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Positive odd-periodic solutions for a system of second-order ordinary differential equations with derivative terms

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Abstract

In this paper, we study the existence of positive odd 2π-periodic solutions for second-order ordinary differential equations

$$\begin{cases} -u''(t) = f(t,u(t),v(t),u'(t)), & t \in [0,2\pi], \\ -v''(t) = g(t,u(t),v(t),v'(t)), & t \in [0,2\pi], \\ u(0) = u(2\pi), u'(0) = u'(2\pi), \\ v(0) = v(2\pi), v'(0) = v'(2\pi), \end{cases}$$

where $f,g : [0,2\pi] \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^+$ are continuous, and f,g are 2π-periodic in t. Under the conditions that nonlinear terms $f(t,x,y,p)$ and $g(t,x,y,q)$ may be superlinear or sublinear growth on x,y,p and q as $|(x,y,p)| \to 0$, $|(x,y,q)| \to 0$ or $|(x,y,p)| \to \infty$, $|(x,y,q)| \to \infty$. The existence results of positive periodic solutions are obtained, our proof is based on the fixed point index theory in cones. Finally, two examples are given to illustrate the applicability of the conclusions of this paper.

Mathematics Subject Classification: 34B18; 34K13; 34C25

Keywords: Second-order ordinary differential equations; Odd periodic solutions; Cone; Fixed point index

1 Introduction and main results

In recent years, the study the existence of periodic solutions of second-order differential equations has been attracting the attention of many mathematicians, see for instance the papers [1, 3, 4, 6–18]. In particular, in [12], the authors considered the existence of positive periodic solutions for the boundary value problems(BVP) of second-order ordinary differential equations

$$\begin{cases} x'' + a_1(t)x = f_1(x, y), & t \in \mathbb{R}, \\ y'' + a_2(t)x = f_2(x, y), & t \in \mathbb{R}, \\ x(0) = x(1), \ x'(0) = x'(1), \\ y(0) = y(1), \ y'(0) = y'(1), \end{cases}$$

where $f_i : \mathbb{R}^2 \to \mathbb{R}_+$ are continuous functions, $i = 1, 2$. The authors proved the existence of positive periodic solutions by applying Leray–Schauder fixed-point theorem.

It is worth noting that in [12], the authors considered the nonlinear terms f, g do not contain derivative terms u', v'. However, to the best of our knowledge, there are no references researching the existence of positive odd 2π-periodic solutions for nonlinear terms f, g contain derivative terms u', v'. The purpose of the current article is to investigate the existence of positive periodic solutions for second-order ordinary differential equations

$$\begin{cases} -u''(t) = f(t, u(t), v(t), u'(t)), & t \in [0, 2\pi], \\ -v''(t) = g(t, u(t), v(t), v'(t)), & t \in [0, 2\pi], \\ u(0) = u(2\pi), u'(0) = u'(2\pi), \\ v(0) = v(2\pi), v'(0) = v'(2\pi), \end{cases} \tag{1.1}$$

where $f, g : [0, 2\pi] \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^+$ are continuous. We consider the case that $f(t, x, y, p)$ and $g(t, x, y, q)$ may be superlinear growth on x, y, p and q. In order to study the existence results of positive odd 2π-periodic solutions for BVP (1.1), we need that $f(t, x, y, p)$ and $g(t, x, y, q)$ satisfy the following Nagumo condition on p and q:

(F1) Let $f, g : [0, \pi] \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^+$ be continuous and 2π-periodic in t, and f, g be odd functions in (t, x, y), that is

$$f(t + 2\pi, x, y, p) = f(t, x, y, p), \;\; f(-t, -x, -y, p) = -f(t, x, y, p),$$
$$g(t + 2\pi, x, y, q) = g(t, x, y, q), \;\; g(-t, -x, -y, q) = -g(t, x, y, q).$$

(F2) For any given $M > 0$, there exists a positive continuous function $H_M(\rho)$ on $\mathbb{R}^+$ satisfying

$$\int_0^{+\infty} \frac{\rho d\rho}{H_M(\rho) + 1} = +\infty,$$

such that for any $(t, x, y, p) \in [0, \pi] \times [-M, M] \times [-M, M] \times \mathbb{R}$ and $(t, x, y, q) \in [0, \pi] \times [-M, M] \times [-M, M] \times \mathbb{R}$, we have

$$|f(t, x, y, p)| \le H_M(|\rho|), \;\; |g(t, x, y, q)| \le H_M(|\rho|).$$

Firstly, we consider the case that $f(t, x, y, p), g(t, x, y, q)$ may be superlinear growth on x, y, p and q as $|(x, y, p)| \to 0$, $|(x, y, q)| \to 0$ or $|(x, y, p)| \to \infty$, $|(x, y, q)| \to \infty$. In this case, we obtain the following result.

Theorem 1.1 *Let $f, g : [0, \pi] \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^+$ be continuous. If f, g satisfy the assumption* (*F*1), (*F*2) *as well as the following conditions*:

(*F*3) *There exist $a_i, b_i, c_i, \delta > 0, i = 1, 2$, satisfying $a + b + c < 1$, where $a = a_1 + a_2$, $b = b_1 + b_2$, $c = c_1 + c_2$, such that*

$$f(t, x, y, p)$$
$$\le a_1 x + b_1 y + c_1 |p|, (t, x, y, p) \in [0, \pi] \times [0, +\infty) \times [0, +\infty) \times \mathbb{R}, |(x, y, p, q)| < \delta,$$

$$g(t,x,y,q) \\ \leq a_2x + b_2y + c_2|q|, (t,x,y,q) \in [0,\pi] \times [0,+\infty) \times [0,+\infty) \times \mathbb{R}, |(x,y,p,q)| < \delta.$$

$(F4)$ *There exist* $d_i, e_i, C_i, \delta > 0, i = 1,2$, *satisfying* $m > 1$, *where* $m = \min\{d,e\}, d = d_1 + d_2, e = e_1 + e_2$, *such that*

$$f(t,x,y,p) \\ \geq d_1x + e_1y - C_1, (t,x,y,p) \in [0,\pi] \times [0,+\infty) \times [0,+\infty) \times \mathbb{R}, |(x,y,p,q)| > H$$

$$g(t,x,y,q) \\ \geq d_2x + e_2y - C_2, (t,x,y,q) \in [0,\pi] \times [0,+\infty) \times [0,+\infty) \times \mathbb{R}.|(x,y,p,q)| > H$$

Then BVP (1.1) *has at least one positive odd* 2π*-periodic solution.*

Secondly, we consider the case that $f(t,x,y,p), g(t,x,y,q)$ may be sublinear growth on x, y, p and q as $|(x,y,p)| \to 0$, $|(x,y,q)| \to 0$ or $|(x,y,p)| \to \infty$, $|(x,y,q)| \to \infty$. In this case, we obtain the following result.

