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Optimal harvest strategies with catch-dependent pricing for chub mackerel in South Korea

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Abstract

Chub mackerel (Scomber japonicus) is a key commercial species in South Korea. However, the catch volume of chub mackerel has experienced significant fluctuations over the past few decades, with current trends indicating a decline. Despite regulatory measures such as closed seasons, resource depletion remains a concern, thereby highlighting the requirement for effective management strategies. Numerous previous studies have proposed optimal harvest strategies by assuming constant prices. However, as large catches of mackerel tend to have lower prices, it is crucial to develop optimal harvest strategies that account for this decrease. Thus, we aim to develop a monthly optimal harvest strategy for chub mackerel that considers catch-dependent pricing. We define logarithmic, rational, and irrational catch-dependent price functions and their corresponding objective functions. In addition, we develop an optimal control system based on a discrete age-structured model. We use Pontryagin's maximum principle to prove the necessary conditions for the optimal harvest strategy under the three catch-dependent pricing functions and perform simulations using the forward–backward sweep method. We compare the optimal harvest strategies under the three catch-dependent pricing scenarios with those under a constant price. The optimal harvest strategies with the rational and irrational price functions are similar to those with a constant price, where the fishing effort increases immediately after spawning and then gradually decreases. In contrast, the optimal harvest strategy with the logarithmic price function involves a gradual increase in fishing effort from July immediately after the spawning period, with the maximum effort in June before the next spawning season. In addition, we compare the effects of monthly closed seasons across the four pricing scenarios. A closed season in July immediately after spawning provides the highest resource recovery efficiency. In contrast, a closed season in June provides the highest catch and profit efficiencies. As the cost per unit of effort increases, the fishing effort, catch, and profit decrease, while the biomass increases, and the profit decrease is smallest under the logarithmic price function. Our method can improve monthly optimal harvest strategies for other species using catch-dependent pricing functions as well as significantly contribute to enhancing fishers' profit.

Keywords: Chub mackerel; Optimal harvest strategy; Catch-dependent pricing; Closed season

1 Introduction

Mackerel are widely distributed in the temperate and subtropical regions of the Pacific, Atlantic, and Indian oceans [1–3]. Chub mackerel is a turbulent and migratory species that lives in the coastal waters of South Korea. They migrate to the northern coast from spring to summer, then to the southern coast from autumn to winter for wintering, and spawn in the eastern waters of Jeju Island in April and June [4, 5].

Chub mackerel is an important commercial fish species in South Korea. The catch of chub mackerel in South Korea continuously increased from 38,256 tons in 1970 to more than 400,000 tons since the mid-1990s. However, after reaching the maximum, the catch decreased to approximately 100,000–200,000 tons in the early 2000s. Since then, the annual catch has shown increasing and decreasing trends, with an overall decline [6]. There are three broad categories of control over commercial fishing: catch restrictions, effort restrictions, and spatial access control [7]. The Korean government has implemented a total allowable catch, closed season, and regulations on the catch of immature fish to control the decline in the resources of club mackerel. Overharvesting and unreasonable harvesting policies have several detrimental effects, such as ecological destruction and species extinction. Therefore, optimal harvest strategies must be established to maximize catch returns and maintain the amount of resources [8].

Optimal fishery harvesting and bioeconomical optimal fishery control have been widely studied. There are several models with different assumptions for optimal control. These are categorized into continuous [9–11] and discrete [12–15] models based on time. Wang and Wang [11] studied the optimal harvesting strategy for a single fish species in continuous time. Li and Yakubu [12] considered fish exploitation levels and recruitment dynamics at discrete times. In addition, size-structured [16–19] and age-structured models [15, 20–23] have been developed according to the categorization of objects based on size and age, respectively. Age-structured models were developed first. Getz [23] used an age-structured model with harvesting and spawning seasons. Jang and Cho [15] considered closed seasons in addition to harvesting and spawning seasons. Tahvonen [20] described the history of the development of age-structured models. Kato [18, 19] investigated nonlinear and linear size-based population models. In addition to the abovementioned models, various other types of optimal fishery harvesting strategies exist, such as stochastic [8, 24–28] and spatial [29–31] methods. Liu et al. [25, 26] developed a two-species stochastic model, and Zhu and Meng [28] extended it to an n-species stochastic model.

Few studies have considered the changes in price according to supply and demand for optimal control. Kilkki et al. [32] changed supply and demand by controlling the price. Finck et al. [33] applied real-time pricing based on supply and demand. Kebai and Senfa [34] minimized the difference between supply and demand. However, these studies did not consider optimal fishery harvesting. Studies on optimal fishery harvesting based on price changes mainly consider randomness in price, and not variations in price based on supply and demand. Andersen [35] and Lewis [36] studied an optimal harvest model under price uncertainty. Hanson and Ryan [37] considered significant inflationary price fluctuations. Brites and Braumann [38] discussed the growth dynamics of harvested fish populations considering linear price changes. However, previous studies have not proposed optimal fishing strategies based on price changes caused by fluctuations in the supply or demand of fish species.

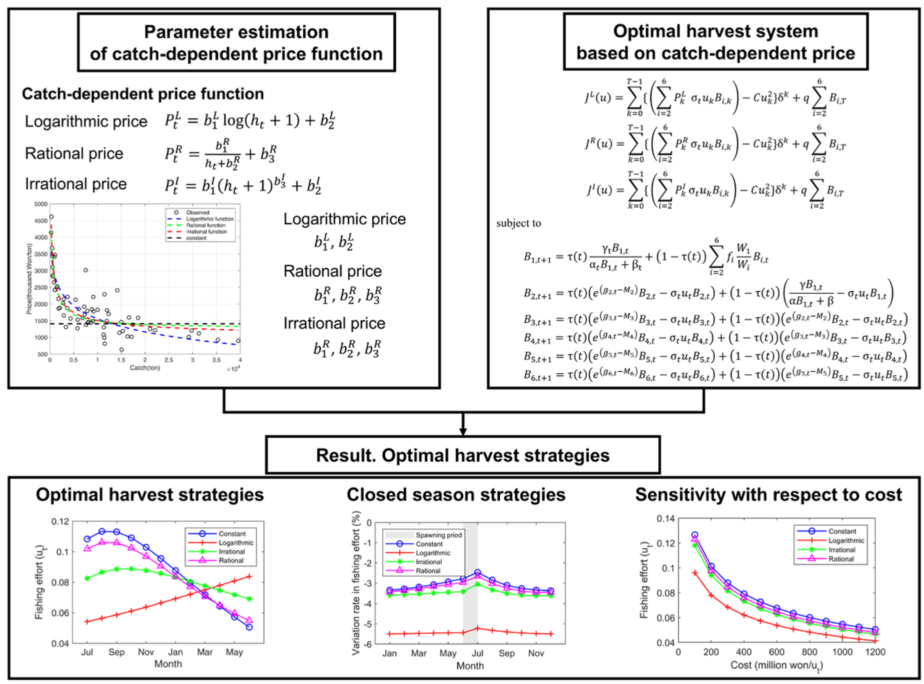

Figure 1 A schematic diagram illustrating optimal harvesting strategies for a discrete age-structured mackerel model under catch-dependent pricing

In this study, we aim to develop a model that incorporates the catch-dependent price to address the limitations of previous studies. In addition, we estimate the monthly fishing effort required to maximize fishing profit based on the catch-dependent price of chub mackerel in South Korea. We propose a discrete age-structured optimal control system to develop optimal harvest strategies. We present three catch-dependent price functions to incorporate the catch-dependent prices into the objective function of the system. Subsequently, we develop optimal harvest strategies based on these functions using Pontryagin's maximum principle and compare the strategies. In addition, we use the functions to determine the most suitable month for imposing a closed fishing season for resource recovery in South Korea. This study provides strategies for optimizing fishing effort to maximize profit while promoting sustainable practices by incorporating catch-dependent pricing and age-structured population dynamics. These findings can guide policymakers in implementing effective management strategies, such as determining the best timing for closed fishing seasons, thereby supporting resource recovery and the long-term viability of mackerel stock in South Korea.

2 Materials and methods

An optimal control system, provided by a discrete age-structured model, was used to describe the optimal harvest based on catch-dependent pricing for chub Mackerel in Korea. The process is summarized as follows: First, three catch-dependent price functions were defined, and their parameters were estimated via nonlinear regression using catch and price data. Second, based on the estimated three catch-dependent price functions, an optimal harvesting system was employed to present the optimal harvesting strategies for each function. Third, the optimal harvest strategies and closed-season policies under the

Table 1 Parameters of age-structured model

Parameter	Description	Value	Source
K	von Bertalanffy growth parameter	2.19	Hwang et al., 2008 [39]
t_0	Theoretical age when size is zero	−0.0035	Hwang et al., 2008 [39]
L_∞	Asymptotic size	34.1	Hwang et al., 2008 [39]
L_t	Length at time t	$L_\infty\left(1-e^{-K(t-t_0)}\right)$	Hwang et al., 2008 [39]
W_t	Weight at time t	$0.0012 \times L_t^{3.697}$	Hwang et al., 2008 [39]
f_i	Fecundity at age i	$f_i = 54.771 \times W_i$	Cha et al., 2002 [40]
M_i	Mortality at age i	$M_i = \frac{16.25}{12\times L_i}$	Jang and Cho, 2022 [15]
D	Density-dependent mortality	9.4×10^{-7}	Jang and Cho, 2022 [15]
g_i	Growth rate at age i	$log\frac{W_{i+1}}{W_i}$	Jang and Cho, 2022 [15]
σ_t	Selectivity at time t	1	Assumed
u_t	Fishing effort at time t	Estimated	
$c_{i,t}$	Catch rate at age i	$c_{i,t} = u_t \times \sigma_i$	
$B_{i,t}$	Biomass at time t	Estimated	
P	Average price	1408.1 (10^3 won/ton)	KOSIS [6]
C	Cost	3×10^8 (won/u_t)	Jang and Cho 2022 [15]
δ	Discount rate	1.001^{-1}	Jang and Cho, 2022 [15]
h_t	Total harvest at time t	$h_t = \sum c_{i,t} \times B_{i,t}$	

three catch-dependent price scenarios were compared with those under a constant price scenario. Finally, a sensitivity analysis of the optimal harvest strategies concerning costs was conducted for each scenario (See Fig. 1).

2.1 Age-structured model

We used the discrete age-structured biomass model of chub mackerel constructed by Jang and Cho [15]. They divided the population of chub mackerel into six age groups based on its life cycle of six years. As juvenile fish face a lack of food in limited spaces, they have a density-dependent natural mortality rate. They constructed their model by dividing the life cycle of mackerel into a spawning season and normal season. The model is expressed as follows:

$$
\begin{aligned}
B_{1,t+1} &= \tau(t)\left(\frac{\gamma_1 B_{1,t}}{\alpha_1 B_{1,t}+\beta_1} - c_{1,t}B_{1,t}\right) + (1-\tau(t))\sum_{i=2}^{6} f_i \frac{W_1}{W_i} B_{i,t} \\
B_{2,t+1} &= \tau(t)\left(e^{(g_{2,t}-M_{2,t})}B_{2,t} - c_{2,t}B_{2,t}\right) + (1-\tau(t))\left(\frac{\gamma B_{1,t}}{\alpha B_{1,t}+\beta} - c_{1,t}B_{1,t}\right) \\
B_{3,t+1} &= \tau(t)\left(e^{(g_{3,t}-M_{3,t})}B_{3,t} - c_{3,t}B_{3,t}\right) + (1-\tau(t))\left(e^{(g_{2,t}-M_{2,t})}B_{2,t} - c_{2,t}B_{2,t}\right) \\
B_{4,t+1} &= \tau(t)\left(e^{(g_{4,t}-M_{4,t})}B_{4,t} - c_{4,t}B_{4,t}\right) + (1-\tau(t))\left(e^{(g_{3,t}-M_{3,t})}B_{3,t} - c_{3,t}B_{3,t}\right) \\
B_{5,t+1} &= \tau(t)\left(e^{(g_{5,t}-M_{5,t})}B_{5,t} - c_{5,t}B_{5,t}\right) + (1-\tau(t))\left(e^{(g_{4,t}-M_{4,t})}B_{4,t} - c_{4,t}B_{4,t}\right) \\
B_{6,t+1} &= \tau(t)\left(e^{(g_{6,t}-M_{6,t})}B_{6,t} - c_{6,t}B_{6,t}\right) + (1-\tau(t))\left(e^{(g_{5,t}-M_{5,t})}B_{5,t} - c_{5,t}B_{5,t}\right),
\end{aligned}
$$

where $\tau(t) = \begin{cases} 1, \text{if } t \; mod \; 12 \neq 0 \\ 0, \text{if } t \; mod \; 12 = 0. \end{cases}$

The descriptions and values of the variables used in the model are presented in Table 1.

