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Analysis of a diffusive brucellosis model with partial immunity and stage structure in heterogeneous environment

Tingting Zheng[1,2], Yicheng Hao[3], Yantao Luo[3,4*] and Zhidong Teng[1,2]

*Correspondence:
luoyantaoxj@163.com
[3]College of Mathematics and System Science, Xinjiang University, Urumqi 830017, P.R. China
[4]Xinjiang Key Laboratory of Applied Mathematics, Urumqi 830017, P.R. China
Full list of author information is available at the end of the article

Abstract

In this paper, in order to study comprehensive effect of stage-structure, incomplete immunity and spatial diffusion on the transmission dynamics of sheep brucellosis, we formulate a reaction-diffusion brucellosis model with partial immunity and stage structure in heterogeneous environment. Firstly, the well-posedness of the system is investigated, including the existence of global solution and its ultimate boundedness, and then the basic reproduction number R_0 is defined using the next generation operator. Further, the threshold criteria on the global dynamics of the model are established in terms of R_0 in two special cases. That is, if $R_0 < 1$, the disease-free steady state is globally asymptotically stable, while if $R_0 > 1$, the model is uniformly persistent and there at least exists a endemic steady state. Furthermore, for the homogeneous space and heterogeneous diffusion model, by constructing suitable Lyapunov functions, we obtain the global asymptotic stability for the disease-free steady-state when $R_0 \leq 1$ and the global asymptotic stability endemic steady states when $R_0 > 1$. Finally, two simulation examples are given to verify our theoretical results.

Keywords: Sheep brucellosis; Reaction-diffusion; Incomplete immunity; Stage-structure; Threshold dynamics

1 Introduction

Brucellosis is an infectious disease caused by bacteria of the genus Brucella. The disease is primarily transmitted to humans through contact with infected animals or their products, such as milk and meat. Common species of Brucella include Brucella abortus (primarily infecting cattle), Brucella melitensis (primarily infecting goats and sheep), and Brucella suis (primarily infecting pigs) [1]. The early symptoms of brucellosis include fever, night sweats, fatigue, and joint pain. These symptoms may persist for weeks to months, and some patients may develop a chronic form of the disease, affecting multiple organs [2]. Currently, brucellosis remains an important public health issue worldwide, especially in areas with frequent agricultural activities, such as the Middle East, North Africa, and South America [3].

As one of the powerful tools for studying infectious diseases, infectious disease dynamics can reveal patterns of disease prevalence and predict trends in disease outbreaks with real data, and then provide a theoretical foundation and quantitative support for decision-makers. Certainly, from the perspective of transmission dynamics, many researchers have also studied brucellosis and achieved significant results [4–12] and their references. For example, Hou et al. [4] constructed a human-sheep coupled SEIVB model to study the epidemic trend of brucellosis in sheep in Inner Mongolia, and their research results showed that combining environmental disinfection with inoculation of susceptible sheep was effective way to control the spread of brucellosis in Inner Mongolia. Li et al. [5] proposed a multi-group SEIRV dynamical model with bidirectional mixed cross-infection between cattle and sheep investigated the influence of cross-infection of mixed feeding on the brucellosis. Sun et al. [7] established a brucellosis model with direct or indirect infection caused by herd introduction, and analyzed the global dynamics of the model. In addition, noise in the environment is ubiquitous, and noise disturbances in the environment may trigger stress responses in animals, leading to a decline in immune system function and increasing the risk of infection. If the stress level of livestock increases, they may be more susceptible to Brucella infection [10, 11]. For instance, Chen et.al [10] explored the influence of state changes on brucellosis by a stochastic brucellosis model with semi-Markovian switchings. Wang et al. [11] formulated a stochastic brucellosis model that incorporates vaccination and environmental pollution transmission, and obtained the persistence in the mean of disease and the existence of a stationary distribution.

In fact, there is a very clear stage structure for brucellosis in sheep populations, where the probability of infection in lambs (from birth to 6 months) is low, while adult fattening sheep (after 6 months) are easily infected [13, 14]. Hence, some researchers incorporating stage-structure into dynamic model of brucellosis. For instance, Bai et al. [8] constructed a two-stage sheep-environment coupled transmission dynamics model based on the transmission characteristics of brucellosis and obtained mixed control is more beneficial to reduce the number of exposed sheep, infected sheep, and brucellosis in environment. Wang and Abdurahman [15] considered a multi-stage sheep brucellosis model with incomplete immunity and environmental white noise and get the threshold dynamics for deterministic model and stochastic version of the model, respectively. Recently, based on the data of human brucellosis cases in China from 2006 to 2020, Ma et al. [16] found that human brucellosis cases are widely distributed in northern, northeastern and western pastoral areas of China, and scattered in other regions And the incidence rate of human brucellosis in China has obvious regional differences, with the passage of time, there is a trend of spreading from north to south, from east to west, and from pastoral areas to rural and urban areas. In view of above facts, Liu et al. [12] proposed a reaction-diffusion brucellosis model with spatiotemporal heterogeneity and nonlocal delay to investigate the complex transmission process of brucellosis due to animal transportation and livestock. Overall, based on the above discussion, although there have been fruitful achievements in the study of dynamic models of brucellosis, including stochastic brucellosis models, stage structure models, environmental transmission brucellosis models, etc., it is not difficult to find that the research on reaction-diffusion brucellosis dynamic models of spatial heterogeneity is still in its infancy. Hence, in this paper, we intend to investigate the comprehensive impact of stage structure, vaccination, environmental transmission, and spatial heterogeneity on the spread of brucellosis. In order to model stage-structure, susceptible sheep are divided into susceptible lambs and susceptible adult sheep, denoted by S_j and S_a, respectively. V represents the density of vaccinated sheep and I represents the density of infectious sheep.

And the amount of Brucella in the environment is measured by W. Based on the complex transmission of Brucella between sheep and environment, the following model is formulated,

$$
\begin{cases}
\dfrac{\partial S_j}{\partial t} = \nabla\cdot(D_1(x)\nabla S_j) + A_j(x) - (d(x)+m(x))S_j(x) - \varepsilon(x)S_j(\beta(x)I + \beta_1(x)W),\\
\dfrac{\partial S_a}{\partial t} = \nabla\cdot(D_2(x)\nabla S_a) + m(x)S_j - S_a(\beta(x)I+\beta_1(x)W)\\
\qquad - (d(x)+\theta(x))S_a + \xi(x)V,\\
\dfrac{\partial V}{\partial t} = \nabla\cdot(D_3(x)\nabla V) + \theta(x)S_a - \eta(x)V(\beta(x)I+\beta_1(x)W) - (d(x)+\xi(x))V,\\
\dfrac{\partial I}{\partial t} = \nabla\cdot(D_4(x)\nabla I) + (\varepsilon(x)S_j + S_a + \eta(x)V)(\beta(x)I+\beta_1(x)W) - (d(x)+\alpha(x))I,\\
\dfrac{\partial W}{\partial t} = k(x)I - (\mu(x)+\delta(x))W,
\end{cases}
\tag{1.1}
$$

for $t > 0$ and $x \in \Omega$ (the habitat Ω is a bounded domain), with the Neumann boundary condition

$$
\frac{\partial S_j(t,x)}{\partial\vartheta} = \frac{\partial S_a(t,x)}{\partial\vartheta} = \frac{\partial V(t,x)}{\partial\vartheta} = \frac{\partial I(t,x)}{\partial\vartheta} = 0,\ x\in\partial\Omega,\ t\geq 0,
\tag{1.2}
$$

where ϑ is the outward normal to $\partial\Omega$, and the initial condition is

$$
\begin{aligned}
&S_j(0,x) = S_{j0}(x) > 0, S_a(0,x) = S_{a0}(x) > 0, V(0,x) = V_0(x) \geq 0,\\
&\qquad I(0,x) = I_0(x)\geq 0, W(0,x) = W_0(x) \geq 0,\ x\in\Omega.
\end{aligned}
\tag{1.3}
$$

Here, $D_1(x)$, $D_2(x)$, $D_3(x)$ and $D_4(x)$ denote the diffusion rate of S_j, S_a, I and W at position x, respectively. The meanings of the other parameters in model (1.1) are shown in Table 1. All the location-dependent parameters of model (1.1) are continuous, strictly positive and uniformly bounded on Ω.

The structure of this paper is as follows. In Sect. 2, the well-posedness of the system is presented. In Sect. 3, the key index-basic reproduction number is discussed. In Sect. 4,

Table 1 The meanings of parameters in model (1.1)

Symbol	Meanings
$A_j(x)$	The constant input of S_j at position x
$\beta(x)$	The infection rate of infected sheep to adult sheep at position x
$\beta_1(x)$	The environmental infection rate of Brucella to adult sheep at position x
$m(x)$	The transition rate from lamb to adult sheep at position x
$d(x)$	The natural death rate of the flock at position x
$\varepsilon(x)$	The ratio coefficient of adult sheep infection rate to lamb infection rate at position x ($0< \varepsilon < 1$)
$\theta(x)$	The vaccination rate of the adult sheep at position x
$\xi(x)$	The vaccine failure rate at position x
$\eta(x)$	The ineffective vaccination rate at position x
$\alpha(x)$	The slaughter rate of infected sheep at position x
$k(x)$	The brucella shedding rate in infected sheep at position x
$\delta(x)$	The natural decay rate of Brucella in the environment at position x
$\mu(x)$	The disinfection rate at position x

we study the long-time threshold dynamics including extinction and persistence. And the global stability for endemic equilibrium is studied under homogeneous space and heterogeneous diffusion environment in Sect. 5. In Sect. 6, we give some numerical simulations to verify theoretical results. Finally, a brief summary is given in Sect. 7.

2 Well-posedness of the model

We first define the functional space for model (1.1) by $\mathbb{X} := C(\overline{\Omega}, R^5)$ be the Banach space with the supremum norm $\|\cdot\|_{\mathbb{X}}$. Define its cone by $\mathbb{X}_+ := C(\overline{\Omega}, R^5_+)$, then $(\mathbb{X}, \mathbb{X}_+)$ is a strongly ordered Banach space. For a bounded function $f(x)$ defined on Ω, we denote $\bar{f} = \max_{x\in\Omega} f(x)$ and $\underline{f} = \min_{x\in\Omega} f(x)$. In this section, we aim to prove that the solution of the model (1.1) exist globally for $t \in [0,\infty)$ in $\mathbb{X}_+$, and model (1.1) admits a compact global attractor.