Theorem 1.2 *Let* $f, g : [0,\pi] \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^+$ *be continuous functions. If* f, g *satisfy the assumption* $(F1)$, $(F2)$ *and the following conditions*:

$(F5)$ *There exist* $d_i, e_i, \delta' > 0, i = 3,4$, *satisfying* $m' > 1$, *where* $m' = \min\{d', e'\}$, $d' = d_3 + d_4$, $e' = e_3 + e_4$, *such that*

$$f(t,x,y,p) \geq d_3x + e_3y, (t,x,y,p) \in [0,\pi] \times [0,+\infty) \times [0,+\infty) \times \mathbb{R}, |(x,y,p,q)| < \delta',$$

$$g(t,x,y,q) \geq d_4x + e_4y, (t,x,y,q) \in [0,\pi] \times [0,+\infty) \times [0,+\infty) \times \mathbb{R}, |(x,y,p,q)| < \delta'.$$

$(F6)$ *There exist* $a_i, b_i, c_i, H, i = 3,4$, *satisfying* $a' + b' + c' < 1$, *where* $a' = a_3 + a_4$, $b' = b_3 + b_4$, $c' = c_3 + c_4$, *such that*

$$f(t,x,y,p) \\ \leq a_3x + b_3y + c_3|p|, (t,x,y,p) \in [0,\pi] \times [0,+\infty) \times [0,+\infty) \times \mathbb{R}, |(x,y,p,q)| > H,$$

$$g(t,x,y,q) \\ \leq a_4x + b_4y + c_4|q|, (t,x,y,q) \in [0,\pi] \times [0,+\infty) \times [0,+\infty) \times \mathbb{R}, |(x,y,p,q)| > H.$$

Then BVP (1.1) *has at least one positive odd* 2π*-periodic solution.*

It is worth noting that in Theorems 1.1 and Theorems 1.2, we use the inequality conditions (F3)–(F4) and (F5)-(F6) to describe the superlinear and sublinear growth of the nonlinearity f and g, respectively. Our inequality conditions are weaker than the usual conditions described by the corresponding upper and lower limits.

The remainder part of this paper is organized as follows. In Sect. 2, for the convenience of the readers we collect some general results and given some notations. In Sect. 3, under suitable hypotheses, we apply the fixed point index theory in cones to obtain the existence results of positive periodic solutions for BVP (1.1). In Sect. 4, we apply the previous results to some examples to demonstrate the applicability of our main results.

2 Preliminaries

Throughout this paper, let $C_{2\pi}(\mathbb{R})$ denote the Banach space of all continuous 2π-periodic function $u(t)$ with norm $||u||_C = \max|u(t)|$, let $C^1_{2\pi}(R)$ be the Banach space of all continuous differentiable 2π-periodic function $u(t)$ with norm $||u||_{C^1} = \max\{||u||_C, ||u'||_C\}$. Generally, $C^n_{2\pi}(\mathbb{R})$ denotes nth-order continuous differentiable 2π-periodic function space. Set $I = [0,\pi]$, then $C(I)$ denote the Banach space of all continuous function $u(t)$ on I, and $C^n(I)$ denote the Banach space of all nth-order continuously differentiable $u(t)$ on I. Let $L^2(I)$ denote the Hilbert space of locally square integrable function $u(t)$ on I with interior product $(u,v) = \int_0^\pi u(t)v(t)dt$ and the normal $||u||_2 = (\int_0^\pi |u(t)|^2dt)^{\frac{1}{2}}$. Let $X \times Y$ denote the Banach space formed by the product space of X and Y according to normal $||(u,v)|| = \max\{||.||_X, ||.||_Y\}$.

If $(u,v) \in C^2_{2\pi}(\mathbb{R}) \times C^2_{2\pi}(\mathbb{R})$ is the odd 2π-periodic solution of the BVP (1.1), then by the oddity of (u,v), then

$$u(0) = u(\pi) = 0, \quad v(0) = v(\pi) = 0.$$

We constraint of (u,v) on $[0,\pi]$ is a solution of the second-order boundary value problem

$$\begin{cases} -u'' = f(t,u(t),v(t),u'(t)), & t \in [0,\pi], \\ -v'' = g(t,u(t),v(t),v'(t)), & t \in [0,\pi], \\ u(0) = u(\pi) = 0, \\ v(0) = v(\pi) = 0. \end{cases} \tag{2.1}$$

On the contrary, if $(u,v) \in C^2[0,\pi] \times C^2[0,\pi]$ is the solution of BVP (2.1), then it is assumed that the odd continuation of (F1), (u,v) with a period of 2π is the odd 2π-periodic solution of BVP(1.1). To prove Theorem 1.1–1.2, we consider BVP (2.1).

Given $h \in C(I)$, we consider the linear boundary value problem (LBVP)

$$\begin{cases} -u'' = h(t), & t \in [0,1], \\ u(0) = u(\pi) = 0. \end{cases} \tag{2.2}$$

It is well known that LBVP (2.2) has a unique solution expressed by

$$u(t) = \int_0^\pi G(t,s)h(s)ds := Sh(t), \tag{2.3}$$

where $G(t,s)$ is the corresponding Green function given by

$$G(t,s) = \begin{cases} t(1-\dfrac{s}{\pi}), & 0 \le t \le s \le 1, \\ s(1-\dfrac{t}{\pi}), & 0 \le s \le t \le 1. \end{cases} \tag{2.4}$$

From (2.3), we have

$$u'(t) = \int_t^\pi (1-\frac{s}{\pi})h(s)ds - \frac{1}{\pi}\int_0^t sh(s)ds, \quad t \in I. \tag{2.5}$$

According to (2.5) and (2.2), we easily see that the solution operator of LBVP (2.2) $S: C(I) \to C^2(I)$ is a linear bounded operator. By compactness of the embedding $C^2(I) \hookrightarrow C^1(I)$, we obtain that $S: C(I) \to C^1(I)$ is a completely continuous linear operator.