2.2 Catch-dependent pricing

Numerous previous studies have developed optimal harvest strategies by assuming constant fish prices [9–15]. However, prices are determined by supply and demand. As the demand for chub mackerel is constant in Korea, we considered that fish prices would be more affected by supply than by demand. Therefore, we proposed logarithmic, rational, and irrational functions for the relationship between the supply (h_t = monthly catch at time t) and price (P_t = monthly price per catch at time t). The functions are expressed as follows:

$$\text{Logarithmic price function:} \quad P_t^L = b_1^L \, log \; (h_t + 1) + b_2^L$$

$$\text{Rational price function:} \quad P_t^R = \frac{b_1^R}{h_t + b_2^R} + b_3^R$$

$$\text{Irrational price function:} \quad P_t^I = \; b_1^I \, (h_t + 1)^{b_3^I} + b_2^I.$$

We collected the monthly catch (tons) and price per catch (1000 won/ton) of chub mackerel in South Korea from July 2017 to June 2022, provided by the Korean Statistical Information Service [6]. The parameters for the logarithmic, rational, and irrational functions were estimated via nonlinear regression using the *fitnlm* function in MATLAB.

2.3 Optimal harvest system based on catch-dependent pricing

The control variable for fishery management is fishing effort, which is a measure of the strength of fishery operations [41]. Jang and Cho proposed an optimal harvest system for mackerel by assuming a constant price (P^{Con}) [15]. In contrast, we propose an optimal harvest strategy by incorporating three different catch-dependent pricing scenarios into the objective function of the optimal harvest system. The optimal harvest strategy under the assumption of a constant price inevitably involves as much harvest as possible during periods of resource abundance because the price remains constant regardless of the timing or quantity of the harvest. Therefore, under the constant price assumption, the optimal harvest strategy is estimated to involve harvesting intensively during periods of resource abundance and high growth rates, such as those immediately after spawning. However, under catch-dependent pricing, the price decreases as the harvest increases, which can alter the optimal harvest strategy with the aim of maximizing profit. Therefore, we compared the optimal harvest strategies and maximum profits obtained using the three catch-dependent pricing scenarios with the previous results obtained under the constant price assumption.

We constructed an optimal harvest system with monthly prices in the form of logarithmic and rational functions according to the monthly catch. Based on the fishing effort u_t and a selectivity (σ_t) of 1, the monthly catch can be expressed as follows:

$$h_t = \sum_{i=2}^{6} \sigma_t u_t B_{i,t}.$$

The fishing effort ($u_t \in [0, 1]$) was defined as a control variable, and an objective function was designed to maximize the catch profit. The objective functions based on the logarithmic, rational, and irrational price functions are represented by $J^L\,(u)$, $J^R\,(u)$, and $J^I\,(u)$, respectively, and the optimal harvest system is expressed as follows:

$$J^L(u) = \sum_{k=0}^{T-1}\{\left(\sum_{i=2}^{6} P_k^L \sigma_t u_k B_{i,k}\right) - Cu_k^2\}\delta^k + q\sum_{i=2}^{6} B_{i,T}$$

$$J^R(u) = \sum_{k=0}^{T-1}\{\left(\sum_{i=2}^{6} P_k^R \sigma_t u_k B_{i,k}\right) - Cu_k^2\}\delta^k + q\sum_{i=2}^{6} B_{i,T}$$

$$J^I(u) = \sum_{k=0}^{T-1}\{\left(\sum_{i=2}^{6} P_k^I \sigma_t u_k B_{i,k}\right) - Cu_k^2\}\delta^k + q\sum_{i=2}^{6} B_{i,T}$$

subject to

$$B_{1,t+1} = \tau(t)\frac{\gamma_t B_{1,t}}{\alpha_t B_{1,t} + \beta_t} + (1-\tau(t))\sum_{i=2}^{6} f_i \frac{W_1}{W_i} B_{i,t} \tag{1}$$

$$B_{2,t+1} = \tau(t)\left(e^{(g_{2,t}-M_2)}B_{2,t} - \sigma_t u_t B_{2,t}\right) + (1-\tau(t))\left(\frac{\gamma B_{1,t}}{\alpha B_{1,t} + \beta} - \sigma_t u_t B_{1,t}\right)$$

$$B_{3,t+1} = \tau(t)\left(e^{(g_{3,t}-M_3)}B_{3,t} - \sigma_t u_t B_{3,t}\right) + (1-\tau(t))\left(e^{(g_{2,t}-M_2)}B_{2,t} - \sigma_t u_t B_{2,t}\right)$$

$$B_{4,t+1} = \tau(t)\left(e^{(g_{4,t}-M_4)}B_{4,t} - \sigma_t u_t B_{4,t}\right) + (1-\tau(t))\left(e^{(g_{3,t}-M_3)}B_{3,t} - \sigma_t u_t B_{3,t}\right)$$

$$B_{5,t+1} = \tau(t)\left(e^{(g_{5,t}-M_5)}B_{5,t} - \sigma_t u_t B_{5,t}\right) + (1-\tau(t))\left(e^{(g_{4,t}-M_4)}B_{4,t} - \sigma_t u_t B_{4,t}\right)$$

$$B_{6,t+1} = \tau(t)\left(e^{(g_{6,t}-M_6)}B_{6,t} - \sigma_t u_t B_{6,t}\right) + (1-\tau(t))\left(e^{(g_{5,t}-M_5)}B_{5,t} - \sigma_t u_t B_{5,t}\right)$$

where $\tau(t) = \begin{cases} 1, \text{ if } t \text{ } mod \text{ } 12 \neq 0 \\ 0, \text{ if } t \text{ } mod \text{ } 12 = 0 \end{cases}$, $P_k^L = b_1^L log\left(\sum_{j=2}^{6}\sigma_k u_k B_{j,k} + 1\right) + b_2^L$, $P_k^R = \frac{b_1^R}{\sum_{j=2}^{6}\sigma_k u_k B_{j,k} + b_2^R} + b_3^R$, $P_k^I = b_1^I\left(\sum_{j=2}^{6}\sigma_k u_k B_{j,k} + 1\right)^{b_3^I} + b_2^I$, $\alpha_t = D\left(1 - e^{g_{1,t}-M_1}\right)$, $\beta_t = \left(g_{1,t} - M_1\right)e^{g_{1,t}-M_1}$, and $\gamma_t = g_{1,t} - M_1$.

The optimal control function (u_t^*) must be determined such that

$$J\left(u^*\right) = \max\{J(u)\},$$

subject to system (1). We define Hamiltonian functions with objective functions $J^L(u)$, $J^R(u)$, and $J^I(u)$ as H_t^L, H_t^R, and H_t^R, respectively, as follows:

$$H_t^L = J_t^L(u,B) + \sum_{i=1}^{6}\lambda_{i,t+1}B_{i,t+1}$$

$$H_t^R = J_t^R(u,B) + \sum_{i=1}^{6}\lambda_{i,t+1}B_{i,t+1}$$

$$H_t^I = J_t^I(u,B) + \sum_{i=1}^{6}\lambda_{i,t+1}B_{i,t+1},$$

where $J_t^L(u,B) = \{\left(\sum_{i=2}^{6} P_t^L \sigma_t u_t B_{i,t}\right) - Cu_t^2\}\delta^t$, $J_t^R(u,B) = \{\left(\sum_{i=2}^{6} P_t^R \sigma_t u_t B_{i,t}\right) - Cu_t^2\}\delta^t$, $J_t^I(u,B) = \{\left(\sum_{i=2}^{6} P_t^I \sigma_t u_t B_{i,t}\right) - Cu_t^2\}\delta^t$, and $\lambda_{i,t}$ denotes adjoint variables for each age

$i = 1, 2, \ldots, 6$ at time $t = 1, 2, \ldots, T-1$. We constructed the necessary conditions of u_t^* for three objective functions ($J^L(u)$, $J^R(u)$, and $J^I(u)$) with catch-dependent pricing using the discrete Pontryagin's maximum principle [42] in the form of the following three theorems:

Theorem 1 *For the optimal control function u_t^{L*} and corresponding solutions $B_{i,t}$ ($i = 1, 2, \ldots, 6$) for system (1) with objective function $J^L(u)$, there exist adjoint variables $\lambda_{i,t}^L$ for $t = 1, 2, \ldots, T-1$ such that*

$$\begin{aligned}
\lambda_{1,t}^L &= \lambda_{1,t+1}^L \tau(t) \frac{\gamma_t \beta_t}{(\alpha_1 B_{1,t} + \beta_t)^2} + \lambda_{2,t+1}^L (1 - \tau(t)) \frac{\gamma_t \beta_t}{(\alpha_1 B_{1,t} + \beta_t)^2} \\
\lambda_{i,t}^L &= \left\{ b_1^L \frac{\sigma_t u_t \sum_{j=2}^6 B_{j,t}}{\sigma_t u_t \sum_{j=2}^6 B_{j,t} + 1} + b_2^L + b_1^L \log\left(\sum_{j=2}^6 \sigma_t u_t B_{j,t} \right) \right\} \sigma_t u_t \delta_t \\
&\quad + \lambda_{i,t+1}^L \tau(t) \left(e^{g_{i,t} - M_i} - \sigma_t u_t \right) + \lambda_{i+1,t+1}^L (1 - \tau(t)) \left(e^{(g_{i,t} - M_i)} - \sigma_t u_t \right) \\
&\quad + \lambda_{1,t+1}^L (1 - \tau(t)) f_i \frac{W_1}{W_i}, \qquad i = 2, 3, 4, 5 \\
\lambda_{6,t}^L &= \left\{ b_1^L \frac{\sigma_t u_t \sum_{j=2}^6 B_{j,t}}{\sigma_t u_t \sum_{j=2}^6 B_{j,t} + 1} + b_2^L + b_1^L \log\left(\sum_{j=2}^6 \sigma_t u_t B_{j,t} \right) \right\} \sigma_t u_t \delta^t \\
&\quad + \lambda_{6,t+1}^L \tau(t) \left(e^{(g_{6,t} - M_t)} - \sigma_t u_t \right) + \lambda_{1,t+1}^L (1 - \tau(t)) f_6 \frac{W_1}{W_6} \\
\lambda_{i,T+1}^L &= q, \quad i = 2, 3, 4, 5, 6,
\end{aligned}$$

and u satisfies u_t^{L*} at

$$\begin{aligned}
&\left\{ \left(\frac{\sigma_t u_t \sum_{j=2}^6 B_{j,t}}{\sigma_t u_t \sum_{j=2}^6 B_{j,t} + 1} + \log\left(\sigma_t u_t \sum_{j=2}^6 B_{j,t} + 1 \right) \right) b_1^L \sum_{i=2}^6 \sigma_t B_{i,t} - 2C u_t \right\} \delta^t \\
&= -b_2^L \sum_{i=2}^6 \sigma_t B_{i,t} \delta^t + \sum_{i=2}^6 \lambda_{i,t+1} \tau(t) B_{i,t} + \sum_{i=3}^6 \lambda_{i,t+1} (1 - \tau(t)) B_{i-1,t}.
\end{aligned}$$