Let $\Gamma_n(t) : C(\overline{\Omega}, R) \to C(\overline{\Omega}, R)$ for $n = 1,2,3,4$ be the C_0-semigroup associated with $\nabla \cdot (D_n(x)\nabla) - \pi_n(x)$ subjects to the Neumann boundary condition, where $\pi_1 = d(x) + m(x)$, $\pi_2 = d(x) + \theta(x)$, $\pi_3 = d(x) + \xi(x)$ and $\pi_4 = d(x) + \alpha(x)$. Then, we have

$$\Gamma_n(t)\psi = \int_{\Omega} T_n(t,x,y)\psi(y)\mathrm{d}y,\ t > 0,\ \psi \in C(\overline{\Omega}, R), \tag{2.1}$$

where $T_n(t,x,y)$ represents the Green function associated with $\nabla \cdot (D_n(x)\nabla) - \pi_n(x)$ subjects to the Neumann boundary condition. Based on [17, Corollary 7.2.3], we can obtain that $\Gamma_n(t)$ for $n = 1,2,3,4$ are compact and strongly positive for each $t > 0$. Hence, there exist constants $M_n > 0$ such that $\|\Gamma_n(t)\| \le M_n e^{\gamma_n t}$ for all $t \ge 0$, where $\gamma_n < 0$ is the principal eigenvalue of $\nabla \cdot (D_n(x)\nabla) - \pi_n(x)$ subjects to the Neumann boundary condition. Furthermore, for any $t \ge 0$, we define the operator $\Gamma_5(t) : C(\overline{\Omega}, R) \to C(\overline{\Omega}, R)$ by

$$\Gamma_5(t)\phi = e^{-(\mu(x)+\delta(x))t}\phi(x)\ \text{ for all }\ t \ge 0,\ \phi \in C(\overline{\Omega}, R).$$

Moreover, we denote by $Z(t,\cdot,\psi) = (S_j(t,\cdot,\psi), S_a(t,\cdot,\psi), V(t,\cdot,\psi), I(t,\cdot,\psi), W(t,\cdot,\psi))$ the solution of model (1.1) with the initial function $\psi = (\psi_1, \psi_2, \dots, \psi_5)$ and define

$$\begin{aligned}
H_1(\psi)(x) =& A_j(x) - \varepsilon(x)\psi_1(x)(\beta(x)\psi_4(x) + \beta_1(x)\psi_5(x)),\\
H_2(\psi)(x) =& m(x)\psi_1(x) - \psi_2(x)(\beta(x)\psi_4(x) + \beta_1(x)\psi_5(x)) + \xi(x)\psi_3(x),\\
H_3(\psi)(x) =& \theta(x)\psi_2(x) - \eta(x)\psi_3(x)(\beta(x)\psi_4(x) + \beta_1(x)\psi_5(x)),\\
H_4(\psi)(x) =& (\varepsilon(x)\psi_1(x) + \psi_2(x) + \eta(x)\psi_3(x))(\beta(x)\psi_4(x) + \beta_1(x)\psi_5(x))\\
H_5(\psi)(x) =& k(x)\psi_4(x),
\end{aligned}$$

then model (1.1) can be rewritten as the following integral equation

$$\begin{cases} Z(t,\cdot,\psi) = \Gamma(t)\psi + \displaystyle\int_0^t \Gamma(t-s)H(Z(s,\cdot,\psi))\mathrm{d}s, \\ Z(0) = \psi, \end{cases} \tag{2.2}$$

where $H = (H_1, H_2, H_3, H_4, H_5)^T$ and $\Gamma = \mathrm{diag}(\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5)$.

Lemma 2.1 *For any initial function* $\psi = (\psi_1, \psi_2, \psi_3, \psi_4, \psi_5) \in \mathbb{X}_+$, *model* (1.1) *has a unique nonnegative mild solution* $Z(t,\cdot,\psi) = (S_j(t,\cdot,\psi), S_a(t,\cdot,\psi), V(t,\cdot,\psi), I(t,\cdot,\psi), W(t,\cdot,\psi)) \in \mathbb{X}_+$ *defined on the interval* $[0,\tau_\infty)$ *with* $\tau_\infty \le \infty$. *Moreover, this solution is a classical solution.*

Proof The existence and uniqueness of solution $Z(t,\cdot,\psi)$ on the interval $[0,\tau_\infty)$ are obtained based on the standard basic theory of partial differential equations. Now, we only need to prove the nonnegativity of the solution. In fact, for any $\psi \in \mathbb{X}_+$ and $h \ge 0$, we have

$$\begin{aligned}
&\psi(x) + hH(\psi)(x)\\
&= \begin{pmatrix}
\psi_1(x) + h[A_j(x) - \varepsilon(x)S_j(\beta(x)I + \beta_1(x)W)]\\
\psi_2(x) + h[m(x)S_j - S_a(\beta(x)I + \beta_1(x)W) + \xi(x)V]\\
\psi_3(x) + h[\theta(x)S_a - \eta(x)V(\beta(x)I + \beta_1(x)W)]\\
\psi_4(x) + h[(\varepsilon(x)S_j + S_a + \eta(x)V)(\beta(x)I + \beta_1(x)W)]\\
\psi_5(x) + hk(x)I
\end{pmatrix}\\
&\ge \begin{pmatrix}
\psi_1(x) - h\varepsilon(x)S_j(\beta(x)I + \beta_1(x)W)\\
\psi_2(x) - h[S_a(\beta(x)I + \beta_1(x)W)]\\
\psi_3(x) - h[\eta(x)V(\beta(x)I + \beta_1(x)W)]\\
\psi_4(x)\\
\psi_5(x)
\end{pmatrix}.
\end{aligned}$$

Therefore, for any given $\psi \in \mathbb{X}_+$, there is a sufficiently small number $h_1 = h(\psi) > 0$ such that $\psi + hH(\psi) \in \mathbb{X}_+$ for all $h \in [0, h_1]$. Then, we have

$$\lim_{h\to 0^+} \frac{1}{h}\mathrm{dist}(\psi + hH(\psi), \mathbb{X}_+) = 0, \ \psi \in \mathbb{X}_+. \tag{2.3}$$

Consequently, from [18, Corollary 4], we easily obtain that the solution $Z(t,\cdot,\psi)$ on $[0,t_\infty)$ is nonnegative. This completes the proof. □

Theorem 2.1 *For any initial function* $\psi = (\psi_1, \psi_2, \psi_3, \psi_4, \psi_5) \in \mathbb{X}_+$, *model* (1.1) *has a unique classical solution* $Z(t,\cdot,\psi) = (S_j(t,\cdot,\psi), S_a(t,\cdot,\psi), V(t,\cdot,\psi), I(t,\cdot,\psi), W(t,\cdot,\psi)) \in \mathbb{X}_+$ *defined on* $[0,\infty)$, *and the solution is also ultimately bounded.*

Proof Suppose that $t_\infty < \infty$, then from [18, Theorem 2], we have $\|Z(t,\cdot,\psi)\| \to \infty$ as $t \to t_\infty$.

For any $t \in [0,t_\infty)$ and $x \in \overline{\Omega}$, we have

$$\frac{\partial S_j}{\partial t} \le \nabla\cdot(D_1(x)\nabla S_j) + \bar{A}_j - (\underline{d} + \underline{m})S_j(x), \ x \in \Omega, \ t \in [0,t_\infty). \tag{2.4}$$

By the comparison principle and [19, Lemma 1], it follows that there exist a constant $N > 0$ such that $S_j(t,x) \le N$, for all $x \in \overline{\Omega}$ and $t \in [0,t_\infty)$.

Set

$$K(t) = \int_\Omega (S_j(t,x) + S_a(t,x) + V(t,x) + I(t,x))\mathrm{d}x.$$

From the divergence theorem [20], we have

$$\int_\Omega \nabla \cdot D_1(x)\nabla S_j \mathrm{d}x = 0,\ \int_\Omega \nabla \cdot D_2(x)\nabla S_a \mathrm{d}x = 0,$$
$$\int_\Omega \nabla \cdot D_3(x)\nabla V \mathrm{d}x = 0, \int_\Omega \nabla \cdot D_4(x)\nabla I \mathrm{d}x = 0.$$

It follows that

$$\begin{aligned}\frac{\mathrm{d}K}{\mathrm{d}t} &= \int_\Omega (A_j(x) - d(x)(S_j(t,x) + S_a(t,x) + V(t,x) + I(t,x)) - \alpha(x)I)\mathrm{d}x \\ &\leq \bar{A}_j|\Omega| - \underline{d}K,\ t \in [0, \tau_\infty),\end{aligned} \tag{2.5}$$

which implies $K(t) \leq \max\{K(0), \frac{\bar{A}_j|\Omega|}{\underline{d}}\} := N_1$ for all $t \in [0, \tau_\infty)$, where $|\Omega|$ is the Lebesgue measure of Ω. Consequently, we also have

$$\int_\Omega S_j(t,x)dx \leq N_1,\ \int_\Omega S_a(t,x)dx \leq N_1,\ \int_\Omega V(t,x)dx \leq N_1,\ \int_\Omega I(t,x)dx \leq N_1 \tag{2.6}$$

for all $t \in [0, \tau_\infty)$.

On the other hand, set $O(t) = \int_\Omega W(t,x)\mathrm{d}x$, from the last equation of model (1.1), one has

$$\begin{aligned}\frac{\mathrm{d}O}{\mathrm{d}t} &= \int_\Omega \frac{\partial}{\partial t} W(t,x)\mathrm{d}x = \int_\Omega [\bar{k}I - (\underline{\mu} + \underline{\delta})W]\mathrm{d}x \\ &\leq \bar{k}N_1 - (\underline{\mu} + \underline{\delta})O(t),\ t \in [0, \tau_\infty),\end{aligned} \tag{2.7}$$

which implies that

$$\int_\Omega W(t,x)\mathrm{d}x \leq \max\{\int_\Omega W(0,x)dx, \frac{\bar{k}N_1}{\underline{\mu} + \underline{\delta}}\} := N_2 \text{ for all } t \in [0, \tau_\infty). \tag{2.8}$$

Secondly, for each $n = 1,2,3,4$, let τ_{ni} $(i = 1,2,\ldots)$ be the eigenvalues of $\nabla \cdot (D_n(x)\nabla) - \pi_n(x)(n = 1,2,3,4)$ subject to the Neumann boundary condition with the eigenfunction $\phi_{ni}(x)$, which satisfying $\tau_{n1} > \tau_{n2} \geq \tau_{n3} \geq \cdots \geq \tau_{ni} \geq \cdots$. From [21, Chap. 5], we can represent

$$T_n(t,x,y) = \sum_{i\geq 1} e^{\tau_{ni}t}\phi_{ni}(x)\phi_{ni}(y).$$

Since $\{\phi_{ni}(x)\}_{i=1}^\infty$ is uniformly bounded on $\bar{\Omega}$, there exists $\omega > 0$ such that

$$T_n(t,x,y) \leq \omega \sum_{i\geq 1} e^{\tau_{ni}t},\ t > 0,\ x,y \in \overline{\Omega}.$$

For each $n = 1,2,3,4$, let ρ_{nj} be the eigenvalue of $\nabla \cdot (D_n(x)\nabla) - \underline{\pi}_n$ subject to the Neumann boundary condition, which satisfying $\rho_{n1} = -\underline{\pi}_n > \rho_{n2} \geq \rho_{n3} \geq \cdots \geq \rho_{nj} \geq \cdots$. Then, by

[21, Theorem 2.4.7], we know that $\rho_{nj} \geq \tau_{nj}$ for all $j \in \mathbb{N}_+$. Since ρ_{nj} decreases like $-j^2$, there exist $\omega_k > 0$ such that

$$T_n(t,x,y) \leq \omega \sum_{j\geq 1} e^{-\rho_{nj}t} \leq \omega_n e^{-\rho_{n1}t} = \omega_n e^{-\underline{\pi}_n t},\ t > 0,\ x, y \in \overline{\Omega}.$$