Lemma 2.1 *For every $h \in C(I)$, LBVP (2.2) has a unique solution $u = Sh$ satisfies*

$$||u||_2 \leq ||u'||_2 \leq ||u''||_2. \tag{2.6}$$

Proof Let $h \in C(I)$, since orthogonal function system $\{\sin k\pi t | k = 1, 2, \ldots\} \subset L^2(I)$ is a complete orthogonal system, it can be expressed by the Fourier sine series expansion

$$u(t) = \sum_{k=1}^{\infty} b_k \sin k\pi t, \tag{2.7}$$

where $b_k = \frac{2}{\pi} \int_0^{\pi} u(t) \sin k\pi t dt, k = 1, 2, \ldots$. It then follows from Parseval equality that

$$||u||_2^2 = \frac{\pi}{2} \sum_{k=1}^{\infty} |b_k|^2. \tag{2.8}$$

On the other hand, since $u' \in L^2(I)$ is an even function, it can be expressed by the Fourier cosine series expansion

$$u'(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos k\pi t, \tag{2.9}$$

where $a_0 = \frac{2}{\pi} \int_0^{\pi} u'(t) dt = u(0) - u(\pi) = 0,\ a_k = \frac{2}{\pi} \int_0^{\pi} u'(t) \cos k\pi t dt = 2kb_k, k = 1, 2, \ldots,$

By (2.7), (2.9) and the Parseva equality, we obtain

$$||u'||_2^2 = \frac{\pi}{2} \sum_{k=1}^{\infty} |2kb_k|^2 \geq \frac{\pi}{2} \sum_{k=1}^{\infty} |b_k|^2 = ||u||_2^2,$$

$$||u''||_2^2 = \frac{\pi}{2} \sum_{k=1}^{\infty} |2k^2\pi b_k|^2 \geq \frac{\pi}{2} \sum_{k=1}^{\infty} |2kb_k|^2 = ||u'||_2^2.$$

Hence (2.5) holds. □

Lemma 2.2 *Let $h \in C^+(I)$. Then the solution $u = Sh$ of LBVP (2.2) has the following properties*:

(a) $u(t) \geq \frac{1}{\pi^2} t(\pi - t) ||u||_C, t \in I$;

(b) $||u||_C \leq \frac{\pi^2}{4} \int_0^{\pi} u(t) \sin t dt$;

(c) *There exists $\xi \in (0, \pi)$ such that $u'(\xi) = 0$, $u'(t) \geq 0$ for $t \in [0, \xi]$, and $u'(t) \leq 0$ for $t \in [\xi, \pi]$. Moreover, $||u'||_C = \max\{u'(0), -u'(\pi)\}$.*

Proof Let $h \in C^+(I)$ and $u = Sh$. From the expression (2.4), we easily see that the Green function $G(t,s)$ has the following properties:

(i) $0 \leq G(t,s) \leq G(s,s)\ \forall t, s \in I$;

(ii) $G(t,s) \geq \frac{1}{\pi} G(s,s) G(t,t)\ \forall t, s \in I$.

By virtue of (2.3) and the properties (i) and (ii), for every $t \in I$, we have

$$u(t) = \int_0^\pi G(t,s)h(s)ds \geq \frac{1}{\pi}G(t,t)\int_0^\pi G(s,s)h(s)ds \geq \frac{1}{\pi^2}t(\pi - t)||u||_C. \tag{2.10}$$

Hence, the conclusion of Lemma 2.2(a) holds.

Multiply the inequality (2.10) by $\sin t$ and integrating over I, we get

$$\int_0^\pi u(t)\sin t ds \geq \frac{1}{\pi^2}||u||_C \int_0^\pi t(\pi - t)dt = \frac{2}{\pi^2}||u||_C.$$

Hence, the conclusion of Lemma 2.2(b) holds.

From (2.5), we see that $u'(0) \geq 0$ and $u'(\pi) \leq 0$. Since $u''(t) = -h(t) \leq 0$ for every $t \in I$, it follows that $u'(t)$ is a monotone decreasing function on I. So, we conclude that there exists $\xi \in (0,\pi)$ such that $u'(\xi) = 0$, $u'(t) \geq 0$ for $t \in [0,\xi]$, and $u'(t) \leq 0$ for $t \in [\xi,\pi]$. Moreover, we get

$$||u'||_C = \max_{t\in I}|u'(t)| = \max\{u'(0), -u'(\pi)\}.$$

Hence, the conclusion of Lemma 2.2(c) holds.

Now, we define a closed convex cone K in $C^1(I)$

$$K = \{u \in C^1(I)|u(t) \geq 0, t \in I\}. \tag{2.11}$$

For every $(u,v) \in K \times K$, set

$$F(u,v)(t) := f(t,u(t),v(t),u'(t)), \quad G(u,v)(t) := f(t,u(t),v(t),v'(t)), t \in I. \tag{2.12}$$

Then F and $G : K \times K \to C^+(I)$ are continuous and it maps every bounded in $K \times K$ into bounded set in $C^+(I)$. We define three mappings

$$A : K^2 \to K^2,\ A_1,\ A_2 : K \times K \to K,$$

by

$$A = (A_1,A_2),\ A_1 = S \circ F,\ A_2 = S \circ G. \tag{2.13}$$

By the complete continuity of operator $S : C(I) \to C^1(I)$, we get $A : K^2 \to K^2$ is a completely continuous mapping. By the definitions of S and K, the odd 2π-periodic solution of BVP(1.1) is equivalent to the nonzero fixed point of A. We will find the nonzero fixed of A by using the fixed point index theory in cones.

To find the nonzero fixed point of A defined by (2.11), we recall some concepts and conclusions on the fixed point index in [2, 5]. Let E be a Banach space and $K \subset E$ be a closed convex cone in E. Assume Ω is a bounded open subset of E with boundary $\partial\Omega$, and $K \cap \Omega \neq \emptyset$. Let $A : K \cap \overline{\Omega} \to K$ be a completely continuous mapping. If $Au \neq u$, for every $u \in K \cap \partial\Omega$, then the fixed point index $i(A, K \cap \Omega, K)$ has definition. One important fact is that if $i(A, K \cap \Omega, K) \neq 0$, then A has a fixed point in $K \cap \Omega$. □

Lemma 2.3 ([5]) *Let Ω be a bounded open subset of E with $0 \in \Omega$, and $A : K \cap \overline{\Omega} \to K$ be a completely continuous mapping. If $\mu Au \neq u$ for every $u \in K \cap \partial\Omega$ and $0 < \mu \leq 1$, then $i(A, K \cap \Omega, K) = 1$.*

Lemma 2.4 ([5]) *Let Ω be a bounded open subset of E and $A : K \cap \overline{\Omega} \to K$ be a completely continuous mapping. If there exists $v_0 \in K \setminus \{0\}$ such that $u - Au \neq \tau v_0$ for every $u \in K \cap \partial\Omega$ and $\tau \geq 0$, then $i(A, K \cap \Omega, K) = 0$.*

Lemma 2.5 ([5]) *Let Ω be a bounded open subset of E, and $A, A_1 : K \cap \overline{\Omega} \to K$ be two completely continuous mappings. If $(1-s)Au + sA_1u \neq u$ for every $u \in K \cap \partial\Omega$ and $0 \leq s \leq 1$, then $i(A, K \cap \Omega, K) = i(A_1, K \cap \Omega, K)$.*

3 Proof of main results

In this section, we will use the fixed-point index theory in cones to prove Theorem 1.1 and Theorem 1.2.