Theorem 2 *For the optimal control function u_t^{R*} and corresponding solutions $B_{i,t}$ ($i = 1, 2, \ldots, 6$) of system (1) with objective function $J^R(u)$, there exist adjoint variables $\lambda_{i,t}^R$ for $t = 1, 2, \ldots, T-1$ such that*

$$\begin{aligned}
\lambda_{1,t}^R &= \lambda_{1,t+1}^R \tau(t) \frac{\gamma_t \beta_t}{(\alpha_1 B_{1,t} + \beta_t)^2} + \lambda_{2,t+1}^R (1 - \tau(t)) \frac{\gamma_t \beta_t}{(\alpha_1 B_{1,t} + \beta_t)^2} \\
\lambda_{i,t}^R &= \left\{ b_1^R \sigma_t u_t + \frac{b_2^R b_3^R u_t}{\left(\sum_{j=2}^6 u_t B_{j,t} + b_3^R \right)^2} \right\} \delta^t + \lambda_{i,t+1}^R \tau(t) \left(e^{g_{i,t} - M_i} - \sigma_t u_t \right) \\
&\quad + \lambda_{i+1,t+1}^R (1 - \tau(t)) \left(e^{g_{i,t} - M_i} - \sigma_t u_t \right) + \lambda_{1,t+1}^R (1 - \tau(t)) f_i \frac{W_1}{W_i}, \quad i = 2, 3, 4, 5 \\
\lambda_{6,t}^R &= \left\{ b_1^R \sigma_t u_t + \frac{b_2^R b_3^R u_t}{\left(\sum_{j=2}^6 u_t B_{j,t} + b_3^R \right)^2} \right\} \delta^t + \lambda_{6,t+1}^R \tau(t) \left(e^{g_{6,t} - M_6} - \sigma_t u_t \right)
\end{aligned}$$

$$+\lambda^R_{1,t+1}(1-\tau(t))f_6\frac{W_1}{W_6}$$

$$\lambda^R_{i,T+1}=q,\quad i=2,3,4,5,6,$$

and u satisfies u^{L*}_t *at*

$$\left\{\frac{b^R_2 b^R_3\sum_{j=2}^6\sigma_t B_{j,t}}{\left(\sum_{j=2}^6 u_t B_{j,t}+b^R_3\right)^2}-2Cu_t\right\}\delta^t$$
$$=\sum_{i=2}^6\lambda^R_{i,t+1}\tau(t)B_{i,t}+\sum_{i=3}^6\lambda^R_{i,t+1}(1-\tau(t))B_{i-1,t}-b_1\sum_{i=2}^6 B_{i,t}\delta^t.$$

Theorem 3 *For the optimal control function* u^{I*}_t *and corresponding solutions* $B_{i,t}$ $(i=1,2,\dots,6)$ *of system* (1) *with objective function* $J^I(u)$, *there exist adjoint variables* $\lambda^I_{i,t}$ *for* $t=1,2,\dots,T-1$ *such that*

$$\lambda^I_{1,t}=\lambda^I_{1,t+1}\tau(t)\frac{\gamma_t\beta_t}{(\alpha_1 B_{1,t}+\beta_t)^2}+\lambda^I_{2,t+1}(1-\tau(t))\frac{\gamma_t\beta_t}{(\alpha_1 B_{1,t}+\beta_t)^2}$$

$$\lambda^I_{i,t}=b^I_1\sigma_t u_t\left(\sum_{j=2}^6\sigma_t u_t B_{j,t}+1\right)^{b^I_3-1}\left\{\left(b^I_3+1\right)\left(\sum_{j=1}^6\sigma_t u_t B_{j,t}\right)+1\right\}\delta^t+b^I_2\sigma_t u_t\delta^t$$
$$+\lambda^I_{i,t+1}\tau(t)\left(e^{g_{i,t}-M_i}-\sigma_t u_t\right)+\lambda^I_{i+1,t+1}(1-\tau(t))\left(e^{(g_{i,t}-M_i)}-\sigma_t u_t\right)$$
$$+\lambda^I_{1,t+1}(1-\tau(t))f_i\frac{W_1}{W_i},\qquad i=2,3,4,5$$

$$\lambda^I_{6,t}=b^I_1\sigma_t u_t\left(\sum_{j=2}^6\sigma_t u_t B_{j,t}+1\right)^{b^I_3-1}\left\{\left(b^I_3+1\right)\left(\sum_{j=1}^6\sigma_t u_t B_{j,t}\right)+1\right\}\delta^t+b^I_2\sigma_t u_t\delta^t$$
$$+\lambda^I_{6,t+1}\tau(t)\left(e^{(g_{6,t}-M_t)}-\sigma_t u_t\right)+\lambda^I_{1,t+1}(1-\tau(t))f_6\frac{W_1}{W_6}$$

$$\lambda^L_{i,T+1}=q,\quad i=2,3,4,5,6,$$

and u satisfies u^{L*}_t *at*

$$\left\{b^I_1\left(\sum_{j=2}^6\sigma_t B_{j,t}\right)\left(b^I_3+1+\sum_{j=2}^6\sigma_t u_t B_{j,t}\right)\left(\sum_{j=2}^6\sigma_t u_t B_{j,t}+1\right)^{b^I_3-1}-2Cu_t\right\}\delta^t$$
$$=-b^I_2\left(\sum_{i=2}^6\sigma_t B_{i,t}\right)+\sum_{i=2}^6\lambda^I_{i,t+1}\tau(t)\sigma_t B_{i,t}+\sum_{i=3}^6\lambda^I_{i,t+1}(1-\tau(t))\sigma_t B_{i-1,t}.$$

Proofs of Theorems 1, 2, *and* 3 Appendix. □

The optimal harvest strategies (u^{L*}_t, u^{R*}_t, and u^{I*}_t) that satisfy the necessary conditions of the three theorems using the forward–backward sweep method [42] can be numerically calculated. However, in the case of Theorem 3, it is difficult to calculate the values of b^I_3

other than −0.5 numerically. Therefore, the optimal harvest strategy (u_t^{I*}) for Theorem 3 is calculated under the assumption that $b_3^I = -0.5$.

2.4 Comparison of optimal harvest strategies based on catch-dependent pricing

We derived the necessary conditions for the optimal harvest strategies for mackerel based on three different catch-dependent price functions. We used these conditions to simulate the optimal harvest strategies over a 10-year period for each function. We used the simulation results to derive the 10-year monthly averages for the optimal fishing effort (u_t), total biomass ($\sum B_{i,t}$), catch (h_t), and profit ($\sum_{k=0}^{T-1}\{\big(\sum_{i=2}^{6} P_k^L\ \sigma_t u_k B_{i,k}\big) - Cu_k^2\}\delta^k$). Subsequently, we compared the optimal harvest strategies based on these three catch-dependent prices with the strategy proposed by Jang and Cho [15], which assumes a fixed price based on the monthly average price from July 2017 to June 2022.

The Korean government designates a one-month closed fishing season for mackerel between April and June, and May was designated as the closed season in 2023. In the proposed optimal harvest strategy system (1), the optimal fishing strategy for a closed season in month t was estimated by adjusting the selectivity to 0 in month t ($\sigma_t = 0$). We compared the variation in the optimal fishing effort, total biomass, catch, and profit when a closed season was designated for each month from January to December with a no-closed-season scenario for four catch-dependent prices. In addition, to assess the effectiveness of implementing a closed season, we compared the profit efficiency (profit per unit effort), fishing efficiency (catch per unit effort), and resource recovery efficiency (resource recovery rate relative to profit reduction) with those of a no-closed-season scenario.

2.5 Sensitivity analysis with respect to cost

The cost associated with production typically affects optimal strategies, and our optimal harvest strategy also incorporates the cost per unit effort into the objective function. While prices can be reasonably estimated using historical data, costs are challenging to estimate owing to various contributing factors. Therefore, constant cost was used in this study. We conducted a sensitivity analysis for costs ranging from 100 to 1200 million won/u_t (with a default value of 300 million) to investigate how changes in costs affect the optimal harvesting strategy.

3 Numerical results

3.1 Catch-dependent pricing

Table 2 shows the results of the nonlinear regression analysis of the logarithmic, rational, and irrational functions for the mackerel catch and price data from July 2017 to June

Table 2 Estimated parameters for nonlinear regression of logarithmic, rational, and irrational functions for mackerel catch and price data from July 2017 to June 2022

Functions	Parameters	Value	P-value	R^2	Adj. R^2
P_t^L	b_1^L	−564.64	<0.001	0.724	0.720
	b_2^L	6758.3	<0.001		
P_t^R	b_1^R	2.2982e + 06	<0.001	0.757	0.750
	b_2^R	507.69	0.0339		
	b_3^R	1273.8	<0.001		
P_t^I	b_1^I	57,540	<0.001	0.769	0.765
	b_2^I	932.03	<0.001		

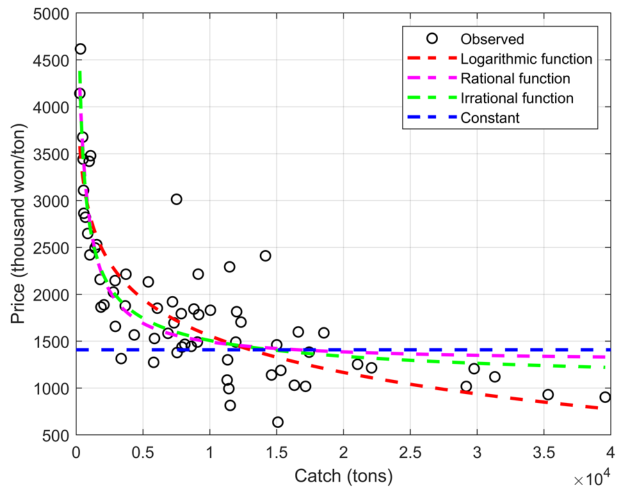

Figure 2 Comparison of monthly catch-dependent price functions for mackerel estimated via nonlinear regression

2022. The p-values for all parameters of these three functions were less than 0.05, indicating statistical significance at a 95% confidence level. The R^2 values for the logarithmic, rational, and irrational functions were estimated to be 0.724, 0.757, and 0.769, respectively. The catch-dependent pricing of mackerel was accurately estimated because the R^2 values of all functions were high, and the irrational function showed the highest R^2 value. This suggests that all three functions are potentially useful as catch-dependent pricing functions for mackerel. Figure 2 shows the average price for mackerel from July 2017 to June 2022 and the catch-dependent prices estimated using the nonlinear regression of the three functions. The prices estimated using the logarithmic, rational, and irrational functions were higher than the average price when the monthly catch was less than approximately 13,000, 16,600, and 14,600 tons, respectively. In particular, when the logarithmic function was used, the decrease in price was more sensitive compared with the rational and irrational functions when the monthly catch was more than 15,000 tons.