Then, by (2.2), (2.6) and (2.8), we have

$$\begin{aligned}
S_a(t,x) =&\Gamma_2(t)S_a(0,x) + \int_0^t \Gamma_2(t-s)(m(x)S_j - S_a(\beta(x)I + \beta_1(x)W) + \xi(x)V)\mathrm{d}s \\
\leq& M_2 e^{\gamma_2 t}\|S_a(0,\cdot)\|_Y + \int_0^t \int_\Omega T_2(t-s,x,y)(m(x)S_j + \xi(x)V)\mathrm{d}y\mathrm{d}s \\
\leq& M_2 e^{\gamma_2 t}\|S_a(0,\cdot)\|_Y + \int_0^t \omega_2 e^{-(\underline{d}+\underline{\theta})(t-s)}(\bar{m} + \bar{\xi})N_1 \mathrm{d}s \\
=& M_2 e^{\gamma_2 t}\|S_a(0,\cdot)\|_Y + \omega_2(\bar{m} + \bar{\xi})N_1 \frac{1 - e^{-(\underline{d}+\underline{m})t}}{(\underline{d} + \underline{m})} \\
\leq& M_2\|S_a(0,\cdot)\|_Y + \frac{\omega_2(\bar{m} + \bar{\xi})N_1}{(\underline{d} + \underline{m})} \triangleq N_3,\ t \in [0, \tau_\infty).
\end{aligned}$$

Furthermore, from (2.2), (2.5) and (2.7), and combine the above results, using the same argument, we can obtain that

$$\begin{aligned}
V(t,x) =&\Gamma_3(t)V(0,x) + \int_0^t \Gamma_3(t-s)(\theta(x)S_a - \eta(x)V(\beta(x)I + \eta(x)\beta_1(x)W))\mathrm{d}s \\
\leq& M_3 e^{\gamma_3 t}\|V(0,\cdot)\|_Y + \frac{\omega_3\bar{\theta}N_3}{(\underline{d} + \underline{\xi})} \triangleq N_4,\ t \in [0, \tau_\infty), \\
I(t,x) =&\Gamma_4(t)I(0,x) + \int_0^t \Gamma_4(t-s)((\varepsilon(x)S_j + S_a + \eta(x)V)(\beta(x)I + \beta_1(x)W)\mathrm{d}s \\
\leq& M_4 e^{\gamma_4 t}\|I(0,\cdot)\|_Y + \frac{\omega_4((\bar{\varepsilon}N + N_3 + \bar{\eta}N_5)(\bar{\beta}N_1 + \bar{\beta}_1 N_4)}{(\underline{d} + \underline{\alpha})} \triangleq N_5,\ t \in [0, \tau_\infty),
\end{aligned}$$

and

$$\begin{aligned}
W(t,x) =& \Gamma_5(t)W(0,x) + \int_0^t \Gamma_5(t-s)k(x)I\mathrm{d}s \leq e^{-(\underline{\mu}+\underline{\delta})t}\|W(0,\cdot)\|_Y \\
&+ \frac{\bar{k}N_5}{(\underline{\mu} + \underline{\delta})} \triangleq N_6,\ t \in [0, \tau_\infty).
\end{aligned}$$

From the above discussions, we finally obtain that solution $Z(t,x)$ for $t \in [0, t_\infty)$ and $x \in \Omega$ is bounded. It contradicts the fact that $\|Z(t,\cdot,\psi)\| \to \infty$ as $t \to \tau_\infty$. This implies that solution $Z(t,x)$ is defined for all $t \in [0,\infty)$.

From the inequalities (2.4), (2.5) and (2.7), we further have $\limsup_{t\to\infty} S_j(t,x) \leq \frac{\bar{A}_j}{\underline{d}+\underline{m}}$ uniformly for $x \in \bar{\Omega}$, $\limsup_{t\to\infty} K(t) \leq \frac{\bar{A}_j|\Omega|}{\underline{d}}$ and $\limsup_{t\to\infty} O(t) \leq \frac{\bar{k}\bar{A}_j|\Omega|}{\underline{d}(\underline{\mu}+\underline{\delta})}$. Therefore, $S_j(t,x)$ is ultimately bounded, and there exist constants $\mathcal{N}_1 > 0$, $\mathcal{N}_2 > 0$ which are independent of any initial value $\psi \in \mathbb{X}_+$, and an enough large time $t_1 > 0$ such that $K(t) \leq \mathcal{N}_1$ and $O(t) \leq \mathcal{N}_2$ for all $t \geq t_1$.

For any $t \geq t_1$, we have

$$
\begin{aligned}
S_a(t,x) =& \Gamma_2(t)S_a(t_1,x) + \int_{t_1}^{t} \Gamma_2(t-s)(m(x)S_j - S_a(\beta(x)I + \beta_1(x)W) + \xi(x)V)\mathrm{d}s \\
\leq& M_2 e^{\gamma_2(t-t_1)}\|S_a(t_1,\cdot)\|_Y + \int_{t_1}^{t}\int_{\Omega} T_2(t-s,x,y)(m(x)S_j + \xi(x)V)\mathrm{d}y\mathrm{d}s \\
\leq& M_2 e^{\gamma_2(t-t_1)}\|S_a(t_1,\cdot)\|_Y + \int_{t_1}^{t} \omega_2 e^{-(\underline{d}+\underline{\theta})(t-s)}(\bar{m} + \bar{\xi})\mathcal{N}_1 \mathrm{d}s \\
\leq& M_2 e^{\gamma_2(t-t_1)}\|S_a(t_1,\cdot)\|_Y + \frac{\omega_2(\bar{m} + \bar{\xi})\mathcal{N}_1}{(\underline{d} + \underline{m})}.
\end{aligned}
$$

Thus, $\limsup_{t\to\infty}\|S_a(t,x)\| \leq \frac{\omega_2(\bar{m}+\bar{\xi})\mathcal{N}_1}{(\underline{d}+\underline{m})}$, which shows that $S_a(t,x)$ is ultimately bounded. By the same method, we can show that $V(t,x)$, $I(t,x)$ and $W(t,x)$ also are ultimately bounded. □

Corollary 1 *All nonnegative solutions $Z(t,\cdot,\psi)$ of model (1.1) generate a solution semiflow $\Phi(t): \mathbb{X}_+ \to \mathbb{X}_+$ with $\Phi(t)\psi = Z(t,\cdot,\psi)$ for any initial value $\psi \in \mathbb{X}_+$ and $t \geq 0$.*

Now, we shall show that model (1.1)–(1.3) possesses a global compact attractor. The main difficulty here lies in showing the the asymptotic smoothness of $\Phi(t)$ instead of weak compactness, and this is caused by the fact that there are no diffusion terms in the last equation of model (1.1). We identify the κ-contraction condition by following the procedures in [22, Theorem 3]. The following lemma is to deal with the non-compactness of semiflow $\Phi(t)$.

Lemma 2.2 *$\Phi(t)$ is a κ-contracting, i.e., for any bounded set $\mathbb{B} \subseteq \mathbb{X}_+$,*

$$\lim_{t\to\infty} \kappa(\Phi(t)\mathbb{B}) = 0,$$

where $\kappa(\cdot)$ is defined by $\kappa(\mathbb{B}) := \inf\{r : \mathbb{B}$ has a finite cover of diameter $< r\}$ is the Kuratowski measure of noncompactness.

Proof Let $u(t,\cdot,\phi) = (S_j(t,\cdot,\psi), S_a(t,\cdot,\psi), V(t,\cdot,\psi), I(t,\cdot,\psi), W(t,\cdot,\psi))$ be the solution of model (1.1) with initial condition $\psi = (\psi_1(x),\psi_2(x),\psi_3(x),\psi_4(x),\psi_5(x)) \in \mathbb{X}_+$. Obviously, Φ can be decomposed as $\Phi(t) = \Phi_1(t) + \Phi_2(t)$, $t \geq 0$, where

$$\Phi_1(t)\psi = \left(S_j(t,\cdot,\psi), S_a(t,\cdot,\psi), V(t,\cdot,\psi), I(t,\cdot,\psi), \int_0^t e^{-(\mu(\cdot)+\delta(\cdot))(t-s)}k(\cdot)I(s,\cdot,\psi)\mathrm{d}s\right)$$

and

$$\Phi_2(t)\psi = \left(0,0,0,0,e^{-(\mu(\cdot)+\delta(\cdot))t}\psi_5\right).$$

For any bounded set $\mathbb{B} \subseteq \mathbb{X}_+$, since the first four equations in model (1.1) have the diffusion terms, we directly obtain that for any $t > 0$, the sets $\{S_j(t,\cdot,\psi) : \psi \in \mathbb{B}\}$, $\{S_a(t,\cdot,\psi) : \psi \in \mathbb{B}\}$, $\{V(t,\cdot,\psi) : \psi \in \mathbb{B}\}$ and $\{I(t,\cdot,\psi) : \psi \in \mathbb{B}\}$ are precompact in $C(\Omega, R_+)$. On the other hand, by [23, Lemma 2.5], it follows that the set $\{\int_0^t e^{-(\mu(\cdot)+\delta(\cdot))(t-s)}k(\cdot)I(s,\cdot,\psi)ds : \psi \in \mathbb{B}\}$ is precompact in $C(\Omega, R_+)$. Hence, we have $\kappa(\Phi_1(t)\mathbb{B}) = 0$, $t > 0$. In addition,

$$\kappa(\Phi_2(t)\mathbb{B}) \le e^{-(\mu(\cdot)+\delta(\cdot))t}\kappa(\mathbb{B}) \le e^{-(\underline{\mu}+\underline{\delta})t}\kappa(\mathbb{B}),\ t \ge 0.$$

Therefore, we have

$$\kappa(\Phi(t)\mathbb{B}) \le \kappa(\Phi_1(t)\mathbb{B}) + \kappa(\Phi_2(t)\mathbb{B}) \le e^{-(\underline{\mu}+\underline{\delta})t}\kappa(\mathbb{B}).$$

Consequently, $\Phi(t)$ is a κ-contracting. □

Based on Theorem 2.1, Lemma 2.2 and Definition 2.1.1 in [24], Theorem 3.4.6 in [25], we easily obtain the existence of compact attractor for model (1.1) in $\mathbb{X}_+$.

Theorem 2.2 *The solution semiflow $\Phi(t)$ of model (1.1) admits a global compact attractor in $\mathbb{X}_+$.*

3 Basic reproduction number

In this section, we will define the basic reproduction number of model (1.1). By setting $I(t,x) = W(t,x) = 0$ in model (1.1), one has

$$\begin{cases} \dfrac{\partial S_j}{\partial t} = \nabla\cdot(D_1(x)\nabla S_j) + A_j(x) - (d+m)S_j(x),\ x\in\Omega,\ t>0,\\ \dfrac{\partial S_a}{\partial t} = \nabla\cdot(D_2(x)\nabla S_a) + m(x)S_j - (d(x)+\theta(x))S_a + \xi(x)V,\ x\in\Omega,\ t>0,\\ \dfrac{\partial V}{\partial t} = \nabla\cdot(D_3(x)\nabla V) + \theta(x)S_a - (d(x)+\xi(x))V,\ x\in\Omega,\ t>0,\\ \dfrac{\partial S_j}{\partial \nu} = \dfrac{\partial S_a}{\partial \nu} = \dfrac{\partial V}{\partial \nu} = 0,\ x\in\partial\Omega,\ t>0. \end{cases} \tag{3.1}$$

From [19, Lemma 1], we have the following results.

