that moderate levels of environmental noise can enhance microorganism growth has profound biological implications.

The demonstrations of the stationary distribution and microorganism extinction extend the model based on ordinary differential equations. In natural environments, the growth rate of microorganisms is influenced by multiple factors, including environmental noises, the rate of nutrient uptake, and the efficiency of nutrient conversion by microorganisms. Additionally, the flocculation processes also play a significant role in microorganism growth. While this study incorporates white noise, real-world microbial environments often involve more complex noise types, such as lévy noise, telegraph noise, and colored noise. This study has not considered these aspects, and we will investigate these issues as part of future work.

Author contributions
All authors contributed equally to the research of this result. All authors have read and approved the final manuscript version.

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Data availability
Not applicable.

Declarations

Competing interests
The authors declare no competing interests.

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