3.2 Comparison of optimal harvest strategies based on catch-dependent pricing

We compared the optimal harvest strategies for chub mackerel under four pricing scenarios, i.e., one with constant pricing and three with catch-dependent pricing. Figure 3 shows the monthly average optimal fishing effort (Fig. 3.A), total biomass (Fig. 3.B), catch (Fig. 3.C), and profit (Fig. 3.D) over the next 10 years for the four scenarios. Under the rational and irrational pricing scenarios, the fishing effort increased in July immediately after the mackerel spawning season. This trend is similar to the constant pricing scenario. However, under the logarithmic pricing scenario, the fishing effort decreased immediately after the spawning season and then gradually increased, with the maximum effort occurring during the spawning season. The monthly average total biomass, catch, and profit were the highest in July for all four scenarios. Thereafter, they gradually decreased and reached the minimum values in June during the spawning season. The variation in the monthly average catch and profit was the highest under the constant pricing scenario and lowest under the logarithmic pricing scenario.

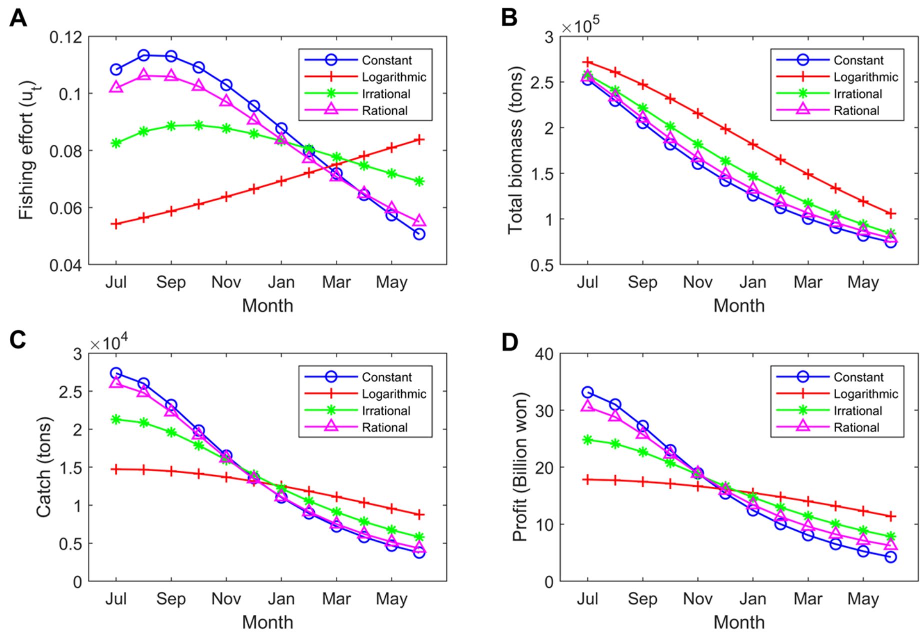

Figure 3 Monthly averages of optimal fishing effort, total biomass, catch, and profit over 10 years for four pricing scenarios

Table 3 Monthly averages of fishing effort, total biomass, catch, and profit over a period of 10 years for four pricing scenarios

Function	u_t	Total biomass (tons)	Catch (tons)	Profit (billion won)
Constant	0.0878	146,330	13,991	16.27
Logarithmic	0.0683	189,967	12,424	15.35
Rational	0.0815	151,764	13,485	16.51
Irrational	0.0846	161,967	13,785	16.13

Table 3 shows the 10-year monthly averages of the fishing effort, total biomass, catch, and profit for each pricing scenario. The average catch and fishing effort under the rational pricing scenario were slightly lower than those under the constant pricing scenario, but the profit was the highest at 16.51 billion won. In the logarithmic pricing scenario, the average catch and profit were the lowest at 12,424 tons and 15.35 billion won, respectively, whereas the total biomass was the highest at 189,967 tons. In addition, the average fishing effort was the lowest at 0.0683, resulting in the highest catch and profit per unit fishing effort at 181,903 tons/u_t and 224.74 billion won/u_t, respectively. This indicates that the optimal harvest strategy under the logarithmic pricing scenario is the most effective in terms of fishing efficiency and resource stability.

3.3 Variation in optimal harvest strategies with monthly closed seasons

To analyze the effects of monthly closed seasons, the average fishing effort, total biomass, catch, and profit were compared over a 10-year period when a closed season was implemented from January to December. Figure 4 shows the variations in the average fishing effort, total biomass, catch, and profit compared with the optimal fishing strategy without a closed season for each month of the closed season. For all catch-dependent price scenarios, a closed season in July immediately after the spawning season resulted in the

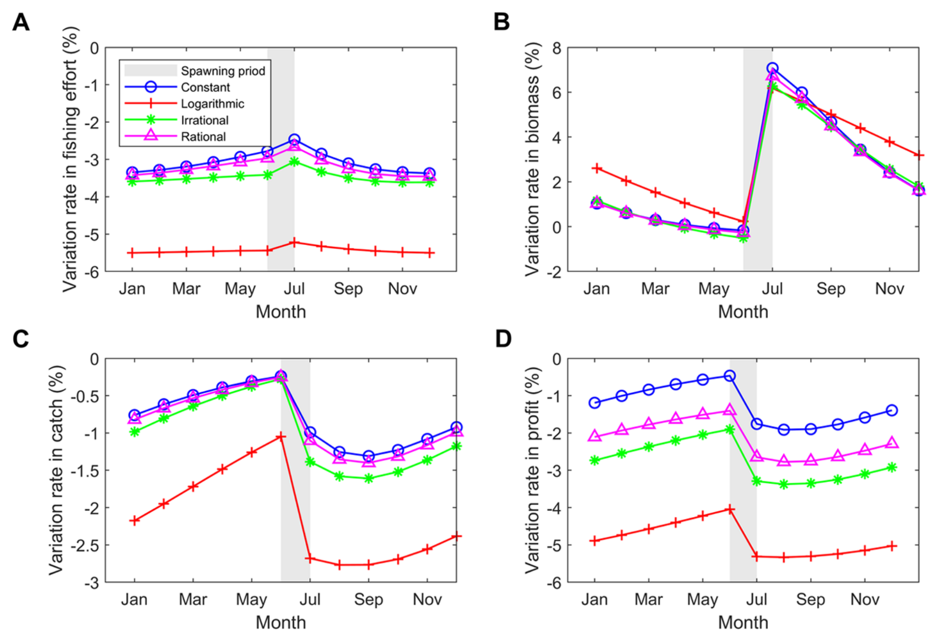

Figure 4 Variation rate for fishing effort, total biomass, catch, and profit for each monthly closed season compared with the optimal fishing strategy without a closed season

largest reduction in profit (2%–5%) and catch (0%–3%) but provided the most significant resource recovery effect (6%–8%). Conversely, a closed season in June during the spawning season resulted in the smallest reduction in catch (0%–1.5%) and profit (0%–4%) but provided the minimal resource recovery effect (0%–1%).

Figure 5 shows the variation rates in the profit, fishing, and resource recovery efficiencies for each monthly closed season compared with the scenario without a closed season for all pricing scenarios. The profit and fishing efficiencies represent the profit and catch per unit effort, respectively. As shown in Fig. 4, although the profit and catch for the scenario with a closed season were lower than those without a closed season, the profit and fishing efficiencies mostly increased. In the case of constant pricing (Fig. 5.A), the profit and fishing efficiencies increased by 2.456% and 2.789% in April and March, respectively. In the catch-dependent pricing scenarios, the profits and fishing efficiencies were the highest in June, with increases of 1.477% and 4.644% for the logarithmic scenario (Fig. 5.B), 1.566% and 3.248% for the irrational scenario (Fig. 5.C), and 1.612% and 2.800% for the rational scenario (Fig. 5.D), respectively. The resource recovery efficiency was the highest in July for all pricing scenarios, with increases of 4.026% (constant), 1.162% (logarithmic), 1.906% (irrational), and 2.538% (rational) compared with the optimal fishing strategy without a closed season. This indicates that, although a closed season can improve the profit, fishing, and resource recovery efficiencies, the effects depend on the catch-dependent pricing scenarios.

3.4 Sensitivity analysis with respect to cost

We compared the optimal harvest strategies for mackerel under four pricing scenarios based on changes in costs. Figure 6 shows the monthly average optimal fishing effort (Fig. 6.A), total biomass (Fig. 6.B), catch (Fig. 6.C), and profit (Fig. 6.D) over the next 10

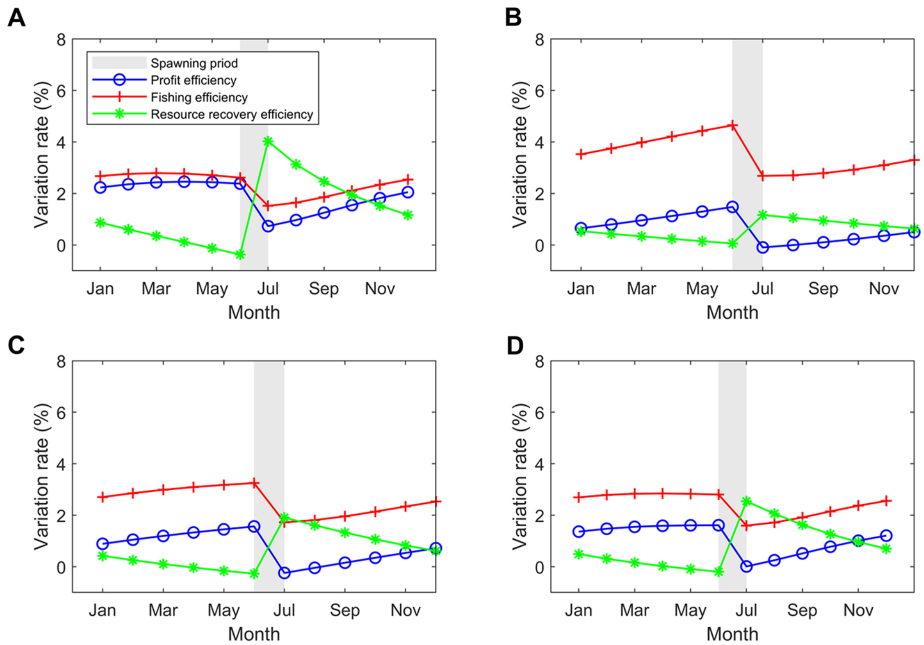

Figure 5 Comparison of variation rates for profit, fishing, and resource recovery efficiencies for different monthly closed seasons and pricing scenarios. *A*. Constant price. *B*. Logarithmic price function. *C*. Irrational price function. *D*. Rational price function