Lemma 3.1 *Suppose that $\xi(x) = 0$ or $D_2(x) = D_3(x)$, then model (3.1) admits a unique positive steady state $(S_j^0(x), S_a^0(x), V^0(x))$, satisfying*

$$\begin{cases} \nabla\cdot(D_1(x)\nabla S_j^0) + A_j(x) - (d+m)S_j^0(x) = 0,\ x\in\Omega,\\ \nabla\cdot(D_2(x)\nabla S_a^0) + m(x)S_j^0 - (d(x)+\theta(x))S_a^0 + \xi(x)V^0 = 0,\ x\in\Omega,\\ \nabla\cdot(D_3(x)\nabla V^0) + \theta(x)S_a^0 - (d(x)+\xi(x))V^0 = 0,\ x\in\Omega,\\ \dfrac{\partial S_j^0}{\partial \nu} = \dfrac{\partial S_a^0}{\partial \nu} = \dfrac{\partial V^0}{\partial \nu} = 0,\ x\in\partial\Omega, \end{cases}$$

which is globally asymptotically stable in $C(\overline{\Omega}, R_+^3)$. Moreover, when $A_j(\cdot)\equiv A_j$, $d(\cdot)\equiv d$, $m(\cdot)\equiv m$, $\xi(\cdot)\equiv\xi$ and $\theta(\cdot)\equiv\theta$ are positive constants, we also have $S_j^0(x)\equiv\frac{A_j}{d+m}$, $S_a^0(x)\equiv\frac{A_jm(d+\xi)}{d(d+m)(d+\theta+\xi)}$, $V^0(x)\equiv\frac{A_jm\theta}{d(d+m)(d+\theta+\xi)}$.

Proof For the first equation of system (3.1), we can obtain that there exists a unique globally stable solution $S_j^0(x)$ that satisfies

$$\nabla \cdot (D_1(x)\nabla S_j^0) + A_j(x) - (d+m)S_j^0(x) = 0,\ x \in \Omega.$$

When $\xi(x) = 0$, from the second equation of system (3.1), we have the following limit equation

$$\begin{cases} \dfrac{\partial S_a}{\partial t} = \nabla \cdot (D_2(x)\nabla S_a) + m(x)S_j^0(x) - (d(x)+\theta(x))S_a,\ x \in \Omega, \\ \dfrac{\partial S_a}{\partial \nu} = 0,\ x \in \partial\Omega,\ t > 0. \end{cases}$$

From [19, Lemma 1], we can easily obtain that there exists a unique globally stable solution $S_a^0(x)$ satisfying

$$\nabla \cdot (D_2(x)\nabla S_a^0) + m(x)S_j^0 - (d(x)+\theta(x))S_a^0 = 0,\ x \in \Omega,$$

and then using similar arguments to the third equation of system (3.1), we can easily get that there exists a unique global stability solution $V^0(x)$ satisfying

$$\nabla \cdot (D_3(x)\nabla V^0) + \theta(x)S_a^0 - d(x)V^0 = 0,\ x \in \Omega.$$

Therefore, we finally obtain that system (3.1) has a unique positive steady state $(S_j^0(x), S_a^0(x), V^0(x))$, which is globally asymptotically stable in $C(\overline{\Omega}, R_+^3)$.

When $D_2(x) = D_3(x)$, let $Q(x,t) = S_a(x,t) + V(x,t)$, then from the second and the third equation of system (3.1), we have the following limit system

$$\begin{cases} \dfrac{\partial Q}{\partial t} = \nabla \cdot (D_2(x)\nabla Q) + m(x)S_j^0(x) - d(x)Q(x,t),\ x \in \Omega,\ t > 0, \\ \dfrac{\partial Q}{\partial \nu} = 0,\ x \in \partial\Omega,\ t > 0. \end{cases}$$

From [19, Lemma 1], we can easily obtain that there exists a unique globally stable solution $Q^0(x)$ satisfying

$$\nabla \cdot (D_2(x)\nabla Q^0) + m(x)S_j^0 - d(x)Q^0 = 0,\ x \in \Omega.$$

Let $S_a(t,x) = Q^0(x) - V(t,x)$, we have the following limit system

$$\begin{cases} \dfrac{\partial V}{\partial t} = \nabla \cdot (D_3(x)\nabla V) + \theta(x)(Q^0 - V) - (d(x)+\xi(x))V,\ x \in \Omega,\ t > 0, \\ \dfrac{\partial V}{\partial \nu} = 0,\ x \in \partial\Omega,\ t > 0. \end{cases}$$

Using similar arguments as in the above, we get that there exists a unique globally stable solution $V^0(x)$ satisfying

$$\nabla \cdot (D_3(x)\nabla V^0) + \theta(x)(Q^0 - V^0) - (d(x)+\xi(x))V^0 = 0,\ x \in \Omega.$$

From the third equation of system (3.1), we have the following limit equation

$$\begin{cases} \dfrac{\partial S_a}{\partial t} = \nabla \cdot (D_2(x)\nabla S_a) + m(x)S_j^0 - (d(x)+\theta(x))S_a + \xi(x)V^0, \ x \in \Omega, \ t > 0, \\ \dfrac{\partial S_a}{\partial \nu} = 0, \ x \in \partial\Omega, \ t > 0. \end{cases}$$

It is easy to prove that this equation has a globally stable solution $S_a^0(x) = Q^0(x) - V^0(x)$, satisfying

$$\nabla \cdot (D_2(x)\nabla S_a^0) + m(x)S_j^0 - (d(x)+\theta(x))S_a^0 + \xi(x)V^0 = 0, \ x \in \Omega.$$

Therefore, we finally obtain that system (3.1) also has a unique positive steady state $(S_j^0(x), S_a^0(x), V^0(x))$, which is globally asymptotically stable in $C(\overline{\Omega}, R_+^3)$. □

From Lemma 3.1, there exists a unique disease-free steady state $U_0(x) = (S_j^0(x), S_a^0(x), V^0, 0, 0)$ for model (1.1). Linearizing model (1.1) at $U_0(x)$ to obtain

$$\begin{cases} \dfrac{\partial \omega_4}{\partial t} = \nabla \cdot (D_4(x)\nabla \omega_4) + (\varepsilon S_j^0 + S_a^0 + \eta V^0)(\beta\omega_4 + \beta_1\omega_5) \\ \qquad - (d+\alpha)\omega_4, \ x \in \Omega, \ t > 0, \\ \dfrac{\partial \omega_5}{\partial t} = k\omega_4 - (\mu+\delta)\omega_5, \ x \in \Omega, \ t > 0, \\ \dfrac{\partial \omega_4}{\partial \nu} = 0, \ x \in \partial\Omega. \end{cases} \tag{3.2}$$

Let $(\omega_4, \omega_5) = (e^{\lambda t}\psi_4(x), e^{\lambda t}\psi_5(x))$, then we obtain the eigenvalue problem

$$\begin{cases} \lambda\psi_4(x) = \nabla \cdot (D_4(x)\nabla\psi_4(x)) + (\varepsilon(x)S_j^0(x) + S_a^0(x) + \eta(x)V^0(x))(\beta(x)\psi_4(x) \\ \qquad + \beta_1(x)\psi_5(x)) - (d(x)+\alpha(x))\psi_4(x), \ x \in \Omega, \\ \lambda\psi_5(x) = k(x)\psi_4(x) - (\mu(x)+\delta(x))\psi_5(x), \ x \in \Omega, \\ \dfrac{\partial \psi_4(x)}{\partial \nu} = 0, \ x \in \partial\Omega. \end{cases} \tag{3.3}$$

From [17, Theorem 7.6.1], we have the following result.

Lemma 3.2 *Problem* (3.3) *has a unique principal eigenvalue* $\lambda_0 = \lambda_0(S_j^0(x), S_a^0(x), V^0(x))$ *with positive eigenvector* $(\psi_2(x), \psi_3(x))$.

Let $T(t)$ be the C^0-semigroup generated by the following system

$$\begin{cases} \dfrac{\partial \omega_4}{\partial t} = \nabla \cdot (D_4(x)\nabla \omega_4) - (d+\alpha)\omega_4, \ x \in \Omega, \ t > 0, \\ \dfrac{\partial \omega_5}{\partial t} = k\omega_4 - (\mu+\delta)\omega_5, \ x \in \Omega, \ t > 0, \\ \dfrac{\partial \omega_4}{\partial \nu} = 0, \ x \in \partial\Omega. \end{cases}$$

Further, we define

$$F(x) = \begin{pmatrix} \beta(x)(\varepsilon(x)S_j^0(x) + S_a^0(x) + \eta(x)V^0(x)) & \beta_1(x)(\varepsilon(x)S_j^0(x) + S_a^0(x) + \eta(x)V^0(x)) \\ 0 & 0 \end{pmatrix}.$$

Denote by $\psi(x) = (\psi_4(x), \psi_(x))$ the initial infection distribution, then as time evolves, the distribution of those infective numbers is $T(t)\psi$ due to the mobility, and transfer of individuals in infected compartments. By the definition of $F(x)$, it follows that the distribution of new infection at time t is $F(x)T(t)\psi(x)$. Thus, $\int_0^{+\infty} F(x)T(t)\psi(x)\mathrm{d}t$ represents the distribution of total new infective numbers. Define the operator

$$\mathscr{L}(\psi)(x) = \int_0^{+\infty} F(x)T(t)\psi(x)\mathrm{d}t. \tag{3.4}$$

Obviously, $\mathscr{L}$ is a continuous and positive operator that maps the initial infection distribution ψ to the distribution of the total infective members produced during the infection period. Based on [26–29], we define the basic reproduction number of model (1.1) as follows,

$$R_0 = r(\mathscr{L}),$$

where $r(\cdot)$ is the spectral radius of $\mathscr{L}$. Furthermore, the arguments similar to those in [29, Theorem 3.1] imply the following results.

Lemma 3.3 *sign*$(R_0 - 1) =$ *sign*(λ_0).

Proof Define operator $\mathscr{B}$ as follows

$$\mathscr{B} = \begin{pmatrix} \nabla \cdot (D_4(x)\nabla) - (d(x) + \alpha(x)) & 0 \\ k(x) & -(\mu(x) + \delta(x)) \end{pmatrix}.$$

It is evident that $\mathscr{B}$ is resolvent-positive with $s(\mathscr{B}) < 0$ and is the generator of the semigroup $T(t)$, where $s(\cdot)$ denotes the spectral bound of an operator. Moreover, by the proof of Lemma 3.2, we see that $Z = F + \mathscr{B}$ is resolvent-positive, and thus, it follows from [29, Theorem 3.1] that $\lambda_0 = s(Z) = s(F + \mathscr{B})$ has the same sign as $r(F(-\mathscr{B})^{-1}) - 1$. Since $\mathscr{L} = F(-\mathscr{B})^{-1}$ and $R_0 = r(\mathscr{L})$, which completes the proof. □

4 Threshold dynamics of the model

Firstly, the following results mainly follow from Lemma 3.1.