Proof of Theorem 1.1 Set $E = C^1(I)$, it is easy to verify that $K \in E$ is the closed convex cone defined by (2.11) and $A = (S \circ F, S \circ G) : K^2 \to K^2$ is the completely continuous mapping defined by (2.13). Then the odd 2π-periodic solution of BVP (1.1) is equivalent to the nontrivial fixed point of A. Let $0 < r < R < +\infty$ and set

$$\begin{aligned} \Omega_1 &= \{(u,v) \in C^1(I) \times C^1(I) \big| ||(u,v)||_{C^1} < r\}, \\ \Omega_2 &= \{(u,v) \in C^1(I) \times C^1(I) \big| ||(u,v)||_{C^1} < R\}. \end{aligned} \tag{3.1}$$

Next, we prove that A has a fixed point in $K^2 \cap (\Omega_2/\overline{\Omega_1})$, where r is small enough and R is large enough. Let $0 < r < \frac{\delta}{2}$, where δ is the positive constant in the condition (F3), we will prove that A satisfies the condition of Lemma 2.3 in $K^2 \cap (\Omega_2/\overline{\Omega_1})$, namely

$$(u,v) \neq \mu A(u,v),\ 0 < \mu \leq 1,\ (u,v) \in K^2 \cap \partial\Omega_1. \tag{3.2}$$

In fact, if (3.2) dose not hold, there exists $(u_0, v_0) \in K^2 \cap \partial\Omega_1$ and $\mu_0 \in (0,1)$ such that $(u_0, v_0) = \mu_0 A(u_0, v_0)$. Since $u_0 = S(\mu_0 F(u_0, v_0))$, $v_0 = S(\mu_0 G(u_0, v_0))$, according to the definition of S, u_0 is the unique solution of LBVP (2.2) for $h = \mu_0 F(u_0, v_0) \in C^+(I)$. Hence, $(u_0, v_0) \in C^2(I) \times C^2(I)$ satisfies the following differential equations

$$\begin{cases} -u_0''(t) = \mu_0 f(t, u_0(t), v_0(t), u_0'(t)), & t \in [0,\pi], \\ -v_0''(t) = \mu_0 g(t, u_0(t), v_0(t), v_0'(t)), & t \in [0,\pi], \\ u_0(0) = u_0(\pi) = 0, \\ v_0(0) = v_0(\pi) = 0. \end{cases} \tag{3.3}$$

Since $(u_0, v_0) \in K^2 \cap \partial\Omega_1$, by the definition of K and Ω_1, we have

$$u_0 \geq 0, v_0 \geq 0,\ |(u_0(t), u_0(t), v_0'(t), u_0'(t))| = 2||(u,v)||_{C^1} < \delta.$$

In view of (F3), we get

$$f(t, u_0(t), v_0(t), u_0'(t)) \leq a_1 u_0(t) + b_1 v_0(t) + c_1 |u_0'(t)|,$$

$$g(t,u_0(t),v_0(t),v_0'(t)) \le a_2u_0(t) + b_2v_0(t) + c_2|v_0'(t)|.$$

From (3.3), we have

$$\begin{aligned} -u_0''(t) - v_0''(t) &= \mu_0 f(t,u_0(t),v_0(t),u_0'(t)) + \mu_0 g(t,u_0(t),v_0(t),v_0'(t)) \\ &\le f(t,u_0(t),v_0(t),u_0'(t)) + g(t,u_0(t),v_0(t),v_0'(t)). \end{aligned} \tag{3.4}$$

Take the norm $||\cdot||_2$ on both sides of (3.4), then by Lemma 2.1, we get

$$\begin{aligned} &||u_0'||_2 + ||v_0'||_2 \\ &\quad\le ||u_0''||_2 + ||v_0''||_2 = ||f(t,u_0(t),v_0(t),u_0'(t))||_2 + ||g(t,u_0(t),v_0(t),v_0'(t))||_2 \\ &\quad\le a_1||u_0||_2 + b_1||v_0||_2 + c_1||u_0'||_2 + a_2||u_0||_2 + b_2||v_0||_2 + c_2||v_0'||_2 \\ &\quad\le a_1||u_0'||_2 + b_1||v_0'||_2 + c_1||u_0'||_2 + a_2||u_0'||_2 + b_2||v_0'||_2 + c_2||v_0'||_2 \\ &\quad\le (a+b+c)(||u_0'||_2 + ||v_0'||_2). \end{aligned} \tag{3.5}$$

From (3.5), we have

$$(1-(a+b+c))(||u_0'||_2 + ||v_0'||_2) \le 0.$$

We get from $a+b+c<1$ that

$$||u_0'||_2 + ||v_0'||_2 = 0,$$

which means that

$$(u_0', v_0') = (0,0),\ u_0 = c_0,\ v_0 = c_0.$$

Furthermore, we obtain that $u_0(0)=0,\ u_0(\pi)=0;\ v_0(0)=0,\ v_0(\pi)=0$, then $u_0 \equiv 0,\ v_0 \equiv 0$, which contradict to the $||(u_0,v_0)|| = r$. Hence (3.2) holds, that is, A satisfies the condition of Lemma 2.3 in $K^2 \cap \partial\Omega_1$. By Lemma 2.3, we have

$$i(A, K^2 \cap \Omega_1, K^2) = 1. \tag{3.6}$$

On the other hand, we show that when R is large enough

$$i(A, K^2 \cap \Omega_2, K^2) = 0. \tag{3.7}$$

Now we define two cone mappings $F_1, G_1 : K^2 \to C^+(I)$ as follows

$$F_1(u,v)(t) := F(u,v)(t) + C_1,\quad G_1(u,v)(t) := G(u,v)(t) + C_2,\ t \in I, \tag{3.8}$$

we set

$$A' = (A_1', A_2'),\ A_1' = S \circ F_1(u,v),\ A_2' = S \circ G_1(u,v), \tag{3.9}$$

where C_1, C_2 are the positive constants in the condition (F4). Then $A' : K^2 \to K^2$ is completely continuous.