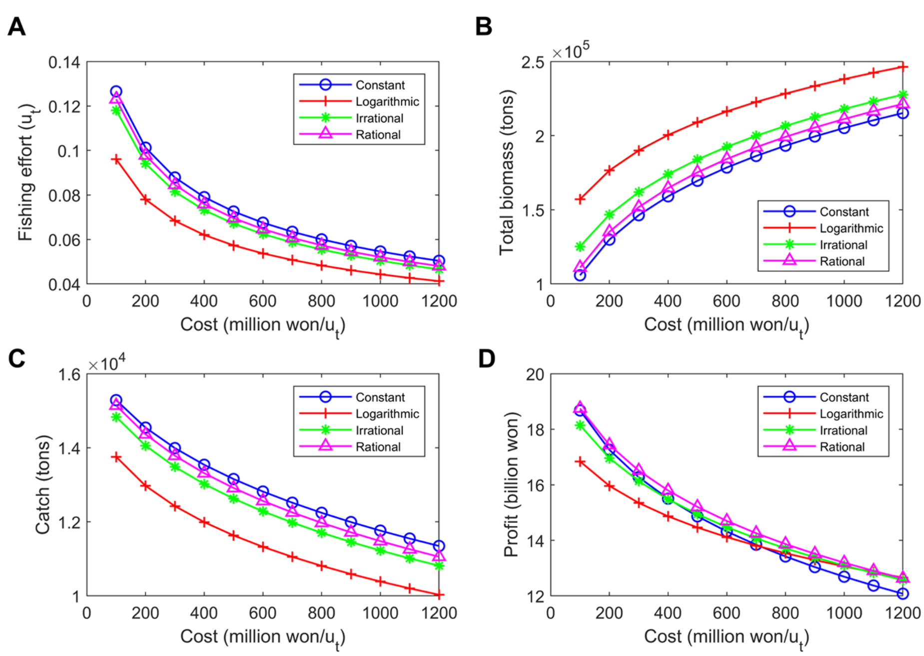

Figure 6 Monthly averages of optimal fishing effort, total biomass, catch, and profit over 10 years for four pricing scenarios as costs varied from 100 to 1200 million

years as costs varied from 100 to 1200 million. In all four scenarios, a decrease in costs increased the monthly average fishing effort, catch, and profit while reducing the total biomass, whereas an increase in costs raised the total biomass but reduced the other variables, with the magnitude of these changes diminishing as the costs increased. A notable observation is that, when the catch-dependent price function was a logarithmic function, the profit, which was the lowest among the four scenarios at the default cost of 300 million, became the highest at approximately 1.266 billion in monthly average profit when the cost reached 1200 million. This result occurred because, as shown in Fig. 2, the logarithmic function exhibited the highest price sensitivity to catch quantity. Furthermore, as indicated in Fig. 6.C, the monthly average catch was approximately 11,000 tons when the cost was 1.2 billion, during which the logarithmic price function reached its highest level.

4 Discussion and conclusion

This study has proposed an optimal harvest strategy by analyzing the catch and price data of chub mackerel in Korea using various catch-dependent pricing functions. We proposed three catch-dependent pricing functions: logarithmic, rational, and irrational. We used these functions to formulate an objective function to maximize fishing profits and utilized a discrete age-structured mackerel model, as suggested in a previous study [15], to propose an optimal harvest strategy system. We employed Pontryagin's maximum principle to derive the necessary conditions for the optimal fishing strategy under the three catch-dependent pricing functions and compared the optimal strategies for constant and catch-dependent pricing.

We compared the optimal harvest strategies under one constant pricing scenario and three catch-dependent pricing scenarios. The optimal harvest strategy for the rational and irrational price scenarios was similar to that for the constant price scenario, where the fishing effort increased immediately after the spawning season and then gradually decreased. In contrast, in the logarithmic scenario, the optimal harvest strategy involved a gradual increase in fishing effort after the spawning season, with a peak immediately before the next spawning period. In the constant price scenario, the 10-year average profit efficiency and total biomass were the lowest at 185.3 billion won/u_t and 146,330 tons, respectively, compared with the optimal strategies with the catch-dependent pricing scenarios. This suggested that failing to account for appropriate catch-dependent pricing not only led to inefficient harvesting strategies but also hindered resource recovery. We also compared the effects of monthly closed seasons under the four pricing scenarios. All scenarios showed that a closed season in July immediately after the spawning period provided the maximum resource recovery efficiency. Conversely, a closed season in June immediately before the spawning period yielded the highest profit efficiency. As the cost per unit of effort increased, the monthly average fishing effort, catch, and profit decreased, while the total biomass quantity increased. In the case of monthly average profit, the extent of the decrease varied depending on the catch-dependent pricing scenario, with the smallest decrease occurring when the price function was logarithmic.

This study has several limitations. First, although the supply aspect of pricing (catch) was considered according to the laws of supply and demand, the demand aspect was not addressed. In Korea, mackerel is a commercially popular species throughout the year. Thus, we did not consider demand because we expected demand variations to be minimal and unlikely to affect the price significantly. Second, we assumed that the cost per unit of fishing effort was fixed and did not account for price volatility owing to rising costs. To address this limitation, the changes in the average fishing effort, catch, biomass, and profit were presented through the cost sensitivity analysis when a constant cost increased or decreased. In addition, as the government of South Korea provides subsidies to prevent price increases owing to rising costs such as fuel, the volatility resulting from cost increases is expected to be low.

Despite these limitations, this study successfully developed an optimal monthly harvest strategy for chub mackerel in Korea under various catch-dependent pricing scenarios. This study demonstrates that, even for fish species with similar life histories, harvest strategies should vary depending on market prices influenced by catch. If the demand is sufficient, even if the catch is high, an optimal harvest strategy based on the constant price assumed in previous studies [9–15] may be sufficient. However, for species with insufficient demand, prices inevitably decrease as catch increases. The estimated profit from optimal harvest strategies incorporating catch-dependent pricing is comparable to that under constant pricing, while maintaining higher resource levels and ensuring resource stability. In contrast, if optimal harvest strategies are designed based on constant pricing but the actual prices decrease with increased catch, fishers' profit may decline and resource stability may deteriorate owing to overharvesting. Our research on optimal harvest strategies using catch-dependent pricing not only addresses these issues but also highlights how optimal strategies can vary depending on the specific catch-dependent pricing function that is applied. In addition, the effects of closed seasons differed based on the pricing scenario. In Korea, the fishing population has steadily decreased owing to aging and workforce attrition, underscoring the necessity of governmental support and resource management policies [43, 44]. We believe that this study can be used to develop improved monthly optimal harvest strategies for other fish species using catch-dependent pricing functions, and can contribute significantly to improving fishers' profit. The stability of fishers' profit could potentially lead to changes in the economic dependence of the community on fishing and the creation of additional employment opportunities [45]. Furthermore, we hope that this study will provide an alternative for fishery managers and policymakers to improve harvest efficiency and resource recovery through monthly closed seasons.

Appendix

We proceed to prove Theorems 1, 2, and 3. Each proof is detailed below:

Theorem 1 *For the optimal control function u_t^{L*} and corresponding solutions $B_{i,t}$ ($i = 1,2,\ldots,6$) for system (1) with objective function $J^L(u)$, there exist adjoint variables $\lambda_{i,t}^L$ for $t = 1,2,\ldots,T-1$ such that*

$$\lambda_{1,t}^L = \lambda_{1,t+1}^L \tau(t) \frac{\gamma_t \beta_t}{(\alpha_1 B_{1,t} + \beta_t)^2} + \lambda_{2,t+1}^L (1-\tau(t)) \frac{\gamma_t \beta_t}{(\alpha_1 B_{1,t} + \beta_t)^2}$$

$$\begin{aligned}\lambda_{i,t}^L = {} & \left\{ b_1^L \frac{\sigma_t u_t \sum_{j=2}^6 B_{j,t}}{\sigma_t u_t \sum_{j=2}^6 B_{j,t} + 1} + b_2^L + b_1^L \log\left(\sum_{j=2}^6 \sigma_t u_t B_{j,t}\right)\right\} \sigma_t u_t \delta_t \\ & + \lambda_{i,t+1}^L \tau(t) \left(e^{g_{i,t}-M_i} - \sigma_t u_t\right) + \lambda_{i+1,t+1}^L (1-\tau(t)) \left(e^{(g_{i,t}-M_i)} - \sigma_t u_t\right) \\ & + \lambda_{1,t+1}^L (1-\tau(t)) f_i \frac{W_1}{W_i}, \qquad i = 2,3,4,5\end{aligned}$$

$$\lambda_{6,t}^L = \left\{ b_1^L \frac{\sigma_t u_t \sum_{j=2}^6 B_{j,t}}{\sigma_t u_t \sum_{j=2}^6 B_{j,t} + 1} + b_2^L + b_1^L \log\left(\sum_{j=2}^6 \sigma_t u_t B_{j,t}\right)\right\} \sigma_t u_t \delta^t$$

$$+ \lambda^L_{6,t+1}\tau(t)\left(e^{(g_{6,t}-M_t)} - \sigma_t u_t\right) + \lambda^L_{1,t+1}(1-\tau(t))f_6\frac{W_1}{W_6}$$

$$\lambda^L_{i,T+1} = q, \quad i = 2,3,4,5,6,$$

and u satisfies u^{L*}_t *at*

$$\left\{\left(\frac{\sigma_t u_t \sum_{j=2}^6 B_{j,t}}{\sigma_t u_t \sum_{j=2}^6 B_{j,t} + 1} + \log\left(\sigma_t u_t \sum_{j=2}^6 B_{j,t} + 1\right)\right) b^L_1 \sum_{i=2}^6 \sigma_t B_{i,t} - 2Cu_t\right\}\delta^t$$
$$= -b^L_2 \sum_{i=2}^6 \sigma_t B_{i,t}\delta^t + \sum_{i=2}^6 \lambda_{i,t+1}\tau(t)B_{i,t} + \sum_{i=3}^6 \lambda_{i,t+1}(1-\tau(t))B_{i-1,t}.$$

Proof of Theorem 1 According to the discrete-time Pontryagin's maximum principle [17], the Hamiltonian H^L_t at time t is defined as

$$H^L_t = J^L_t(u,B) + \sum_{i=2}^6 \lambda^L_{i,t+1} B_{i,t+1},$$

where $J^L_t(u,B) = \left\{\left(\sum_{i=2}^6 P^L \sigma_t u_t B_{i,t}\right) - Cu_t^2\right\}\delta^t$.

For $t = 1,2,\ldots,T$, expanding this definition yields

$$\begin{aligned}
H^L_t &= J^L_t(u,B) + \sum_{i=2}^6 \lambda^L_{i,t+1} B_{i,t+1} \\
&= \left\{\left(\sum_{i=2}^6 P^L_t \sigma_t u_t B_{i,t}\right) - Cu_t^2\right\}\delta^t + \sum_{i=2}^6 \lambda^L_{i,t+1} B_{i,t+1} \\
&= \{\left(b^L_1 \log\left(\sigma_t u_t \sum_{j=2}^6 B_{j,t} + 1\right) + b^L_2\right)\sum_{i=2}^6 \sigma_t u_t B_{i,t} - Cu_t^2\}\delta^t \\
&+ \lambda_{1,t+1}\left(\tau(t)\frac{\gamma_t B_{1,t}}{\alpha_t B_{1,t} + \beta_t} + (1-\tau(t))\sum_{i=2}^6 f_i \frac{W_{1,t}}{W_{i,t}} B_{i,t}\right) \\
&+ \lambda_{2,t+1}\left(\tau(t)\left(e^{g_{2,t}-M_2} - \sigma_t u_t\right)B_{2,t} + \left(1 - \tau(t)\frac{\gamma_1 B_{1,t}}{\alpha_{1,t}B_{1,t} + \beta_{1,t}}\right)\right) \\
&+ \sum_{i=3}^6 \lambda_{i,t+1}(\tau(t)\left(e^{g_{i,t}-M_i} - \sigma_t u_t\right)B_{i,t} + (1-\tau(t))\left(e^{g_{i-1,t}-M_{i-1}} - \sigma_t u_t\right)B_{i-1,t}).
\end{aligned}$$

We apply the three necessary conditions (adjoint, transversality, and optimality) from Pontryagin's maximum principle:

$$\lambda^L_{i,t} = \frac{\partial H^L_t}{\partial B_{i,t}}, \qquad i = 1,2,\ldots,6 \qquad \text{(Adjoint condition)}$$
$$\lambda^L_{i,T} = \phi'\left(x^*_T\right), \qquad i = 1,2,\ldots,6 \qquad \text{(Transversality condition)}$$
$$\frac{\partial H^L_t}{\partial u_t} = 0 \text{ at } u^* \qquad \text{(Optimality condition).}$$

These conditions are applied for $t = 1,2,\ldots,T-1$.