Lemma 4.1 *Suppose that $\xi(x) = 0$ or $D_2(x) = D_3(x)$, then for any nonnegative solution $Z(t,x) = (S_j(t,x), S_a(t,x), V(t,x), I(t,x), W(t,x))$ of model (1.1), one has*

$$\limsup_{t\to\infty} S_j(t,x) \le S_j^0(x),\ \limsup_{t\to\infty} S_a(t,x) \le S_a^0(x),\ \limsup_{t\to\infty} V(t,x) \le V^0(x),$$

uniformly for $x \in \Omega$.

Proof Directly from model (1.1), we obtain

$$\begin{cases} \dfrac{\partial S_j}{\partial t} \leq \nabla \cdot (D_1(x)\nabla S_j) + A_j(x) - (d+m)S_j(x),\ x \in \Omega,\ t > 0, \\ \dfrac{\partial S_a}{\partial t} \leq \nabla \cdot (D_2(x)\nabla S_a) + m(x)S_j - (d(x)+\theta(x))S_a + \xi(x)V,\ x \in \Omega,\ t > 0, \\ \dfrac{\partial V}{\partial t} \leq \nabla \cdot (D_3(x)\nabla V) + \theta(x)S_a - (d(x)+\xi(x))V,\ x \in \Omega,\ t > 0, \\ \dfrac{\partial S_j}{\partial \nu} = \dfrac{\partial S_a}{\partial \nu} = \dfrac{\partial V}{\partial \nu} = 0,\ x \in \partial\Omega,\ t > 0. \end{cases} \tag{4.1}$$

By the comparison principle [30] and Lemma 3.1, we immediately obtain that $\limsup_{t\to\infty} S_j(t,x) \leq S_j^0(x)$, $\limsup_{t\to\infty} S_a(t,x) \leq S_a^0(x)$ and $\limsup_{t\to\infty} V(t,x) \leq V^0(x)$. □

Lemma 4.2 *Assume that* $(S_j(t,x), S_a(t,x), V(t,x), I(t,x), W(t,x))$ *is the solution of model* (1.1) *with nonnegative initial value* $(S_j(0,x), S_a(0,x), V(0,x), I(0,x), W(0,x))$ *and* $I(0,x) \not\equiv 0$ *for* $x \in \Omega$. *Then* $(S_j(t,x), S_a(t,x), V(t,x), I(t,x), W(t,x))$ *is positive for all* $t > 0$ *and* $x \in \Omega$.

Proof In fact, the positivity of $S_j(t,x)$, $S_a(t,x)$, $V(t,x)$ and $I(t,x)$ can be obtained by the maximum principle (See [21]). Furthermore, the positivity of $W(t,x)$ can be obtain from the inequality

$$\frac{\partial W(t,x)}{\partial t} > -(\mu(x) + \delta(x))W(t,x)$$

for all $t > 0$ and $x \in \Omega$. □

4.1 Extinction of disease

Theorem 4.1 *Assume that* $\xi(x) = 0$ *or* $D_2(x) = D_3(x)$. *If* $R_0 < 1$, *then* $U_0(x)$ *is globally asymptotically stable in* $\mathbb{X}_+$.

Proof Let $(S_j(t,x), S_a(t,x), V(t,x), I(t,x), W(t,x))$ is any solution of model (1.1) with the nonnegative initial function $(S_j(0,x), S_a(0,x), V(0,x), I(0,x), W(0,x))$. If $I(t,x) \equiv 0$ for all $t \geq 0$ and $x \in \Omega$. From the fifth equation of model (1.1) we further have that $W(t,x) \to 0$ as $t \to \infty$ uniformly for $x \in \Omega$.

Now, we assume that there are $t^* \geq 0$ and $x^* \in \Omega$ such that $I(t^*,x^*) > 0$. Then from Lemma 4.2 we have that $(S_j(t,x), S_a(t,x), V(t,x), I(t,x), W(t,x))$ is positive for all $t > t^*$ and $x \in \Omega$. From Lemma 4.1, we have

$$\limsup_{t\to\infty} S_j(t,x) \leq S_j^0(x),\ \limsup_{t\to\infty} S_a(t,x) \leq S_a^0(x),\ \limsup_{t\to\infty} V(t,x) \leq V^0(x),$$

uniformly for $x \in \overline{\Omega}$, which implies that for any $\delta > 0$ there exists a $t_2 > t^*$ such that $S_j(t,x) \leq S_j^0(x) + \delta$, $S_a(t,x) \leq S_a^0(x) + \delta$ and $V(t,x) \leq V^0(x) + \delta$ for any $t \geq t_2$ and $x \in \Omega$. Hence, we can obtain the following differential inqualities from the forth and fifth equations of model (1.1)

$$\begin{cases} \dfrac{\partial I}{\partial t} \le \nabla\cdot(D_4(x)\nabla I) + \varepsilon(x)(\beta(x)(S_j^0(x)+\delta)I + \beta_1(x)(S_j^0(x)+\delta)W) \\ \qquad + \beta(x)(S_a^0(x)+\delta)I + \beta_1(x)(S_a^0(x)+\delta)W + \eta(x)\beta(x)(V^0(x)+\delta)I \\ \qquad + \eta(x)\beta_1(x)(V^0(x)+\delta)W - (d(x)+\alpha(x))I,\ x\in\Omega,\ t>t_2, \\ \dfrac{\partial W}{\partial t} = k(x)I - (\mu(x)+\delta(x))W,\ x\in\Omega,\ t>t_2, \\ \dfrac{\partial I}{\partial \nu} = 0,\ x\in\partial\Omega. \end{cases} \tag{4.2}$$

The corresponding comparison system is

$$\begin{cases} \dfrac{\partial P}{\partial t} = \nabla\cdot(D_4(x)\nabla P) + \varepsilon(x)(\beta(x)(S_j^0(x)+\delta)P + \beta_1(x)(S_j^0(x)+\delta)Q) \\ \qquad + \beta(x)(S_a^0(x)+\delta)P + \beta_1(x)(S_a^0(x)+\delta)Q + \eta(x)\beta(x)(V^0(x)+\delta)P \\ \qquad + \eta(x)\beta_1(x)(V^0(x)+\delta)Q - (d(x)+\alpha(x))P,\ x\in\Omega,\ t>t_2, \\ \dfrac{\partial Q}{\partial t} = k(x)P - (\mu(x)+\delta(x))Q,\ x\in\Omega,\ t>t_2, \\ \dfrac{\partial P}{\partial \nu} = 0,\ x\in\partial\Omega. \end{cases} \tag{4.3}$$

Further, consider the eigenvalue problem

$$\begin{cases} \lambda\psi_4(x) = \nabla\cdot(D_4(x)\nabla\psi_4(x)) + (\varepsilon(x)(S_j^0(x)+\delta) + (S_a^0(x)+\delta) \\ \qquad + \eta(x)(V^0(x)+\delta))(\beta(x)\psi_4(x) \\ \qquad + \beta_1(x)\psi_5(x)) - (d(x)+\alpha(x))\psi_4(x),\ x\in\Omega, \\ \lambda\psi_5(x) = k(x)\psi_4(x) - (\mu(x)+\delta(x))\psi_5(x),\ x\in\Omega, \\ \dfrac{\partial\psi_4(x)}{\partial \nu} = 0,\ x\in\partial\Omega. \end{cases} \tag{4.4}$$

Let λ_1 be the principle eigenvalue of problem (4.4) with the strictly positive eigenfunction $(\psi_4(x),\psi_5(x))$. When $R_0 < 1$, it follows from Lemma 3.3 that $\lambda_0 < 0$. By the continuous dependence of principle eigenvalue with respect to δ, there is an enough small $\delta > 0$ such that $\lambda_1 < 0$. Choose a constant $\delta_1 > 0$ such that $\delta_1(\psi_4(x),\psi_5(x)) \ge (I(t_2,x), W(t_2,x))$. The comparison principle implies that

$$(I(t_2,x), W(t_2,x)) \le \delta_1(\psi_4(x),\psi_5(x))e^{\lambda_1(t-t_2)},\ t\ge t_2. \tag{4.5}$$

Obviously, we have that $(I(t,x), W(t,x)) \to 0$ uniformly on $x\in\overline{\Omega}$ as $t\to\infty$.

Furthermore, we get the limit system as follows

$$\begin{cases} \dfrac{\partial S_j}{\partial t} =\nabla \cdot (D_1(x)\nabla S_j) + A_j(x) - (d+m)S_j(x), \ x \in \Omega, \ t > 0, \\ \dfrac{\partial S_a}{\partial t} =\nabla \cdot (D_2(x)\nabla S_a) + m(x)S_j - (d(x)+\theta(x))S_a + \xi(x)V, \ x \in \Omega, \ t > 0, \\ \dfrac{\partial V}{\partial t} =\nabla \cdot (D_3(x)\nabla V) + \theta(x)S_a - (d(x)+\xi(x))V, \ x \in \Omega, \ t > 0, \\ \dfrac{\partial S_j}{\partial \nu} = \dfrac{\partial S_a}{\partial \nu} = \dfrac{\partial V}{\partial \nu} = 0, \ x \in \partial\Omega, \ t > 0. \end{cases} \tag{4.6}$$

It follows from the theory of asymptotically autonomous semiflows [31, Corollary 4.3] and Lemma 3.1, $S_j(t,x) \to S_j^0(x)$, $S_a(t,x) \to S_a^0(x)$ and $V(t,x) \to V^0(x)$ uniformly on $x \in \overline{\Omega}$ as $t \to \infty$. Therefore, we obtain $U_0(x)$ is globally asymptotically stable. □

4.2 Uniform persistence of disease

Theorem 4.2 *If $R_0 > 1$, then there is constant $\overline{\zeta} > 0$ such that for any initial value $\psi = (\psi_1, \psi_2, \psi_3, \psi_4, \psi_5) \in \mathbb{X}_+$ with $\psi_4 \not\equiv 0$ and $\psi_5 \not\equiv 0$, the solution $Z(t,\cdot,\psi) = (S_j(t,\cdot,\psi), S_a(t,\cdot,\psi), V(t,\cdot,\psi), I(t,\cdot,\psi), W(t,\cdot,\psi))$ of model (1.1) satisfies*

$$\begin{aligned} &\liminf_{t\to\infty} S_j(t,\cdot,\psi) \ge \overline{\zeta}, \ \liminf_{t\to\infty} S_a(t,\cdot,\psi) \ge \overline{\zeta}, \ \liminf_{t\to\infty} V(t,\cdot,\psi) \ge \overline{\zeta}, \\ &\liminf_{t\to\infty} I(t,\cdot,\psi) \ge \overline{\zeta}, \ \liminf_{t\to\infty} W(t,\cdot,\psi) \ge \overline{\zeta} \end{aligned}$$

uniformly for $x \in \overline{\Omega}$.

Proof Firstly, from the ultimate boundedness of solutions for model (1.1) (see Theorem 2.1), we easily obtain that $S_j(t,x)$, $S_a(t,x)$ and $V(t,x)$ are positive for all $t > 0$ and $x \in \Omega$ and uniformly persistent.