Firstly, we show that (3.7) holds. We choose $e_0 = \sin \pi t \in K \setminus \{0\}$. Since $-e_0''(t) = \pi^2 e_0(t)$, from the definition of S it follows that $S(\pi^2 e_0(t)) = e_0$. By Lemma 2.2(a), $e_0 \in K \setminus \{0\}$. We show that A' satisfies the condition of Lemma 2.4 in $K^2 \cap \partial\Omega_2$ for this e_0, namely

$$(u,v) - A'(u,v) \neq \tau(e_0,e_0), \quad \tau \geq 0, \quad (u,v) \in K^2 \cap \partial\Omega_2. \tag{3.10}$$

In fact, if (3.10) dose not hold, there exist $(u_1,v_1) \in K^2 \cap \partial\Omega_2$ and $\tau_0 \geq 0$ such that $(u_1,v_1) - A'(u_1,v_1) = \tau(e_0,e_0)$. Since

$$\begin{aligned} u_1 &= A_1'(u_1,v_1) + \tau_0 e_0 = S(F_1(u_1,v_1) + \tau_0 \pi^2 e_0), \\ v_1 &= A_2'(u_1,v_1) + \tau_0 e_0 = S(G_1(u_1,v_1) + \tau_0 \pi^2 e_0), \end{aligned}$$

according to the definition of S, u_1 is the unique solution of LBVP (2.2) for $h = F_1(u,v) + \tau_0 \pi^2 e_0 \in C^+(I)$. Hence, $(u_1,v_1) \in C^2(I) \times C^2(I)$ satisfies the differential equations

$$\begin{cases} -u_1''(t) = f(t,u_1(t),v_1(t),u_1'(t)) + C_1 + \tau_0\pi^2 \sin \pi t, & t \in I, \\ -v_1''(t) = g(t,u_1(t),v_1(t),v_1'(t)) + C_2 + \tau_0\pi^2 \sin \pi t, & t \in I, \\ u_1(0) = u_1(\pi) = 0, \\ v_1(0) = v_1(\pi) = 0. \end{cases} \tag{3.11}$$

From (3.11) and (F4), we see that

$$\begin{aligned} &- u_1''(t) - v_1''(t) \\ &\quad = f(t,u_1(t),v_1(t),u_1'(t)) + C_1 + \tau_0\pi^2 \sin \pi t + g(t,u_1(t),v_1(t),v_1'(t)) + C_2 \\ &\qquad + \tau_0 \pi^2 \sin \pi t \\ &\quad \geq d_1 u_1(t) + e_1 v_1(t) + d_2 u_1(t) + e_2 v_1(t) + 2\tau_0 \pi^2 \sin \pi t \\ &\quad \geq (d+e)(u_1(t) + v_1(t)). \end{aligned} \tag{3.12}$$

Multiplying (3.12) by $\sin t$ and integrating over I, we get

$$- \int_0^\pi u_1''(t) \sin t dt - \int_0^\pi v_1''(t) \sin t dt \geq (d+e) \int_0^\pi (u_1(t) + v_1(t)) \sin t dt. \tag{3.13}$$

Then using integration by parts for the left side, we have

$$- \int_0^\pi u_1''(t) \sin \pi t dt - \int_0^\pi v_1''(t) \sin \pi t dt = \int_0^\pi (u_1(t) + v_1(t)) \sin t dt. \tag{3.14}$$

Taking (3.13) and (3.14) into account, we obtain that

$$\int_0^\pi (u_1(t) + v_1(t)) \sin t dt \geq (d+e) \int_0^\pi (u_1(t) + v_1(t)) \sin t dt. \tag{3.15}$$

Since $\int_0^\pi |u_1(t)|\sin t dt \geq \frac{4}{\pi^2}||u_1||_C > 0$, $\int_0^\pi |v_1(t)|\sin t dt \geq \frac{4}{\pi^2}||v_1||_C > 0$ by Lemma 2.2(b), from (3.15) it follows that $(d+e) \leq 1$, which contradicts to the (F4). Hence (3.10) holds, namely A' satisfies the condition of Lemma 2.4 in $K^2 \cap \Omega_2$. By Lemma 2.4, we have

$$i(A', K^2 \cap \Omega_2, K^2) = 0. \tag{3.16}$$

Next, we show that A and A' satisfy the condition of Lemma 2.5 in $K^2 \cap \Omega_2$, when R is large enough, that is

$$(1-s)A(u,v) + sA'(u,v) \neq (u,v), \ \ (u,v) \in K^2 \cap \partial\Omega_2, \ 0 \leq s \leq 1. \tag{3.17}$$

In fact, if (3.17) dose not hold, there exist $(u_2, v_2) \in K^2 \cap \partial\Omega_2$ and $0 \leq s_0 \leq 1$ such that $(1-s_0)A(u_2,v_2) + s_0A'(u_2,v_2) = (u_2,v_2)$. Since $u_2 = S((1-s_0)F(u_2,v_2) + s_0F_1(u_2,v_2)) \in C^+(I)$, by the definition of S, u_2 is the unique solution of LBVP (2.2) for $h = (1-s_0)F(u_2,v_2) + s_0F_1(u_2,v_2) \in C^+(I)$. Hence, $(u_2,v_2) \in C^2(I) \times C^2(I)$ satisfies the differential equations

$$\begin{cases} -u_2''(t) = f(t, u_2(t), v_2(t), u_2'(t)) + s_0C_1, & t \in I, \\ -v_2''(t) = g(t, u_2(t), v_2(t), v_2'(t)) + s_0C_2, & t \in I, \\ u_2(0) = u_2(\pi) = 0, \\ v_2(0) = v_2(\pi) = 0. \end{cases} \tag{3.18}$$

From (3.18) and (F4), we see that

$$\begin{aligned} -u_2''(t) - v_2''(t) &= f(t, u_2(t), v_2(t), u_2'(t)) + s_0C_1 + g(t, u_2(t), v_2(t), v_2'(t)) + s_0C_2 \\ &\geq d_1u_2(t) + e_1v_2(t) - (1-s_0)C_1 + d_2u_2(t) + e_2v_2(t) - (1-s_0)C_2 \\ &\geq (d_1+d_2)u_2(t) + (e_1+e_2)v_2(t) - (C_1+C_2) \\ &\geq du_2(t) + ev_2(t) - (C_1+C_2). \end{aligned} \tag{3.19}$$

Multiplying (3.19) by $\sin t$ and integrating over I, we get

$$\begin{aligned} &\int_0^\pi u_2(t)\sin t dt + \int_0^\pi v_2(t)\sin t dt \\ &\quad \geq d\int_0^\pi u_2(t)\sin t dt + e\int_0^\pi v_2(t)\sin t dt - 2(C_1+C_2) \\ &\quad \geq m\int_0^\pi (u_2(t)+v_2(t))\sin t dt - 2(C_1+C_2). \end{aligned}$$

From this inequality, it follows that

$$\int_0^\pi (u_2(t)+v_2(t))\sin t dt \leq \frac{2(C_1+C_2)}{m-1}. \tag{3.20}$$

By Lemma 2.2(b), we obtain

$$||u_2||_C + ||v_2||_C \leq \frac{\pi^2}{4}\int_0^\pi u_2(t)\sin t dt + \frac{\pi^2}{4}\int_0^\pi v_2(t)\sin t dt. \tag{3.21}$$