First, the adjoint condition allows us to obtain the adjoint variables $\lambda^L_{i,t}$ implicitly. For each $t = 1, 2, \dots, T-1$ and $i = 1, 2, \dots, 6$, we define the adjoint variable $\lambda^L_{i,t}$ as

$$\lambda^L_{i,t} = \frac{\partial H^L_t}{\partial B_{i,t}}.$$

In the process of computing these partial derivatives, note the following relationships regarding $B_{j,t}$. First,

$$\frac{\partial B_{j,t}}{\partial B_{1,t}} = \begin{cases} 0, & \text{if } j \neq 1, \\ 1, & \text{if } j = 1, \end{cases}$$

which indicates that $B_{j,t}$ is not directly affected by $B_{1,t}$ unless $j = 1$. Second, for $\gamma_t B_{1,t}(\alpha_t \times B_{1,t} + \beta_t)^{-1}$, we obtain

$$\frac{\partial(\gamma_t B_{1,t}(\alpha_t B_{1,t} + \beta_t)^{-1})}{\partial B_{j,t}} = \begin{cases} 0, & \text{if } j \neq 1 \\ \gamma_t B_{1,t}(\alpha_t B_{1,t} + \beta_t)^{-2}, & \text{if } j = 1. \end{cases}$$

Substituting H^L_t into the above expression and differentiating with respect to $B_{i,t}$ yields the recursive relationship for $\lambda^L_{i,t}$:

$$\begin{aligned}
\lambda^L_{1,t} &= \frac{\partial H^L_t}{\partial B_{1,t}} \\
&= \frac{\partial \left\{ \left(\sum_{j=2}^6 P^L \sigma_t u_t B_{k,t} \right) - C u_t^2 \right\} \delta^t + \sum_{j=2}^6 \lambda^L_{j,t+1} B_{j,t+1}}{\partial B_{1,t}} \\
&= \lambda^L_{1,t+1} \tau(t) \frac{\gamma_t \beta_t}{(\alpha_1 B_{1,t} + \beta_t)^2} + \lambda^L_{2,t+1} (1 - \tau(t)) \frac{\gamma_t \beta_t}{(\alpha_1 B_{1,t} + \beta_t)^2} \\
\lambda^L_{i,t} &= \frac{\partial H^L_t}{\partial B_{i,t}} \\
&= \frac{\partial \left\{ \left(\sum_{j=2}^6 P^L_t \sigma_t u_t B_{j,t} \right) - C u_t^2 \right\} \delta^t + \sum_{j=2}^6 \lambda^L_{j,t+1} B_{j,t+1}}{\partial B_{i,t}} \\
&= \left\{ b^L_1 \frac{\sigma_t u_t \sum_{j=2}^6 B_{j,t}}{\sigma_t u_t \sum_{j=2}^6 B_{j,t} + 1} + b^L_2 + b^L_1 \log \left(\sum_{j=2}^6 \sigma_t u_t B_{j,t} \right) \right\} \sigma_t u_t \delta_t \\
&+ \lambda^L_{i,t+1} \tau(t) \left(e^{g_{i,t} - M_i} - \sigma_t u_t \right) + \lambda^L_{i+1,t+1} (1 - \tau(t)) \left(e^{(g_{i,t} - M_i)} - \sigma_t u_t \right) \\
&+ \lambda^L_{1,t+1} (1 - \tau(t)) f_i \frac{W_1}{W_i}, \qquad i = 2, 3, 4, 5 \\
\lambda^L_{6,t} &= \frac{\partial H^L_t}{\partial B_{i,t}} \\
&= \frac{\partial \left\{ \left(\sum_{j=2}^6 P^L \sigma_t u_t B_{j,t} \right) - C u_t^2 \right\} \delta^t + \sum_{j=2}^6 \lambda^L_{j,t+1} B_{j,t+1}}{\partial B_{6,t}} \\
&= \left\{ b^L_1 \frac{\sigma_t u_t \sum_{j=2}^6 B_{j,t}}{\sigma_t u_t \sum_{j=2}^6 B_{j,t} + 1} + b^L_2 + b^L_1 \log \left(\sum_{j=2}^6 \sigma_t u_t B_{j,t} \right) \right\} \sigma_t u_t \delta^t
\end{aligned}$$

$$+ \lambda^L_{6,t+1}\tau(t)\left(e^{(g_{6,t}-M_t)} - \sigma_t u_t\right) + \lambda^L_{1,t+1}(1-\tau(t))f_6\frac{W_1}{W_6}.$$

Second, the terminal state $t = T + 1$ is derived through the transversality condition

$$\lambda^L_{i,T+1} = q, \quad i = 2,3,4,5,6.$$

Third, we determine u_t that satisfies the optimality condition

$$\begin{aligned}
0 &= \frac{\partial H^L_t}{\partial u_t} \\
&= \left\{ b^L_1 \frac{\sigma_t \sum_{j=2}^6 B_{j,t}}{\sigma_t u_t \sum_{j=2}^6 B_{j,t} + 1} u_t \sum_{i=2}^{6} \sigma_t B_{i,t} + \left(b^L_1 \log\left(\sigma_t u_t \sum_{j=2}^{m} B_{j,t} + 1\right) + b^L_2 \right) \right. \\
&\quad \left. \times \sum_{i=2}^{m} \sigma_t B_{i,t} - 2C u_t \right\} \delta^t \\
&\quad - \sum_{i=2}^{6} \lambda_{i,t+1}\tau(t) B_{i,t} - \sum_{i=3}^{6} \lambda_{i,t+1}(1-\tau(t)) B_{i-1,t}.
\end{aligned}$$

We isolate all terms containing u_t on the left side and move the remainder to the right side:

$$\begin{aligned}
&\left\{ \left(\frac{\sigma_t u_t \sum_{j=2}^6 B_{j,t}}{\sigma_t u_t \sum_{j=2}^6 B_{j,t} + 1} + \log\left(\sigma_t u_t \sum_{j=2}^{6} B_{j,t} + 1\right) \right) b^L_1 \sum_{i=2}^{6} \sigma_t B_{i,t} - 2C u_t \right\} \delta^t \\
&= -b^L_2 \sum_{i=2}^{6} \sigma_t B_{i,t}\delta^t + \sum_{i=2}^{6} \lambda_{i,t+1}\tau(t) B_{i,t} + \sum_{i=3}^{6} \lambda_{i,t+1}(1-\tau(t)) B_{i-1,t}.
\end{aligned}$$

This expression clearly separates the u_t-dependent part from the part that does not depend on u_t, making it straightforward to solve for u_t.

Hence, for $t = 1,2,\ldots,T-1$, there exist adjoint variables $\lambda^L_{i,t}$ for $i = 1,2,\ldots,6$ and $t = 1,2,\ldots,T-1$ such that

$$\begin{aligned}
\lambda^L_{1,t} &= \lambda^L_{1,t+1}\tau(t)\frac{\gamma_t\beta_t}{(\alpha_1 B_{1,t}+\beta_t)^2} + \lambda^L_{2,t+1}(1-\tau(t))\frac{\gamma_t\beta_t}{(\alpha_1 B_{1,t}+\beta_t)^2} \\
\lambda^L_{i,t} &= \left\{ b^L_1 \frac{\sigma_t u_t \sum_{j=2}^6 B_{j,t}}{\sigma_t u_t \sum_{j=2}^6 B_{j,t} + 1} + b^L_2 + b^L_1 \log\left(\sum_{j=2}^{6} \sigma_t u_t B_{j,t}\right) \right\} \sigma_t u_t \delta_t \\
&\quad + \lambda^L_{i,t+1}\tau(t)\left(e^{g_{i,t}-M_i} - \sigma_t u_t\right) + \lambda^L_{i+1,t+1}(1-\tau(t))\left(e^{(g_{i,t}-M_i)} - \sigma_t u_t\right) \\
&\quad + \lambda^L_{1,t+1}(1-\tau(t))f_i\frac{W_1}{W_i}, \qquad i = 2,3,4,5 \\
\lambda^L_{6,t} &= \left\{ b^L_1 \frac{\sigma_t u_t \sum_{j=2}^6 B_{j,t}}{\sigma_t u_t \sum_{j=2}^6 B_{j,t} + 1} + b^L_2 + b^L_1 \log\left(\sum_{j=2}^{6} \sigma_t u_t B_{j,t}\right) \right\} \sigma_t u_t \delta^t \\
&\quad + \lambda^L_{6,t+1}\tau(t)\left(e^{(g_{6,t}-M_t)} - \sigma_t u_t\right) + \lambda^L_{1,t+1}(1-\tau(t))f_6\frac{W_1}{W_6}
\end{aligned}$$

$$\lambda_{i,T+1}^{L} = q, \quad i = 2,3,4,5,6,$$

and u satisfies u_t^{L*} at

$$\left\{ \left(\frac{\sigma_t u_t \sum_{j=2}^{6} B_{j,t}}{\sigma_t u_t \sum_{j=2}^{6} B_{j,t} + 1} + \log \left(\sigma_t u_t \sum_{j=2}^{6} B_{j,t} + 1 \right) \right) b_1^L \sum_{i=2}^{6} \sigma_t B_{i,t} - 2Cu_t \right\} \delta^t$$
$$= -b_2^L \sum_{i=2}^{6} \sigma_t B_{i,t} \delta^t + \sum_{i=2}^{6} \lambda_{i,t+1} \tau(t) B_{i,t} + \sum_{i=3}^{6} \lambda_{i,t+1} (1 - \tau(t)) B_{i-1,t}. \qquad \square$$

Theorems 2 and 3 are quite similar to Theorem 1. The proofs of Theorems 2 and 3 proceed in a manner entirely analogous to that of Theorem 1. Therefore, much of the proof process is omitted. Theorem 2 utilizes the rational price function P_t^R, while Theorem 3 applies the irrational price function P_t^I.