Define the set

$$C_0 = \left\{\psi = (\psi_1, \psi_2, \psi_3, \psi_4, \psi_5) \in \mathbb{X}_+ : \psi_4 \not\equiv 0, \psi_5 \not\equiv 0\right\}.$$

We have

$$\partial C_0 := \mathbb{X}_+ \setminus C_0 = \left\{\psi = (\psi_1, \psi_2, \psi_3, \psi_4, \psi_5) \in \mathbb{X}_+ : \psi_4 \equiv 0 \text{ or } \psi_5 \equiv 0\right\}.$$

Clearly, C_0 is a positively invariant set for the solution semiflow $\Phi(t)$ of model (1.1). Let

$$M_\partial := \left\{\psi \in \mathbb{X}_+ : \Phi(t)\psi \in \partial C_0 \text{ for all } t \ge 0\right\}, \ M_1 = \left\{U_0(x)\right\},$$

and $\omega(\psi)$ be the omega limit set of solution $\Phi(t)\psi$, where $U_0(x)$ is the disease-free steady state and $\Phi(t)$ is the solution semiflow of model (1.1). We give the following two claims:

Claim 1. $\cup_{\psi\in M_\partial}\omega(\psi) = M_1$. For any $t \ge 0$, we have $\Phi(t)U_0(x) = U_0(x)$. Hence, $M_1 \subset \cup_{\psi\in M_\partial}\omega(\psi)$. Now, we prove $\cup_{\psi\in M_\partial}\omega(\psi) \subset M_1$. Since $\Phi(t)\psi \in \partial C_0$ for any given $\psi \in M_\partial$

and $t \geq 0$, then $I(t,\cdot,\psi) \equiv 0$ or $W(t,\cdot,\psi) \equiv 0$ for all $t \geq 0$. If $I(t,\cdot,\psi) \equiv 0$, we immediately obtain $W(t,\cdot,\psi) \equiv 0$ from the fourth equation of model (1.1). Thus, from the equations of $S_j(t,x)$, $S_a(t,x)$ and $V(t,x)$ in model (1.1), we acquire

$$\begin{cases} \dfrac{\partial S_j(t,x)}{\partial t} = \nabla \cdot (D_1(x)\nabla S_j) + A_j(x) - (d(x) + m(x))S_j(x), \\ \dfrac{\partial S_a(t,x)}{\partial t} = \nabla \cdot (D_2(x)\nabla S_a) + m(x)S_j - (d(x) + \theta(x))S_a + \xi(x)V, \\ \dfrac{\partial V(t,x)}{\partial t} = \nabla \cdot (D_3(x)\nabla V) + \theta(x)S_a - (d(x) + \xi(x))V, \\ \dfrac{\partial S_j(t,x)}{\partial \nu} = \dfrac{\partial S_a(t,x)}{\partial \nu} = \dfrac{\partial V(t,x)}{\partial \nu} = 0,\ x \in \partial\Omega. \end{cases} \tag{4.7}$$

From Lemma 3.1, we get $\lim_{t\to\infty} S_j(t,x) = S_j^0(x)$, $\lim_{t\to\infty} S_a(t,x) = S_a^0(x)$ and $\lim_{t\to\infty} V(t,x) = V^0(x)$ for $x \in \Omega$. This implies that $\omega(\psi) = U_0(x)$.

If $W(t,\cdot,\psi) \equiv 0$ for $t \geq 0$, then from the fifth equation of model (1.1), we have $I(t,\cdot,\psi) \equiv 0$. Similarly, we also get $\lim_{t\to\infty} S_j(t,x) = S_j^0(x)$, $\lim_{t\to\infty} S_a(t,x) = S_a^0(x)$ and $\lim_{t\to\infty} V(t,x) = V^0(x)$ for $x \in \Omega$. This also shows $\omega(\psi) = U_0(x)$.

Based on the above discussion, it follows that $\cup_{\psi\in M_\partial}\omega(\psi) \subset M_1$. Therefore, $\cup_{\psi\in M_\partial}\omega(\psi) = M_1$, and then **Claim 1** holds.

Claim 2. $U_0(x)$ is uniform weak repeller for set C_0. That is, there is a constant $\eta > 0$ such that for any $\psi \in C_0$ solution $u(t,\cdot)$ with initial value $u(0,\cdot) = \psi$ satisfies

$$\limsup_{t\to\infty} \|u(t,\cdot) - U_0(x)\| \geq \eta.$$

Supposing that **Claim 2** is not true. Then for any constant $\eta > 0$ there exists a $\psi \in C_0$ such that $\limsup_{t\to\infty} \|u(t,\cdot) - U_0(x)\| < \delta$. This implies that there exists an enough large $t_3 > 0$ such that

$$\begin{aligned} &0 < S_j^0(x) - \delta < S_j(t,x),\ 0 < S_a^0(x) - \delta < S_a(t,x),\ 0 < V^0(x) - \delta < V(t,x), \\ &I(t,x) < \delta,\ W(t,x) < \delta \ \text{ for all } t \geq t_3,\ x \in \overline{\Omega}. \end{aligned}$$

Therefore, from model (1.1), we obtain

$$\begin{cases} \dfrac{\partial I(t,x)}{\partial t} \geq \nabla \cdot (D_4(x)\nabla I) + \varepsilon(x)(\beta(x)(S_j^0(x) - \delta)I + \beta_1(x)(S_j^0(x) - \delta)W \\ \qquad + \beta(x)(S_a^0(x) - \delta)I + \beta_1(x)(S_a^0(x) - \delta)W + \eta(x)\beta(x)(V^0(x) - \delta)I \\ \qquad + \eta(x)\beta_1(x)(V^0(x) - \delta)W - (d(x) + \alpha(x))I,\ x \in \Omega,\ t > t_3, \\ \dfrac{\partial W(t,x)}{\partial t} = k(x)I - (\mu(x) + \delta(x))W,\ x \in \Omega,\ t > t_3, \\ \dfrac{\partial I(t,x)}{\partial \nu} = 0,\ t > t_3,\ x \in \partial\Omega. \end{cases} \tag{4.8}$$

We have the comparison system

$$\begin{cases} \dfrac{\partial y_1(t,x)}{\partial t} =\nabla\cdot(D_4(x)\nabla y_1)+\varepsilon(x)(\beta(x)(S_j^0(x)-\delta)y_1+\beta_1(x)(S_j^0(x)-\delta)y_2 \\ \qquad +\beta(x)(S_a^0(x)-\delta)y_1+\beta_1(x)(S_a^0(x)-\delta)y_2+\eta(x)\beta(x)(V^0(x)-\delta)y_1 \\ \qquad +\eta(x)\beta_1(x)(V^0(x)-\delta)y_2-(d(x)+\alpha(x))y_1,\ x\in\Omega,\ t>t_3, \\ \dfrac{\partial y_2(t,x)}{\partial t} =k(x)y_1-(\mu(x)+\delta(x))y_2,\ x\in\Omega,\ t>t_3, \\ \dfrac{\partial y_1}{\partial\nu}=\dfrac{\partial y_2}{\partial\nu}=0,\ t>t_3,\ x\in\partial\Omega. \end{cases} \tag{4.9}$$

For any initial value $\psi=(\psi_1,\psi_2,\psi_3,\psi_4,\psi_5)\in\mathbb{X}_+$ with $\psi_4(x)\not\equiv 0$ and $\psi_5(x)\not\equiv 0$, from the parabolic maximum principle [21], we can obtain $I(t,x)>0$ and $W(t,x)>0$ for all $t>0$ and $x\in\Omega$. By Lemma 3.3, we have $\lambda_0>0$ when $R_0>1$.

We consider the following eigenvalue problem associated with system (4.9)

$$\begin{cases} \lambda\xi_1(x)=\nabla\cdot(D_4(x)\nabla\xi_1(x))+\varepsilon(x)(\beta(x)(S_j^0(x)-\delta)\xi_1(x) \\ \qquad +\beta_1(x)(S_j^0(x)-\delta)\xi_2(x)+\beta(x)(S_a^0(x)-\delta)\xi_1(x) \\ \qquad +\beta_1(x)(S_a^0(x)-\delta)\xi_2(x)+\eta(x)\beta(x)(V^0(x)-\delta)\xi_1(x) \\ \qquad +\eta(x)\beta_1(x)(V^0(x)-\delta)\xi_2(x)-(d(x)+\alpha(x))\xi_1(x),\ x\in\Omega, \\ \lambda\xi_2(x)=\nabla\cdot(D_5(x)\nabla\xi_2(x))+k(x)\xi_1(x)-(\mu(x)+\delta(x))\xi_2(x),\ x\in\Omega, \\ \dfrac{\partial\xi_1(x)}{\partial\nu}=\dfrac{\partial\xi_2(x)}{\partial\nu}=0,\ x\in\partial\Omega. \end{cases} \tag{4.10}$$

Let $\lambda_0(\delta)$ be the principle eigenvalue of problem (4.10) with the strictly positive eigenfunction $(\xi_1(x),\xi_2(x))$ for $x\in\overline{\Omega}$ (See Lemma 3.2). Since $\lim_{\delta\to 0}\lambda_0(\delta)=\lambda_0$, there exists an enough small constant $\delta\in(0,1)$ such that $\lambda_0(\delta)>0$.

It is obvious that system (4.9) has the solution $(y_1(t,x),y_2(t,x))=e^{\lambda_0(\delta)(t-t_3)}(\xi_1(x),\xi_2(x))$. Due to $(I(t,x),W(t,x))>0$ for all $t>0$ and $x\in\overline{\Omega}$, there is a constant $\rho>0$ such that $(I(t_3,x),W(t_3,x))\geq\rho(\xi_1(x),\xi_2(x))$ for $x\in\overline{\Omega}$. Note that $\rho(y_1(t,x),y_2(t,x))$ is also the solution of system (4.9). According to the comparison principle and equation (4.8), we can obtain $(I(t,x),W(t,x))\geq\rho(y_1(t,x),y_2(t,x))$ for all $t>t_3$ and $x\in\overline{\Omega}$. Owing to $\lambda_0(\delta)>0$, we get $\lim_{t\to\infty}y_i(t,x)=\infty$ $(i=1,2)$. This implies $\lim_{t\to\infty}I(t,x)=\infty$ and $\lim_{t\to\infty}W(t,x)=\infty$, which is a contradiction with the boundedness of $(I(t,x),W(t,x))$. Therefore, **Claim 2** holds.

Define a continuous function $l:X^+\to R^+$ as follows:

$$l(\psi)=\min\left\{\min_{x\in\overline{\Omega}}\psi_3(x),\min_{x\in\overline{\Omega}}\psi_4(x)\right\},\ \psi\in X^+.$$

Obviously, $l^{-1}(0,+\infty)\subset C_0$. By the parabolic maximum principle [21], we can obtain that $I(t,x)>0$ and $W(t,x)>0$ for all $t\geq 0$ and $x\in\overline{\Omega}$ when $\psi_4(x)\not\equiv 0$ and $\psi_5(x)\not\equiv 0$. Thus, if $l(\psi)>0$, then $l(\Phi(t)\psi)>0$. By the definition of the generalized distance function (See [32, Theorem 3]), we know that l is a generalized distance function for semiflow $\Phi(t):X^+\to X^+$.