From (3.20) and (3.21), we have

$$||u_2||_C + ||v_2||_C \le \frac{\pi^2(C_1 + C_2)}{2(m-1)} := M.$$

From this inequality, it follows that

$$||u_2||_C \le M,\ ||v_2||_C \le M. \tag{3.22}$$

For this $M > 0$, by the assumption (F2), there is a positive continuous function $G_M \in C(R^+, R^+)$ satisfying (1.6) such that (1.7) holds. According to (3.21) and (F2), we get

$$f(t, u_2(t), v_2(t), u_2'(t)) \le G_M(|u_2'(t)|),\ \ g(t, u_2(t), v_2(t), v_2'(t)) \le G_M(|v_2'(t)|).$$

Combining with (3.18), we have

$$-u_2''(t) \le H_M(|u_2'(t)|) + C_1,\ \ -v_2''(t) \le H_M(|u_2'(t)|) + C_2. \tag{3.23}$$

By (1.6), we easily obtain

$$\int_0^{+\infty} \frac{\rho_1 d\rho_1}{H_M(\rho_1) + C_1} = +\infty,\ \int_0^{+\infty} \frac{\rho_2 d\rho_2}{H_M(\rho_2) + C_2} = +\infty.$$

Hence there exist two constants $M_1 > M, M_2 > M$ such that

$$\int_0^{M_1} \frac{\rho_1 d\rho_1}{H_M(\rho_1) + C_1} > M,\ \int_0^{M_2} \frac{\rho_2 d\rho_2}{H_M(\rho_2) + C_2} > M. \tag{3.24}$$

By Lemma 2.2(c), there exists $\xi \in (0, \pi)$ such that $u_2'(\xi) = 0,\ v_2'(\xi) = 0, u_2'(\xi) \ge 0,\ v_2'(\xi) \ge 0$ for $t \in [0, \xi]$, $u_2'(\xi) \le 0,\ v_2'(\xi) \le 0$ for $t \in [\xi, \pi]$, and $||u_2'||_C = \max\{u_2'(0), -u_2'(\pi)\}$, $||v_2'||_C = \max\{v_2'(0), -v_2'(\pi)\}$. Hence $||u_2'||_C = u_2'(0)$, $||u_2'||_C = -u_2'(\pi)$ or $||v_2'||_C = v_2'(0)$, $||v_2'||_C = -v_2'(\pi)$. We only discuss the cases of that $||u_2'||_C = u_2'(0)$, $||v_2'||_C = v_2'(0)$, and the other cases are similar.

Since $u_2'(t) \ge 0, v_2'(t) \ge 0$ for $t \in [0, \xi]$, multiplying both sides of the inequality (3.23) by $u_2'(t)$, $v_2'(t)$, we obtain that

$$\frac{u_2''(t)u_2'(t)}{G_M(u_2'(t)) + C_1} \le u_2'(t),\ \ \frac{v_2''(t)v_2'(t)}{G_M(v_2'(t)) + C_2} \le v_2'(t), t \in [0, \xi].$$

Integrating both sides of these inequations over $[0, \xi]$ and taking $\rho_1 = u'(t), \rho_2 = v'(t)$ for the left side, we have

$$\int_0^{u_2'(0)} \frac{\rho_1 d\rho_1}{H_M(\rho_1) + C_1} \le u_2(\xi) \le ||u_2||_C,\ \int_0^{v'(0)} \frac{\rho_2 d\rho_2}{H_M(\rho_2) + C_2} \le v_2(\xi) \le ||v_2||_C.$$

From these inequations and (3.22), we have

$$\int_0^{||u_2'||_C} \frac{\rho_1 d\rho_1}{H_M(\rho_1) + C_1} \le M,\ \int_0^{||v_2'||_C} \frac{\rho_2 d\rho_2}{H_M(\rho_2) + C_2} \le M. \tag{3.25}$$

Using these inequations and (3.24), we conclude that

$$||u_2'||_C \le M_1,\ ||v_2'||_C \le M_2. \tag{3.26}$$

Hence,

$$||u_2||_{C^1} = \max\{||u_2||_C, ||u_2'||_C\} \le M_1,\ ||v_2||_{C^1} = \max\{||v_2||_C, ||v_2'||_C\} \le M_2. \tag{3.27}$$

Let $R > M_1, R > M_2$. Since $(u_2, v_2) \in \partial\Omega_2$, by the definition of Ω_2, $||u_2, v_2)||_{C^1} = R > \max\{M_1, M_2\}$, this contradicts to (3.27). Hence, (3.17) holds, namely A and A' satisfy the condition of Lemma 2.5 in $K^2 \cap \Omega_2$. By Lemma 2.5, we have

$$i(A, K^2 \cap \Omega_2, K^2) = i(A', K^2 \cap \Omega_2, K^2). \tag{3.28}$$

From (3.16) and (3.28) it follows that (3.7) holds.

It then follows from the additivity of fixed-point index, (3.6) and (3.7) that

$$i(A, K^2 \cap (\Omega_2 \backslash \overline{\Omega_1}), K^2) = i(A, K^2 \cap \Omega_2, K^2) - i(A, K^2 \cap \Omega_1, K^2) = -1.$$

Then A has a fixed point in $K^2 \cap (\Omega_2 \backslash \overline{\Omega_1})$, which is a positive solution of BVP (2.2), it is assumed that the odd continuation of (F1), (u^*, v^*) with a period of 2π is the odd 2π-periodic solution of BVP(1.1). The proof of Theorem 1.1 is completed. □

Proof of Theorem 1.2 Let $\Omega_1, \Omega_2 \subset C^1(I)$ be defined by (3.1). We prove that the mapping $A = (S \circ F, S \circ G) : K^2 \to K^2$ defined by (2.13) has a fixed point in $K^2 \cap (\Omega_2 / \overline{\Omega_1})$ when r is small enough and R large enough.

Let $r \in (0, \delta')$, where δ' is the positive constant in the condition (F5). Choose $u_0 = v_0 = \sin t$. Then $u_0, v_0 \in K \setminus \{0\}$. We show that A satisfies the condition of Lemma 2.4 in $K^2 \cap \partial\Omega_1$, namely

$$(u, v) - A(u, v) \ne \tau(u_0, v_0),\ \ \forall (u, v) \in K^2 \cap \partial\Omega_1,\ \ \tau \ge 0. \tag{3.29}$$

In fact, if (3.29) does not hold, there exist $(u_3, v_3) \in K^2 \cap \partial\Omega_1$ and $\tau_0 \ge 0$ such that $(u_3, v_3) - A(u_3, v_3) = \tau_0(u_0, v_0)$. Since $u_3 = S(F(u_3, v_3) + \tau_0 u_0), v_3 = S(G(u_3, v_3) + \tau_0 v_0)$, by the definition of S, u_3 is the unique solution of LBVP (2.2) for $h = F(u_3, v_3) + \tau_0 v_0$. Hence, $(u_3, v_3) \in C^2(I) \times C^2(I)$ satisfies the differential equations

$$\begin{cases} -u_3''(t) = f(t, u_3(t), v_3(t), u_3'(t)) + \tau_0 u_0(t), & t \in [0, \pi], \\ -v_3''(t) = g(t, u_3(t), v_3(t), v_3'(t)) + \tau_0 v_0(t), & t \in [0, \pi], \\ u_3(0) = u_3(\pi) = 0, \\ v_3(0) = v_3(\pi) = 0. \end{cases} \tag{3.30}$$