Theorem 2 *For the optimal control function u_t^{R*} and corresponding solutions $B_{i,t}$ ($i = 1,2,\dots,6$) of system (1) with objective function $J^R(u)$, there exist adjoint variables $\lambda_{i,t}^R$ for $t = 1,2,\dots,T-1$ such that*

$$\lambda_{1,t}^R = \lambda_{1,t+1}^R \tau(t) \frac{\gamma_t \beta_t}{(\alpha_1 B_{1,t} + \beta_t)^2} + \lambda_{2,t+1}^R (1 - \tau(t)) \frac{\gamma_t \beta_t}{(\alpha_1 B_{1,t} + \beta_t)^2}$$
$$\lambda_{i,t}^R = \left\{ b_1^R \sigma_t u_t + \frac{b_2^R b_3^R u_t}{\left(\sum_{j=2}^{6} u_t B_{j,t} + b_3^R \right)^2} \right\} \delta^t + \lambda_{i,t+1}^R \tau(t) \left(e^{g_{i,t} - M_i} - \sigma_t u_t \right)$$
$$+ \lambda_{i+1,t+1}^R (1 - \tau(t)) \left(e^{g_{i,t} - M_i} - \sigma_t u_t \right) + \lambda_{1,t+1}^R (1 - \tau(t)) f_i \frac{W_1}{W_i}, \quad i = 2,3,4,5$$
$$\lambda_{6,t}^R = \left\{ b_1^R \sigma_t u_t + \frac{b_2^R b_3^R u_t}{\left(\sum_{j=2}^{6} u_t B_{j,t} + b_3^R \right)^2} \right\} \delta^t + \lambda_{6,t+1}^R \tau(t) \left(e^{g_{6,t} - M_6} - \sigma_t u_t \right)$$
$$+ \lambda_{1,t+1}^R (1 - \tau(t)) f_6 \frac{W_1}{W_6}$$
$$\lambda_{i,T+1}^R = q, \quad i = 2,3,4,5,6,$$

and u satisfies u_t^{L*} at

$$\left\{ \frac{b_2^R b_3^R \sum_{j=2}^{6} \sigma_t B_{j,t}}{\left(\sum_{j=2}^{6} u_t B_{j,t} + b_3^R \right)^2} - 2Cu_t \right\} \delta^t$$
$$= \sum_{i=2}^{6} \lambda_{i,t+1}^R \tau(t) B_{i,t} + \sum_{i=3}^{6} \lambda_{i,t+1}^R (1 - \tau(t)) B_{i-1,t} - b_1 \sum_{i=2}^{6} B_{i,t} \delta^t.$$

Proof of Theorem 2 According to the discrete-time Pontryagin's maximum principle [17], the Hamiltonian H_t^L at time t is defined as

$$H_t^R = J_t^R(u,B) + \sum_{i=2}^{6} \lambda_{i,t+1}^R B_{i,t+1}$$

$$
\begin{aligned}
&= \left\{ \left(\sum_{i=2}^{6} P_t^R \sigma_t u_t B_{i,t} \right) - C u_t^2 \right\} \delta^t + \sum_{i=2}^{6} \lambda_{i,t+1}^R B_{i,t+1} \\
&= \left\{ b_1^R \sum_{i=2}^{6} \sigma_t u_t B_{i,t} + \frac{b_2^R \sum_{i=2}^{6} \sigma_t u_t B_{i,t}}{\sum_{j=2}^{6} \sigma_t u_t B_{j,t} + b_3^R} - C u_t \right\} \delta^t \\
&+ \lambda_{1,t+1} \left(\tau\,(t)\, \frac{\gamma_t B_{1,t}}{\alpha_t B_{1,t} + \beta_t} + (1 - \tau\,(t)) \sum_{i=2}^{6} f_i \frac{W_{1,t}}{W_{i,t}} B_{i,t} \right) \\
&+ \lambda_{2,t+1} \left(\tau\,(t) \left(e^{g_{2,t} - M_2} - \sigma_t u_t \right) B_{2,t} + \left(1 - \tau\,(t)\, \frac{\gamma_1 B_{1,t}}{\alpha_{1,t} B_{1,t} + \beta_{1,t}} \right) \right) \\
&+ \sum_{i=3}^{6} \lambda_{i,t+1} (\tau\,(t) \left(e^{g_{i,t} - M_i} - \sigma_t u_t \right) B_{i,t} + (1 - \tau(t)) \left(e^{g_{i-1,t} - M_{i-1}} - \sigma_t u_t \right) B_{i-1,t}).
\end{aligned}
$$

We apply the three necessary conditions (adjoint, transversality, and optimality) from Pontryagin's maximum principle. First, the adjoint condition allows us to obtain the adjoint variables $\lambda_{i,t}^R$ implicitly:

$$
\begin{aligned}
\lambda_{1,t}^R &= \frac{\partial H_t^R}{\partial B_{1,t}} = \lambda_{1,t+1}^R \tau\,(t)\, \frac{\gamma_t \beta_t}{(\alpha_1 B_{1,t} + \beta_t)^2} + \lambda_{2,t+1}^R (1 - \tau\,(t))\, \frac{\gamma_t \beta_t}{(\alpha_1 B_{1,t} + \beta_t)^2} \\
\lambda_{i,t}^R &= \frac{\partial H_t^R}{\partial B_{i,t}} \\
&= \left\{ b_1^R \sigma_t u_t + \frac{b_2^R b_3^R u_t}{\left(\sum_{j=2}^{m} u_t B_{j,t} + b_3^R \right)^2} \right\} \delta^t + \lambda_t^R \tau\,(t) \left(e^{g_{i,t} - M_i} - \sigma_t u_t \right) \\
&+ \lambda_{i+1,t+1}^R (1 - \tau\,(t)) \left(e^{(g_{i,t} - M_i)} - \sigma_t u_t \right) + \lambda_{1,t+1}^R (1 - \tau\,(t)) f_i \frac{W_1}{W_i}, \quad i = 2, 3, 4, 5 \\
\lambda_{i,t}^R &= \frac{\partial H_t^R}{\partial B_{i,t}} \\
&= \left\{ b_1^R \sigma_t u_t + \frac{b_2^R b_3^R u_t}{\left(\sum_{j=2}^{6} u_t B_{j,t} + b_3^R \right)^2} \right\} \delta^t + \lambda_{6,t+1}^L \tau\,(t) \left(e^{(g_{6,t} - M_t)} - \sigma_t u_t \right) \\
&+ \lambda_{1,t+1}^L (1 - \tau\,(t)) f_6 \frac{W_1}{W_6}.
\end{aligned}
$$

Second, the terminal state $t = T + 1$ is derived through the transversality condition

$$
\lambda_{i,T+1}^R = q, \quad i = 2, 3, 4, 5, 6.
$$

Third, we determine u_t that satisfies the optimality condition

$$
\begin{aligned}
0 &= \frac{\partial H_t^R}{\partial u_t} \\
&= \left\{ b_1^R \sum_{i=2}^{6} \sigma_t B_{i,t} + \frac{b_2^R \sum_{i=2}^{6} \sigma_t B_{i,t}}{\sum_{j=2}^{6} \sigma_t u_t B_{j,t} + b_3^R} - \frac{b_2^R \sum_{i=2}^{6} \sigma_t u_t B_i}{\left(\sum_{j=2}^{6} \sigma_t u_t B_{j,t} + b_3^R \right)^2} \sum_{i=2}^{6} \sigma_t B_{i,t} - 2 C u_t \right\} \delta^t
\end{aligned}
$$

$$-\sum_{i=2}^{6}\lambda_{i,t+1}\tau(t)B_{i,t}-\sum_{i=3}^{6}\lambda_{i,t+1}(1-\tau(t))B_{i-1,t}.$$

We isolate all terms containing u_t on the left side and move the remainder to the right side:

$$\left\{\frac{b_2^R b_3^R\sum_{i=2}^{6}\sigma_t B_{i,t}}{\left(\sum_{j=2}^{6}u_t B_{j,t}+b_3^R\right)^2}-2Cu_t\right\}\delta^t$$
$$=\sum_{i=2}^{6}\lambda_{i,t+1}^R\tau(t)B_{i,t}+\sum_{i=3}^{6}\lambda_{i,t+1}^R(1-\tau(t))B_{i-1,t}-b_1\sum_{i=2}^{6}B_{i,t}\delta^t.$$

This expression clearly separates the u_t-dependent part from the part that does not depend on u_t, making it straightforward to solve for u_t.

Hence, for $t=1,2,\dots,T-1$, there exist adjoint variables $\lambda_{i,t}^R$ for $t=1,2,\dots,T-1$ such that

$$\lambda_{1,t}^R=\lambda_{1,t+1}^R\tau(t)\frac{\gamma_t\beta_t}{(\alpha_1B_{1,t}+\beta_t)^2}+\lambda_{2,t+1}^R(1-\tau(t))\frac{\gamma_t\beta_t}{(\alpha_1B_{1,t}+\beta_t)^2}$$
$$\lambda_{i,t}^R=\left\{b_1^R\sigma_t u_t+\frac{b_2^R b_3^R u_t}{\left(\sum_{j=2}^{6}u_t B_{j,t}+b_3^R\right)^2}\right\}\delta^t+\lambda_{i,t+1}^R\tau(t)\left(e^{g_{i,t}-M_i}-\sigma_t u_t\right)$$
$$+\lambda_{i+1,t+1}^R(1-\tau(t))\left(e^{g_{i,t}-M_i}-\sigma_t u_t\right)+\lambda_{1,t+1}^R(1-\tau(t))f_i\frac{W_1}{W_i},\quad i=2,3,4,5$$
$$\lambda_{6,t}^R=\left\{b_1^R\sigma_t u_t+\frac{b_2^R b_3^R u_t}{\left(\sum_{j=2}^{6}u_t B_{j,t}+b_3^R\right)^2}\right\}\delta^t+\lambda_{6,t+1}^R\tau(t)\left(e^{g_{6,t}-M_6}-\sigma_t u_t\right)$$
$$+\lambda_{1,t+1}^R(1-\tau(t))f_6\frac{W_1}{W_6}$$
$$\lambda_{i,T+1}^R=q,\quad i=2,3,4,5,6,$$

and u satisfies u_t^{L*} at

$$\left\{\frac{b_2^R b_3^R\sum_{i=2}^{6}\sigma_t B_{i,t}}{\left(\sum_{j=2}^{6}u_t B_{j,t}+b_3^R\right)^2}-2Cu_t\right\}\delta^t$$
$$=\sum_{i=2}^{6}\lambda_{i,t+1}^R\tau(t)B_{i,t}+\sum_{i=3}^{6}\lambda_{i,t+1}^R(1-\tau(t))B_{i-1,t}-b_1\sum_{i=2}^{6}B_{i,t}\delta^t.$$

□

Theorem 3 *For the optimal control function u_t^{I*} and corresponding solutions $B_{i,t}$ ($i=1,2,\dots,6$) of system* (1) *with objective function $J^I(u)$, there exist adjoint variables $\lambda_{i,t}^I$ for $t=1,2,\dots,T-1$ such that*

$$\lambda_{1,t}^I=\lambda_{1,t+1}^I\tau(t)\frac{\gamma_t\beta_t}{(\alpha_1B_{1,t}+\beta_t)^2}+\lambda_{2,t+1}^I(1-\tau(t))\frac{\gamma_t\beta_t}{(\alpha_1B_{1,t}+\beta_t)^2}$$

$$\lambda_{i,t}^{I} = b_1^I\sigma_t u_t\left(\sum_{j=2}^{6}\sigma_t u_t B_{j,t}+1\right)^{b_3^I-1}\left\{\left(b_3^I+1\right)\left(\sum_{j=1}^{6}\sigma_t u_t B_{j,t}\right)+1\right\}\delta^t + b_2^I\sigma_t u_t\delta^t$$
$$+\lambda_{i,t+1}^{I}\tau(t)\left(e^{g_{i,t}-M_i}-\sigma_t u_t\right)+\lambda_{i+1,t+1}^{I}(1-\tau(t))\left(e^{(g_{i,t}-M_i)}-\sigma_t u_t\right)$$
$$+\lambda_{1,t+1}^{I}(1-\tau(t))f_i\frac{W_1}{W_i},\qquad i=2,3,4,5$$
$$\lambda_{6,t}^{I} = b_1^I\sigma_t u_t\left(\sum_{j=2}^{6}\sigma_t u_t B_{j,t}+1\right)^{b_3^I-1}\left\{\left(b_3^I+1\right)\left(\sum_{j=1}^{6}\sigma_t u_t B_{j,t}\right)+1\right\}\delta^t + b_2^I\sigma_t u_t\delta^t$$
$$+\lambda_{6,t+1}^{I}\tau(t)\left(e^{(g_{6,t}-M_t)}-\sigma_t u_t\right)+\lambda_{1,t+1}^{I}(1-\tau(t))f_6\frac{W_1}{W_6}$$
$$\lambda_{i,T+1}^{L}=q,\quad i=2,3,4,5,6,$$

and u satisfies u_t^{L*} at

$$\left\{b_1^I\left(\sum_{j=2}^{6}\sigma_t B_{j,t}\right)\left(b_3^I+1+\sum_{j=2}^{6}\sigma_t u_t B_{j,t}\right)\left(\sum_{j=2}^{6}\sigma_t u_t B_{j,t}+1\right)^{b_3^I-1}-2Cu_t\right\}\delta^t$$
$$=-b_2^I\left(\sum_{i=2}^{6}\sigma_t B_{i,t}\right)+\sum_{i=2}^{6}\lambda_{i,t+1}^{I}\tau(t)\sigma_t B_{i,t}+\sum_{i=3}^{6}\lambda_{i,t+1}^{I}(1-\tau(t))\sigma_t B_{i-1,t}.$$