From the discussion in **Claim 1** and **Claim 2**, we can see that $U_0(x)$ is isolated invariant set in X^+ and $W^s(U_0(x)) \cap C_0 = \emptyset$, where $W^s(U_0(x))$ is the stable set of $U_0(x)$. Thus, $W^s(U_0(x)) \cap p^{-1}(0,\infty) = \emptyset$. It follows from [32, Theorem 3] that there is a constant $\rho_3 > 0$ such that $\liminf_{t\to\infty} p(\Phi(t)\psi) \geq \rho_3$ for all $\psi \in C_0$. This completes the proof. □

Corollary 2 *When* $R_0 > 1$, *model* (1.1) *has at least one endemic equilibrium* $P^*(x) = (S_j^*(x), S_a^*(x), V^*(x), I^*(x), W^*(x))$.

Remark 4.1 We here have proved the existence of endemic equilibrium, but its uniqueness and stability are still an open question. In the following section, we will prove the global asymptotic stability of endemic equilibrium in the spatial homogeneous environment.

5 Global stability in spatial homogeneous environment

In this section, we consider a special case of model (1.1) to establish the complete results for the global stability of model (1.1). This special case exactly is the spatial homogeneous environment. That is, all parameters of model (1.1) are positive constants. As a result, model (1.1) becomes into

$$\begin{cases} \dfrac{\partial S_j}{\partial t} = D_1\Delta S_j + A_j - (d+m)S_j - \varepsilon(\beta S_j I + \beta_1 S_j W),\ t\geq 0,\ x\in\Omega, \\ \dfrac{\partial S_a}{\partial t} = D_2\Delta S_a + mS_j - \beta S_a I - \beta_1 S_a W - (d+\theta)S_a + \xi V,\ t\geq 0,\ x\in\Omega, \\ \dfrac{\partial V}{\partial t} = D_3\Delta V + \theta S_a - \eta\beta VI - \eta\beta_1 VW - (d+\xi)V,\ t\geq 0,\ x\in\Omega, \\ \dfrac{\partial I}{\partial t} = D_4\Delta I + \varepsilon(\beta S_j I + \beta_1 S_j W) + \beta S_a I + \beta_1 S_a W \\ \qquad + \eta\beta VI + \eta\beta_1 VW - (d+\alpha)I,\ t\geq 0,\ x\in\Omega, \\ \dfrac{\partial W}{\partial t} = kI - (\mu+\delta)W,\ t\geq 0,\ x\in\Omega. \end{cases} \tag{5.1}$$

Obviously, model (5.1) has always disease-free equilibrium $P^0 = (S_j^0, S_a^0, V^0, 0, 0)$ with $S_j^0 = \frac{A_j}{d+m}$, $S_a^0 = \frac{A_j m(d+\xi)}{d(d+m)(d+\theta+\xi)}$ and $V^0 = \frac{A_j m\theta}{d(d+m)(d+\theta+\xi)}$. Then, we can get that model (5.1) has the basic reproduction number

$$R_0 = \frac{\varepsilon S_j^0 + S_a^0 + \eta V^0}{d+\alpha}\left(\beta + \frac{k\beta_1}{\mu+\delta}\right).$$

From Corollary 2, we can obtain that when $R_0 > 1$, model (5.1) has an endemic equilibrium $P^* = (S_j^*, S_a^*, V^*, I^*, W^*)$ satisfying the following equations

$$\begin{cases} A_j - (d+m)S_j^* - \varepsilon(\beta S_j^* I^* + \beta_1 S_j^* W^*) = 0, \\ mS_j^* - \beta S_a^* I^* - \beta_1 S_a^* W^* - (d+\theta)S_a^* + \xi V^* = 0, \\ \theta S_a^* - \eta\beta V^* I^* - \eta\beta_1 V^* W^* - (d+\xi)V^* = 0, \\ \varepsilon(\beta S_j^* I^* + \beta_1 S_j^* W^*) + \beta S_a^* I^* + \beta_1 S_a^* W^* + \eta\beta V^* I^* + \eta\beta_1 V^* W^* - (d+\alpha)I^* = 0, \\ kI^* - (\mu+\delta)W^* = 0. \end{cases} \tag{5.2}$$

Theorem 5.1 *If $R_0 \le 1$, then the disease-free equilibrium $P^0 = (S_j^0, S_a^0, V^0, 0, 0)$ of model (5.1) is globally asymptotically stable.*

Proof If $R_0 < 1$, we can directly obtain that disease-free equilibrium $P^0 = (S_j^0, S_a^0, V^0, 0, 0)$ of model (5.1) is globally asymptotically stable from Theorem 4.1.

Now, we consider the case of $R_0 = 1$. Choose the Lyapunov function

$$L_1(t) = \int_\Omega \Bigg(S_j - S_j^0 - S_j^0 \ln \frac{S_j}{S_j^0} + S_a - S_a^0 - S_a^0 \ln \frac{S_a}{S_a^0} + V - V^0 - V^0 \ln \frac{V}{V^0} + I + \frac{\varepsilon\beta_1 S_j^0 + \beta_1 S_a^0 + \eta\beta_1 V^0}{\mu + \delta} W \Bigg) \mathrm{d}x.$$

Then along any positive solution $(S_j(t,x), S_a(t,x), V(t,x), I(t,x), W(t,x))$ of model (5.1), combining the divergence theorem and the Neumann boundary conditions, we can get

$$\begin{aligned} \frac{\mathrm{d}L_1(t)}{\mathrm{d}t} \le & \int_\Omega \Bigg((d+\alpha)(R_0 - 1)I + dS_j^0 (2 - \frac{S_j^0}{S_j} - \frac{S_j}{S_j^0}) + dS_a^0 (3 - \frac{S_j^0}{S_j} - \frac{S_a}{S_a^0} - \frac{S_j S_a^0}{S_j^0 S_a}) \\ & + dV^0 (4 - \frac{S_j^0}{S_j} - \frac{S_j S_a^0}{S_j^0 S_a} - \frac{S_a V^0}{S_a^0 V} - \frac{V}{V^0}) + \xi V^0 (2 - \frac{S_a^0 V}{S_a V^0} - \frac{S_a V^0}{S_a^0 V}) \Bigg) \mathrm{d}x \\ \le & 0. \end{aligned} \tag{5.3}$$

Obviously, $\frac{\mathrm{d}L_1(t)}{\mathrm{d}t} \le 0$ when $R_0 = 1$, and $\frac{\mathrm{d}L_1(t)}{\mathrm{d}t} \equiv 0$ implies that $S_j(t,x) \equiv S_j^0$, $S_a(t,x) \equiv S_a^0$ and $V(t,x) \equiv V^0$. Directly from model (5.1), it follows that $I(t,x) \equiv 0$ and $W(t,x) \equiv 0$. Thus, by LaSalle's invariable principle, we get that the equilibrium P^0 is globally asymptotically stable when $R_0 = 1$. This completes the proof. □

Theorem 5.2 *The endemic equilibrium $P^* = (S_j^*, S_a^*, V^*, I^*, W^*)$ of model (5.1) is global asymptotically stable if $R_0 > 1$.*

Proof Define a Lyapunov function as follows:

$$\begin{aligned} L_2(t) = & \int_\Omega \Bigg(S_j - S_j^* - S_j^* \ln \frac{S_j}{S_j^*} + S_a - S_a^* - S_a^* \ln \frac{S_a}{S_a^*} + V - V^* - V^* \ln \frac{V}{V^*} + I - I^* \\ & - I^* \ln \frac{I}{I^*} + \frac{\varepsilon\beta_1 S_j^* W^* + \beta_1 S_a^* W^* + \eta\beta_1 V^* W^*}{\mu + \delta} (W - W^* - W^* \ln \frac{W}{W^*}) \Bigg) \mathrm{d}x. \end{aligned}$$

Calculating the time derivative of L_2, we have

$$\begin{aligned} \frac{\mathrm{d}L_2(t)}{\mathrm{d}t} = & \int_\Omega \Bigg\{ (1 - \frac{S_j^*}{S_j})[D_1 \Delta S_j + A_j - (d+m)S_j - \varepsilon(\beta S_j I + \beta_1 S_j W)] \\ & + (1 - \frac{S_a^*}{S_a})[D_2 \Delta S_a + mS_j - \beta S_a I - \beta_1 S_a W - (d+\theta)S_a + \xi V] \\ & + (1 - \frac{V^*}{V})[D_3 \Delta V + \theta S_a - \eta\beta VI - \eta\beta_1 VW - (d+\xi)V] \end{aligned}$$

$$
\begin{aligned}
&+ (1-\frac{I^*}{I})[D_4\Delta I + \varepsilon(\beta S_j I + \beta_1 S_j W) + \beta S_a I + \beta_1 S_a W \\
&+ \eta\beta VI + \eta\beta_1 VW - (d+\alpha)I] \\
&+ \frac{\varepsilon\beta_1 S_j^* W^* + \beta_1 S_a^* W^* + \eta\beta_1 V^* W^*}{\mu+\delta}(1-\frac{W^*}{W})[kI - (\mu+\delta)W]\Big\}\mathrm{d}x \\
= &\int_\Omega \Big\{ -D_1 S_j^* \frac{||\nabla S_j||^2}{S_j^2} - D_2 S_a^* \frac{||\nabla S_a||^2}{S_a^2} - D_3 V^* \frac{||\nabla V||^2}{V^2} - D_4 I^* \frac{||\nabla I||^2}{I^2} \\
&- \frac{\varepsilon\beta_1 S_j^* W^* + \beta_1 S_a^* W^* + \eta\beta_1 V^* W^*}{\mu+\delta} D_5 W^* \frac{||\nabla W||^2}{W^2} \\
&+ \varepsilon\beta_1 S_j^* W^*(3 - \frac{S_j^*}{S_j} - \frac{IW^*}{I^* W} - \frac{S_j W I^*}{S_j^* W * I}) + (dS_a^* + \beta S_a^* I^*)(3 - \frac{S_j^*}{S_j} - \frac{S_a}{S_a^*} - \frac{S_j S_a^*}{S_a S_j^*}) \\
&+ (dV^* + \eta\beta V^* I^*)(4 - \frac{S_j^*}{S_j} - \frac{S_j S_a^*}{S_j^* S_a} - \frac{S_a V^*}{S_a^* V} - \frac{V}{V^*}) \\
&+ \xi V^*(2 - \frac{S_a V^*}{S_a^* V} - \frac{S_a^* V}{S_a V^*}) + (dS_j^* + \varepsilon\beta S_j^* I^*)(2 - \frac{S_j}{S_j^*} - \frac{S_j^*}{S_j}) \\
&+ \beta_1 S_a^* W^*(4 - \frac{S_j^*}{S_j} - \frac{S_j S_a^*}{S_j^* S_a} - \frac{IW^*}{I^* W} - \frac{S_a W I^*}{S_a^* W^* I}) \\
&+ \eta\beta_1 V^* W^*(5 - \frac{S_j^*}{S_j} - \frac{S_j S_a^*}{S_j^* S_a} - \frac{S_a V^*}{S_a^* V} - \frac{IW^*}{I^* W} - \frac{VWI^*}{V^* W^* I})\Big\}\mathrm{d}x \le 0.
\end{aligned}
$$

Thus, we know that $\frac{\mathrm{d}L_2(t)}{\mathrm{d}t} \le 0$. Furthermore, we know that $\frac{\mathrm{d}L_2(t)}{\mathrm{d}t} = 0$ if and only if $S_j = S_j^*$, $S_a = S_a^*$, $V = V^*$, $I = I^*$ and $W = W^*$. Thus, by LaSalle's invariable principle, it is clear that endemic equilibrium P^* is globally asymptotically stable. This completes the proof. □

6 Numerical simulations

In this section, we mainly give two examples to verify our theoretical results for the spatial heterogeneous model (1.1) and the spatial homogeneous model (5.1).