From $(u_3, v_3) \in K^2 \cap \partial\Omega_1$, the definitions of K and Ω_1, we see that

$$u_3 \ge 0, v_3 \ge 0,\ ||(u, v)||_{C^1} = r < \delta'.$$

From (3.30) and (F5), we get

$$\begin{aligned}-u_3''(t)-v_3''(t)&=f(t,u_3(t),v_3(t),u_3'(t))+g(t,u_3(t),v_3(t),v_3'(t))+\tau_0v_0(t)+\tau_0u_0(t)\\&\geq d_3u_3(t)+e_3v_3(t)+d_4u_3(t)+e_4v_3(t)+\tau_0v_0(t)+\tau_0u_0(t)\\&\geq (d_3+d_4)u_3(t)+(e_3+e_4)v_3(t)\\&\geq d'u_3(t)+e'v_3(t).\end{aligned}$$

Multiplying this inequality by $\sin t$ and integrating over I, then using integration by parts for the left side, we have

$$\begin{aligned}\int_0^\pi u_3(t)\sin tdt+\int_0^\pi v_3(t)\sin tdt&\geq d'\int_0^\pi u_3(t)\sin tdt+e'\int_0^\pi v_3(t)\sin tdt\\&\geq m'(\int_0^\pi u_3(t)\sin tdt+\int_0^\pi v_3(t)\sin tdt).\end{aligned}\tag{3.31}$$

We deduce from Lemma 2.2(b) that

$$\int_0^\pi |u_3(t)|\sin tdt\geq\frac{4}{\pi^2}||u_3||_C>0,\ \int_0^\pi |v_3(t)|\sin tdt\geq\frac{4}{\pi^2}||v_3||_C>0.$$

From (3.31), we see that $m\leq 1$, which contradicts to the assumption in (F5). Hence (3.29) holds. In view of Lemma 2.4, we have

$$i(A,K^2\cap\Omega_1,K^2)=0.\tag{3.32}$$

Let $R>\delta$ be large enough. We show that A satisfies the condition of Lemma 2.3 in $K^2\cap\partial\Omega_2$, namely

$$(u,v)\neq\mu A(u,v),\ \ \forall(u,v)\in K^2\cap\partial\Omega_2,\ \ 0<\mu\leq 1.\tag{3.33}$$

In fact, if (3.33) dose not hold, there exist $(u_4,v_4)\in\partial\Omega_2\cap K^2$ and $\mu_1\in(0,1)$ such that $(u_4,v_4)=\mu_1A(u_4,v_4)$. Since $u_4=S(\mu_0F(u_4,v_4))$, $v_4=S(\mu_1G(u_4,v_4))$, by the definition of S, u_4 is the unique solution of LBVP(2.2) for $h=\mu_0F(u_4,v_4)\in C^+(I)$. Hence, $(u_4,v_4)\in C^2(I)\times C^2(I)$ satisfies the differential equations

$$\begin{cases}-u_4''(t)=\mu_1f(t,u_4(t),v_4(t),u_4'(t)), & t\in[0,\pi],\\-v_4''(t)=\mu_1g(t,u_4(t),v_4(t),v_4'(t)), & t\in[0,\pi],\\u_4(0)=u_4(\pi)=0,\\v_4(0)=v_4(\pi)=0.\end{cases}\tag{3.34}$$

Since $(u_4,v_4)\in\partial\Omega_2\cap K^2$, by the definitions of K and Ω_2, we have

$$u_4\geq 0,v_4\geq 0,\ ||(u,v)||_{C^1}=R.$$

From (3.34) and (F6), we get

$$\begin{aligned} -u_4''(t) - v_4''(t) &= \mu_1 f(t, u_4(t), v_4(t), u_4'(t)) + \mu_1 g(t, u_4(t), v_4(t), v_4'(t)) \\ &\leq f(t, u_4(t), v_4(t), u_4'(t)) + g(t, u_4(t), v_4(t), v_4'(t)). \end{aligned} \tag{3.35}$$

Take the norm $||\cdot||_2$ on both sides of (3.35), we deduce from Lemma 2.1 that

$$\begin{aligned} ||u_4'||_2 + ||v_4'||_2 \leq ||u_4''||_2 + ||v_4''||_2 =& ||f(t, u_4(t), v_4(t), u_4'(t))||_2 + ||g(t, u_4(t), v_4(t), v_4'(t))||_2 \\ \leq& a_3||u_4||_2 + b_3||v_4||_2 + c_3||u_4'||_2 + a_4||u_4||_2 \\ &+ b_4||v_4||_2 + c_4||v_4'||_2 \\ \leq& a_3||u_4'||_2 + b_3||v_4'||_2 + c_3||u_4'||_2 + a_4||u_4'||_2 \\ &+ b_4||v_4'||_2 + c_4||v_4'||_2 \\ \leq& (a' + b' + c')(||u_4'||_2 + ||v_4'||_2), \end{aligned} \tag{3.36}$$

According to $a' + b' + c' < 1$, we obtain that

$$||u_4'||_2 + ||v_4'||_2 = 0, (u_4', v_4') = (0, 0), u_4 = c_0, \ v_4 = c_0.$$

Furthermore, we have $u_4(0) = 0, \ u_4(\pi) = 0$ and $v_4(0) = 0, \ v_4(\pi) = 0$, then $u_4 \equiv 0, \ v_4 \equiv 0$, which contradict to the $||(u_0, v_0)|| = R$. Hence (3.33) holds, namely A satisfies the condition of Lemma 2.3 in $K^2 \cap \partial\Omega_2$. By Lemma 2.3, we have

$$i(A, K^2 \cap \Omega_2, K^2) = 1. \tag{3.37}$$

Using (3.32) and (3.37), we get

$$i(A, K^2 \cap (\Omega_2 \backslash \overline{\Omega_1}), K^2) = i(A, K^2 \cap \Omega_2, K^2) - i(A, K^2 \cap \Omega_1, K^2) = 1.$$

Then A has a fixed point in $K^2 \cap (\Omega_2 \backslash \overline{\Omega_1})$, which is a positive solution of BVP(2.2), it is assumed that the odd continuation of (F1), (u^*, v^*) with a period of 2π is the odd 2π-periodic solution of systems (1.1). The proof of Theorem 1.2 is completed. □

4 Example

In this section, we apply the main results of this paper to two concrete examples to obtain the existence of positive odd 2π-periodic solutions of the systems, which further illustrates the applicability of these conclusions.