Proof of Theorem 3 According to the discrete-time Pontryagin's maximum principle [17], the Hamiltonian H_t^L at time t is defined as

$$H_t^I = J_t^I(u,B)+\sum_{i=2}^{6}\lambda_{i,t+1}^{I}B_{i,t+1}$$
$$=\left\{\left(\sum_{i=2}^{6}P_t^I\sigma_t u_t B_{i,t}\right)-Cu_t^2\right\}\delta^t+\sum_{i=2}^{6}\lambda_{i,t+1}^{I}B_{i,t+1}$$
$$=\{b_1^I\left(\sum_{i=2}^{6}\sigma_t u_t B_{i,t}\right)\left(\sum_{j=2}^{6}\sigma_t u_t B_{j,t}+1\right)^{b_2^I}+b_3^I\left(\sum_{i=2}^{6}\sigma_t u_t B_{i,t}\right)-Cu_t^2\}\delta^t$$
$$+\lambda_{1,t+1}\left(\tau(t)\frac{\gamma_t B_{1,t}}{\alpha_t B_{1,t}+\beta_t}+(1-\tau(t))\sum_{i=2}^{6}f_i\frac{W_{1,t}}{W_{i,t}}B_{i,t}\right)$$
$$+\lambda_{2,t+1}\left(\tau(t)\left(e^{g_{2,t}-M_2}-\sigma_t u_t\right)B_{2,t}+\left(1-\tau(t)\frac{\gamma_1 B_{1,t}}{\alpha_{1,t}B_{1,t}+\beta_{1,t}}\right)\right)$$
$$+\sum_{i=3}^{6}\lambda_{i,t+1}\left(\tau(t)\left(e^{g_{i,t}-M_i}-\sigma_t u_t\right)B_{i,t}+(1-\tau(t))\left(e^{g_{i-1,t}-M_{i-1}}-\sigma_t u_t\right)B_{i-1,t}\right).$$

We apply the three necessary conditions from Pontryagin's maximum principle. First, the adjoint condition allows us to obtain the adjoint variables $\lambda_{i,t}^I$ implicitly:

$$\lambda_{1,t}^{I}=\frac{\partial H_t^I}{\partial B_{1,t}}$$

$$
\begin{aligned}
&= \lambda^I_{1,t+1}\tau(t)\frac{\gamma_t\beta_t}{(\alpha_1 B_{1,t}+\beta_t)^2} + \lambda^I_{2,t+1}(1-\tau(t))\frac{\gamma_t\beta_t}{(\alpha_1 B_{1,t}+\beta_t)^2}\\
\lambda^I_{i,t} &= \frac{\partial H^I_t}{\partial B_{i,t}}\\
&= b^I_1\sigma_t u_t\left(\sum_{j=2}^{6}\sigma_t u_t B_{j,t}+1\right)^{b^I_2-1}\left\{\sum_{i=1}^{6}\sigma_t u_t B_{i,t}+1+b^I_2\left(\sum_{j=1}^{6}\sigma_t u_t B_{j,t}\right)\right\}\delta^t\\
&\quad + b^I_3\sigma_t u_t\delta^t\\
&\quad + \lambda^I_{i,t+1}\tau(t)\left(e^{g_{i,t}-M_i}-\sigma_t u_t\right) + \lambda^I_{i+1,t+1}(1-\tau(t))\left(e^{(g_{i,t}-M_i)}-\sigma_t u_t\right)\\
&\quad + \lambda^I_{1,t+1}(1-\tau(t))f_i\frac{W_1}{W_i}\\
&= b^I_1\sigma_t u_t\left(\sum_{j=2}^{6}\sigma_t u_t B_{j,t}+1\right)^{b^I_2-1}\left\{\left(b^I_2+1\right)\left(\sum_{j=1}^{6}\sigma_t u_t B_{j,t}\right)+1\right\}\delta^t + b^I_3\sigma_t u_t\delta^t\\
&\quad + \lambda^I_{i,t+1}\tau(t)\left(e^{g_{i,t}-M_i}-\sigma_t u_t\right) + \lambda^I_{i+1,t+1}(1-\tau(t))\left(e^{(g_{i,t}-M_i)}-\sigma_t u_t\right)\\
&\quad + \lambda^I_{1,t+1}(1-\tau(t))f_i\frac{W_1}{W_i}, \quad i=2,3,4,5\\
\lambda^I_{6,t} &= \frac{\partial H^I_t}{\partial B_{i,t}}\\
&= b^I_1\sigma_t u_t\left(\sum_{j=2}^{6}\sigma_t u_t B_{j,t}+1\right)^{b^I_2-1}\left\{\left(b^I_2+1\right)\left(\sum_{j=1}^{6}\sigma_t u_t B_{j,t}\right)+1\right\}\delta^t + b^I_3\sigma_t u_t\delta^t\\
&\quad + \lambda^I_{6,t+1}\tau(t)\left(e^{(g_{6,t}-M_t)}-\sigma_t u_t\right) + \lambda^I_{1,t+1}(1-\tau(t))f_6\frac{W_1}{W_6}.
\end{aligned}
$$

Second, the terminal state $t = T + 1$ is derived through the transversality condition

$$
\lambda^I_{i,T+1} = q, \quad i = 2,3,4,5,6.
$$

Third, we determine u_t that satisfies the optimality condition

$$
\begin{aligned}
0 &= \frac{\partial H^I_t}{\partial u_t}\\
&= \left\{b^I_1\left(\sum_{j=2}^{6}\sigma_t B_{j,t}\right)\left(\sum_{j=2}^{6}\sigma_t u_t B_{j,t}+1\right)^{b^I_2} + b^I_1 b^I_2\left(\sum_{j=2}^{6}\sigma_t B_{j,t}\right)\right.\\
&\quad\left.\times\left(\sum_{j=2}^{6}\sigma_t u_t B_{j,t}+1\right)^{b^I_2-1}\right\}\delta^t\\
&\quad + b^I_3\left(\sum_{i=2}^{6}\sigma_t B_{i,t}\right)\delta^t - 2Cu_t\delta^t - \sum_{i=2}^{6}\lambda^I_{i,t+1}\tau(t)B_{i,t} - \sum_{i=3}^{6}\lambda^I_{i,t+1}(1-\tau(t))B_{i-1,t}.
\end{aligned}
$$

We isolate all terms containing u_t on the left side and move the remainder to the right side:

$$\left\{ b_1^I \left(\sum_{j=2}^{6} \sigma_t B_{j,t} \right) \left(b_3^I + 1 + \sum_{j=2}^{6} \sigma_t u_t B_{j,t} \right) \left(\sum_{j=2}^{6} \sigma_t u_t B_{j,t} + 1 \right)^{b_3^I - 1} - 2C u_t \right\} \delta^t$$
$$= -b_2^I \left(\sum_{i=2}^{6} \sigma_t B_{i,t} \right) + \sum_{i=2}^{6} \lambda_{i,t+1}^I \tau(t) \sigma_t B_{i,t} + \sum_{i=3}^{6} \lambda_{i,t+1}^I (1 - \tau(t)) \sigma_t B_{i-1,t}.$$

This expression clearly separates the u_t-dependent part from the part that does not depend on u_t, making it straightforward to solve for u_t.

Hence, for $t = 1, 2, \ldots, T-1$, there exist adjoint variables $\lambda_{i,t}^I$ for $t = 1, 2, \ldots, T-1$ such that

$$\lambda_{1,t}^I = \lambda_{1,t+1}^I \tau(t) \frac{\gamma_t \beta_t}{(\alpha_1 B_{1,t} + \beta_t)^2} + \lambda_{2,t+1}^I (1 - \tau(t)) \frac{\gamma_t \beta_t}{(\alpha_1 B_{1,t} + \beta_t)^2}$$

$$\lambda_{i,t}^I = b_1^I \sigma_t u_t \left(\sum_{j=2}^{6} \sigma_t u_t B_{j,t} + 1 \right)^{b_3^I - 1} \left\{ \left(b_3^I + 1 \right) \left(\sum_{j=1}^{6} \sigma_t u_t B_{j,t} \right) + 1 \right\} \delta^t + b_2^I \sigma_t u_t \delta^t$$
$$+ \lambda_{i,t+1}^I \tau(t) \left(e^{g_{i,t} - M_i} - \sigma_t u_t \right) + \lambda_{i+1,t+1}^I (1 - \tau(t)) \left(e^{(g_{i,t} - M_i)} - \sigma_t u_t \right)$$
$$+ \lambda_{1,t+1}^I (1 - \tau(t)) f_i \frac{W_1}{W_i}, \quad i = 2, 3, 4, 5$$

$$\lambda_{6,t}^I = b_1^I \sigma_t u_t \left(\sum_{j=2}^{6} \sigma_t u_t B_{j,t} + 1 \right)^{b_3^I - 1} \left\{ \left(b_3^I + 1 \right) \left(\sum_{j=1}^{6} \sigma_t u_t B_{j,t} \right) + 1 \right\} \delta^t + b_2^I \sigma_t u_t \delta^t$$
$$+ \lambda_{6,t+1}^I \tau(t) \left(e^{(g_{6,t} - M_t)} - \sigma_t u_t \right) + \lambda_{1,t+1}^I (1 - \tau(t)) f_6 \frac{W_1}{W_6}$$

$$\lambda_{i,T+1}^L = q, \quad i = 2, 3, 4, 5, 6,$$

and u satisfies u_t^{L*} at

$$\left\{ b_1^I \left(\sum_{j=2}^{6} \sigma_t B_{j,t} \right) \left(b_3^I + 1 + \sum_{j=2}^{6} \sigma_t u_t B_{j,t} \right) \left(\sum_{j=2}^{6} \sigma_t u_t B_{j,t} + 1 \right)^{b_3^I - 1} - 2C u_t \right\} \delta^t$$
$$= -b_2^I \left(\sum_{i=2}^{6} \sigma_t B_{i,t} \right) + \sum_{i=2}^{6} \lambda_{i,t+1}^I \tau(t) \sigma_t B_{i,t} + \sum_{i=3}^{6} \lambda_{i,t+1}^I (1 - \tau(t)) \sigma_t B_{i-1,t}. \qquad \square$$

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GC and GJ retrieved and analyzed data; GC and GJ developed the model structure and simulated the models using the data. All authors contributed to writing and revising the subsequent versions of the manuscript. All authors have read and approved the final manuscript.

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Availability of data and materials

The datasets analyzed during the current study are available in the Korean Statistical Information Service (KOSIS) repository:
https://kosis.kr/statHtml/statHtml.do?orgId=101&tblId=DT_1EW0001&vw_cd=MT_ETITLE&list_id=K2_7&scrId=&language=en&seqNo=&lang_mode=en&obj_var_id=&itm_id=&conn_path=MT_ETITLE&path=%252Feng%252FstatisticsList%252FstatisticsListIndex.do.

Declarations

Competing interests

The authors declare that they have no competing interests.

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