Example 1 In order to verify Theorem 4.1, we set the diffusion rate $D_2(x) = D_3(x) = 0.4 \times (1 + 0.05\sin(2\pi x))$, $A_j(x) = 160 \times (1 + 0.05\sin(2\pi x))$, $\beta_1(x) = 1.2 \times 10^{-5}(1 + 0.5\sin(2\pi x))$, $\alpha(x) = 0.2 \times (1 + 0.05\sin(2\pi x))$ and the other parameters in model (1.1) as follows:

$$
\begin{aligned}
m(x) &= 0.8 \times (1 + 0.05\sin(2\pi x)), d(x) = 0.00274 \times (1 + 0.05\sin(2\pi x)), \varepsilon(x) \\
&= 0.5 \times (1 + 0.04\sin(2\pi x)), \\
\beta(x) &= 1.8 \times 10^{-5}(1 + 0.5\sin(2\pi x)), \mu(x) = 2.3 \times (1 + 0.04\sin(2\pi x)), \theta(x) \\
&= 0.8 \times (1 + 0.05\sin(2\pi x)), \\
\eta(x) &= 0.1 \times (1 + 0.05\sin(2\pi x)), \xi(x) = 0.056 \times (1 + 0.05\sin(2\pi x)), k(x) \\
&= 2 \times (1 + 0.05\sin(2\pi x)), \\
\delta(x) &= 8 \times (1 + 0.05\sin(2\pi x)), D_1(x) = 0.32 \times (1 + 0.05\sin(2\pi x)), D_4(x) \\
&= 0.4 \times (1 + 0.05\sin(2\pi x)).
\end{aligned}
$$

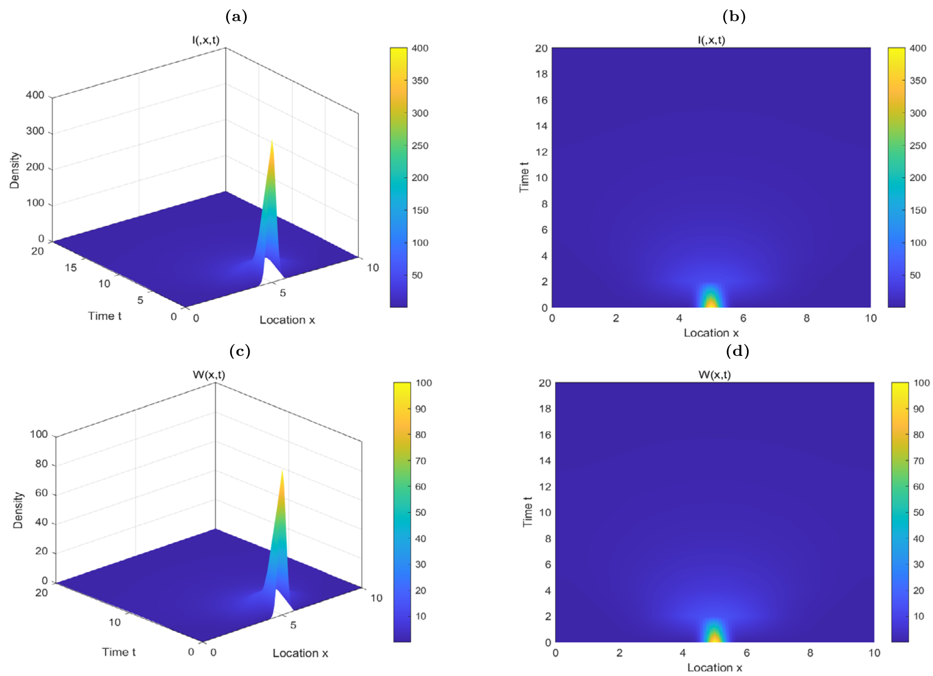

Figure 1 The global asymptotically stability of the disease-free steady state $U_0(x) = (S_j^0(x), S_a^0(x), V^0(x), 0, 0)$ with initial values $S_j(0,x) = 320 \times e^{-10(x-5)^2}$, $S_a(0,x) = 32{,}000 \times e^{-10(x-5)^2}$, $V(0,x) = 500 \times e^{-10(x-5)^2}$, $I(0,x) = 400 \times e^{-10(x-5)^2}$, $W(0,x) = 100 \times e^{-10(x-5)^2}$

Using above parameters, we can easily calculate the basic reproduction number $\mathcal{R}_0 \approx 0.9535 < 1$, and the corresponding spatiotemporal evolution solution trajectory is shown in the Fig. 1. From Fig. 1, we can find that the disease-free steady state $U_0(x)$ is globally asymptotically stable, which is consist with the conclusion in Theorem 4.1.

In addition, although we have theoretically demonstrated that there at least exits one endemic steady-state in spatially heterogeneous model, we can only provide its global asymptotic stability in heterogeneous diffusion spatially homogeneous model. Therefore, we speculate that the endemic steady-state solution in spatially heterogeneous models is unique and globally asymptotically stable. Therefore, we intend to show the spatiotemporal evolution behavior of the solution under spatially heterogeneous environment with the help of Matlab.

Example 2 Keep the other parameters unchanged, we just change:

$$\beta(x) = 5.8 \times 10^{-5}(1 + 0.5\sin(2\pi x)), \beta_1(x) = 4.2 \times 10^{-5}(1 + 0.5\sin(2\pi x)),$$

Using above parameters, we can easily calculate the basic reproduction number $\mathcal{R}_0 \approx 3.1027 < 1$, and the corresponding spatiotemporal evolution solution trajectory is shown in the Fig. 2. From Fig. 2, we can find that the endemic steady state $P^*(x)$ is globally asymptotically stable.

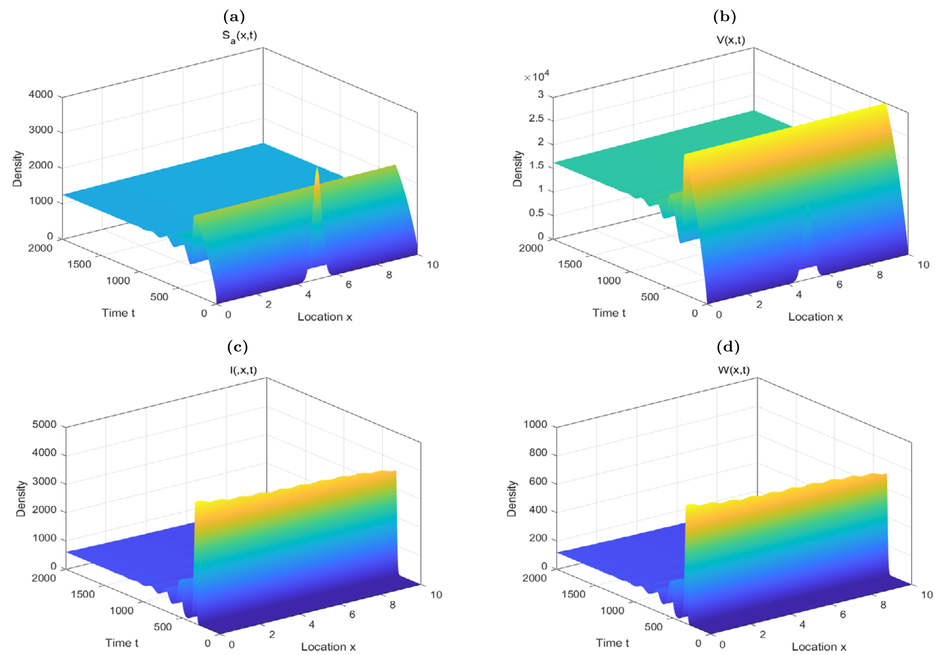

Figure 2 The global asymptotically stability of the disease-free steady state $P^*(x)$ with initial values $S_j(0,x) = 320 \times e^{-10(x-5)^2}$, $S_a(0,x) = 32{,}000 \times e^{-10(x-5)^2}$, $V(0,x) = 500 \times e^{-10(x-5)^2}$, $I(0,x) = 400 \times e^{-10(x-5)^2}$, $W(0,x) = 100 \times e^{-10(x-5)^2}$

7 Conclusion

In this paper, we investigate the global dynamics of a reaction-diffusion brucellosis model with partial immunity and stage structure in heterogeneous environment. Firstly, we obtain the global well-posedness and dissipativity of the model by the semigroup theory. Then, we discuss the existence of global compact attractor. Due to the facts that there are no diffusion terms in the last equation of model (1.1), the asymptotic smoothness of solution map $\Phi(t)$ instead of weak compactness cannot be obtained. We identify the κ-contraction condition by following the procedures in [22, Theorem 3]. Moreover, the basic reproduction number R_0 is defined as the spectral radius of the next infection operator that determines the extinction and persistence of disease. As a whole, the following is our main threshold dynamics results:

(a) Extinction: we obtain the global asymptotically stable of the disease-free steady state for spatially heterogeneous model in the following two conditions: (A) the vaccine failure rate $\xi(x) = 0$, (B) the diffusion rate $D_2(x) = D_3(x)$ is same. That is to say, if $R_0 < 1$ and condition (A) or (B) hold, the unique disease-free steady state is global asymptotically stable and there is no endemic steady state, which demonstrates the disease is extinction.

(b) Persistence: if $R_0 > 1$, the disease is persistent and there is at least one endemic steady state, which need neither (A) nor (B) hold. Moreover, we investigate the homogeneous space and heterogeneous diffusion model (all parameters are constants except the diffusion rate $D_i(x)$) and further obtain that when $R_0 \le 1$ and (A) or (B) hold, the unique disease-free steady state is global asymptotically stable and when $R_0 > 1$, the disease is persistent and there is at least one endemic steady state, which need neither (A) nor (B) hold. In particular, for the homogeneous space and heterogeneous diffusion model, the endemic steady state is global asymptotically stable when $R_0 > 1$.

Author contributions
All authors contributed to the study conception and design. All authors read and approved the final manuscript.

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Data availability
The manuscript has no associated data.

Declarations

Competing interests
The authors declare no competing interests.

Author details
[1]College of Medical Engineering and Technology, Xinjiang Medical University, Urumqi 830017, P.R. China. [2]Institute of Medical Engineering Interdisciplinary Research, Xinjiang Medical University, Urumqi 830017, P.R. China. [3]College of Mathematics and System Science, Xinjiang University, Urumqi 830017, P.R. China. [4]Xinjiang Key Laboratory of Applied Mathematics, Urumqi 830017, P.R. China.

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