Example 4.1 Consider the following boundary value problems of second-order ordinary differential equations:

$$\begin{cases} -u''(t) = u^3(t) + v^3(t) + u'^2(t)\sin t, & t \in \mathbb{R}, \\ -v''(t) = u^3(t) + 2v^3(t) + v'^2(t)\sin t, & t \in \mathbb{R}, \\ u(0) = u(2\pi), u'(0) = u'(2\pi), \\ v(0) = v(2\pi), v'(0) = v'(2\pi), \end{cases} \tag{4.1}$$

It is clear that nonlinear terms

$$\begin{cases} f(t,x,y,p) = x^3(t) + y^3(t) + p^2(t)\sin t, & t \in \mathbb{R}, \\ g(t,x,y,q) = x^3(t) + 2y^3(t) + q^2(t)\sin t, & t \in \mathbb{R}. \end{cases} \tag{4.2}$$

It is not difficult to see that functions f,g satisfies assumption (F1) and (F2). To verify that condition (F3), let $a = b = c = d = \frac{1}{8}$ and $\delta = \frac{1}{7}$. When $(t,x,y,p) \in I \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}, |(x,y,p,q)| < \delta$, by definition (4.2), we have

$$\begin{aligned} f(t,x,y,p) + g(t,x,y,q) &= x^3(t) + y^3(t) + \sin tp^2(t) + x^3(t) + 2y^3(t) + \sin tq^2(t) \\ &\le |x|^3 + |y|^3 + |p|^2 + |x|^3 + 2|y|^3 + |q|^2 \\ &\le 2|x|^3 + 3|y|^3 + |p|^2 + |q|^2 \\ &\le 2|(x,y,p,q)|^3 + 3|(x,y,p,q)|^3 + |(x,y,p,q)|^2 + |(x,y,p,q)|^2 \\ &\le 7|(x,y,p,q)|^2 \\ &\le 7|(x,y,p,q)|(|x| + |y| + |p| + |q|) \\ &\le a|x| + b|y| + c|p| + d|q|. \end{aligned}$$

Hence f,g satisfies the conditions (F3), when $(t,x,y,p) \in I \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}$, using (4.1), we have

$$\begin{aligned} f(t,x,y,p) + g(t,x,y,q) &= x^3(t) + y^3(t) + \sin tp^2(t) + x^3(t) + 2y^3(t) + \sin tq^2(t) \\ &\ge x^3(t) + y^3(t) - 2 + x^3(t) + 2y^3(t) - 3 \\ &\ge 2x^3(t) + 3y^3(t) \\ &\ge x(t) + y(t) - 5, \end{aligned}$$

then f,g satisfies the conditions (F4). Then, Theorem 1.1 guarantees that there exists an odd 2π-periodic solution (u^*,v^*) of BVP (4.1).

Example 4.2 Consider the following boundary value problems of second-order ordinary differential equations:

$$\begin{cases} -u''(t) = \sqrt[3]{u(t)} + \sqrt[3]{v(t)} + \sqrt{|u'(t)|}\sin t, & t \in \mathbb{R}, \\ -v''(t) = \sqrt[3]{u(t)} + \sqrt[3]{v(t)} + \sqrt{|v'(t)|}\sin t, & t \in \mathbb{R}, \\ u(0) = u(2\pi), u'(0) = u'(2\pi), \\ v(0) = v(2\pi), v'(0) = v'(2\pi), \end{cases} \tag{4.3}$$

It is easily seen that nonlinear terms

$$\begin{cases} f(t,x,y,p) = \sqrt[3]{x} + \sqrt[3]{y} + \sqrt{|p|}\sin t, & t \in \mathbb{R}, \\ g(t,x,y,q) = \sqrt[3]{x} + \sqrt[3]{y} + \sqrt{|q|}\sin t, & t \in \mathbb{R}. \end{cases} \tag{4.4}$$

By some simple calculations, we obtain that nonlinear terms f,g satisfy assumption (F1). To verify condition (F5), let $d' = e' = 18$ and $\delta' = \frac{1}{27}$, then when $(t,x,y,p) \in I \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}, 0 < |(x,y,p,q)| < \delta$, from (4.4), we get

$$\begin{aligned} f(t,x,y,p) + g(t,x,y,q) &= \sqrt[3]{x} + \sqrt[3]{y} + \sqrt{|p|}\sin t + \sqrt[3]{x} + \sqrt[3]{y} + \sqrt{|q|}\sin t \\ &\geq 2\sqrt[3]{x} + 2\sqrt[3]{y} \\ &\geq \frac{2x}{|(x,y,p,q)|^{\frac{2}{3}}} + \frac{2y}{|(x,y,p,q)|^{\frac{2}{3}}} \\ &\geq \frac{2x}{\delta'^{\frac{2}{3}}} + \frac{2y}{\delta'^{\frac{2}{3}}} \\ &= 18x + 18y. \end{aligned}$$

Hence f,g satisfies the condition (F5). To verify condition (F6), let $a' = b' = c' = d' = \frac{1}{5}$ and $H = 30$. When $(t,x,y,p) \in I \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}, |(x,y,p,q)| > H$, according to (4.4), we have

$$\begin{aligned} f(t,x,y,p) + g(t,x,y,q) &= \sqrt[3]{x} + \sqrt[3]{y} + \sqrt{|p|}\sin t + \sqrt[3]{x} + \sqrt[3]{y} + \sqrt{|q|}\sin t \\ &\leq 4|(x,y,p,q)|^{\frac{1}{3}} + 2|(x,y,p,q)|^{\frac{1}{2}} \\ &\leq 6|(x,y,p,q)|^{\frac{1}{2}} \\ &\leq \frac{6|(x,y,p,q)|}{|(x,y,p,q)|^{\frac{1}{2}}} \\ &\leq \frac{1}{30}|(x,y,p,q)| \leq a'|x| + b'|y| + c'|p| + d'|q|. \end{aligned}$$

Then f,g satisfies the conditions (F6). From Theorem 1.2, we conclude that there exists a odd 2π-periodic solution (u^*,v^*) of BVP (4.3).

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Author contributions
The main idea of this paper was proposed YY and XLH. YY prepared the manuscript initially and performed all the steps of the proofs in this research. All authors read and approved the final manuscript.

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Data availability
The datasets used or analyzed during the current study are available from the corresponding author on reasonable request.

Declarations

Consent for publication
Manuscript is approved by all authors for publication. I would like to declare on behalf of my co-authors that the work described was original research that has not been published previously, and is not under consideration for publication elsewhere.

Competing interests
The authors declare that there are no conflicts of interest regarding the publication of this paper.

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